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On Normal Subgroups Lattice of Dihedral Group

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ABSTRACT

In this paper, we obtain subgroup and normal subgroup lattices of dihedral group D_{2p}. we also give the Hasse diagrams of these lattices of D_{2p}.

MSC..

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1. Introduction

The subgroups lattice of a group G is the lattice whose elements are the subgroups of G, with the partially ordered relation being set inclusion. In this lattice, the join of two subgroups is the subgroup generated by their union, and, the meet of two subgroups is their intersection.

Lattice theoretic information a propos the lattice of subgroups can be used to infer information about the original group. The origin of the subject may be traced back to Dedekind. But real history of

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the theory of subgroup lattices began in 1928 with a paper of Ada Rottländer. Subsequently many group theorists have worked in this field, most notably Baer, Sadovskii, Suzuki, and Zacher [1, 2, 3].

A dihedral group is the group of symmetries of a regular polygon, including both rotations and reflections. Dihedral groups are among the simplest examples of finite groups, and they play an important role in group theory, geometry, and chemistry. The number of subgroups of a Dihedral group $\boldsymbol{D_n}$ is given by the formula, $\boldsymbol{d(n)+\sigma(n)}$, where $\boldsymbol{d(n)}$ is the number of divisors of $\boldsymbol{n}$ and $\boldsymbol{\sigma(n)}$ is the sum of the divisors of $\boldsymbol{n}$ [4, 5, 6]. Several papers have treated with subgroup lattices. Accordingly in [7] discussed subgroup lattice in symmetric group S4, while [8] studied subgroup lattice of quasidihedral group. In particular, they seem to have structures that can be described by Hasse diagrams. As well as j. Alperin, D. J. Benson and J. H. Conway were examining Hasse diagrams is some detail, see for instance [9, 10].

Therefore, we examine in this paper the subgroups lattice and the normal subgroups lattice of the dihedral group also we depict it graphically by means of the Hasse diagramas as we mentioned in the abstract.

2. Preliminaries

We recall some definitions and results that will be used latter.

Definition **2.1:[10]** Let n be a positive integer greater than or equal 3. The group of all symmetries of the regular polygon with n sides, including both rotations and reflections, is called a dihedral group and denoted by D_n. The set of rotations is generated by r-counter clockwise rotation with angle $2\pi/n$ of order n, and every element sr^j in the set of reflections is of order 2 and generates the subgroup $\{1, sr^j\}$,where 1 is the identity element in D_n. The $2n$ elements in D_n can be written as $\{1, r, r^2, \dots, r^{n-1}, s, sr, \dots, sr^{n-1}\}$. In general, $D_n = \{ s^j r^k : 0 \le k \le n-1,\ 0 \le j \le 1\}$ which has the properties $r^n = 1$, $sr^k s = r^{-k}$, $(sr^k)^2 = 1$, for all $0 \le k \le n-1$. The composition of two elements of D_n is given by :

$r^i r^k = r^{i+k}, r^i sr^k = sr^{k-i}, sr^i r^k = sr^{i+k}, sr^i sr^k = r^{k-i}$, for all $0 \le i, k \le n-1$.

Theorem **2.2:**(Lagrange)**[3]** The order and index of any subgroup of a finite group divides the order of the group.

***Corollary* 2.3:[3]** If G is a group of order n, then the order of any element $a \in G$ is a factor of n.

***Definition* 2.4:[11]** A subgroup H of a group G is said to be a normal subgroup of G if for every $a \in G$ and $h \in H, aha^{-1} \in G$.

***Definition* 2.5:[6]** A finite group of order p^r, where p is a prime and r is a positive integer is called a p-group.

***Definition* 2.6:[6]** Let G be a finite group, p is a prime and p^r is the highest power of p dividing order G. Then a subgroup H of order p^r is called a sylow p −subgroup of G.

***Theorem* 2.7:*(Sylow)*[8]** If G is a finite group of order $p^n q$, where $n \geq 1$ and p does not divide q, then G contains a subgroup of order p^i for each i, where $1 < i < n$ and every subgroup of order p^i is a normal subgroup of order p^{i+1}, for all $1 < i < n$.

***Theorem* 2.8:*(Sylow)*[6]** If p divides the order of the group G, then the number of sylow p −subgroups is congruent 1 modulo p and divides the order of G.

***Theorem* 2.9:[9]** There is a unique sylow p −subgroup of a finite group G if and only if it is normal.

***Definition* 2.10:[15]** A partial ordered on a nonempty set W is a binary relation $\leq$ on W that is reflexive, antisymmetric and transitive. The pair $< W, \leq >$ is called a partially ordered set or poset. A poset $< W, \leq >$ is totally ordered if every $x, y \in W$ are comparable, that is $x \leq y$ or $y \leq x$. A nonempty subset V of W is a chain in W if V is totally ordered set by $\leq$.

***Definition* 2.11:[15]** Let $< W, \leq >$ be a poset and let $V \subseteq W$. An upper bound for V is an element $x \in W$ for which $v \leq x, \forall v \in V$. The least upper bound of V is called the supremum or join of V. A lower bound for V is an element $x \in W$ for which $x \leq v,\ \forall v \in V$. The greatest lower bound of V is called the infimum or meet of V. A Poset $< W, \leq >$is called a lattice if every pair x, y elements of W has a supremum and an infimum.

Note that the set of all of subgroups of the group G under the partially ordered relation set inclusion is a lattice. This lattice is called the subgroups lattice of the group G.

***Definition* 2.12:[6]** A Hasse diagram is the graphical representation of a poset. The elements of the poset are represented by points and the relation between elements, if there be any, is represented by a segment joining them.

3. Main Results

In this section, we study the subgroups lattice and normal subgroups lattice and normal subgroup lattice of the dihedral group.

3.1. The subgroups lattice of D_{2p}.

The order of the group D_{2p} is 4p. The elements their orders and the number of elements that have the same order of the dihedral group D_{2p} are shown in the following table.

Elements	Order	Number of elements
1	1	1
r^p,s,sr,..., sr^{2p-1}	2	2p+1
r^2, r^4, ..., r^{2p-2}	p	p-1
r, r^3,...,r^{p-2}, r^{p+2},r^{p+4}, r^{2p-1}	$2p$	p-1

The number of subgroups of the dihedral group D_{2p} equal $d(2p)+\sigma(2p)=4+1+2+p+2p=7+3p$.

According to the Theorem 2.2, nontrivial subgroups of D_{2p} have orders $1,2,4,p,2p$, and $4p$. The subgroup of D_{2p} of order 1 is the trivial subgroup $p_1=\{1\}$.

Subgroups of order 2**:**

Let W be an arbitrary subgroup of D_{2p} of order 2. Then W is cyclic Since 2 is a prime number. Therefore, W must contain an element of D_{2p} of order 2. Thus, there are $2p+1$ subgroups of D_{2p} of order 2, these subgroups are $W_1=\{1,r^p\}$, $W_2=\{1,s\}$, $W_3=\{1,sr\}$, $W_{2p+1}=\{1,sr^{2p-1}\}$.

Subgroups of order p**:**

The unique subgroup of D_{2p} of order p is $Z_1=\{1,r^2,r^4,\dots,r^{2p-2}\}$, so it contains all elements of $D_{2p}$of order p.

Subgroups of order 4**:**

If E is a subgroup of D_{2p} of order 4, then by Corollary 2.3 the elements of E must have order 1, 2 or 4. But there are no elements of order 4, then three elements of E of order 2. Since $r^p sr^k=sr^{k-p}=sr^{k+p}$ [since $r^p=r^{-p}$], $r^p sr^{p+k}=sr^{p+k-p}=sr^k$, $sr^k r^p=sr^{p+k}$, and $sr^{p+k}r^p=sr^{2p+k}=sr^k$ [since $r^{2p}=1$], for all $0\le k\le 2p-1$. Thus there are p subgroups of D_{2p} of order 4. These subgroups are $E_1=\{1,r^p,s,sr^p\}$, $E_2=\{1,r^p,sr,sr^{p+1}\}$, …, $E_p=\{1,r^p,sr^{p-1},sr^{2p-1}\}$.

Subgroups of order $2p$**:**

The subgroups $Q_1 = \{1, r, r^2, \dots, r^{2p-1}\}$, $Q_2 = \{1, r^2, r^4, \dots, r^{2p-2}, s, sr^2, \dots, sr^{2p-2}\}$, and $Q_3 = \{1, r^2, r^4, \dots, r^{2p-2}, sr, sr^3, \dots, sr^{2p-1}\}$ are the subgroups of order $2p$.

Subgroups of order $4p$**:**

The whole group is the only subgroup of order $4p$.

Therefore, the Hasse diagram of subgroups lattice of D_{2p} is given in Fig.1.

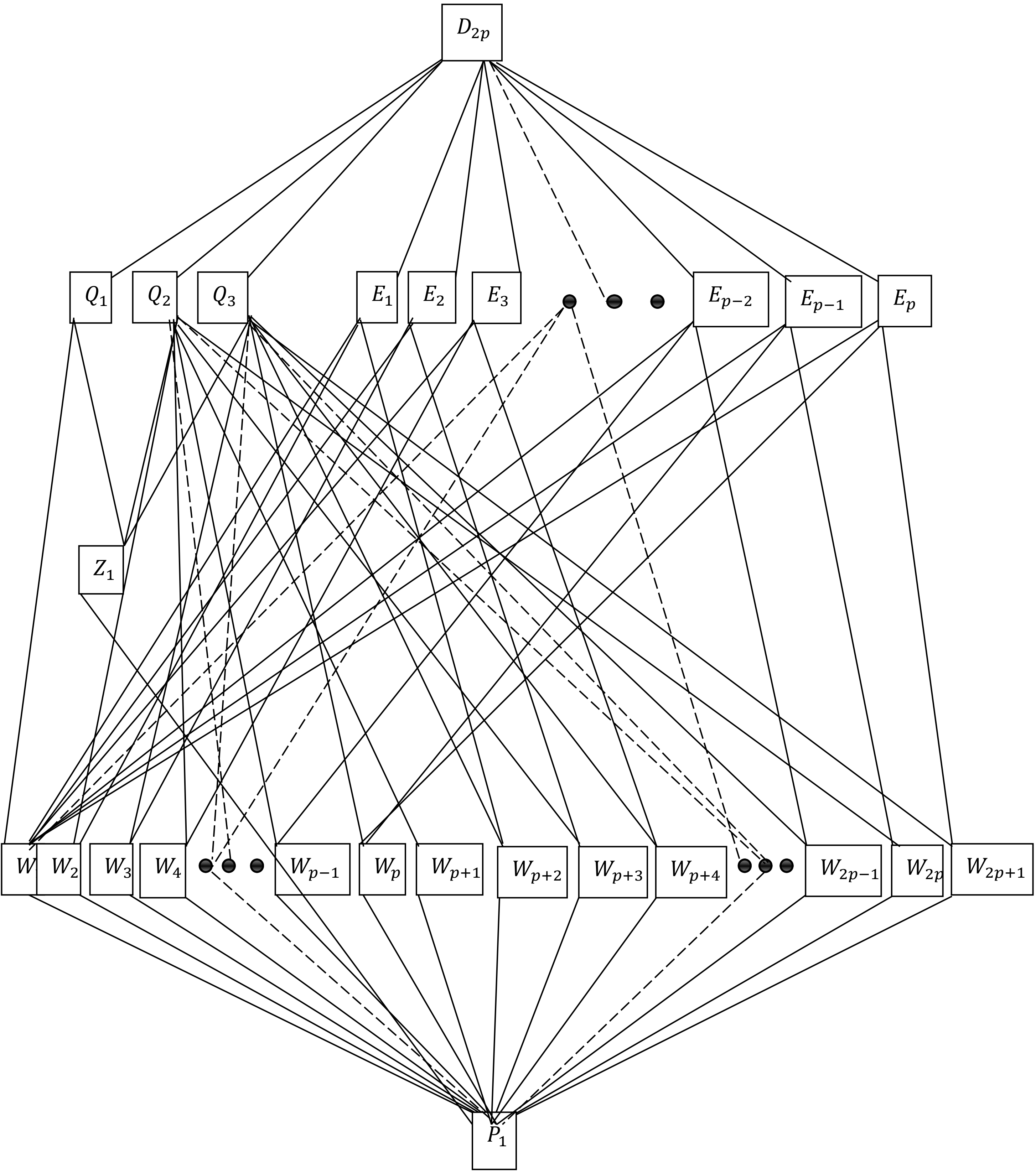

Fig. 1- The Hasse diagram of subgroups Lattice of the group D_{2p}.

3.2. The Normal subgroups lattice of D_{2p}.

The trivial subgroups p_1 and the whole group D_{2p} are normal subgroups of D_{2p}. Among the subgroups of order 2, the subgroup W_1 is a normal subgroups of D_{2p}, since $r(sr^k)r^{-1} = sr^{k-2}$, for all $0 \leq k \leq 2p - 1$, therefore the subgroups $W_2, W_3, \dots, W_{2p+1}$ of D_{2p} are not normal. According to the Theorem 2.8 and Theorem 2.9 the subgroup Z_1 of order p is normal. But the subgroups E_1, E_2, …, E_p are not normal. Finally, since the index of each subgroup of order $2p$ $(Q_1, Q_2, and\ Q_3)$ is 2, then they are normal subgroups of D_{2p}. So, according to these results, the Hasse diagram of Norml subgroups lattice of D_{2p} is given as in Fig. 2.

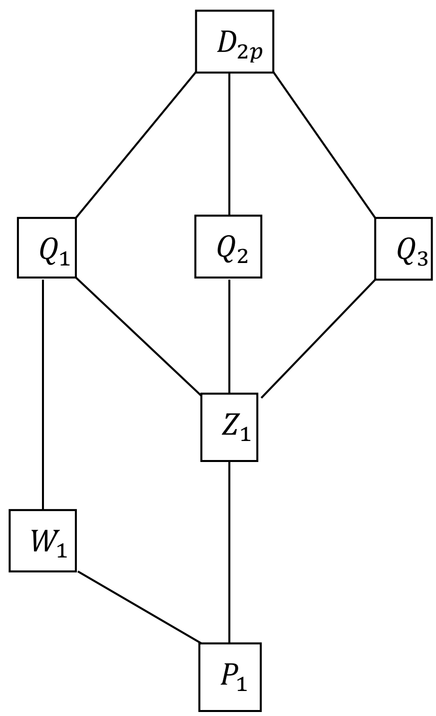

Fig. 2- The Hasse diagram of normal subgroups Lattice of the group D_{2p}.

3.3 Example.

a. The subgroups lattice of D_6, thus $D_6 = \{1, r, r^2, r^3, r^4, r^5, s, sr, sr^2, sr^3, sr^4, sr^5\}$ with relations $r^6 = 1,\ s^2 = 1, srs = r^{-1}$, so
$1 = (1), r = (123456), r^2 = (135)(246),\ r^3 = (14)(25)(36), r^4 = (153)(264),\ r^5 = (165432),\ s = (26)(35),\ sr = (13)(46), sr^2 = (15)(24), sr^3 = (12)(36)(45),\ sr^4 = (14)(23)(56), sr^5 = (16)(25)(34)$. The order of elements of D_6 are shown as follows.

Elements	**Order**
1	1
$r^3, s, sr, sr^2, sr^3, sr^4, sr^5$	2
$r^2,\ r^4$	3
$r,\ r^5$	6

According to the Theorem (2.2), nontrivial subgroups of D_6 must have order 1, 2, 3, 4, 6, 12. obviously, the subgroup of D_6 of order 1 is the trivial subgroup $P_1 = \{1\}$. And the Subgroups of D_6 of order 2 are $W_1 = \{1, r^3\}, W_2 = \{1, s\}, W_3 = \{1, sr\}, W_4 = \{1, sr^2\}, W_5 = \{1, sr^3\}, W_6 = \{1, sr^4\}$, $W_7 = \{1, sr^5\}$, so there is only one subgroup of D_6 of order 3 is $Z_1 = \{1, r^2, r^4\}$, and we get the subgroups of order 4 containing r^3 are $E_1 = \{1, r^3, s, sr^3\}$, $E_2 = \{1, r^3, sr, sr^4\}$ and $E_3 = \{1, r^3, sr^2, sr^5\}$, so that gives one subgroup of order 6: $Q_1 = \{1, r, r^2, r^3, r^4, r^5\}$. So there exist others subgroups contain two elements of order 3 and three elements of order 2. These are the only possibilities: $Q_2 = \{1, r^2, r^4, s, sr^2, sr^4\}$, $Q_3 = \{1, r^2, r^4, sr, sr^3, sr^5\}$. Finally The whole group is the only subgroup of order 12.

According to this result, we have the Hasse diagram of subgroups lattice of D_6 is shown as Fig. 3 below.

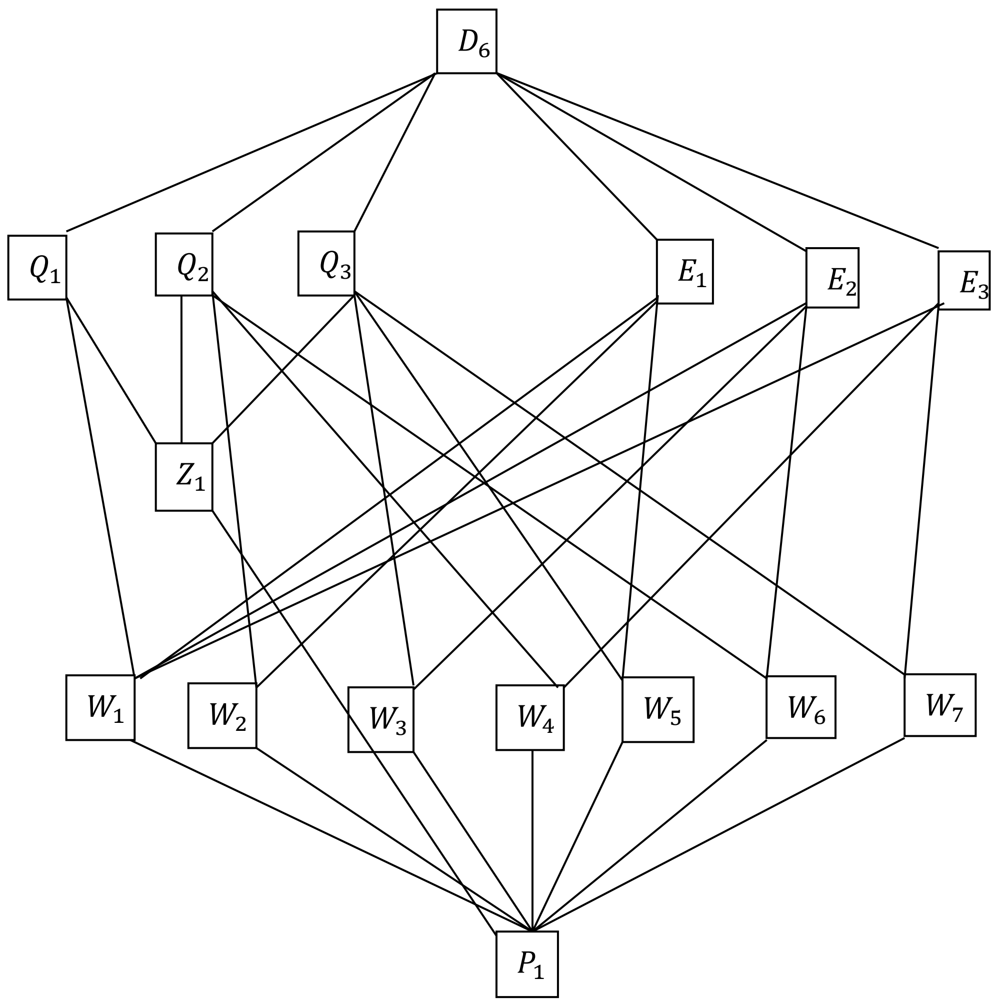

Fig. 3- The Hasse diagram of subgroups Lattice of D_6.

b. The normal subgroups lattice of D_6 are P_1 , W_1, Q_1, Q_2, Q_3 and the whole group D_6 . Hence, we have the Hasse diagram of normal subgroups lattice of D_6 is shown as Fig. 4.

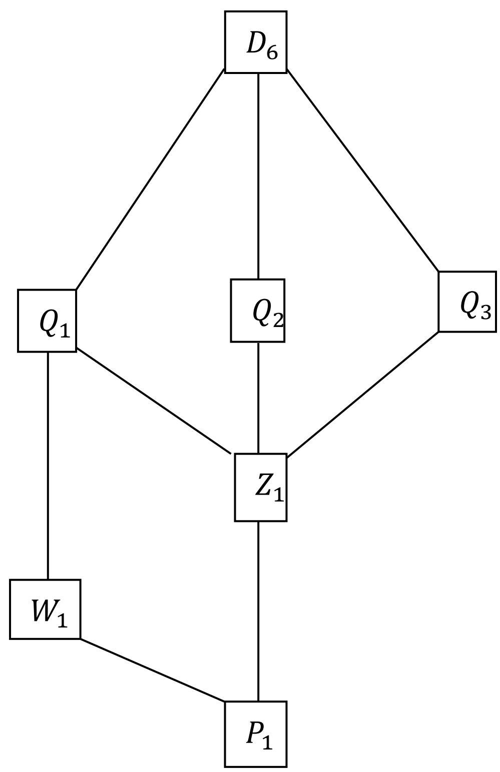

Fig. 4- The Hasse diagram of normal subgroups Lattice of the group D_6.

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