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New Concept on Fourth-Order Differential Subordination and Superordination with Some Results for Multivalent Analytic Functions

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ABSTRACT

In this paper, we introduce new concept that is fourth-order differential subordination and superordination associated with differential linear operator $I_p(n,\lambda)$ in open unit disk. Also, we obtain some new results.

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1 . Introduction

Let $\mathcal{H}(U)$ be the class of functions which are analytic in the open unit disk $U=\{z : z\in\mathbb{C} : |z|<1\}$. For $n\in\mathbb{N}=\{1,2,3,\ldots\}$,and $a\in\mathbb{C}$, let $\mathcal{H}[a,n]=\{f\in\mathcal{H}(U) : f(z)=a+a_nz^n+a_{n+1}z^{n+1}+\ldots\}$,and also let $\mathcal{H}_1=[1,1]$. Let Σ_p denote the class of all analytic functions of the form :

$$f(z)=z^p+\sum_{k=p+1}^{\infty}a_kz^k\,. \qquad (1.1)$$

We consider a linear operator $I_p(n,\lambda)$ on the class Σ_p of multivalent functions by the infinite series

$$I_p(n,\lambda)f(z) = z^p + \sum_{k=p+1}^{\infty} \left(\frac{k+\lambda}{p+\lambda}\right)^n a_k z^k , \qquad (\lambda > -p). \tag{1.2}$$

The operator $I_p(n,\lambda)$ was studied by [2] .It is easily verified from (1.2) that

$$z\left[I_p(n,\lambda)f(z)\right]' = (p+\lambda)I_p(n+1,\lambda)f(z) - \lambda I_p(n,\lambda) . \tag{1.3}$$

For several past years, there are many authors introduce and dealing with the theory of second-order differential subordination and superordination for example([$1-3$, 6, 8,9,10,13, 16]), recently years, the many authors discussed the theory of third-order differential subordination and superordination for example([4, 5,11,12,17,18,19]). In the present paper, we investigated to the fourth-order. In 2011,Antonino and Miller [4] extended the theory of second-order differential subordination in the open unit disk introduced by Miller and Mocanu [14] to the third-order case, now, we extend this to fourth-order differential subordination. They determined properties of functions$p$ that satisfy the following fourth-order differential subordination :

$$\{\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z), z^4p''''(z); z) : z \in U\} \subset \Omega .$$

In 2014, Tang et al [19] extended the theory of second-order differential superordination in the open unit disk introduced by Miller and Mocanu [15] to third-order case, now we extend this to fourth-order differential superordination. They determined properties of functions p that satisfy the following fourth-order differential superordination :

$$\Omega \subset \{\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z), z^4p''''(z); z) : z \in U\} .$$

To prove our main results, we need the basic concepts in theory of the fourth-order.

Definition 1.1. [14]. Let $f(z)$ and $F(z)$ be members of the analytic function class $\mathcal{H}(U)$. The function $f(z)$ is said to be subordinate to $F(z)$ or $F(z)$ is superordinate to $f(z)$ if there exists a Schwarz function $\mathrm{w}(z)$ analytic in U with $\mathrm{w}(0) = 0$ and $|\mathrm{w}(z)| < 1$ $(z \in U)$, and such that $f(z) = F(\mathrm{w}(z))$. In such case, we write

$$f \prec F, \text{ or } f(z) \prec F(z).$$

If $F(z)$ is univalent in U, then $f(z) \prec F(z)$ if and only if $f(0) = F(0)$ and $f(U) \subset F(U)$.

Definition 1.2.[4]. Let $\mathcal{Q}$ denote the set of functions q that are analytic and univalent on the set $\overline{U}\backslash \mathrm{E}(q)$, where

$$\mathrm{E}(q) = \{\zeta : \zeta \in \partial U \text{ and } \lim_{z\to\zeta} q(z) = \infty\},$$

is such that $\min |q'(\zeta)| = \rho > 0$ for $\zeta \in \partial U \backslash \mathrm{E}(q)$. Further let the subclass of $\mathcal{Q}$ for which $q(0) = a$ be denoted by $\mathcal{Q}(a)$ and $\mathcal{Q}(1) = \mathcal{Q}_1$.

Definition 1.3. Let $\psi : \mathbb{C}^5 \times U \longrightarrow \mathbb{C}$ and the function $h(z)$ be univalent in U. If the function $p(z)$ is analytic in U satisfies the following fourth-order differential subordination :

$$\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z), z^4p''''(z); z) \prec h(z), \tag{1.4}$$

then $p(z)$ is called a solution of the differential subordination . A univalent function $q(z)$is called a dominant of the solutions of the differential subordination or more simply a dominant if $p(z) \prec q(z)$ for all $p(z)$ satisfying (1.4) . A dominant $\tilde{q}(z)$ that satisfies $\tilde{q}(z) \prec q(z)$ for all dominants $q(z)$ of (1.4) is said to be the best dominant.

Definition 1.4. Let Ω be a set in $\mathbb{C}, q \in \mathcal{Q}$and $n \in \mathbb{N}\backslash\{2\}$. The class of admissible functions $\Psi_n[\Omega, q]$ consists of those functions $\psi : \mathbb{C}^5 \times U \longrightarrow \mathbb{C}$ that satisfy the following admissibility condition:

$$\psi(r, s, t, w, b; z) \notin \Omega,$$

whenever

$$r = q(\zeta) \quad , \qquad s = \kappa\zeta q'(\zeta) \quad , \qquad \Re\left(\frac{t}{s}+1\right) \geq \kappa\Re\left(\frac{\zeta q''(\zeta)}{q'(\zeta)}+1\right),$$

and

$$\Re\left(\frac{w}{s}\right) \geq \kappa^2\Re\left(\frac{\zeta^2 q'''(\zeta)}{q'(\zeta)}\right), \qquad \Re\left(\frac{b}{s}\right) \geq \kappa^3\Re\left(\frac{\zeta^3 q''''(\zeta)}{q'(\zeta)}\right),$$

where $z \in U, \zeta \in \partial U \backslash \mathrm{E}(q)$, and $\kappa \geq n$.

Theorem 1.5.[7]. Let $p \in \mathcal{H}[a,n]$ with $n \in \mathbb{N}\backslash\{2\}$. Also, let $q \in \mathcal{Q}(a)$ and satisfy the following conditions:

$$\Re\left(\frac{\zeta^2 q'''(\zeta)}{q'(\zeta)}\right) \geq 0, \qquad \left|\frac{z^2 p''(z)}{q'(\zeta)}\right| \leq \kappa^2,$$

where $z \in U, \zeta \in \partial U \backslash \mathrm{E}(q)$ and $\kappa \geq n$. If Ω a set in $\mathbb{C}$, $\psi \in \Psi_n[\Omega, q]$ and

$$\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z), z^4p''''(z); z) \in \Omega,$$

then $p(z) \prec q(z)$, $(z \in U)$.

Definition 1.6. Let $\psi : \mathbb{C}^5 \times U \longrightarrow \mathbb{C}$ and the function $h(z)$ be analytic in U. If the functions $p(z)$ and

$$\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z), z^4p''''(z); z),$$

are univalent in U and satisfy the following fourth-order differential superordination:

$$h(z) \prec \psi(p(z), zp'(z), z^2p''(z), z^3p'''(z), z^4p''''(z); z), \tag{1.5}$$

then $p(z)$ is called a solution of the differential superordination. An analytic function $q(z)$ is called a subordinat of the solutions of the differential superordination or more simply a subordinant if $q(z) \prec p(z)$ for all $p(z)$ satisfying (1.5). A univalent subordinant $\tilde{q}(z)$ that satisfies the condition $q(z) \prec \tilde{q}(z)$ for all subordinants $q(z)$ of (1.5) is said to be the best subordinant. We note that the best subordinant is unique up to a rotation of U.

Definition 1.7. Let Ω be a set in $\mathbb{C}$,$q(z) \in \mathcal{H}[a,n]$and $q'(z) \neq 0$. The class of admissible functions$\Psi'_n[\Omega, q]$ consists of those functions $\psi : \mathbb{C}^5 \times \overline{U} \longrightarrow \mathbb{C}$ that satisfy the following admissibility condition:

$$\psi(r, s, t, w, b; \zeta) \in \Omega,$$

whenever

$$r = q(z) \quad , \qquad s = \frac{zq'(z)}{m} \quad , \qquad \Re\left(\frac{t}{s}+1\right) \leq \frac{1}{m}\Re\left(\frac{z\,q''(z)}{q'(z)}+1\right),$$

and

$$\Re\left(\frac{w}{s}\right) \leq \frac{1}{m^2}\Re\left(\frac{z^2 q'''(z)}{q'(z)}\right), \quad \Re\left(\frac{b}{s}\right) \leq \frac{1}{m^3}\Re\left(\frac{z^3 q''''(z)}{q'(z)}\right),$$

where $z \in U, \zeta \in \partial U$, and $m \geq n \geq 3$.

Theorem1.8.[7]. Let $q(z) \in \mathcal{H}[a,n]$ and $\psi \in \Psi'_n[\Omega, q]$. If

$$\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z), z^4p''''(z); z)$$

is univalent in U and $p(z) \in Q(a)$ satisfy the following conditions:

$$\Re\left(\frac{z^2q'''(z)}{q'(z)}\right) \geq 0, \qquad \left|\frac{z^2p''(z)}{q'(z)}\right| \leq \frac{1}{m^2},$$

where $z \in U, \zeta \in \partial U$ and $m \geq n \geq 3$,
then

$$\Omega \subset \{\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z), z^4p''''(z); z): z \in U\},$$

implies that

$$q(z) \prec p(z), \quad (z \in U).$$

2. Fourth-Order Differential Subordination with $I_p(n, \lambda)$

We first define the following class of admissible functions, which are required in proving the differential subordination theorem involving the operator $I_p(n, \lambda)$ defined by (1.2).

Definition 2.1. Let Ω be a set in $\mathbb{C}$, and let $q \in \mathcal{Q}_1 \cap \mathcal{H}_1$. The class of admissible functions $\Phi_I[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^5 \times U \longrightarrow \mathbb{C}$ that satisfy the following admissibility condition:

$$\phi(u, v, x, y, g; z) \notin \Omega,$$

whenever

$$u = q(\zeta), \quad v = \frac{\kappa\zeta q'(\zeta) + \lambda q(\zeta)}{p + \lambda}, \qquad \Re\left\{\frac{(p+\lambda)^2x - \lambda^2u}{(p+\lambda)v - \lambda u} - 2\lambda\right\} \geq \kappa\Re\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right\},$$

$$\Re\left\{\frac{(p+\lambda)^2[(p+\lambda)y - (3\lambda+3)x] + (2\lambda^3+3\lambda^2)u}{(p+\lambda)v - \lambda u} + (3\lambda^2 + 6\lambda + 2)\right\} \geq \kappa^2\Re\left\{\frac{\zeta^2q'''(\zeta)}{q'(\zeta)}\right\},$$

and

$$\Re\left\{\frac{(p+\lambda)[(p+\lambda)^3g - (p+\lambda)^2(4\lambda+6)y + (p+\lambda)(8\lambda^2+18\lambda+11)x}{(p+\lambda)v - \lambda u}\right.$$
$$\left.\frac{-(8\lambda^3+18\lambda^2+22\lambda+6)v] + (3\lambda^4+6\lambda^3+11\lambda^2+6\lambda)u}{(p+\lambda)v - \lambda u}\right\} \geq \kappa^3\Re\left\{\frac{\zeta^3q''''(\zeta)}{q'(\zeta)}\right\},$$

where $z \in U, \lambda > -p,\ \zeta \in \partial U \backslash \mathrm{E}(q)$ and $\kappa \geq 3$.

Theorem 2.2. Let $\phi \in \Phi_I[\Omega, q]$. If the functions $f(z) \in \Sigma_p$ and $q \in \mathcal{Q}_1$ satisfy the following conditions :

$$\Re\left(\frac{\zeta^2q'''(\zeta)}{q'(\zeta)}\right) \geq 0, \qquad \left|\frac{I_p(n+2,\lambda)f(z)}{q'(\zeta)}\right| \leq \kappa^2, \tag{2.1}$$

and

$$\{\phi(I_p(n,\lambda)f(z),\ I_p(n+1,\lambda)f(z),\ I_p(n+2,\lambda)f(z), I_p(n+3,\lambda)f(z), I_p(n+4,\lambda)f(z); z): z \in U\} \subset \Omega, \tag{2.2}$$

then $\quad I_p(n,\lambda)f(z) \prec q(z),\ (z \in U).$

Proof. Define the analytic function $p(z)$ in U by

$$p(z) = I_p(n,\lambda)f(z). \tag{2.3}$$

Then, differentiating (2.3) with respect to z and using (1.3), we have

$$I_p(n+1,\lambda)f(z) = \frac{zp'(z)+\lambda p(z)}{p+\lambda}. \tag{2.4}$$

Further computations show that

$$I_p(n+2,\lambda)f(z) = \frac{z^2p''(z) + (2\lambda+1)zp'(z) + \lambda^2 p(z)}{(p+\lambda)^2}, \tag{2.5}$$

$$I_p(n+3,\lambda)f(z) = \frac{z^3p'''(z) + (3\lambda+3)z^2p''(z) + (3\lambda^2+3\lambda+1)zp'(z) + \lambda^3 p(z)}{(p+\lambda)^3}. \tag{2.6}$$

and

$$I_p(n+4,\lambda)f(z) = \frac{z^4p''''(z) + (4\lambda+6)z^3p'''(z) + (4\lambda^2+12\lambda+7)z^2p''(z) + (4\lambda^3+4\lambda^2+4\lambda+1)zp'(z) + \lambda^4 p(z)}{(p+\lambda)^4}. \tag{2.7}$$

Define the transformation from $\mathbb{C}^5$ to $\mathbb{C}$ by

$$u(r,s,t,w,b) = r, \quad v(r,s,t,w,b) = \frac{s+\lambda r}{p+\lambda}, \quad x(r,s,t,w,b) = \frac{t+(2\lambda+1)s+\lambda^2 r}{(p+\lambda)^2},$$

$$y(r,s,t,w,b) = \frac{w+(3\lambda+3)t+(3\lambda^2+3\lambda+1)s+\lambda^3 r}{(p+\lambda)^3},$$

and

$$g(r,s,t,w,b) = \frac{b+(4\lambda+6)w+(4\lambda^2+12\lambda+7)t+(4\lambda^3+4\lambda^2+4\lambda+1)s+\lambda^4 r}{(p+\lambda)^4}. \tag{2.8}$$

Let

$$\psi(r,s,t,w,b;z) = \phi(u,v,x,y,g;z)$$

$$= \phi\left(r, \frac{s+\lambda r}{p+\lambda}, \frac{t+(2\lambda+1)s+\lambda^2 r}{(p+\lambda)^2}, \frac{w+(3\lambda+3)t+(3\lambda^2+3\lambda+1)s+\lambda^3 r}{(p+\lambda)^3}, \frac{b+(4\lambda+6)w+(4\lambda^2+12\lambda+7)t+(4\lambda^3+4\lambda^2+4\lambda+1)s+\lambda^4 r}{(p+\lambda)^4}; z\right). \tag{2.9}$$

The proof will make use of Theorem 1.5. Using equations (2.3) to (2.7), we have from (2.9) that

$$\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z), z^4p''''(z); z) =$$

$$\phi\big(I_p(n,\lambda)f(z), I_p(n+1,\lambda)f(z), I_p(n+2,\lambda)f(z), I_p(n+3,\lambda)f(z), I_p(n+4,\lambda)f(z); z\big). \quad (2.10)$$

Hence (2.2) becomes

$$\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z), z^4p''''(z); z) \in \Omega.$$

We note that

$$\frac{t}{s}+1=\frac{(p+\lambda)^2x-\lambda^2u}{(p+\lambda)v-\lambda u}-2\lambda,$$

$$\frac{w}{s}=\frac{(p+\lambda)^2[(p+\lambda)y-(3\lambda+3)x]+(2\lambda^3+3\lambda^2)u}{(p+\lambda)v-\lambda u}+(3\lambda^2+6\lambda+2),$$

and

$$\frac{b}{s}=\frac{(p+\lambda)[(p+\lambda)^3g-(p+\lambda)^2(4\lambda+6)y+(p+\lambda)(8\lambda^2+18\lambda+11)x}{(p+\lambda)v-\lambda u}$$
$$\frac{-(8\lambda^3+18\lambda^2+22\lambda+6)v]+(3\lambda^4+6\lambda^3+11\lambda^2+6\lambda)u}{(p+\lambda)v-\lambda u}.$$

Therefore, the admissibility condition for $\phi \in \Phi_I[\Omega, q]$ in Definition 2.1 is equivalent to the admissibility condition for $\psi \in \Psi_3[\Omega, q]$ as given in Definition 1.4 with $n = 3$. Therefore , by using (2.1) and Theorem 1.5, we obtain

$$p(z) = I_p(n,\lambda)f(z) \prec q(z).$$

The next Corollary is an extension of Theorem 2.2 to the case where the behavior of $q(z)$ on ∂U is not known.

Corollary 2.3. Let $\Omega \subset \mathbb{C}$, and let the function $q(z)$ be univalent in U with $q(0) = 1$. Let $\phi \in \Phi_I[\Omega, q_\rho]$ for some $\rho \in (0,1)$, where $q_\rho(z) = q(\rho z)$. If the function $f(z) \in \Sigma_p$ and $q_\rho(z)$ satisfy the following conditions :

$$\Re\left(\frac{\zeta^2 q_\rho'''(z)}{q_\rho'(z)}\right) \geq 0, \qquad \left|\frac{I_p(n+2,\lambda)f(z)}{q_\rho'(z)}\right| \leq \kappa^2, \qquad \big(z \in U, \zeta \in \partial U\backslash \mathrm{E}(q_\rho)\big) \quad (2.11)$$

and

$$\phi\big(I_p(n,\lambda)f(z), I_p(n+1,\lambda)f(z), I_p(n+2,\lambda)f(z), I_p(n+3,\lambda)f(z), I_p(n+4,\lambda)f(z); z\big) \in \Omega,$$

then

$$I_p(n,\lambda)f(z) \prec q(z), \qquad (z \in U).$$

Proof. By using Theorem 2.2, yields $I_p(n,\lambda)f(z) \prec q_\rho(z)$. Then we obtain the result from $q_\rho(z) \prec q(z)$, $(z \in U)$. If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(U)$ for some conformal mapping $h(z)$ of U onto Ω. In this case, the class $\Phi_I[h(U), q]$ is written as $\Phi_I[h, q]$. The following two results are immediate consequence of Theorem 2.2 and Corollary 2.3.

Theorem 2.4. Let $\phi \in \Phi_I[h, q]$. If the function $f \in \Sigma_p$ and $q \in Q_1$ satisfy the condition (2.1) and

$$\phi\big(I_p(n,\lambda)f(z), I_p(n+1,\lambda)f(z), I_p(n+2,\lambda)f(z), I_p(n+3,\lambda)f(z), I_p(n+4,\lambda)f(z); z\big) \prec h(z), \quad (2.12)$$

then

$$I_p(n,\lambda)f(z) \prec q(z), \qquad (z \in U).$$

Corollary 2.5. Let $\Omega \subset \mathbb{C}$ and let the function q be univalent in U with $q(0) = 1$. Let $\phi \in$

$\Phi_I[h, q_\rho]$ for some $\rho \in (0,1)$, where $q_\rho(z) = q(\rho z)$. If the function $f \in \Sigma_p$ and q_ρ satisfy the condition (2.11), and

$$\phi\big(I_p(n,\lambda)f(z), I_p(n+1,\lambda)f(z), I_p(n+2,\lambda)f(z), I_p(n+3,\lambda)f(z), I_p(n+4,\lambda)f(z); z\big) \prec h(z), \tag{2.13}$$

then

$$I_p(n,\lambda)f(z) \prec q(z)\,, \qquad (z \in U).$$

Our next theorem yields the best dominant of the differential subordination (2.12).

Theorem 2.6. Let the function h be univalent in U. Also let $\phi : \mathbb{C}^5 \times U \longrightarrow \mathbb{C}$ and suppose that the differential equation

$$\phi\Bigg(q(z), \frac{zq'(z) + \lambda q(z)}{p+\lambda}, \frac{z^2q''(z) + (2\lambda+1)zq'(z) + \lambda^2 q(z)}{(p+\lambda)^2},$$

$$\frac{z^3q'''(z) + (3\lambda+3)z^2q''(z) + (3\lambda^2+3\lambda+1)zq'(z) + \lambda^3 q(z)}{(p+\lambda)^3}, \frac{z^4q''''(z) + (4\lambda+6)z^3q'''(z) +}{(p+\lambda)^4}$$

$$\frac{(4\lambda^2+12\lambda+7)z^2q''(z) + (4\lambda^3+4\lambda^2+4\lambda+1)zq'(z) + \lambda^4 q(z)}{(p+\lambda)^4}; z\Bigg) = h(z), \tag{2.14}$$

has a solution $q(z)$ with $q(0) = 1$ and satisfies the condition (2.1). If the function $f \in \Sigma_p$ satisfies condition (2.12) and

$$\phi\big(I_p(n,\lambda)f(z), I_p(n+1,\lambda)f(z), I_p(n+2,\lambda)f(z), I_p(n+3,\lambda)f(z), I_p(n+4,\lambda)f(z); z\big)$$

is analytic in U, then $I_p(n,\lambda)f(z) \prec q(z)$, and $q(z)$ is the best dominant.

Proof. By using Theorem 2.2, that $q(z)$ is a dominant of (2.12). Since $q(z)$ satisfy (2.14), it is also a solution of (2.12) and therefore $q(z)$ will be dominated by all dominants. Hence $q(z)$ is the best dominant.

In the special case $q(z) = Mz$, $M > 0$, and in view of Definition 2.1 , the class of admissible functions $\Phi_I[\Omega, q]$, denoted by $\Phi_I[\Omega, M]$ is defined below.

Definition 2.7. Let Ω be a set in $\mathbb{C}$, and $M > 0$.The class of admissible functions $\Phi_I[\Omega, M]$ consists of those functions $\phi : \mathbb{C}^5 \times U \longrightarrow \mathbb{C}$ that satisfy the admissibility condition:

$$\phi\Bigg(Me^{i\theta}, \frac{\kappa+\lambda}{p+\lambda}Me^{i\theta}, \frac{L + [(2\lambda+1)\kappa + \lambda^2]Me^{i\theta}}{(p+\lambda)^2}, \frac{N + (3\lambda+3)L + [(3\lambda^2+3\lambda+1)\kappa + \lambda^3]Me^{i\theta}}{(p+\lambda)^3}$$

$$\frac{A + (4\lambda+6)N + (4\lambda^2+12\lambda+7)L + [(4\lambda^3+4\lambda^2+4\lambda+1)\kappa + \lambda^4]Me^{i\theta}}{(p+\lambda)^4}; z\Bigg) \notin \Omega, \tag{2.15}$$

where $p > -\lambda, z \in U$, $\Re\big(Le^{-i\theta}\big) \geq (\kappa-1)\kappa M$, $\Re\big(Ne^{-i\theta}\big) \geq 0$ and $\Re\big(Ae^{-i\theta}\big) \geq 0$ for all $\theta \in \mathbb{R}$ and $\kappa \geq 3$.

Corollary 2.8. Let $\phi \in \Phi_I[\Omega, M]$. If the function $f \in \Sigma_p$ satisfies the following conditions:

$$\big|I_p(n+2,\lambda)f(z)\big| \leq \kappa^2 M \qquad (\kappa \geq 3\,; M > 0),$$

and

$$\phi\big(I_p(n,\lambda)f(z), I_p(n+1,\lambda)f(z), I_p(n+2,\lambda)f(z), I_p(n+3,\lambda)f(z), I_p(n+4,\lambda)f(z); z\big) \in \Omega,$$

then

$$\left|I_p(n,\lambda)f(z)\right| < M.$$

In the special case $\Omega = q(U) = \{\omega : |\omega| < M\}$, the class $\Phi_I[\Omega, M]$ is simply denoted by $\Phi_I[M]$.

Corollary 2.9. Let $\kappa \geq 3$, $\lambda > -p$ and $M > 0$. If the function $f \in \Sigma_p$ satisfies $\left|I_p(n+2,\lambda)f(z)\right| \leq \kappa^2 M$,

and

$$\left|(p+\lambda)^4 I_p(n+4,\lambda)f(z) - \lambda(p+\lambda)^3 I_p(n+3,\lambda)f(z)\right| < (|\lambda^3+\lambda^2+3\lambda+1| + 2|\lambda^2+9\lambda+7|)3M,$$

then

$$\left|I_p(n,\lambda)f(z)\right| < M.$$

Proof. Let

$$\phi(u,v,x,y,g;z) = (p+\lambda)^4 g - \lambda(p+\lambda)^3 y, \quad \Omega = h(U),$$

where

$$h(z) = (|\lambda^3+\lambda^2+3\lambda+1| + 2|\lambda^2+9\lambda+7|)3Mz\,, M > 0.$$

Using Corollary 2.8, we need to show that $\phi \in \Phi_{I,1}[\Omega\,, M]$. Since

$$\left|\phi\left(Me^{i\theta}, \frac{\kappa+\lambda}{p+\lambda}Me^{i\theta}, \frac{L+[(2\lambda+1)\kappa+\lambda^2]Me^{i\theta}}{(p+\lambda)^2}, \frac{N+(3\lambda+3)L+[(3\lambda^2+3\lambda+1)\kappa+\lambda^3]Me^{i\theta}}{(p+\lambda)^3}, \frac{A+(4\lambda+6)N+(4\lambda^2+12\lambda+7)L+[(4\lambda^3+4\lambda^2+4\lambda+1)\kappa+\lambda^4]Me^{i\theta}}{(p+\lambda)^4}; z\right)\right|$$

$$= \left|A + (3\lambda+6)N + (\lambda^2+9\lambda+7)L + (\lambda^3+\lambda^2+3\lambda+1)\kappa Me^{i\theta}\right|$$

$$= \left|Ae^{-i\theta} + (3\lambda+6)Ne^{-i\theta} + (\lambda^2+9\lambda+7)Le^{-i\theta} + (\lambda^3+\lambda^2+3\lambda+1)\kappa M\right|$$

$$\geq \Re\big(Ae^{-i\theta}\big) + |3\lambda+6|\Re\big(Ne^{-i\theta}\big) + |\lambda^2+9\lambda+7|\Re\big(Le^{-i\theta}\big) + |\lambda^3+\lambda^2+3\lambda+1|\kappa M$$

$$\geq |\lambda^3+\lambda^2+3\lambda+1|\kappa M + |\lambda^2+9\lambda+7|\kappa(\kappa-1)M$$

$$\geq (|\lambda^3+\lambda^2+3\lambda+1| + 2|\lambda^2+9\lambda+7|)3M,$$

whenever $z \in U, \Re\big(Le^{-i\theta}\big) \geq (\kappa-1)\kappa M$, $\Re\big(Ne^{-i\theta}\big) \geq 0$ and $\Re\big(Ae^{-i\theta}\big) \geq 0$ for all $\theta \in \mathbb{R}$ and $\kappa \geq 3$.
The proof is complete.

3. Fourth-Order Differential Superordination with $I_p(n,\lambda)$

In this section, we obtain fourth-order differential superordination and sandwich-type results for multivalent functions associated with the operator $I_p(n,\lambda)$ defined by (1.2). For this aim, the class of admissible functions is given in the following definition.

Definition 3.1. Let Ω be a set in $\mathbb{C}$ and $q \in \mathcal{H}_1$ with $q'(z) \neq 0$. The class of admissible functions $\Phi'_I[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^5 \times \overline{U} \longrightarrow \mathbb{C}$ that satisfy the following admissibility condition:

$$\phi(u, v, x, y, g; \zeta) \in \Omega\,,$$

whenever

$$u = q(z)\ ,\quad v = \frac{zq'(z) + mq(z)}{(p+\lambda)m}\ ,\quad \Re\left\{\frac{(p+\lambda)^2 x - \lambda^2 u}{(p+\lambda)v - \lambda u} - 2\lambda\right\} \leq \frac{1}{m}\Re\left\{\frac{zq''(z)}{q'(z)} + 1\right\},$$

$$\Re\left\{\frac{(p+\lambda)^2[(p+\lambda)y - (3\lambda+3)x] + (2\lambda^3 + 3\lambda^2)u}{(p+\lambda)v - \lambda u} + (3\lambda^2 + 6\lambda + 2)\right\} \leq \frac{1}{m^2}\Re\left\{\frac{z^2 q'''(z)}{q'(z)}\right\},$$

and

$$\Re\left\{\frac{(p+\lambda)[(p+\lambda)^3 g - (p+\lambda)^2(4\lambda+6)y + (p+\lambda)(8\lambda^2 + 18\lambda + 11)x}{(p+\lambda)v - \lambda u}\right.$$

$$\left.\frac{-(8\lambda^3 + 18\lambda^2 + 22\lambda + 6)v] + (3\lambda^4 + 6\lambda^3 + 11\lambda^2 + 6\lambda)u}{(p+\lambda)v - \lambda u}\right\} \leq \frac{1}{m^3}\Re\left\{\frac{z^3 q''''(z)}{q'(z)}\right\},$$

where $z \in U, \zeta \in \partial U,\ \lambda \in \mathbb{C}\backslash\{0, -1, -2, \ldots\}$, and $m \geq 3$.

Theorem 3.2. Let $\phi \in \Phi'_I[\Omega, q]$. If the functions $f(z) \in \Sigma_p$ and $I_p(n,\lambda)f(z) \in \mathcal{Q}_1$ satisfy the following conditions:

$$\Re\left(\frac{z^2 q'''(z)}{q'(z)}\right) \geq 0, \qquad \left|\frac{I_p(n+2,\lambda)f(z)}{q'(z)}\right| \leq \frac{1}{m^2}\,, \tag{3.1}$$

$$\phi\big(I_p(n,\lambda)f(z), I_p(n+1,\lambda)f(z), I_p(n+2,\lambda)f(z), I_p(n+3,\lambda)f(z), I_p(n+4,\lambda)f(z); z\big)$$

is univalent, and

$$\Omega \subset \left\{\phi\big(I_p(n,\lambda)f(z), I_p(n+1,\lambda)f(z), I_p(n+2,\lambda)f(z), I_p(n+3,\lambda)f(z), I_p(n+4,\lambda)f(z); z\big) : z \in U\right\}, \tag{3.2}$$

then

$$q(z) \prec I_p(n,\lambda)f(z).$$

Proof. Let the functions $p(z)$ and ψ be defined by (2.3) and (2.9). Since $\phi \in \Phi'_I[\Omega, q]$. Thus from (2.10) and (3.2) yield

$$\Omega \subset \{\psi(p(z)\,, zp'(z)\,, z^2p''(z)\,, z^3p'''(z), z^4p''''(z); z) : z \in U\}\,.$$

In view from (2.8) that the admissible condition for $\phi \in \Phi'_I[\Omega, q]$ in Definition (3.1) is equivalent the admissible condition for ψ as given in Definition 1.7 with $n = 3$. Hence $\psi \in \Psi'_3[\Omega, q]$, and by using (3.1) and Theorem 1.8 , we have

$$q(z) \prec p(z) = I_p(n,\lambda)f(z).$$

Therefore, we completes the proof of Theorem 3.2.

If $\Omega \neq \mathbb{C}$ is a simply connected domain, and $\Omega = h(U)$ for some conformal mapping $h(z)$ of U onto Ω, in this case the class $\Phi'_I[h(U), q]$ is written as $\Phi'_I[h, q]$. The next Theorem is directly consequence of Theorem 3.2.

Theorem 3.3. Let $\phi \in \Phi'_I[h, q]$. Also, let the function $h(z)$ be analytic in U. If the function $f(z) \in \Sigma_p$, $I_p(n,\lambda)f(z) \in \mathcal{Q}_1$ and $q \in \mathcal{H}_1$ satisfies the condition (3.1),

$$\left\{\phi\left(I_p(n,\lambda)f(z), I_p(n+1,\lambda)f(z), I_p(n+2,\lambda)f(z), I_p(n+3,\lambda)f(z), I_p(n+4,\lambda)f(z); z\right): z \in U\right\}$$

is univalent in U, and

$$h(z) \prec \phi\left(I_p(n,\lambda)f(z), I_p(n+1,\lambda)f(z), I_p(n+2,\lambda)f(z), I_p(n+3,\lambda)f(z), I_p(n+4,\lambda)f(z); z\right), \tag{3.3}$$

then

$$q(z) \prec I_p(n,\lambda)f(z).$$

Theorem 3.4. Let the function h be analytic in U, and let $\phi : \mathbb{C}^5 \times \overline{U} \longrightarrow \mathbb{C}$ and ψ be given by (2.9). Suppose that

the differential equation

$$\psi(q(z), zq'(z), z^2q''(z), z^3q'''(z), z^4q''''(z); z) = h(z), \tag{3.4}$$

has a solution $q(z) \in \mathcal{Q}_1$. If the functions $f \in \Sigma_p$, $I_p(n,\lambda)f(z) \in \mathcal{Q}_1$ and $q \in \mathcal{H}_1$ with $q'(z) \neq 0$ satisfy the condition (2.1) and satisfies the condition (3.1),

$$\left\{\phi\left(I_p(n,\lambda)f(z), I_p(n+1,\lambda)f(z), I_p(n+2,\lambda)f(z), I_p(n+3,\lambda)f(z), I_p(n+4,\lambda)f(z); z\right): \quad z \in U\right\}$$

is univalent in U, and

$$h(z) \prec \phi\left(I_p(n,\lambda)f(z), I_p(n+1,\lambda)f(z), I_p(n+2,\lambda)f(z), I_p(n+3,\lambda)f(z), I_p(n+4,\lambda)f(z); z\right)$$

then

$$q(z) \prec I_p(n,\lambda)f(z),$$

and $q(z)$ is the best subordinant of (3.3).

Proof. The proof is similar to that of Theorem 2.6 and it is being omitted here. By Combining Theorem 2.4 and Theorem 3.3, we obtain the following sandwich type result.

Corollary 3.5. Let the functions $h_1(z)$, $q_1(z)$ be analytic in U and let the function $h_2(z)$ be univalent in U, $q_2(z) \in \mathcal{Q}_1$ with $q_1(0) = q_2(0) = 1$ and $\phi \in \Phi_I[h_2, q_2] \cap \Phi'_I[h_1, q_1]$. If the function $f(z) \in \Sigma_p$, $I_p(n,\lambda)f(z) \in \mathcal{Q}_1 \cap \mathcal{H}$, $\left\{\phi\left(I_p(n,\lambda)f(z), I_p(n+1,\lambda)f(z), I_p(n+2,\lambda)f(z), I_p(n+3,\lambda)f(z), I_p(n+4,\lambda)f(z); z\right): z \in U\right\}$

is univalent in U, and the conditions (2.1) and (3.1) are satisfied,

$$h_1(z) \prec \phi\left(I_p(n,\lambda)f(z), I_p(n+1,\lambda)f(z), I_p(n+2,\lambda)f(z), I_p(n+3,\lambda)f(z), I_p(n+4,\lambda)f(z); z\right) \prec h_2(z),$$

then

$$q_1(z) \prec I_p(n,\lambda)f(z) \prec q_2(z).$$

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