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An algorithm for a modified computation of Dedekind sums

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ABSTRACT

The purpose of this paper is to present an algorithm that compute Dedekind sums *s(a, b) .This algorithm implemented by using essential definition of Dedekind sum .since* these sums arise in the theory of modular forms and in number theory as well as in various other areas of mathematics. These sums are rational numbers and a table of the exact values is given. Some applications for the important of Dedekind sums were presented.

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1. . Introduction

The look of Dedekind sums in a huge range of geometry and number theory are explored and the hopes of illuminating new connections. Dedekind sums are traditional topics of examine that familiarized by using Richard .Dedekind inside the nineteenth century in his research of the η-function [1]. Dedekind sums emerge in lots of fields of mathematic, most unmistakably in combinatorial geometry (lattice point account in polytope) used lattice point enumeration to analyze the enumerative geometry of polytopes, analytic number theory (modular forms) , algorithmic complexity (pseudo-random wide variety turbines) ,topology (signature deformities of manifolds), and algebraic number theory (class numbers)[2,3,4,5] have been investigated.. In [6], Jabuka *et al.* raise the question of when two Dedekind sums$\mathfrak{s}(a_1, b)$ and $\mathfrak{s}(a_2, b)$ are equal. In the same paper, they prove the necessary condition for equality of two Dedekind sums . Girstmair [7] shows that this condition is equivalent to 12$\mathfrak{s}(a_1, b)$−12 $\mathfrak{s}(a_2, b)$∈ Z.. In [8] a new approach for the equivalence of the essential and sufficient condition and it is connected by the equality of two Dedekind sums. Also we can apply the equality of Dedekind sums for large polytope in dilated n^{th}polytope when we need to compute the volume and lattice point of P. To define the Dedekind sums, let

$$((x)) = \begin{cases} x - \lfloor x \rfloor - \frac{1}{2} & if \quad x \notin z \\ 0 & otherwise \end{cases}$$

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In which manner $\lfloor x \rfloor$ the greatest integer less than or equal to x, and it is called sawtooth function. For $a,\in N$ is coprime the Dedekind sum that denoted with $\mathfrak{s}(a,b)$ is defined by means of:

$$\mathfrak{s}(a,b) = \sum_{k \bmod b} \left(\left(\frac{ak}{b}\right)\right)\left(\left(\frac{k}{b}\right)\right).$$

2. Basic concepts

Some basic standards that pertains to the Dedekind sums and properties are given, which can be [1][2]:

Let a and b be coprime, then we've the following residences

i. The Dedekind sums is a periodic function of the variable a, with period b, via periodicity of the Sawtooth. Function. That is,.$\mathfrak{s}(a+kb,b) = \mathfrak{s}(a,b)$ $\quad for\ all\ k$ in Z.

ii. $\mathfrak{s}(1,b) = \frac{-1}{4} + \frac{1}{6b} + \frac{b}{12}$.

iii. $\mathfrak{s}(na,nb) = \mathfrak{s}(a,b)$

iv. $\mathfrak{s}(-a,b) = -\mathfrak{s}(a,b)$

v. $a_1 \equiv a_2 \bmod b$ then $\mathfrak{s}(a_1,b) = \mathfrak{s}(a_2,b)$

vi. $a_1a_2 \equiv 1 \bmod b$ then $\mathfrak{s}(a_1,b) = \mathfrak{s}(a_2,b)$

vii. The most exquisite outcome concerning the Dedekind sum is the famous I reciprocity formulation:

$$\mathfrak{s}(a,b) + \mathfrak{s}(b,a) = \frac{1}{12}\left(\frac{a}{b} + \frac{b}{a} + \frac{1}{ab}\right) - \frac{1}{4}.$$

Consequently, reciprocity formula permits us to calculate s(a,b) in polynomial time analogous in spirit to the Euclidean algorithm..

Example(1) [2]:

Let a= 100 and b = 147, now alternately the usage of the reciprocity method and some properties of Dedekind sums: s(100,147) =577/882. For more information see[2]

A priori, to calculate s(100,147), it takes 147 steps, where as it is able to be computed with 9 step through the reciprocity formulation. This relation will also be quite useful to the reader who wishes values of $\mathfrak{s}(a,b)$ not in the table 1. For example, if 1 <a<10 then $\mathfrak{s}(a,b)$ can be obtained for any b >0 by at most one application of the reciprocity formula.

3. An Algorithm for Computing Dedekind Sums:

To compute the Dedekind sums for a coprime a to b. Algorithm 1 is designed for this purpose

Algorithm 1: (Compute Dedekind sums $\mathfrak{s}(a,b)$)

Input: positive integer numbers a,b where a is coprime to b

Output: Dedekind sums $\mathfrak{s}(a,b)$

For $k = 1$ to b

If $mod\left((\frac{a*k}{b}),1\right) <> 0$ and $mod\left((\frac{k}{b}),1\right) <> 0$

$$\mathfrak{s} = \mathfrak{s} + \left(\left(\frac{a*k}{b}\right) - \left\lfloor\frac{a*k}{b}\right\rfloor - 0.5\right) * \left(\left(\frac{a}{b}\right) - \left\lfloor\frac{a}{b}\right\rfloor - 0.5\right)$$

End if

To compute the Dedekind sums for all $a's$ coprime to b. An algorithm 2 was designed.

Algorithm 2: (Compute Dedekind sums for all $a's$ coprime to b)

Input: positive integer number b.

Output: a , Dedekind sums $\mathfrak{s}(a,b)$

For $i = 1\ to\ b$

If $g.c.d(i,b) = 1$

$a = i$

If a >= (b/2) go to 13

$\mathfrak{s} = 0$

For $k = 1\ to\ b$

If $b \leq (\frac{a}{2})$, then

If $mod\left((\frac{a*k}{b}),1\right) <> 0\ and\ mod\left((\frac{k}{b}),1\right) <> 0$

$$\mathfrak{s} = \mathfrak{s} + \left(\left(\frac{a*k}{b}\right) - \left\lfloor\frac{a*k}{b}\right\rfloor - 0.5\right) * (\left(\frac{a}{b}\right) - \left\lfloor\frac{a}{b}\right\rfloor - 0.5)$$

End if

End if

End for

End if

End if

End for

A MATLAB program is used to run those algorithms. After using a Sieve type method to eliminate from consideration those a except for $1 \leq a \leq [b/2]$ and $g.c.d.(a,b) = 1$. Dedekind sums $\mathfrak{s}(a,b)$ is computed in rational form by using definition of Dedekind sums [2].

The output of the proposed program within MATLAB language according to algorithm 2 is given the value of Dedekind sums in Table 1.

Table 1: Some values of Dedekind sums

a	b	$\mathfrak{s}(a,b)$
1	2	0/1
1	3	1/18

1	4	1/8
1	5	1/5
2	5	0/1
1	6	5/18
1	7	5/14
2	7	1/14
3	7	-1/14
1	8	7/16
3	8	1/16
1	9	14/27
2	9	4/27
4	9	-4/27
1	10	3/5
3	10	0/1

4. Application of Dedekind sum in geometric combinatorial

In this section, we are able to supply a formula for Earhart polynomial , that used to gain results about lattice polyhedra and Dedekind sums. These applications encompass a components for the number of lattice points in an arbitrary lattice tetrahedron[2].

Theorem(1)

Let P be a tetrahedron with vertices (0,0,0), $(a,0,0)$, $(0,b,0)$, $(0,0,c)$ and a, b, c are pairwise relatively prime. Then

$$L_P(t) =$$

$$\frac{abc}{6}t^3+\frac{ab+ac+bc+1}{4}t^2+\left(\frac{1}{12}\left(\frac{bc}{a}+\frac{ac}{b}+\frac{ab}{c}+\frac{1}{abc}\right)+\frac{a+b+c}{4}+\frac{3}{4}-\mathfrak{s}(bc,a)-\mathfrak{s}(ac,b)-\mathfrak{s}(ab,c)\right)t+1. \quad \text{... (1)}$$

Example (2):

To locate the Ehrhart polynomial for the tetrahedron, we use the components given by Theorem (1)and Algorithm(1). let us taken the tetrahedron with vertices (0,0,0), (5,0,0), (0,9,0) and (0,0,11)

Solution: Appling the formula (1) given in theorem (1) and by using the algorithm (1) of Dedekind sums which are

$\mathfrak{s}(99,5) = -\frac{1}{5}$, $\mathfrak{s}(55,9) = \frac{14}{27}$ and $\mathfrak{s}(45,11) = \frac{15}{22}$,

$L_P(t) = \frac{165}{2}t^3 + 50t^2 + \frac{17}{2}t + 1.$

The formula of Example2 is profitable, as it reduces the number of computation of the Dedekind sums, which may be carried out efficaciously. Indeed, recursively by way of making use of the reciprocity formula and the obvious identity

$$\mathfrak{s}(a+kb,b) = \mathfrak{s}(a,b), \qquad \text{for all } k \in Z.$$

5. Conclusion

According to the wide appearance of Dedekind sums in different applications on different subjects, for instance in geometric combinatorics, in particular enumeration of lattice point as shown in theorem1 therefore need to apply this algorithms for facilitate calculation of Dedekind sums. Also we can apply those algorithms and the equality of Dedekind sums for large polytope in dilated n^{th} polytope when we need to compute the volume and lattice point of polytope.

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