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Differential Sandwich Theorems for Univalent Functions Involving a Differential Operator

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ABSTRACT

In the present paper, we obtain some subordination and superordination results involving the differential operator $\mathcal{W}_{\alpha,\beta}^{j,\delta}$ for certain normalized analytic functions in the open unit disk. These results are applied to obtain sandwich results.

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1 . Introduction

Let $H = H(U)$ be the class of analytic functions in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. For n a positive integer and $a \in \mathbb{C}$. Let $H\ [a , n]$ be the subclass of H consisting of functions of the form :

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots \qquad (a \in \mathbb{C}). \tag{1.1}$$

Also, let T be the subclass of H consisting of functions of the form:

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$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.2}$$

Let $f, g \in T$. The function f is said to be subordinate to g, or g is said to be superordinate to f, if there exists a Schwarz function w analytic in U with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$ such that $f(z) = g(w(z))$. In such a case we write $f \prec g$ or $f(z) \prec g(z)(z \in U)$. If g is univalent in U, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.

Let $p, h \in H$ and $\psi(r, s, t; z): \mathbb{C}^3 \times U \longrightarrow \mathbb{C}$. If p and $\psi(p(z), zp'(z), z^2p''(z); z)$ are univalent functions in U and if p satisfies the second-order differential superordination

$$h(z) \prec \psi(p(z), zp'(z), z^2p''(z); z), \tag{1.3}$$

then p is called a solution of the differential superordination (1.3). (If f is subordinate to g, then g is superordinate to f). An analytic function q is called a subordinant of (1.3),if $q \prec p$ for all the functions p satisfying (1.3). An univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all the subordinants q of (1.3) is called the best subordinant. Miller and Macanu [12] have obtained conditions on the functions h, q and ψ for which the following implication holds:

$$h(z) \prec \psi(p(z), zp'(z), z^2p''(z); z) \implies q(z) \prec p(z). \tag{1.4}$$

For $\alpha \in R$, $\beta \geq 0$ $with$ $\alpha + \beta > 0$, $m, \delta \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $f \epsilon \mathcal{A}$. The differential operator

$\mathcal{W}_{\alpha,\beta}^{j,\delta}: T \longrightarrow T$ (see[10]) is defined by

$$\mathcal{W}_{\alpha,\beta}^{j,\delta} f(z) = z + \sum_{n=2}^{\infty} \left[\sum_{m=1}^{j} \binom{j}{m} (-1)^{m+1} \left(\frac{\alpha^m + n\beta^m}{\alpha^m + \beta^m}\right)\right]^{\delta} a_n z^n . \tag{1.5}$$

We note from (1.5) that, we have

$$z\left(\mathcal{W}_{\alpha,\beta}^{j,\delta} f(z)\right)' = \left[\!\left[\sum_{m=1}^{j} \binom{j}{m} (-1)^{m+1} \left(\left(\frac{\alpha}{\beta}\right)^m + 1\right)\right]\!\right] \mathcal{W}_{\alpha,\beta}^{j,\delta+1} f(z) - \left[\!\left[\sum_{m=1}^{j} \binom{j}{m} (-1)^{m+1} \left(\frac{\alpha}{\beta}\right)^m\right]\!\right] \mathcal{W}_{\alpha,\beta}^{j,\delta} f(z) . \tag{1.6}$$

Ali et al. [1] obtained sufficient conditions for certain normalized analytic functions to satisfy

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z)$$

where q_1 and q_2 are given univalent functions in U with $q_1(0) = q_2(0) = 1$. Also, Tuneski [15] obtained a sufficient conditions for starlikeness of f in terms of the quantity $\frac{f''(z)f(z)}{(f'(z))^2}$. Recently, Shanmugam et al. [13,14], Atshan et al. ([2], [3], [4], [5], [6], [7]) ,Goyal et al. [9] also obtained sandwich results for certain classes of analytic functions.The main object of the present paper is to find sufficient conditions for certain normalized analytic functions f to satisfy

$$q_1(z) \prec \left(\frac{\mathcal{W}_{\alpha,\beta}^{j,\delta} f(z)}{z}\right)^{\gamma} \prec q_2(z)$$

and

$$q_1(z) \prec \left(\frac{\mathcal{W}_{\alpha,\beta}^{j,\delta+1} f(z)}{\mathcal{W}_{\alpha,\beta}^{j,\delta} f(z)}\right)^{\gamma} \prec q_2(z)\,,$$

where q_1 and q_2 are given univalent functions in U with $q_1(0) = q_2(0) = 1$.

2. Preliminaries

In order to prove our subordination and superordination results, we need the following definition and lemmas.

Definition 2.1 [11]. Denote by Q the set of all functions f that are analytic and injective on $\overline{U} \setminus E(f)$, where $\overline{U} = U \cup \{ z \in \partial U\}$ *and*

$$\mathrm{E}(f) = \left\{ \zeta \in \partial U : \lim_{z\to\zeta} f(z) = \infty \right\} \tag{2.1}$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(f)$.Further, let the subclass of Q for which $f(z)$= a be denoted by Q(a),Q(0)=Q_0 and Q(1)=Q_1=$\{f \epsilon Q : f(0) = 1\}$.

Lemma 2.1 [11]. Let q be univalent in the unit disk U and let θ and ϕ be analytic in a domain D containing $q(U)$ with $\phi(w) \neq 0$ when $w \in q(U)$. Set $Q(z) = zq'(z)\phi\big(q(z)\big)$ and $h(z) = \theta\big(q(z)\big) + Q(z)$. Suppose that

(i) $Q(z)$ is starlike univalent in U,

(ii) $Re\left\{\frac{zh'(z)}{Q(z)}\right\} > 0$ for $z \in U$.

If p is analytic in U, with $p(0) = q(0), p(U) \subset D$ and

$$\theta\big(p(z)\big) + zp'(z)\phi\big(p(z)\big) \prec \theta\big(q(z)\big) + zq'(z)\phi\big(q(z)\big), \tag{2.2}$$

then $p \prec q$ and q is the best dominant of (2.2).

Lemma 2.2 [12]. Let q be a convex univalent function in U and let $\alpha \in \mathbb{C}$, $\beta \in \mathbb{C} \setminus \{0\}$ with

$$Re\left\{1 + \frac{zq''(z)}{q'(z)}\right\} > \max\left\{0, -Re\left(\frac{\alpha}{\beta}\right)\right\}.$$

If p is analytic in U and

$$\alpha p(z) + \beta zp'(z) \prec \alpha q(z) + \beta zq'(z), \tag{2.3}$$

then $p \prec q$ and q is the best dominant of (2.3).

Lemma 2.3 [12]. Let q be convex univalent in U and let $\beta \in \mathbb{C}$. Further assume that $Re(\beta) > 0$. If $p \in H\,[q(0),1] \cap Q$ and $p(z) + \beta zp'(z)$ is univalent in U, then

$$q(z) + \beta zq'(z) \prec p(z) + \beta zp'(z), \tag{2.4}$$

which implies that $q \prec p$ and q is the best subordinant of (2.4).

Lemma 2.4 [8]. Let q be convex univalent in the unit disk U and let θ and ϕ be analytic in a domain D containing $q(U)$. Suppose that

(i) $Re\left\{\frac{\theta'(q(z))}{\phi(q(z))}\right\} > 0$ for $z \in U$,

(ii) $Q(z) = zq'(z)\phi(q(z))$ is starlike univalent in U.

If $p \in H\,[q(0),1] \cap Q$, with $p(U) \subset D$, $\theta(p(z)) + zp'(z)\phi(p(z))$ is univalent in U and

$$\theta(q(z)) + zq'(z)\phi(q(z)) \prec \theta(p(z)) + zp'(z)\phi(p(z)), \tag{2.5}$$

then $q \prec p$ and q is the best subordinant of (2.5).

3. Subordination Results

Theorem 3.1. Let q be convex univalent in U with $q(0) = 1$, $0 \neq \varepsilon \in \mathbb{C}$, $\gamma > 0$ and suppose that q satisfies

$$Re\left\{1 + \frac{zq''(z)}{q'(z)}\right\} > \max\left\{0, -Re\left(\frac{\gamma}{\varepsilon}\right)\right\}. \tag{3.1}$$

If $f \in T$ satisfies the subordination

$$\left[\!\left[1 - \sum_{m=1}^{j}\binom{j}{m}(-1)^{m+1}\left(\left(\frac{\alpha}{\beta}\right)^m + 1\right)\right]\!\right]\left(\frac{\mathcal{W}_{\alpha,\beta}^{j,\delta}f(z)}{z}\right)^{\gamma} + \sum_{m=1}^{j}\binom{j}{m}(-1)^{m+1}\left(\left(\frac{\alpha}{\beta}\right)^m + 1\right)\left(\frac{\mathcal{W}_{\alpha,\beta}^{j,\delta}f(z)}{z}\right)^{\gamma}\left(\frac{\mathcal{W}_{\alpha,\beta}^{j,\delta+1}f(z)}{\mathcal{W}_{\alpha,\beta}^{j,\delta}f(z)}\right)$$
$$\prec q(z) + \frac{\varepsilon}{\gamma}zq'(z), \tag{3.2}$$

then

$$\left(\frac{\mathcal{W}_{\alpha,\beta}^{j,\delta}f(z)}{z}\right)^{\gamma} \prec q(z) \tag{3.3}$$

and q is the best dominant of (3.2).

Proof. Define the function p by

$$p(z) = \left(\frac{\mathcal{W}_{\alpha,\beta}^{j,\delta}f(z)}{z}\right)^{\gamma}. \tag{3.4}$$

Differentiating (3.4) logarithmically with respect to z, we get

$$\frac{zp'(z)}{p(z)} = \gamma\left(\frac{z\left(\mathcal{W}_{\alpha,\beta}^{j,\delta}f(z)\right)'}{\mathcal{W}_{\alpha,\beta}^{j,\delta}f(z)} - 1\right). \tag{3.5}$$

Now, in view of (1.6), we obtain the following subordination

$$\frac{zp'(z)}{p(z)} = \sum_{m=1}^{j}\binom{j}{m}(-1)^{m+1}\left(\left(\frac{\alpha}{\beta}\right)^{m}+1\right)\left(\frac{\mathcal{W}_{\alpha,\beta}^{j,\delta+1}f(z)}{\mathcal{W}_{\alpha,\beta}^{j,\delta}f(z)} - 1\right).$$

Therefore,

$$\frac{zp'(z)}{\gamma} = \sum_{m=1}^{j}\binom{j}{m}(-1)^{m+1}\left(\left(\frac{\alpha}{\beta}\right)^{m}+1\right)\left(\frac{\mathcal{W}_{\alpha,\beta}^{j,\delta}f(z)}{z}\right)^{\gamma}\left(\frac{\mathcal{W}_{\alpha,\beta}^{j,\delta+1}f(z)}{\mathcal{W}_{\alpha,\beta}^{j,\delta}f(z)} - 1\right).$$

The subordination (3.2) from the hypothesis becomes

$$p(z) + \frac{\varepsilon}{\gamma}zp'(z) \prec q(z) + \frac{\varepsilon}{\gamma}zq'(z).$$

An application of Lemma 2.2 with $\beta = \frac{\varepsilon}{\gamma}$ and $\alpha = 1$, we obtain (3.3).

Putting $q(z) = \left(\frac{1+z}{1-z}\right)^{\sigma}$ $(0 < \sigma \leq 1)$ in Theorem 3.1, we obtain the following corollary:

Corollary 3.1. Let $0 < \sigma \leq 1, 0 \neq \varepsilon \in \mathbb{C}, \gamma > 0$ and

$$Re\left\{\frac{1+2\sigma z+z^2}{1-z^2}\right\} > \max\left\{0, -Re\left(\frac{\gamma}{\varepsilon}\right)\right\}.$$

If $f \in T$ satisfies the subordination

$$\left[\!\left[1-\sum_{m=1}^{j}\binom{j}{m}(-1)^{m+1}\left(\left(\frac{\alpha}{\beta}\right)^m+1\right)\right]\!\right]\left(\frac{\mathcal{W}_{\alpha,\beta}^{j,\delta}f(z)}{z}\right)^{\gamma}+\sum_{m=1}^{j}\binom{j}{m}(-1)^{m+1}\left(\left(\frac{\alpha}{\beta}\right)^m+1\right)\left(\frac{\mathcal{W}_{\alpha,\beta}^{j,\delta}f(z)}{z}\right)^{\gamma}\left(\frac{\mathcal{W}_{\alpha,\beta}^{j,\delta+1}f(z)}{\mathcal{W}_{\alpha,\beta}^{j,\delta}f(z)}\right)$$
$$\prec\left(1+\frac{2\varepsilon\sigma z}{\gamma(1-z^2)}\right)\left(\frac{1+z}{1-z}\right)^{\sigma},$$

then

$$\left(\frac{\mathcal{W}_{\alpha,\beta}^{j,\delta}f(z)}{z}\right)^{\gamma}\prec\left(\frac{1+z}{1-z}\right)^{\sigma}$$

and $q(z)=\left(\frac{1+z}{1-z}\right)^{\sigma}$ is the best dominant.

Theorem 3.2. Let q be convex univalent in U with $q(0)=1, q(z)\neq 0\ (z\in U)$ and assume that q satisfies

$$Re\left\{1+\frac{xm}{\varepsilon}+\frac{y(m+1)}{\varepsilon}q(z)+(m-1)\frac{zq'(z)}{q(z)}+\frac{zq''(z)}{q'(z)}\right\}>0, \tag{3.6}$$

where $x,y,m\in\mathbb{C}, \varepsilon\in\mathbb{C}\setminus\{0\}$ and $z\in U$.

Suppose that $z\big(q(z)\big)^{m-1}q'(z)$ is starlike univalent in U . If $f\in T$ satisfies

$$\Omega(x,y,\gamma,j,\alpha,m,\beta,\varepsilon;z)\prec\big(x+yq(z)\big)\big(q(z)\big)^m+\varepsilon z\big(q(z)\big)^{m-1}q'(z), \tag{3.7}$$

where

$$\Omega(x,y,\gamma,j,\alpha,m,\beta,\varepsilon;z)=x\left(\frac{\mathcal{W}_{\alpha,\beta}^{j,\delta+1}f(z)}{\mathcal{W}_{\alpha,\beta}^{j,\delta}f(z)}\right)^{\gamma m}+y\left(\frac{\mathcal{W}_{\alpha,\beta}^{j,\delta+1}f(z)}{\mathcal{W}_{\alpha,\beta}^{j,\delta}f(z)}\right)^{\gamma(m+1)}+\varepsilon\gamma\left(\frac{\mathcal{W}_{\alpha,\beta}^{j,\delta+1}f(z)}{\mathcal{W}_{\alpha,\beta}^{j,\delta}f(z)}\right)\left(\frac{\mathcal{W}_{\alpha,\beta}^{j,\delta}f(z)}{\mathcal{W}_{\alpha,\beta}^{j,\delta+1}f(z)}-1\right)^{\gamma m},(\gamma>0,z\in U), \tag{3.8}$$

then

$$\left(\frac{\mathcal{W}_{\alpha,\beta}^{j,\delta+1}f(z)}{\mathcal{W}_{\alpha,\beta}^{j,\delta}f(z)}\right)^{\gamma}\prec q(z) \tag{3.9}$$

and q is the best dominant of (3.7).

Proof. Define the function p by

$$p(z)=\left(\frac{\mathcal{W}_{\alpha,\beta}^{j,\delta+1}f(z)}{\mathcal{W}_{\alpha,\beta}^{j,\delta}f(z)}\right)^{\gamma}. \tag{3.10}$$

By setting

$$\theta(w) = (x + yw)w^m \quad and \quad \phi(w) = \varepsilon w^{m-1}, w \neq 0,$$

we see that $\theta(w)$ is analytic in $\mathbb{C}$, $\phi(w)$ is analytic in $\mathbb{C} \setminus \{0\}$ and that $\phi(w) \neq 0, w \in \mathbb{C} \setminus \{0\}$. Also, we get

$$Q(z) = zq'(z)\phi\big(q(z)\big) = \varepsilon z\big(q(z)\big)^{m-1} q'(z)$$

and

$$h(z) = \theta\big(q(z)\big) + Q(z) = \big(x + yq(z)\big)\big(q(z)\big)^m + \varepsilon z\big(q(z)\big)^{m-1} q'(z).$$

It is clear that $Q(z)$ is starlike univalent in U,

$$Re\left\{\frac{zh'(z)}{Q(z)}\right\} = Re\left\{1 + \frac{xm}{\varepsilon} + \frac{y(m+1)}{\varepsilon} q(z) + (m-1)\frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q'(z)}\right\} > 0.$$

By a straightforward computation, we obtain

$$\big(x + yp(z)\big)\big(p(z)\big)^m + \varepsilon z\big(p(z)\big)^{m-1} p'(z) = \Omega(x, y, \gamma, j, \alpha, m, \beta, \varepsilon; z), \tag{3.11}$$

where is given $\Omega(x, y, \gamma, j, \alpha, m, \beta, \varepsilon; z)$ by (3.8).

From (3.7) and (3.11), we have

$$\big(x + yp(z)\big)\big(p(z)\big)^m + \varepsilon z\big(p(z)\big)^{m-1} p'(z) \prec \big(x + yq(z)\big)\big(q(z)\big)^m + \varepsilon z\big(q(z)\big)^{m-1} q'(z). \tag{3.12}$$

Therefore, by Lemma 2.1, we get $p(z) \prec q(z)$. By using (3.10), we obtain the result.

Putting $q(z) = \frac{1+Az}{1+Bz}$ $(-1 \leq B < A \leq 1)$ in Theorem 3.2, we obtain the following corollary:

Corollary 3.2. Let $-1 \leq B < A \leq 1$ and

$$Re\left\{\frac{xm}{\varepsilon} + \frac{y(m+1)(1+Az)}{\varepsilon(1+Bz)} + \frac{1+m(A-B)z-ABz^2}{(1+Az)(1+Bz)}\right\} > 0\,,$$

where $x, y, m \in \mathbb{C}, \varepsilon \in \mathbb{C} \setminus \{0\}$ and $z \in U$. If $f \in T$ satisfies

$$\Omega(x, y, \gamma, j, \alpha, m, \beta, \varepsilon; z) \prec \left(x + y\left(\frac{1+Az}{1+Bz}\right)\right)\left(\frac{1+Az}{1+Bz}\right)^m + \frac{\varepsilon(A-B)(1+Az)^{m-1}z}{(1+Bz)^{m+1}},$$

where is given $\Omega(x, y, \gamma, j, \alpha, m, \beta, \varepsilon; z)$ by (3.8),

then

$$\left(\frac{\mathcal{W}_{\alpha,\beta}^{j,\delta+1} f(z)}{\mathcal{W}_{\alpha,\beta}^{j,\delta} f(z)}\right)^\gamma \prec \frac{1+Az}{1+Bz}$$

and $q(z) = \frac{1+Az}{1+Bz}$ is the best dominant.

4. Superordination Results

Theorem 4.1. Let q be convex univalent in U with $q(0)=1, \gamma>0$ and $Re\{\varepsilon\}>0$. Let $f \in T$ satisfies

$$\left(\frac{\mathcal{W}_{\alpha,\beta}^{j,\delta}f(z)}{z}\right)^{\gamma} \in H[q(0),1] \cap Q$$

and

$$\left[\!\left[1-\varepsilon\sum_{m=1}^{j}\binom{j}{m}(-1)^{m+1}\left(\left(\frac{\alpha}{\beta}\right)^{m}+1\right)\right]\!\right]\left(\frac{\mathcal{W}_{\alpha,\beta}^{j,\delta}f(z)}{z}\right)^{\gamma}+\varepsilon\sum_{m=1}^{j}\binom{j}{m}(-1)^{m+1}\left(\left(\frac{\alpha}{\beta}\right)^{m}+1\right)\left(\frac{\mathcal{W}_{\alpha,\beta}^{j,\delta}f(z)}{z}\right)^{\gamma}\left(\frac{\mathcal{W}_{\alpha,\beta}^{j,\delta+1}f(z)}{\mathcal{W}_{\alpha,\beta}^{j,\delta}f(z)}\right)$$

be univalent in U. If

$$q(z)+\frac{\varepsilon}{\gamma}zq'(z) \prec \left[\!\left[1-\varepsilon\sum_{m=1}^{j}\binom{j}{m}(-1)^{m+1}\left(\left(\frac{\alpha}{\beta}\right)^{m}+1\right)\right]\!\right]\left(\frac{\mathcal{W}_{\alpha,\beta}^{j,\delta}f(z)}{z}\right)^{\gamma}+\varepsilon\sum_{m=1}^{j}\binom{j}{m}(-1)^{m+1}\left(\left(\frac{\alpha}{\beta}\right)^{m}+1\right)\left(\frac{\mathcal{W}_{\alpha,\beta}^{j,\delta}f(z)}{z}\right)^{\gamma}\left(\frac{\mathcal{W}_{\alpha,\beta}^{j,\delta+1}f(z)}{\mathcal{W}_{\alpha,\beta}^{j,\delta}f(z)}\right), \quad (4.1)$$

then

$$q(z) \prec \left(\frac{\mathcal{W}_{\alpha,\beta}^{j,\delta}f(z)}{z}\right)^{\gamma} \tag{4.2}$$

and q is the best subordinant of (4.1).

Proof. Define the function p by

$$p(z)=\left(\frac{\mathcal{W}_{\alpha,\beta}^{j,\delta}f(z)}{z}\right)^{\gamma}. \tag{4.3}$$

Differentiating (4.3) with respect to z logarithmically, we get

$$\frac{zp'(z)}{p(z)}=\gamma\left(\frac{z\left(\mathcal{W}_{\alpha,\beta}^{j,\delta}f(z)\right)'}{\mathcal{W}_{\alpha,\beta}^{j,\delta}f(z)}-1\right). \tag{4.4}$$

After some computations and using (1.6), from (4.4), we obtain

$$\left[\!\left[1-\varepsilon\sum_{m=1}^{j}\binom{j}{m}(-1)^{m+1}\left(\left(\frac{\alpha}{\beta}\right)^{m}+1\right)\right]\!\right]\left(\frac{\mathcal{W}_{\alpha,\beta}^{j,\delta}f(z)}{z}\right)^{\gamma}+\varepsilon\sum_{m=1}^{j}\binom{j}{m}(-1)^{m+1}\left(\left(\frac{\alpha}{\beta}\right)^{m}+1\right)\left(\frac{\mathcal{W}_{\alpha,\beta}^{j,\delta}f(z)}{z}\right)^{\gamma}\left(\frac{\mathcal{W}_{\alpha,\beta}^{j,\delta+1}f(z)}{\mathcal{W}_{\alpha,\beta}^{j,\delta}f(z)}\right)$$

$$=p(z)+\frac{\varepsilon}{\gamma}zp'(z),$$

and now, by using Lemma 2.3, we get the desired result.

Putting $q(z)=\left(\frac{1+z}{1-z}\right)^{\sigma}$ $(0<\sigma\leq 1)$ in Theorem 4.1, we obtain the following corollary:

Corollary 4.1. Let $0<\sigma\leq 1, \gamma>0$ and $Re\{\varepsilon\}>0$. If $f\in T$ satisfies

$$\left(\frac{\mathcal{W}_{\alpha,\beta}^{j,\delta}f(z)}{z}\right)^{\gamma}\in H\,[q(0),1]\cap Q$$

and

$$\left[\!\left[1-\varepsilon\sum_{m=1}^{j}\binom{j}{m}(-1)^{m+1}\left(\left(\frac{\alpha}{\beta}\right)^{m}+1\right)\right]\!\right]\left(\frac{\mathcal{W}_{\alpha,\beta}^{j,\delta}f(z)}{z}\right)^{\gamma}+\varepsilon\sum_{m=1}^{j}\binom{j}{m}(-1)^{m+1}\left(\left(\frac{\alpha}{\beta}\right)^{m}+1\right)\left(\frac{\mathcal{W}_{\alpha,\beta}^{j,\delta}f(z)}{z}\right)^{\gamma}\left(\frac{\mathcal{W}_{\alpha,\beta}^{j,\delta+1}f(z)}{\mathcal{W}_{\alpha,\beta}^{j,\delta}f(z)}\right)$$

be univalent in U. If

$$\left(1+\frac{2\varepsilon\sigma z}{\gamma(1-z^{2})}\right)\left(\frac{1+z}{1-z}\right)^{\sigma}\prec\left[\!\left[1-\varepsilon\sum_{m=1}^{j}\binom{j}{m}(-1)^{m+1}\left(\left(\frac{\alpha}{\beta}\right)^{m}+1\right)\right]\!\right]\left(\frac{\mathcal{W}_{\alpha,\beta}^{j,\delta}f(z)}{z}\right)^{\gamma}+\varepsilon\sum_{m=1}^{j}\binom{j}{m}(-1)^{m+1}\left(\left(\frac{\alpha}{\beta}\right)^{m}+1\right)\left(\frac{\mathcal{W}_{\alpha,\beta}^{j,\delta}f(z)}{z}\right)^{\gamma}\left(\frac{\mathcal{W}_{\alpha,\beta}^{j,\delta+1}f(z)}{\mathcal{W}_{\alpha,\beta}^{j,\delta}f(z)}\right),$$

then

$$\left(\frac{1+z}{1-z}\right)^{\sigma}\prec\left(\frac{\mathcal{W}_{\alpha,\beta}^{j,\delta}f(z)}{z}\right)^{\gamma}$$

and $q(z)=\left(\frac{1+z}{1-z}\right)^{\sigma}$ is the best subordinant.

Theorem 4.2. Let q be convex univalent in U with $q(0)=1$, and assume that q satisfies

$$Re\left\{\frac{xm}{\varepsilon}q'(z)+\frac{y(m+1)}{\varepsilon}q(z)q'(z)\right\}>0, \tag{4.5}$$

where $x,y,m\in\mathbb{C}, \varepsilon\in\mathbb{C}\setminus\{0\}$ and $z\in U$.

Suppose that $z(q(z))^{m-1}q'(z)$ is starlike univalent in U . Let $f\in T$ satisfies

$$\left(\frac{\mathcal{W}_{\alpha,\beta}^{j,\delta+1}f(z)}{\mathcal{W}_{\alpha,\beta}^{j,\delta}f(z)}\right)^{\gamma}\in H\,[q(0),1]\cap Q$$

and $\Omega(x,y,\gamma,j,\alpha,m,\beta,\varepsilon;z)$ is univalent in U, where is given $\Omega(x,y,\gamma,j,\alpha,m,\beta,\varepsilon;z)$ by (3.8). If

$$\left(x+yq(z)\right)\left(q(z)\right)^{m}+\varepsilon z\left(q(z)\right)^{m-1}q'(z)\prec\ \Omega(x,y,\gamma,j,\alpha,m,\beta,\varepsilon;z), \tag{4.6}$$

then

$$q(z) \prec \left(\frac{\mathcal{W}_{\alpha,\beta}^{j,\delta+1} f(z)}{\mathcal{W}_{\alpha,\beta}^{j,\delta} f(z)}\right)^{\gamma} \qquad (4.7)$$

and q is the best subordinant of (4.6).

Proof. Define the function p by

$$p(z) = \left(\frac{\mathcal{W}_{\alpha,\beta}^{j,\delta+1} f(z)}{\mathcal{W}_{\alpha,\beta}^{j,\delta} f(z)}\right)^{\gamma}. \qquad (4.8)$$

By setting

$$\theta(w) = (x + yw)w^m \quad and \quad \phi(w) = \varepsilon w^{m-1}, w \neq 0,$$

we see that $\theta(w)$ is analytic in $\mathbb{C}$, $\phi(w)$ is analytic in $\mathbb{C} \setminus \{0\}$ and that $\phi(w) \neq 0, w \in \mathbb{C} \setminus \{0\}$. Also, we get

$$Q(z) = zq'(z)\phi\big(q(z)\big) = \varepsilon z\big(q(z)\big)^{m-1} q'(z).$$

It is clear that $Q(z)$ is starlike univalent in U,

$$Re\left\{\frac{\theta'(q(z))}{\phi(q(z))}\right\} = Re\left\{\frac{xm}{\varepsilon} q'(z) + \frac{y(m+1)}{\varepsilon} q(z)q'(z)\right\} > 0.$$

By a straight forward computation, we obtain

$$\Omega(x, y, \gamma, j, \alpha, m, \beta, \varepsilon; z) = \big(u + vp(z)\big)\big(p(z)\big)^m + \eta z\big(p(z)\big)^{m-1} p'(z), \qquad (4.9)$$

where $\Omega(x, y, \gamma, j, \alpha, m, \beta, \varepsilon; z)$ is given by (3.8).

From (4.6) and (4.9), we have

$$\big(x + yq(z)\big)\big(q(z)\big)^m + \varepsilon z\big(q(z)\big)^{m-1} q'(z) \prec \big(x + yp(z)\big)\big(p(z)\big)^m + \varepsilon z\big(p(z)\big)^{m-1} p'(z). \qquad (4.10)$$

Therefore, by Lemma 2.4, we get $q(z) \prec p(z)$. By using (4.8), we obtain the result.

5. Sandwich Results

Concluding the results of differential subordination and superordination, we arrive at the following "sandwich results".

Theorem 5.1. Let q_1 be convex univalent in U with $q_1(0) = 1$, $Re\{\varepsilon\} > 0$ and let q_2 be univalent in U, $q_2(0) = 1$ and satisfies (3.1). Let $f \in T$ satisfies

$$\left(\frac{\mathcal{W}_{\alpha,\beta}^{j,\delta}f(z)}{z}\right)^{\gamma} \in H\,[1,1] \cap Q$$

and

$$\left[\!\left[1-\varepsilon\sum_{m=1}^{j}\binom{j}{m}(-1)^{m+1}\left(\left(\frac{\alpha}{\beta}\right)^{m}+1\right)\right]\!\right]\left(\frac{\mathcal{W}_{\alpha,\beta}^{j,\delta}f(z)}{z}\right)^{\gamma}+\varepsilon\sum_{m=1}^{j}\binom{j}{m}(-1)^{m+1}\left(\left(\frac{\alpha}{\beta}\right)^{m}+1\right)\left(\frac{\mathcal{W}_{\alpha,\beta}^{j,\delta}f(z)}{z}\right)^{\gamma}\left(\frac{\mathcal{W}_{\alpha,\beta}^{j,\delta+1}f(z)}{\mathcal{W}_{\alpha,\beta}^{j,\delta}f(z)}\right)$$

be univalent in U. If

$$q_1(z)+\frac{\varepsilon}{\gamma}zq_1{}'(z) \prec \left[\!\left[1-\varepsilon\sum_{m=1}^{j}\binom{j}{m}(-1)^{m+1}\left(\left(\frac{\alpha}{\beta}\right)^{m}+1\right)\right]\!\right]\left(\frac{\mathcal{W}_{\alpha,\beta}^{j,\delta}f(z)}{z}\right)^{\gamma}+\varepsilon\sum_{m=1}^{j}\binom{j}{m}(-1)^{m+1}\left(\left(\frac{\alpha}{\beta}\right)^{m}+1\right)\left(\frac{\mathcal{W}_{\alpha,\beta}^{j,\delta}f(z)}{z}\right)^{\gamma}\left(\frac{\mathcal{W}_{\alpha,\beta}^{j,\delta+1}f(z)}{\mathcal{W}_{\alpha,\beta}^{j,\delta}f(z)}\right) \prec q_2(z)+\frac{\varepsilon}{\gamma}zq_2{}'(z),$$

then

$$q_1(z) \prec \left(\frac{\mathcal{W}_{\alpha,\beta}^{j,\delta}f(z)}{z}\right)^{\gamma} \prec q_2(z)$$

and q_1 and q_2 are, respectively, the best subordinant and the best dominant.

Theorem 5.2. Let q_1 be convex univalent in U with $q_1(0) = q_2(0) = 1.$ suppose q_1 satisfies (4.5) and q_2 satisfies (3.6). Let $f \in T$ satisfies

$$\left(\frac{\mathcal{W}_{\alpha,\beta}^{j,\delta+1}f(z)}{\mathcal{W}_{\alpha,\beta}^{j,\delta}f(z)}\right)^{\gamma} \in H\,[1,1] \cap Q$$

and $\Omega(x,y,\gamma,j,\alpha,m,\beta,\varepsilon;z)$ is univalent in U, where $\Omega(x,y,\gamma,j,\alpha,m,\beta,\varepsilon;z)$ is given by (3.8). If

$$\big(x+yq_1(z)\big)\big(q_1(z)\big)^{m}+\varepsilon z\big(q_1(z)\big)^{m-1}q_1{}'(z) \prec \Omega(x,y,\gamma,j,\alpha,m,\beta,\varepsilon;z) \prec \big(x+yq_2(z)\big)\big(q_2(z)\big)^{m}+\varepsilon z\big(q_2(z)\big)^{m-1}q_2{}'(z),$$

then

$$q_1(z) \prec \left(\frac{\mathcal{W}_{\alpha,\beta}^{j,\delta+1}f(z)}{\mathcal{W}_{\alpha,\beta}^{j,\delta}f(z)}\right)^{\gamma} \prec q_2(z)$$

and q_1 and q_2 are, respectively, the best subordinant and the best dominant.

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