New Concept on Fourth-Order Differential Subordination and Superordination with Some Results for Multivalent Analytic Functions

Waggas Galib Atshan a, Ihsan Ali Abbas b, and Sibel Yalcin c

a Department of Mathematics, College of Science, University of Al-Qadisiyah, Diwaniyah, Iraq, Email: waggashnd@gmail.com, waggas.galib@qu.edu.iq
b Ministry of Education, Education of Al-Qadisiyah, Diwaniyah, Iraq. Email: ihsan.a.abbas@qu.edu.iq
c Department of Mathematics, Faculty of Arts and Science, University of Bursa Uludag, Bursa-Turkey. Email: syalcin@uludag.edu.tr.

ARTICLE INFO
Article history:
Received: 08/02/2020
Revised form:12/02/2020
Accepted: 05/03/2020
Available online: 18/03/2020

Keywords:
Differential subordination, Differential superordination, Multivalent function, Admissible function, Fourth-Order.

ABSTRACT
In this paper, we introduce new concept that is fourth-order differential subordination and superordination associated with differential linear operator \( I_p(n, \lambda) \) in open unit disk. Also, we obtain some new results.

MSC: 30C45, 30C50

DOI: https://doi.org/10.29304/jqcm.2020.12.1.681

1. Introduction

Let \( \mathcal{H}(U) \) be the class of functions which are analytic in the open unit disk \( U = \{ z : z \in \mathbb{C} : |z| < 1 \} \). For \( n \in \mathbb{N} = \{ 1, 2, 3, \ldots, \} \) and \( a \in \mathbb{C} \), let \( \mathcal{H}[a,n] = \{ f \in \mathcal{H}(U) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots \} \), and also let \( \mathcal{H}_1 = [1,1] \).

Let \( \Sigma_p \) denote the class of all analytic functions of the form:

\[
f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k. \tag{1.1}
\]

We consider a linear operator \( I_p(n, \lambda) \) on the class \( \Sigma_p \) of multivalent functions by the infinite series
\[ I_p(n, \lambda)f(z) = z^p + \sum_{k=p+1}^{\infty} \left( \frac{k + \lambda}{p + \lambda} \right)^n a_k z^k, \quad (\lambda > -p). \] 

(1.2)

The operator \( I_p(n, \lambda) \) was studied by [2]. It is easily verified from (1.2) that

\[ z[I_p(n, \lambda)f(z)]' = (p + \lambda)I_p(n + 1, \lambda)f(z) - \lambda I_p(n, \lambda). \] 

(1.3)

For several past years, there are many authors introduce and dealing with the theory of second-order differential subordination and superordination for example(\([1 - 3, 6, 8, 9, 10, 13, 16]\)), recently years, the many authors discussed the theory of third-order differential subordination and superordination for example(\([4, 5, 11, 12, 17, 18, 19]\)). In the present paper, we investigated to the fourth-order. In 2011, Antonino and Miller [4] extended the theory of second-order differential subordination in the open unit disk introduced by Miller and Mocanu [14] to the third-order case, now, we extend this to fourth-order differential subordination. They determined properties of functions \( p \) that satisfy the following fourth-order differential subordination:

\[ \{\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z), z^4p''''(z); z): z \in U\} \subset \Omega. \]

In 2014, Tang et al [19] extended the theory of second-order differential superordination in the open unit disk introduced by Miller and Mocanu [15] to third-order case, now we extend this to fourth-order differential superordination. They determined properties of functions \( p \) that satisfy the following fourth-order differential superordination:

\[ \Omega \subset \{\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z), z^4p''''(z); z): z \in U\}. \]

To prove our main results, we need the basic concepts in theory of the fourth-order.

**Definition 1.1.** [14]. Let \( f(z) \) and \( F(z) \) be members of the analytic function class \( \mathcal{H}(U) \). The function \( f(z) \) is said to be subordinate to \( F(z) \) or \( F(z) \) is superordinate to \( f(z) \) if there exists a Schwarz function \( w(z) \) analytic in \( U \) with \( w(0) = 0 \) and \( |w(z)| < 1 (z \in U) \), and such that \( f(z) = F(w(z)) \). In such case, we write \( f < F \), or \( F(z) < F(z) \).

If \( F(z) \) is univalent in \( U \), then \( f(z) < F(z) \) if and only if \( f(0) = F(0) \) and \( f(U) \subset F(U) \).

**Definition 1.2.** [4]. Let \( Q \) denote the set of functions \( q \) that are analytic and univalent on the set \( U \setminus \mathcal{E}(q) \), where

\[ \mathcal{E}(q) = \{ \zeta: \zeta \in \partial U \text{ and } \lim_{z \to \zeta} q(z) = \infty \}, \]

is such that \( \min \{ \left| q'(\zeta) \right| \} = \rho > 0 \) for \( \zeta \in \partial U \setminus \mathcal{E}(q) \). Further let the subclass of \( Q \) for which \( q(0) = a \) be denoted by \( Q(a) \) and \( Q(1) = Q_1 \).

**Definition 1.3.** Let \( \psi: \mathbb{C}^5 \times U \to \mathbb{C} \) and the function \( h(z) \) be univalent in \( U \). If the function \( p(z) \) is analytic in \( U \) satisfies the following fourth-order differential subordination:

\[ \psi(p(z), zp'(z), z^2p''(z), z^3p'''(z), z^4p''''(z); z) < h(z), \] 

(1.4)

then \( p(z) \) is called a solution of the differential subordination. A univalent function \( q(z) \) is called a dominant of the solutions of the differential subordination or more simply a dominant if \( p(z) < q(z) \) for all \( p(z) \) satisfying (1.4). A dominant \( \bar{q}(z) \) that satisfies \( \bar{q}(z) < q(z) \) for all dominants \( q(z) \) of (1.4) is said to be the best dominant.

**Definition 1.4.** Let \( \Omega \) be a set in \( \mathbb{C}, q \in Q \) and \( n \in \mathbb{N}\setminus\{2\} \). The class of admissible functions \( \Psi_n[\Omega, q] \) consists of those functions \( \psi: \mathbb{C}^5 \times U \to \mathbb{C} \) that satisfy the following admissibility condition:

\[ \psi(r, s, t, w, b; z) \notin \Omega, \]
whenever

\[ r = q(\zeta), \quad s = \kappa q'(\zeta), \quad \Re \left( \frac{t}{s} + 1 \right) \geq \kappa \Re \left( \frac{\zeta q''(\zeta)}{q'(\zeta)} + 1 \right), \]

and

\[ \Re \left( \frac{w}{s} \right) \geq \kappa^2 \Re \left( \frac{\zeta^2 q'''(\zeta)}{q'(\zeta)} \right), \quad \Re \left( \frac{b}{s} \right) \geq \kappa^3 \Re \left( \frac{\zeta^3 q''''(\zeta)}{q'(\zeta)} \right), \]

where \( z \in U, \zeta \in \partial U \backslash E(q) \), and \( \kappa \geq n \).

**Theorem 1.5.**[7] Let \( p \in \mathcal{H}[a, n] \) with \( n \in \mathbb{N} \backslash \{2\} \). Also, let \( q \in Q(a) \) and satisfy the following conditions:

\[ \Re \left( \frac{\zeta^2 q'''(\zeta)}{q'(\zeta)} \right) \geq 0, \quad \left| z^2 p''(z) \right| \leq \kappa^2, \]

where \( z \in U \), \( \zeta \in \partial U \backslash E(q) \) and \( \kappa \geq n \). If \( \Omega \) a set in \( \mathbb{C} \), \( \psi \in \Psi_n[\Omega, q] \) and

\[ \psi(p(z), zp'(z), z^2 p''(z), z^3 p'''(z), z^4 p''''(z); z) \in \Omega, \]

then \( p(z) < q(z), \quad (z \in U) \).

**Definition 1.6.** Let \( \psi : \mathbb{C}^5 \times U \rightarrow \mathbb{C} \) and the function \( h(z) \) be analytic in \( U \). If the functions \( p(z) \) and

\[ \psi(p(z), zp'(z), z^2 p''(z), z^3 p'''(z), z^4 p''''(z); z), \]

are univalent in \( U \) and satisfy the following fourth-order differential superordination:

\[ h(z) < \psi(p(z), zp'(z), z^2 p''(z), z^3 p'''(z), z^4 p''''(z); z), \quad (1.5) \]

then \( p(z) \) is called a solution of the differential superordination. An analytic function \( q(z) \) is called a subordinat of the solutions of the differential superordination or more simply a subordinat if \( q(z) < p(z) \) for all \( p(z) \) satisfying (1.5). A univalent subordinant \( \tilde{q}(z) \) that satisfies the condition \( q(z) < \tilde{q}(z) \) for all subordinants \( q(z) \) of (1.5) is said to be the best subordinat. We note that the best subordinant is unique up to a rotation of \( U \).

**Definition 1.7.** Let \( \Omega \) be a set in \( \mathbb{C} \), \( q(z) \in \mathcal{H}[a, n] \) and \( q'(z) \neq 0 \). The class of admissible functions \( \Psi_n[\Omega, q] \) consists of those functions \( \psi : \mathbb{C}^5 \times \overline{U} \rightarrow \mathbb{C} \) that satisfy the following admissibility condition:

\[ \psi(r, s, t, w, b; \zeta) \in \Omega, \]

whenever

\[ r = q(z), \quad s = \frac{zq'(z)}{m}, \quad \Re \left( \frac{t}{s} + 1 \right) \leq \frac{1}{m} \Re \left( \frac{z q''(z)}{q'(z)} + 1 \right), \]

and

\[ \Re \left( \frac{w}{s} \right) \leq \frac{1}{m^2} \Re \left( \frac{z^2 q'''(z)}{q'(z)} \right), \quad \Re \left( \frac{b}{s} \right) \leq \frac{1}{m^3} \Re \left( \frac{z^3 q''''(z)}{q'(z)} \right), \]

where \( z \in U, \zeta \in \partial U \), and \( m \geq n \geq 3 \).

**Theorem 1.8.**[7] Let \( q(z) \in \mathcal{H}[a, n] \) and \( \psi \in \Psi_n[\Omega, q] \). If
\[ \psi(p(z), zp'(z), z^2p''(z), z^3p'''(z), z^4p''''; z) \]

is univalent in \( U \) and \( p(z) \in Q(a) \) satisfy the following conditions:

\[ \Re \left( \frac{z^2 q'''(z)}{q'(z)} \right) \geq 0, \quad \left| \frac{z^2 p''(z)}{q'(z)} \right| \leq \frac{1}{m^2}, \]

where \( z \in U, \zeta \in \partial U \) and \( m \geq n \geq 3 \), then

\[ \Omega \subset \{ \psi(p(z), zp'(z), z^2p''(z), z^3p'''(z), z^4p''''; z); z \in U \} , \]

implies that

\[ q(z) < p(z), \quad (z \in U). \]

2. Fourth-Order Differential Subordination with \( I_p(n, \lambda) \)

We first define the following class of admissible functions, which are required in proving the differential subordination theorem involving the operator \( I_p(n, \lambda) \) defined by (1.2).

**Definition 2.1.** Let \( \Omega \) be a set in \( \mathbb{C} \), and let \( q \in Q_1 \cap H_1 \). The class of admissible functions \( \Phi_p[\Omega, q] \) consists of those functions \( \Phi : \mathbb{C} \times U \rightarrow \mathbb{C} \) that satisfy the following admissibility condition:

\[ \phi(u, v, x, y, g; z) \notin \Omega, \]

whenever

\[ u = q(z), \quad v = \frac{\kappa q'(z) + \lambda q(z)}{p + \lambda}, \quad \Re \left\{ \frac{(p + \lambda) x - \lambda^2 u}{(p + \lambda) v - \lambda u} - 2 \lambda \right\} \geq \kappa \Re \left\{ \frac{q''(z)}{q'(z)} + 1 \right\}, \]

\[ \Re \left\{ \frac{(p + \lambda)^2 [(p + \lambda) y - (3 \lambda + 3) x] + (2 \lambda^3 + 3 \lambda^2) u}{(p + \lambda) v - \lambda u} + (3 \lambda^2 + 6 \lambda + 2) \right\} \geq \kappa^2 \Re \left\{ \frac{z^2 q'''(z)}{q'(z)} \right\}, \]

and

\[ \Re \left\{ \frac{(p + \lambda)[(p + \lambda)^3 g - (p + \lambda)^2 (4 \lambda + 6) y + (p + \lambda) (8 \lambda^2 + 18 \lambda + 11) x}{(p + \lambda) v - \lambda u} \right. \right. \]

\[ \left. \left. \frac{-(8 \lambda^3 + 18 \lambda^2 + 22 \lambda + 6) u + (3 \lambda^4 + 6 \lambda^3 + 11 \lambda^2 + 6 \lambda) u}{(p + \lambda) v - \lambda u} \right\} \geq \kappa^3 \Re \left\{ \frac{z^3 q'''(z)}{q'(z)} \right\}, \]

where \( z \in U, \lambda > -p, \quad \zeta \in \partial U \setminus \mathbb{E}(q) \) and \( \kappa \geq 3 \).

**Theorem 2.2.** Let \( \phi \in \Phi_p[\Omega, q] \). If the functions \( f(z) \in \Sigma_p \) and \( q \in Q_1 \) satisfy the following conditions:

\[ \Re \left( \frac{\xi q'''(z)}{q'(z)} \right) \geq 0, \quad \left| L_p(n + 2, \lambda) f(z) \right| \leq \kappa^2, \quad (2.1) \]

and

\[ \phi\left( L_p(n, \lambda) f(z), L_p(n + 1, \lambda) f(z), L_p(n + 2, \lambda) f(z), L_p(n + 3, \lambda) f(z), L_p(n + 4, \lambda) f(z); z \in U \right) \subset \Omega, \quad (2.2) \]

then

\[ L_p(n, \lambda) f(z) < q(z), \quad (z \in U). \]

**Proof.** Define the analytic function \( p(z) \) in \( U \) by
\[ p(z) = I_p(n, \lambda) f(z). \] (2.3)

Then, differentiating (2.3) with respect to \( z \) and using (1.3), we have

\[ I_p(n + 1, \lambda) f(z) = \frac{zp'(z) + \lambda p(z)}{p + \lambda}. \] (2.4)

Further computations show that

\[ I_p(n + 2, \lambda) f(z) = \frac{z^2p''(z) + (2\lambda + 1)zp'(z) + \lambda^2 p(z)}{(p + \lambda)^2}, \] (2.5)

\[ I_p(n + 3, \lambda) f(z) = \frac{z^3p'''(z) + (3\lambda + 3)z^2p''(z) + (3\lambda^2 + 3\lambda + 1)zp'(z) + \lambda^3 p(z)}{(p + \lambda)^3}. \] (2.6)

and

\[ I_p(n + 4, \lambda) f(z) = \frac{z^4p''''(z) + (4\lambda + 6)z^3p'''(z) + (4\lambda^2 + 12\lambda + 7)z^2p''(z) + (4\lambda^3 + 4\lambda^2 + 4\lambda + 1)zp'(z) + \lambda^4 p(z)}{(p + \lambda)^4}. \] (2.7)

Define the transformation from \( \mathbb{C}^5 \) to \( \mathbb{C} \) by

\[
\begin{align*}
  u(r, s, t, w, b) &= r, \\
  v(r, s, t, w, b) &= \frac{s + \lambda r}{p + \lambda}, \\
  x(r, s, t, w, b) &= \frac{t + 2\lambda + 1)s + \lambda^2 r}{(p + \lambda)^2}, \\
  y(r, s, t, w, b) &= \frac{w + (3\lambda + 3)t + (3\lambda^2 + 3\lambda + 1)s + \lambda^3 r}{(p + \lambda)^3}, \\
  g(r, s, t, w, b) &= \frac{b + (4\lambda + 6)s + (4\lambda^2 + 12\lambda + 7)t + (4\lambda^3 + 4\lambda^2 + 4\lambda + 1)s + \lambda^4 r}{(p + \lambda)^4}.
\end{align*}
\] (2.8)

Let

\[
\psi(r, s, t, w, b; z) = \phi(u, v, x, y, g; z)
\]

\[
\phi \left( r, \frac{s + \lambda r}{p + \lambda}, \frac{t + 2\lambda + 1)s + \lambda^2 r}{(p + \lambda)^2}, \frac{w + (3\lambda + 3)t + (3\lambda^2 + 3\lambda + 1)s + \lambda^3 r}{(p + \lambda)^3}, \frac{b + (4\lambda + 6)s + (4\lambda^2 + 12\lambda + 7)t + (4\lambda^3 + 4\lambda^2 + 4\lambda + 1)s + \lambda^4 r}{(p + \lambda)^4}; z \right). \] (2.9)

The proof will make use of Theorem 1.5. Using equations (2.3) to (2.7), we have from (2.9) that

\[
\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z), z^4p''''(z); z) =
\]
\[ \phi(I_p(n, \lambda)f(z), I_p(n + 1, \lambda)f(z), I_p(n + 2, \lambda)f(z), I_p(n + 3, \lambda)f(z), I_p(n + 4, \lambda)f(z); z). \] (2.10)

Hence (2.2) becomes

\[ \psi(p(z), zp'(z), z^2p''(z), z^3p'''(z), z^4p''''(z); z) \in \Omega. \]

We note that

\[ \frac{t}{s} + 1 = \frac{(p + \lambda)^2x - \lambda^2u}{(p + \lambda)v - \lambda u} - 2\lambda, \]

and

\[ \frac{w}{s} = \frac{(p + \lambda)^2[(p + \lambda)y - (3\lambda + 3)x] + (2\lambda^3 + 3\lambda^2)u}{(p + \lambda)v - \lambda u} + (3\lambda^2 + 6\lambda + 2), \]

Therefore, the admissibility condition for \( \phi \in \Phi_q[\Omega, q] \) in Definition 2.1 is equivalent to the admissibility condition for \( \psi \in \Psi_q[\Omega, q] \) as given in Definition 1.4 with \( n = 3 \). Therefore, by using (2.1) and Theorem 1.5, we obtain

\[ p(z) = I_p(n, \lambda)f(z) < q(z). \]

The next Corollary is an extension of Theorem 2.2 to the case where the behavior of \( q(z) \) on \( \partial U \) is not known.

**Corollary 2.3.** Let \( \Omega \subset \mathbb{C} \), and let the function \( q(z) \) be univalent in \( U \) with \( q(0) = 1 \). Let \( \phi \in \Phi_q[\Omega, q] \) for some \( \rho \in (0,1) \), where \( q_\rho(z) = q(\rho z) \). If the function \( f(z) \in \Sigma_\rho \) and \( q_\rho(z) \) satisfy the following conditions:

\[ \Re \left( \frac{z^2q''''(z)}{q_\rho'(z)} \right) \geq 0, \quad \left| I_p(n + 2, \lambda)f(z) \right| \leq \kappa^2, \quad \left( z \in U, \zeta \in \partial U \setminus \mathbb{E}(q_\rho) \right) \] (2.11)

and

\[ \phi(I_p(n, \lambda)f(z), I_p(n + 1, \lambda)f(z), I_p(n + 2, \lambda)f(z), I_p(n + 3, \lambda)f(z), I_p(n + 4, \lambda)f(z); z) \in \Omega, \]

then

\[ I_p(n, \lambda)f(z) < q(z), \quad (z \in U). \]

**Proof.** By using Theorem 2.2, yields \( I_p(n, \lambda)f(z) < q_\rho(z) \). Then we obtain the result from \( q_\rho(z) < q(z), (z \in U) \). If \( \Omega \not\subset \mathbb{C} \) is a simply connected domain, then \( \Omega = h(U) \) for some conformal mapping \( h(z) \) of \( U \) onto \( \Omega \). In this case, the class \( \Phi_q[h(U), q] \) is written as \( \Phi_q[h, q] \). The following two results are immediate consequence of Theorem 2.2 and Corollary 2.3.

**Theorem 2.4.** Let \( \phi \in \Phi_q[h, q] \). If the function \( f \in \Sigma_q \) and \( q \in \Omega \) satisfy the condition (2.1) and

\[ \phi(I_p(n, \lambda)f(z), I_p(n + 1, \lambda)f(z), I_p(n + 2, \lambda)f(z), I_p(n + 3, \lambda)f(z), I_p(n + 4, \lambda)f(z); z) < h(z), \] (2.12)

then

\[ I_p(n, \lambda)f(z) < q(z), \quad (z \in U). \]

**Corollary 2.5.** Let \( \Omega \subset \mathbb{C} \) and let the function \( q \) be univalent in \( U \) with \( q(0) = 1 \). Let \( \phi \in \)
\( \Phi_I[h, q_\rho] \) for some \( \rho \in (0,1) \), where \( q_\rho(z) = q(\rho z) \). If the function \( f \in \Sigma_p \) and \( q_\rho \) satisfy the condition (2.11), and

\[
\phi\left( l_p(n, \lambda) f(z), l_p(n + 1, \lambda) f(z), l_p(n + 2, \lambda) f(z), l_p(n + 3, \lambda) f(z), l_p(n + 4, \lambda) f(z); z \right) < h(z),
\]  

then

\[
l_p(n, \lambda) f(z) < q(z), \quad (z \in U).
\]

Our next theorem yields the best dominant of the differential subordination (2.12).

**Theorem 2.6.** Let the function \( h \) be univalent in \( U \). Also let \( \phi : \mathbb{C}^5 \times U \to \mathbb{C} \) and suppose that the differential equation

\[
\phi\left( q(z), \frac{zq'(z) + \lambda q(z)}{p + \lambda}, \frac{zq''(z) + (2\lambda + 1)zq'(z) + \lambda^2 q(z)}{(p + \lambda)^2}, \frac{z^3 q'''(z) + (3\lambda + 3)z^2 q''(z) + (3\lambda^2 + 3\lambda + 1)zq'(z) + \lambda^3 q(z)}{(p + \lambda)^3}, \frac{z^4 q''''(z) + (4\lambda + 1)zq'(z) + \lambda^4 q(z)}{(p + \lambda)^4}; z \right) = h(z),
\]

has a solution \( q(z) \) with \( q(0) = 1 \) and satisfies the condition (2.1). If the function \( f \in \Sigma_p \) satisfies condition (2.12) and

\[
\phi\left( l_p(n, \lambda) f(z), l_p(n + 1, \lambda) f(z), l_p(n + 2, \lambda) f(z), l_p(n + 3, \lambda) f(z), l_p(n + 4, \lambda) f(z); z \right)
\]

is analytic in \( U \), then \( l_p(n, \lambda) f(z) < q(z) \), and \( q(z) \) is the best dominant.

**Proof.** By using Theorem 2.2, that \( q(z) \) is a dominant of (2.12). Since \( q(z) \) satisfy (2.14), it is also a solution of (2.12) and therefore \( q(z) \) will be dominated by all dominants. Hence \( q(z) \) is the best dominant.

In the special case \( q(z) = Mz, M > 0 \), and in view of Definition 2.1, the class of admissible functions \( \Phi_I[\Omega, q] \), denoted by \( \Phi_I[\Omega, M] \) is defined below.

**Definition 2.7.** Let \( \Omega \) be a set in \( \mathbb{C} \), and \( M > 0 \). The class of admissible functions \( \Phi_I[\Omega, M] \) consists of those functions \( \phi : \mathbb{C}^5 \times U \to \mathbb{C} \) that satisfy the admissibility condition:

\[
\phi\left( \frac{M e^{i\theta}}{p + \lambda} \frac{\kappa + \lambda}{p + \lambda} L + \frac{[(2\lambda + 1)\kappa + \lambda^2]}{(p + \lambda)^2} Me^{i\theta}, N + \frac{(3\lambda + 3)L + [(3\lambda^2 + 3\lambda + 1)\kappa + \lambda^3]}{(p + \lambda)^3} Me^{i\theta}\right) \in \Omega,
\]

where \( p > -\lambda, z \in U, \Re\left(Le^{-i\theta}\right) \geq (\kappa - 1)\kappa M, \Re\left(Ne^{-i\theta}\right) \geq 0 \) and \( \Re\left(Ae^{-i\theta}\right) \geq 0 \) for all \( \theta \in \mathbb{R} \) and \( \kappa \geq 3 \).

**Corollary 2.8.** Let \( \phi \in \Phi_I[\Omega, M] \). If the function \( f \in \Sigma_p \) satisfies the following conditions:

\[
|l_p(n + 2, \lambda) f(z)| \leq \kappa^2 M \quad (\kappa \geq 3; M > 0),
\]
and
\[
\phi(I_p(n, \lambda) f(z), I_p(n + 1, \lambda) f(z), I_p(n + 2, \lambda) f(z), I_p(n + 3, \lambda) f(z), I_p(n + 4, \lambda) f(z); z) \in \Omega,
\]
then
\[
|I_p(n, \lambda) f(z)| < M.
\]

In the special case \( \Omega = q(U) = \{ \omega : |\omega| < M \} \), the class \( \Phi_I[\Omega, M] \) is simply denoted by \( \Phi_I[M] \).

**Corollary 2.9.** Let \( \kappa \geq 3, \lambda > -p \) and \( M > 0 \). If the function \( f \in \Sigma_p \) satisfies
\[
|I_p(n + 2, \lambda) f(z)| \leq \kappa^2 M,
\]
and
\[
|\lambda(p + \lambda)^3I_p(n + 3, \lambda) f(z)| < (|\lambda^3 + \lambda^2 + 3\lambda + 1| + 2|\lambda^2 + 9\lambda + 7|)3M,
\]
then
\[
|I_p(n, \lambda) f(z)| < M.
\]

**Proof.** Let
\[
\phi(u, v, x, y, g; z) = (p + \lambda)^4 g - \lambda(p + \lambda)^3 y, \ \Omega = h(U),
\]
where
\[
h(z) = (|\lambda^3 + \lambda^2 + 3\lambda + 1| + 2|\lambda^2 + 9\lambda + 7|)3Mz, M > 0.
\]

Using Corollary 2.8, we need to show that \( \phi \in \Phi_{I, \lambda}[\Omega, M] \). Since
\[
\left| \phi \left( \frac{Me^{i\theta}}{p + \lambda}, \frac{k + \lambda}{p + \lambda}, \frac{L + [(2\lambda + 1)k + \lambda^2]Me^{i\theta}}{(p + \lambda)^2}, \frac{N + (3\lambda + 3)L + [(3\lambda^2 + 3\lambda + 1)k + \lambda^3]Me^{i\theta}}{(p + \lambda)^3} \right) \right| = \left| A + (4\lambda + 6)N + (4\lambda^2 + 12\lambda + 7)L + [(4\lambda^3 + 4\lambda^2 + 4\lambda + 1)k + \lambda^4]Me^{i\theta} \right| \]
\[
= |Ae^{-i\theta} + (3\lambda + 6)Ne^{-i\theta} + (\lambda^2 + 9\lambda + 7)Le^{-i\theta} + (\lambda^3 + \lambda^2 + 3\lambda + 1)\kappa Me^{i\theta}|
\]
\[
\geq \Re(Ae^{-i\theta}) + |3\lambda + 6|\Re(Ne^{-i\theta}) + |\lambda^2 + 9\lambda + 7|\Re(Le^{-i\theta}) + |\lambda^3 + \lambda^2 + 3\lambda + 1|\kappa M
\]
\[
\geq |\lambda^3 + \lambda^2 + 3\lambda + 1|\kappa M + |\lambda^2 + 9\lambda + 7|\kappa(\kappa - 1)M
\]
\[
\geq (|\lambda^3 + \lambda^2 + 3\lambda + 1| + 2|\lambda^2 + 9\lambda + 7|)3M,
\]
whenever \( z \in U, \Re(Le^{-i\theta}) \geq (\kappa - 1)\kappa M, \Re(Ne^{-i\theta}) \geq 0 \) and \( \Re(Ae^{-i\theta}) \geq 0 \) for all \( \theta \in \mathbb{R} \) and \( \kappa \geq 3 \).

The proof is complete.

3. **Fourth-Order Differential Superordination with \( I_p(n, \lambda) \)**

In this section, we obtain fourth-order differential superordination and sandwich-type results for multivalent functions associated with the operator \( I_p(n, \lambda) \) defined by (1.2). For this aim, the class of admissible functions is given in the following definition.
**Definition 3.1.** Let $\Omega$ be a set in $\mathbb{C}$ and $q \in \mathcal{H}_1$ with $q'(z) \not= 0$. The class of admissible functions $\Phi^1_q[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^5 \times \overline{U} \to \mathbb{C}$ that satisfy the following admissibility condition:

$$ \phi(u, v, x, y, g; \zeta) \in \Omega, $$

whenever

$$ u = q(z), \quad v = \frac{zq'(z) + mq(z)}{(p + \lambda)m}, \quad \Re \left\{ \frac{(p + \lambda)^2 x - \lambda^2 u}{(p + \lambda)v - \lambda u} - 2\lambda \right\} \leq \frac{1}{m} \Re \left\{ \frac{zq''(z)}{q'(z)} + 1 \right\}, $$

and

$$ \Re \left\{ \frac{(p + \lambda)^3 g - (p + \lambda)^2 (4\lambda + 6)y + (p + \lambda)(8\lambda^2 + 18\lambda + 11)x}{(p + \lambda)v - \lambda u} \right\} \leq \frac{1}{m^2} \Re \left\{ \frac{z^2 q'''(z)}{q'(z)} \right\}, $$

where $z \in U, \zeta \in \partial U, \lambda \in \mathbb{C}\{0, -1, -2, \ldots\}$, and $m \geq 3$.

**Theorem 3.2.** Let $\phi \in \Phi^1_q[\Omega, q]$. If the functions $f(z) \in \Sigma_p$ and $l_p(n, \lambda)f(z) \in Q_1$ satisfy the following conditions:

$$ \Re \left( \frac{z^2 q'''(z)}{q'(z)} \right) \geq 0, \quad \left| l_p(n + 2, \lambda)f(z) \right| \leq \frac{1}{m}, \tag{3.1} $$

$$ \phi(l_p(n, \lambda)f(z), l_p(n + 1, \lambda)f(z), l_p(n + 2, \lambda)f(z), l_p(n + 3, \lambda)f(z), l_p(n + 4, \lambda)f(z); z) $$

is univalent, and

$$ \Omega \subset \{ \phi(l_p(n, \lambda)f(z), l_p(n + 1, \lambda)f(z), l_p(n + 2, \lambda)f(z), l_p(n + 3, \lambda)f(z), l_p(n + 4, \lambda)f(z); z) : z \in U \}, \tag{3.2} $$

then

$$ q(z) < l_p(n, \lambda)f(z). $$

**Proof.** Let the functions $p(z)$ and $\psi$ be defined by (2.3) and (2.9). Since $\phi \in \Phi^1_q[\Omega, q]$. Thus from (2.10) and (3.2) yield

$$ \Omega \subset \{ \psi(p(z), zp'(z), z^2 p''(z), z^3 p'''(z), z^4 p''''(z); z) : z \in U \}. $$

In view from (2.8) that the admissible condition for $\phi \in \Phi^1_q[\Omega, q]$ in Definition (3.1) is equivalent the admissible condition for $\psi$ as given in Definition 1.7 with $n = 3$. Hence $\psi \in \Psi^1_q[\Omega, q]$, and by using (3.1) and Theorem 1.8, we have

$$ q(z) < p(z) = l_p(n, \lambda)f(z). $$

Therefore, we completes the proof of Theorem 3.2.
If $\Omega \neq \mathbb{C}$ is a simply connected domain, and $\Omega = h(U)$ for some conformal mapping $h(z)$ of $U$ onto $\Omega$, in this case the class $\Phi_f[h(U), q]$ is written as $\Phi_f[h, q]$. The next Theorem is directly consequence of Theorem 3.2.

**Theorem 3.3.** Let $\phi \in \Phi_f[h, q]$. Also, let the function $h(z)$ be analytic in $U$. If the function $f(z) \in \Sigma_p, I_p(n, \lambda)f(z) \in Q_1$ and $q \in \mathcal{H}_1$ satisfies the condition (3.1),

$$\{\phi(I_p(n, \lambda)f(z), I_p(n + 1, \lambda)f(z), I_p(n + 2, \lambda)f(z), I_p(n + 3, \lambda)f(z), I_p(n + 4, \lambda)f(z); z \in U\}$$

is univalent in $U$, and

$$h(z) < \phi(I_p(n, \lambda)f(z), I_p(n + 1, \lambda)f(z), I_p(n + 2, \lambda)f(z), I_p(n + 3, \lambda)f(z), I_p(n + 4, \lambda)f(z); z), \quad (3.3)$$

then

$$q(z) < I_p(n, \lambda)f(z).$$

**Theorem 3.4.** Let the function $h$ be analytic in $U$, and let $\phi : \mathbb{C}^5 \times U \rightarrow \mathbb{C}$ and $\psi$ be given by (2.9). Suppose that the differential equation

$$\psi(q(z), zq'(z), z^2q''(z), z^3q'''(z), z^4q''''(z); z) = h(z), \quad (3.4)$$

has a solution $q(z) \in Q_1$. If the functions $f \in \Sigma_p, I_p(n, \lambda)f(z) \in Q_1$ and $q \in \mathcal{H}_1$ with $q'(z) \neq 0$ satisfy the condition (2.1) and satisfies the condition (3.1),

$$\{\phi(I_p(n, \lambda)f(z), I_p(n + 1, \lambda)f(z), I_p(n + 2, \lambda)f(z), I_p(n + 3, \lambda)f(z), I_p(n + 4, \lambda)f(z); z); \quad z \in U\}$$

is univalent in $U$, and

$$h(z) < \phi(I_p(n, \lambda)f(z), I_p(n + 1, \lambda)f(z), I_p(n + 2, \lambda)f(z), I_p(n + 3, \lambda)f(z), I_p(n + 4, \lambda)f(z); z)$$

then

$$q(z) < I_p(n, \lambda)f(z),$$

and $q(z)$ is the best subordinant of (3.3).

**Proof.** The proof is similar to that of Theorem 2.6 and it is being omitted here. By Combining Theorem 2.4 and Theorem 3.3, we obtain the following sandwich type result.

**Corollary 3.5.** Let the functions $h_1(z), q_1(z)$ be analytic in $U$ and let the function $h_2(z)$ be univalent in $U, q_2(z) \in Q_1$ with $q_1(0) = q_2(0) = 1$ and $\phi \in \Phi_f[h_2, q_2] \cap \Phi_f[h_1, q_1]$. If the function $f(z) \in \Sigma_p, I_p(n, \lambda)f(z) \in Q_1 \cap \mathcal{H}_1 \{\phi(I_p(n, \lambda)f(z), I_p(n + 1, \lambda)f(z), I_p(n + 2, \lambda)f(z), I_p(n + 3, \lambda)f(z), I_p(n + 4, \lambda)f(z); z); z \in U\}$

is univalent in $U$, and the conditions (2.1) and (3.1) are satisfied,

$$h_1(z) < \phi(I_p(n, \lambda)f(z), I_p(n + 1, \lambda)f(z), I_p(n + 2, \lambda)f(z), I_p(n + 3, \lambda)f(z), I_p(n + 4, \lambda)f(z); z) < h_2(z),$$

then

$$q_1(z) < I_p(n, \lambda)f(z) < q_2(z).$$
References


