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Infected Intermediate Predator and Harvest in Food Chain

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ABSTRACT

In this paper, a mathematical model consisting of the food chain model with disease in intermediate predator is proposed and discussed. The food chain model consists of four types: prey, intermediate predator, infected intermediate predator, and top predator. We studied the solutions for the original model and positive and bounded solutions in the sub models. Also found equilibrium points with sufficient and necessary conditions. By using Jacobian matrix and Lyapunov function to provide local and global stability. Can use the harvesting to control the disease and it can be used as tool to prevent disease transformation into an epidemic. Finally, some results were illustrated in numerical simulations.

MSC : 30C45 , 30C50

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1 . Introduction

One of the most important prominent issues in ecosystems is the impact of some infectious diseases in addition to the environmental aspect point. Thus it gives some scientists and researchers in the field of the environment great importance to develop an important tool alongside the experimental environment and describe how the disease spreads in most populations and turns from susceptible to infected populations. When diseases spread among the population, communities compete with other species in the same place for food, survival and predation . This is because no

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group can survive on its own alone, researchers and scientists have presented numerous studies that have been studied in describing population interaction. Researchers first describe Lautka and Volterra in modern mathematical sciences, describing the competition between predators and prey. But most models involve injury of one type of population originated from the classic movement of Kermack and McKendrick [1]. After these two wonderful works, the door become open for researchers to offer many studies in epidemiology and environmental science theory. Even in the last few decades, these models have become important tools for analyzing and understanding the spread of infectious diseases and controlling them. One of the most important studies of the predator model was the study presented by researchers [2]. Many researchers [3, 4, 5, 6] also studied these models with diseases. Also studied [10, 4, 8, 7] the role of disease in destabilizing the system. The atmosphere of the regions is important for biological activities primarily responsible for environmental changes. The coexistence of interacting biological species has been very important in the past few decades, and has been extensively studied using mathematical models by many researchers [11-20] . As a result, many species are extinct and many other extinctions due to the presence of many external forces and influences such as over-exploitation. From this basis, we study the predatory prey, where the predator is at risk of disease and harvest. [12] Krivan proposed a mathematical model and research on the effects of antimicrobial behavior on the predator system. Anti-social behavior has been shown to lead to constant fluctuations and low population density. Chattobadai et al. [13] prey - predator model with some cover on the prey species. There is the observation that the stability of the global system around the positive balance does not necessarily mean continuity of the system. More recently, Kar [21] has proposed a model for the study of predatory prey and independent harvesting on any species. Has shown that the use of harvest and control efforts, it is possible to break the periodic behavior of the system. In the above investigations, the dynamics of the predator living in the unprotected area with prey are also not explicitly studied. The protected area plays a vital and important role in the aquatic environment to protect resources and fisheries from overexploitation [22-26]. In particular, Dubey et al [22] add a proposal from the mathematical model to analyze the dynamics of the fisheries resource system in a two-zone aquatic environment, the first free fishing zone and the second protected area where strict fishing is prohibited. He pointed out that even if the fisheries are continuously exploited in an unprotected area, fish are at an appropriate level of habitat balance. Moreover, harvest is one of the best and most important means of combating and eradicating the disease and the epidemic and preventing its spread among the population. Using this method requires great care and care. Very severe because any misuse may expose species of extinction. There are several important studies that have been studied on this important subject, and the harvest [9] in the predator model has been studied to create a controlled environment while ensuring species survival and continuous harvesting. Continuous harvesting in prey predator model in [5, 29]. This is paper divide into seven sections, section two is the described and developed of the model, section three contained nature of the solutions , section four use a Dynamic of Subsystem to study subsystems of system (1), section five is Existence of Equilibrium points and Stability in the model, section six Numerical Simulation, for Dynamical. Finally, section seven contained Conclusions.

2. The Food Chain Model with Holling Type II and Harvest

$$\begin{cases} \dfrac{dx}{dt} = rx(1-x) - \alpha_0 \dfrac{xy}{1+\mu x} - \alpha_4 \dfrac{xy_1}{1+\mu x} \\ \dfrac{dy}{dt} = \alpha_1 \dfrac{xy}{1+\mu x} - \alpha_2 \dfrac{yz}{1+\mu y} - \dfrac{cyy_1}{y+y_1} + ky_1 - d_1 y \\ \dfrac{dy_1}{dt} = \dfrac{cyy_1}{y+y_1} + \alpha_5 \dfrac{xy_1}{1+\mu x} - ky_1 - qy_1 \\ \dfrac{dz}{dt} = \alpha_3 \dfrac{yz}{1+\mu y} - d_2 z \end{cases} \qquad (1)$$

The following model describes the relation between food chain function with $\alpha_0 \dfrac{xy}{1+\mu x}$ is the Michaelis-Menten type (or Holling type II) functional response, intermediate predator become infected with relative function $\dfrac{cyy_1}{y+y_1}$. Where $x', y', y_1', z' > 0$. x, y, y_1, z denoted susceptible prey, intermediate predator ,infected intermediate predator, and top predator respectively. Parameters denoted as follows, r the rate of growth of susceptible prey, μ is the half saturation constant, rate α_0 is the per capita rate of predation of the intermediate predator, rate α_1 measures the efficiency of biomass conversion from prey to intermediate predator, rate α_2 is the per capita rate of predation of the top predator, rate α_3 measures the efficiency of biomass conversion from intermediate predator to top predator, rate α_4 is the per capita rate of predation of the prey, rate α_5 measures the efficiency of biomass conversion from infected intermediate predator to top predator. Rate c is the contact between susceptible intermediate predator (S. predator) and infected intermediate predator (I. predator) while rate k denoted the transformation from I. intermediate predator to S, intermediate predator, as this model known SIS, d_1, d_2 are natural death of intermediate and top predator respectively. Rate q is harvesting of I. intermediate predator.

3. Nature of Solution

Lemma 1: All the solutions of the system (1) in □ are positive and bounded.

Proof:

Let $M(t)=x(t)+y(t)+y_1(t)+z(\mathrm{t})$ and a constant $\mu>0$. Then

$$\frac{dM}{dt}=\frac{dx}{dt}+\frac{dy}{dt}+\frac{dy_1}{dt}+\frac{dz}{dt}$$

$$\frac{dM}{dt}+\mu x=rx(1-x)-(\alpha_0-\alpha_1)\frac{xy}{1+\mu x}-(\alpha_4-\alpha_5)\frac{xy_1}{1+\mu x}$$

$$-(\alpha_2-\alpha_3)\frac{yz}{1+\mu y}-d_1y-qy_1-d_2z+\mu x$$

Since $\alpha_0>\alpha_1$, $\alpha_2>\alpha_3$ and $\alpha_4>\alpha_5$ [28], then

$$\frac{dM}{dt}+\mu x\le -rx^2+(r+\mu)x$$

$$\frac{dM}{dt}+\mu x\le -\frac{(r+\mu)^2}{4r}$$ say ν, then we get

$$0<M(x,y,y_1,z)\le\frac{\nu}{\mu}\left(1-e^{-\mu t}\right)+e^{-\mu t}(x,y,y_1,z)\big|_{t=0}\ [27]$$

4. Dynamic of Subsystems

In this section we want to study subsystems of system (1). Subsystems obtained in case of extinction one or two population of system (1). Therefore, there are many subsystems, as system (1) as classical model contains prey and predator only, system (1) contains prey and infected predator, system (1) contain all population without top predator, system (1) without disease, finally all population survive.

4.1 System (1) as Classical Model.

System (1) without top predator and disease known classical model or Lotka Volterra equations. In this subsystem the interaction between prey and predator only without any external influence. We describe this interaction as follows:

$$\begin{cases}\dfrac{dx}{dt}=rx(1-x)-\alpha_0\dfrac{xy}{1+\mu x}\\[2mm] \dfrac{dy}{dt}=\alpha_1\dfrac{xy}{1+\mu x}-d_1y\end{cases}\qquad (2)$$

4.1.1. Nature of Solution

Lemma 2: All solutions of subsystem (2) are positive and bounded.

Proof: As lemma 1, see Figure 1.

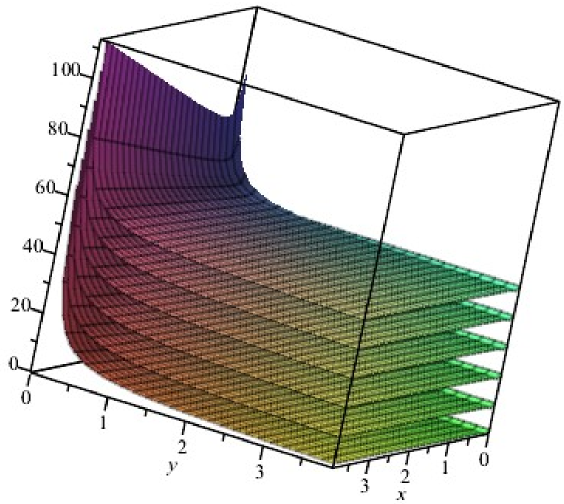

Figure 1. All solutions of subsystem (2) are positive and bounded

Lemma 3: In subsystem (2), $\alpha_1 > d_1$.

Proof: Suppose that $\alpha_1 \le d_1$, then $\alpha_1 y \le d_1 y$. Since carrying capacity of prey is one, then $\frac{x}{1+\mu x} < 1$, hence $\frac{\alpha_1 xy}{1+\mu x} < \alpha_1 y \le d_1 y$ therefore $\alpha_1 \frac{xy}{1+\mu x} \le d_1 y$. This means $\frac{dy}{dt} \le 0$. Which is contradiction, then $\alpha_1 > d_1$

Lemma 4: Subsystem (2) has no periodic orbit in□ ̂ .

Let $H = \frac{1}{xy}$, $h_1 = rx(1-x) - \alpha_0 \frac{xy}{1+\mu x}$ and $h_2 = \alpha_1 \frac{xy}{1+\mu x} - d_1 y$, then

$$Hh_1 = \frac{r}{y}(1-x) - \alpha_0 \frac{1}{1+\mu x}$$

And $Hh_2 = \alpha_1 \frac{1}{1+\mu x} - \frac{d_1}{x}$

$$\nabla(x,y) = \frac{\partial(h_1,H)}{\partial x} + \frac{\partial(h_2,H)}{\partial y}$$

$$\nabla(x,y) = \frac{-r}{y} + \frac{\alpha_0 \mu}{(1+\mu x)^2} < 0$$

Note that $\nabla(\hat{x},\hat{y})$ does not change sign and is not identically zero in □ ̂ . Therefore according to Bendixson-Dulic criterion, there subsystem (2) has no periodic solution.

4.1.2 Equilibrium Point and Stability

Subsystem (2) has three equilibrium points $P_{2,0}(0,0), P_{2,1}(1,0)$ and $P_{2,2}(\hat{x},\hat{y})$, where

$\hat{x} = \frac{}{(\alpha_1 - d_1\mu)}$ and $\hat{y} = \frac{)(1+\mu\hat{x})}{\alpha_0}$, with condition $\alpha_1 > d_1\mu$. Jacobian matrix of subsystem (2) is

$$J_2 = \begin{bmatrix} r - 2rx - \frac{\alpha_0 y}{(1+\mu\hat{x})} & \frac{\alpha_0 x}{(1+\mu x)} \\ \alpha_1 \frac{y}{(1+\mu x)^2} & \frac{\alpha_1 x}{1+\mu x} - d_1 \end{bmatrix}$$

The characteristic equation near $P_{3,2}(\hat{x},\hat{y})$ is $\lambda^2 + \left(-r(1-2\hat{x}) + \frac{\hat{x}}{1+\mu\hat{x}} \right) \quad \alpha_1 \frac{\hat{x}\hat{y}}{(1+\mu\hat{x})}$

and by Routh-Hurwitz criterion it's stability if $\hat{x} < \frac{}{2}$.

Lemma 5: Positive equilibrium point $P_{3,2}(\hat{x},\hat{y})$ of the subsystem (2) is globally asymptotically stable in Int, with conditions $y\hat{x} > x\hat{y}$

Proof:

Suppose $G(x,y) = C_1 G_1(x,y) + C_2 G_2(x,y)$, where

$G_1 = C_1\left(x - \hat{x} - \hat{x} \ln \frac{}{\hat{x}} \right)$ and $G_2 = C_2\left(y - \hat{y} - \hat{y} \ln \frac{}{\hat{y}} \right)$

$$G_1' = C_1(x-\hat{x})\left(-rx - \alpha_0 \frac{}{1+\mu x} + r\hat{x} + \alpha_0 \frac{}{1+\mu\hat{x}} \right)$$

$$= -rC_1(x-\hat{x}) - \alpha_0 C_1 \left(\frac{(x-\hat{x})(y-\hat{y}) - \mu(x\hat{y} - y\hat{x})(x-\hat{x})}{(1+\mu x)(1+\mu\hat{x})} \right)$$

$$G_2' = C_2(y-\hat{y})\left(\alpha_1 \left(\frac{x}{1+\mu x} - \frac{\hat{x}}{1+\mu\hat{x}} \right) \right)$$

$$= C_2 \frac{\alpha_1 (x-\hat{x})(y-\hat{y})}{(1+\mu x)(1+\mu\hat{x})}$$

Choose $C_2 = C_1 \frac{\alpha_0}{\alpha_1}$ then

$$\frac{dG}{dt} = -rC_1(x-\hat{x}) - \alpha_0 C_1 \left(\frac{\mu(y\hat{x} - x\hat{y})(x-\hat{x})}{(1+\mu x)(1+\mu\hat{x})} \right).$$

Figure 2. with $r = 0.78, \alpha_0 = 0.5, \alpha_1 = 0.4, \mu = 0.5, d_1 = 0.2$

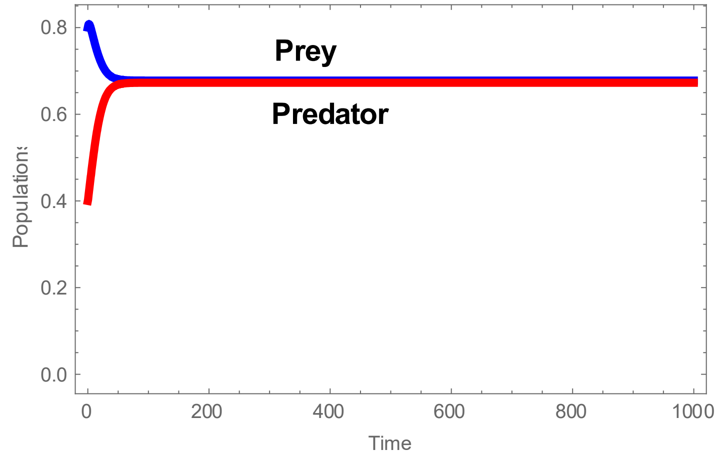

Figure 2. Osculation of prey and predator of subsystem (2).

4.2 Prey and Infected Predator.

When the disease turns into an epidemic and there is no top predator in this case, system (1) becomes follows:

$$\begin{cases} \dfrac{dx}{dt} = rx(1-x) - \alpha_4 \dfrac{xy_1}{1+\mu x} \\ \dfrac{dy_1}{dt} = \alpha_5 \dfrac{xy_1}{1+\mu x} - ky_1 - qy_1 \end{cases} \qquad (3)$$

4.2.1. Nature of Solution

Lemma 6: All solutions of sub system (3) are positive and bounded.

Proof: As lemma 1, see figure 3.

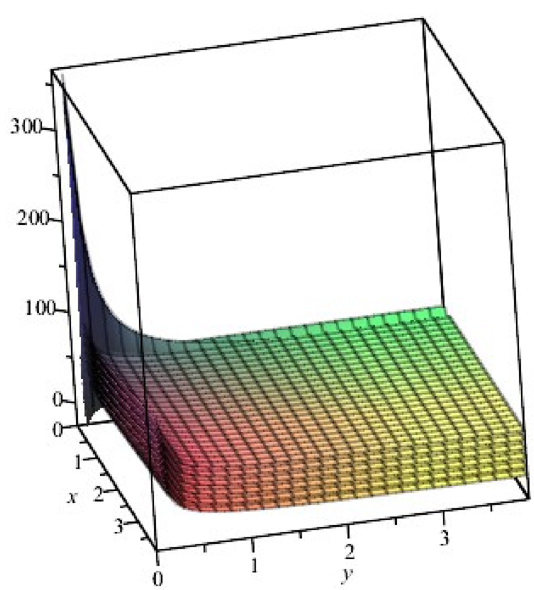

Figure 3. All solutions of subsystem (3) are positive and bounded

Lemma 7: In subsystem (3), $\alpha_5 > k + q$.

Proof: Suppose that $\alpha_5 \le k+q$, then $\alpha_5 y_1 \le (k+q)y_1$. Since the carrying capacity of prey is one this means $\frac{x}{1+\mu x} \le 1$. So

$\alpha_5 \frac{xy_1}{1+\mu x} \le (k+q)y_1$, hence $\frac{dy_1}{dt} \le 0$ which it is contradiction, therefore $\alpha_5 > k+q$.

Lemma 8: Subsystem (3) has no periodic orbit in □$_{(x,y_1)}$.

Proof:

Let $H = \frac{1}{xy_1}$, $Hh_1 = rx(1-x) - \alpha_4 \frac{xy_1}{1+\mu x}$ and $h_2 = \alpha_5 \frac{xy_1}{1+\mu y_1} - ky_1 - qy_1$, then

$Hh_1 = \frac{r}{y_1}(1-x) - \alpha_4 \frac{1}{1+\mu x}$ and $Hh_2 = \alpha_5 \frac{1}{1+\mu x} - \frac{k}{x} - \frac{q}{x}$

$$\nabla(x, y_1) = \frac{\partial(h_1, H)}{\partial x} + \frac{\partial(h_2, H)}{\partial y_1}$$

$$\nabla(x, y_1) = \frac{-r}{y1} + \alpha_4 \frac{\mu}{(1+\mu x)^2} < 0$$

Note that $\nabla(x, y_1)$ does not change sign and is not identically zero in □$_{(x,y_1)}$. Therefore according to Bendixson-Dulic criterion, there subsystem (3) has no periodic solution.

4.2.2 Equilibrium Points and Stability

Subsystem (3) has three equilibrium points $P_{3,0}(0,0), P_{3,1}(1,0)$ and $P_{3,2}(\bar{x}, \bar{y}_1)$ where

$$\bar{x} = \frac{k+q}{\alpha_5 - (k+q)\mu} \text{ and } \bar{y}_1 = \frac{r(1-\bar{x})(1+\mu\bar{x})}{\alpha_4}$$

with condition $\alpha_5 > (k+q)\mu$.

Jacobian matrix of subsystem (3) is

$$J_3 = \begin{bmatrix} r - 2rx - \frac{\alpha_4 y_1}{(1+\mu x)^2} & -\frac{\alpha_4 x}{1+\mu x} \\ \frac{\alpha_5 y_1}{(1+\mu x)^2} & \frac{\alpha_5 x}{1+\mu x} - k - q \end{bmatrix}$$

The characteristic equation near $P_{3,2}(\bar{x}, \bar{y})$ is $\lambda^2 - \left[r(1-2\bar{x}) - \frac{r(1-\bar{x})}{1+\mu\bar{x}} \right]\lambda + \alpha_4\alpha_5 \frac{\bar{x}\bar{y}_1}{(1+\mu\bar{y}_1)^3} = 0$.

By Routh-Hurwitz criterion this point is stability if $\bar{x} \ge \frac{1}{2}$.

Lemma 9: Positive equilibrium point $P_{4,2}(\bar{x}, \bar{y}_1)$ of the subsystem (3) is globally asymptotically stable in Int □$_{(x,y_1)}$, with condition $y\bar{x} > x\bar{y}$.

Proof: As proof in Lemma 4.

See figures 4, 5 with fixed parameters as

$r = 0.78, \alpha_4 = 0.5, \alpha_5 = 0.4, \mu = 0.5, k = 0.2$

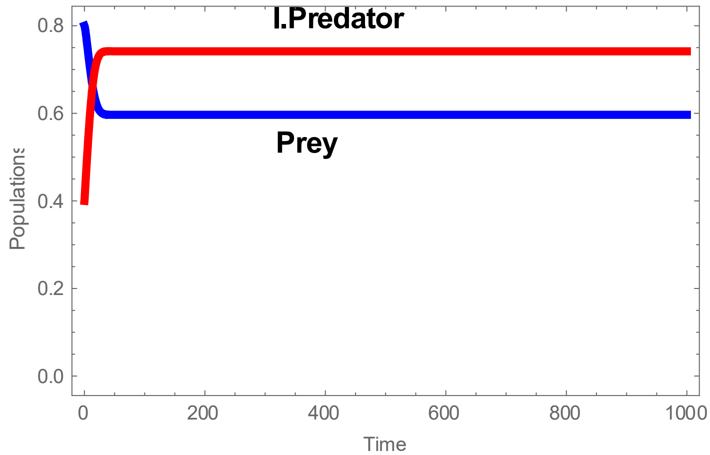

Figure 4. Global stability of subsystem (3) without harvesting

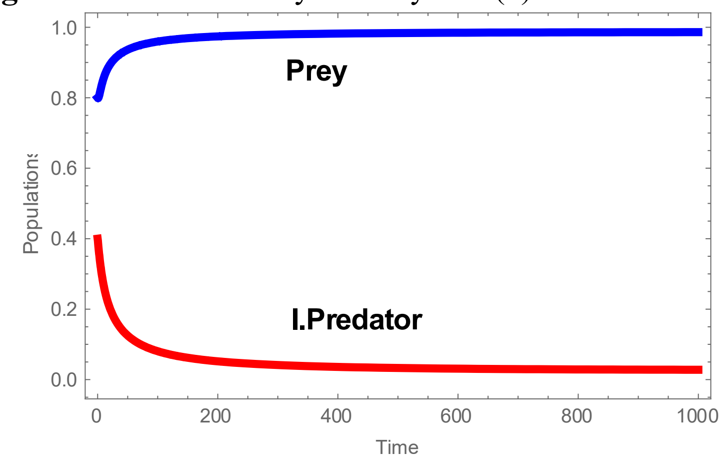

Figure 5. Global stability of subsystem (3) with harvesting *q=0.1*

4.3. In the Absence of Disease

This subsystem without infected predator, contains three population, prey, intermediate predator and top predator respectively. In this case this model known as Food Chain Model. As such model, intermediate predator dependent wholly on prey while top predator wholly dependent on intermediate predator. We can describe that as follows:

$$\begin{cases} \dfrac{dx}{dt} = rx(1-x) - \alpha_0 \dfrac{xy}{1+\mu x} \\ \dfrac{dy}{dt} = \alpha_1 \dfrac{xy}{1+\mu x} - \alpha_2 \dfrac{yz}{1+\mu y} - d_1 y \\ \dfrac{dz}{dt} = \alpha_3 \dfrac{yz}{1+\mu y} - d_2 z \end{cases} \qquad (4)$$

4.3.1. Natural of Solution

Lemma 9: All the solutions of the sub system (4) in □ are positive and bounded.

Proof: Same proof in Lemma 1.

Lemma 10: In subsystem (4) $\alpha_3 > d_2$

Proof: As lemma 6.

4.3.2. Existence of Equilibrium Points and Stability

The model in the system (4) has the following equilibrium points:

- The trivial equilibrium points $P_{4,0}(0,0,0)$ always exists.
- The equilibrium points $P_{4,1}(1,0,0)$ exists on the boundary of the octant.
- The nontrivial equilibrium points $P_{4,2}(\hat{x},\hat{y},\hat{z})$ where

$\hat{x}=\dfrac{\alpha_2\hat{z}+d_1+d_1\mu\hat{y}}{\alpha_1(1+\mu\hat{y})-\alpha_2\mu\hat{z}-d_1\mu(1+\mu\hat{y})}$, $\hat{y}=\dfrac{d_2}{(\alpha_3-d_2\mu)}$ and $\hat{z}=\dfrac{\alpha_3}{\alpha_2 d_2}\left(\dfrac{\alpha_1\hat{x}}{1+\mu\hat{x}}-d_1\right)\hat{y}$ with conditions $\alpha_3>d_2\mu$

$\alpha_1(1+\mu\hat{y})>\alpha_2\mu\hat{z}+d_1\mu(1+\mu\hat{y})$ and $\dfrac{\alpha_1\hat{x}}{1+\mu\hat{x}}>d_1$

The Jacobian matrix is:

$$J_4=\begin{bmatrix} r-2rx-\alpha_0\dfrac{y}{(1+\mu x)^2} & -\alpha_0\dfrac{x}{1+\mu x} & 0 \\ \alpha_1\dfrac{y}{(1+\mu x)^2} & \alpha_1\dfrac{x}{1+\mu x}-\alpha_2\dfrac{z}{(1+\mu y)^2}-d_1 & -\alpha_2\dfrac{y}{1+\mu y} \\ 0 & \alpha_3\dfrac{z}{(1+\mu y)^2} & \alpha_3\dfrac{y}{1+\mu y}-d_2 \end{bmatrix}$$

Jacobian matrix near positive equilibrium point is

$$J_4=\begin{bmatrix} r-2r\hat{x}-\alpha_0\dfrac{\hat{y}}{(1+\mu\hat{x})^2} & -\alpha_0\dfrac{\hat{x}}{1+\mu\hat{x}} & 0 \\ \alpha_1\dfrac{\hat{y}}{(1+\mu\hat{x})^2} & \alpha_1\dfrac{\hat{x}}{1+\mu\hat{x}}-\alpha_2\dfrac{\hat{z}}{(1+\mu\hat{y})^2}-d_1 & -\alpha_2\dfrac{\hat{y}}{1+\mu\hat{y}} \\ 0 & \alpha_3\dfrac{\hat{z}}{(1+\mu\hat{y})^2} & 0 \end{bmatrix}$$

$$\begin{vmatrix} M_1-\lambda & M_2 & 0 \\ M_3 & M_4-\lambda & M_5 \\ 0 & M_6 & -\lambda \end{vmatrix}=0$$

Also we will only study the positive point $P_{4,2}$, the characteristic equation near this point is $\lambda^3+A\lambda^2+B\lambda+C=0$, where

$A=-(M_1+M_4)>0$ if $\hat{x}\geq\dfrac{1}{2}$ and $\alpha_1\dfrac{\hat{x}}{1+\mu\hat{x}}>\alpha_2\dfrac{\hat{z}}{(1+\mu\hat{y})^2}+d_1$

$B=M_1M_4-M_2M_3-M_5M_6$

$C=M_1M_5M_6>0$

By Routh-Hurwitz criterion its stability if $AB-C>0$.

Lemma 11: Equilibrium point $P_{4,2}(\hat{x},\hat{y},\hat{z})$ is globally asymptotically stable provided that the following conditions hold $y\hat{x}>x\hat{y}$ and $\hat{y}z>y\hat{z}$

Proof: As proof in Lemma 4.
See figures 6-a and 6-b with fixed parameters as
$r = 0.78, \alpha_0 = 0.5, \alpha_1 = 0.4, \alpha_3 = 0.35, \alpha_4 = 0.3, \mu = 0.5, d_1 = 0.074, d_2 = 0.07$

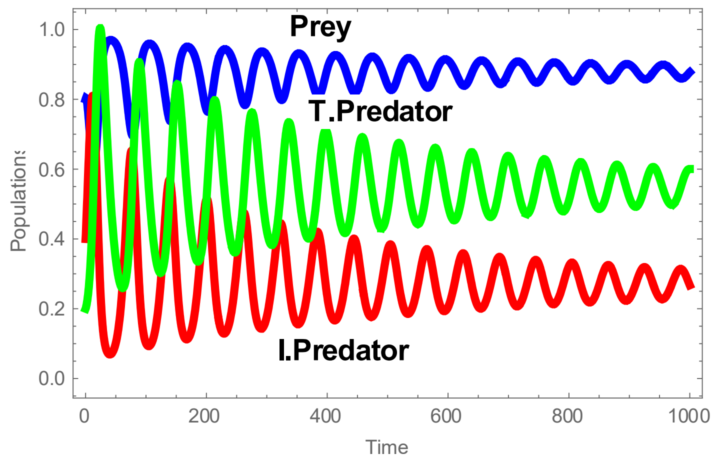

Figure 6-a. Global stability of subsystem (4)

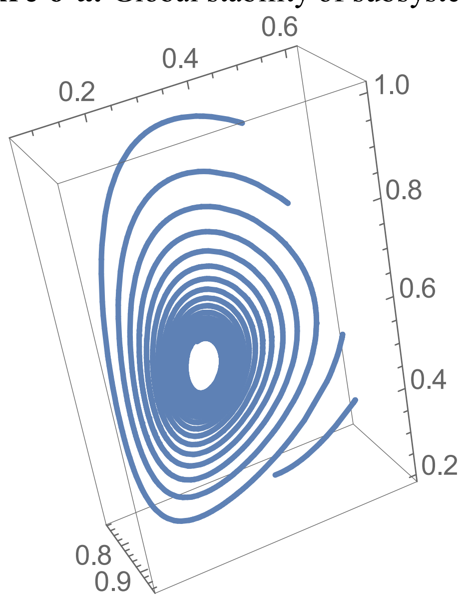

Figure 6-b. Global stability of subsystem (4)

4.4. In the Absence of Top Predator

In absence top predator, System (1) becomes prey, susceptible and infected predator. This model known classical model with disease. Interaction between these populations describe as follows:

$$\begin{cases} \dfrac{dx}{dt} = rx(1-x) - \alpha_0 \dfrac{xy}{1+\mu x} - \alpha_4 \dfrac{xy_1}{1+\mu x} \\ \dfrac{dy}{dt} = \alpha_1 \dfrac{xy}{1+\mu x} - \dfrac{cyy_1}{y+y_1} + ky_1 - d_1 y \\ \dfrac{dy_1}{dt} = \dfrac{cyy_1}{y+y_1} + \alpha_5 \dfrac{xy_1}{1+\mu x} - ky_1 - qy_1 \end{cases} \quad (5)$$

4.4.1Natural of Solution

Lemma 12: All the solutions of the sub system (5) in are positive and bounded.

Proof: As proof in Lemma 1.

4.4.2Existence of Equilibrium Points and Stability

The model in the system (5) has the following equilibrium points:

- The trivial equilibrium points $P_{5,0}(0,0,0)$ always exists.
- The equilibrium points $P_{5,1}(1,0,0)$ exists on the boundary of the octant.
- The nontrivial equilibrium points $P_{5,2}(\bar{\bar{x}}, \bar{\bar{y}}, \bar{\bar{y}}_1)$ where

$$\bar{\bar{x}} = \frac{\left(\dfrac{c\overline{\overline{yy}}_1}{\bar{\bar{y}}+\bar{\bar{y}}_1} - k\bar{\bar{y}}_1 + d_1\bar{\bar{y}}\right)}{\alpha_1\bar{\bar{y}} - \mu\left(\dfrac{c\overline{\overline{yy}}_1}{\bar{\bar{y}}+\bar{\bar{y}}_1} - k\bar{\bar{y}}_1 + d_1\bar{\bar{y}}\right)}, \bar{\bar{y}} = \frac{y_1\left(k+q-\alpha_5\dfrac{\bar{\bar{x}}}{1+\mu\bar{\bar{x}}}\right)}{c-\left(k+q-\alpha_5\dfrac{x}{1+\mu x}\right)} \text{ and}$$

$$\bar{\bar{y}}_1 = \frac{1+\mu\bar{\bar{x}}}{\alpha_4}\left[r(1-\bar{\bar{x}}) - \alpha_0\frac{\bar{\bar{y}}}{1+\mu\bar{\bar{x}}}\right]$$ with conditions

$\alpha_1\bar{\bar{y}} > \mu\left(\dfrac{c\overline{\overline{yy}}_1}{\bar{\bar{y}}+\bar{\bar{y}}_1} - k\bar{\bar{y}}_1 + d_1\bar{\bar{y}}\right), c > \left(k+q-\alpha_5\dfrac{x}{1+\mu x}\right)$ and $r(1-\bar{\bar{x}}) > \alpha_0\dfrac{\bar{\bar{y}}}{1+\mu\bar{\bar{x}}}$. The Jacobian matrix of subsystem (5) near $P_{5,2}$ is :

$$J_5 = \begin{bmatrix} \dfrac{-r\bar{\bar{x}}\left(1-\mu+2\mu\bar{\bar{x}}\right)}{1+\mu\bar{\bar{x}}} - \lambda & -\alpha_0\dfrac{\bar{\bar{x}}}{1+\mu\bar{\bar{x}}} & -\alpha_4\dfrac{\bar{\bar{x}}}{1+\mu\bar{\bar{x}}} \\ \alpha_1\dfrac{\bar{\bar{y}}}{(1+\mu x)^2} & \alpha_1\dfrac{\bar{\bar{x}}}{1+\mu\bar{\bar{x}}} - \dfrac{c\bar{\bar{y}}_1^2}{(\bar{\bar{y}}+\bar{\bar{y}}_1)^2} - d_1 - \lambda & -\dfrac{c\bar{\bar{y}}}{(\bar{\bar{y}}+\bar{\bar{y}}_1)^2} + k \\ \alpha_5\dfrac{\bar{\bar{y}}_1}{(1+\mu\bar{\bar{x}})^2} & \dfrac{c\bar{\bar{y}}_1^2}{(y+\bar{\bar{y}}_1)^2} & \dfrac{c\bar{\bar{y}}^2}{(\bar{\bar{y}}+\bar{\bar{y}}_1)^2} + \alpha_5\dfrac{\bar{\bar{x}}}{(1+\mu\bar{\bar{x}})} - k - q - \lambda \end{bmatrix}$$

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix} = 0$$

Also, we will only study the positive point $P_{5,2}(\bar{\bar{x}}, \bar{\bar{y}}, \bar{\bar{y}}_1)$, the characteristic equation near this point is $\lambda^3 + A\lambda^2 + B\lambda + C = 0$ Where

$A = -a_{11} - a_{22} - a_{33}$

$B = a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33} - a_{23}a_{32} - a_{12}a_{21} - a_{13}a_{31}$

$C = -a_{11}a_{22}a_{33} + a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33} - a_{12}a_{23}a_{31} - a_{13}a_{21}a_{32} + a_{13}a_{31}a_{22}$

By Routh-Hurwitz criterion, and because $AB - C > 0$, its stability

Lemma 13: Equilibrium point $P_5\left(\bar{\bar{x}}, \bar{\bar{y}}, \bar{\bar{y}}_1\right)$ of subsystem (5) is globally stability provided that the following conditions hold $x\bar{\bar{y}} > y\bar{\bar{x}}, x\bar{\bar{y}}_1 > y_1\bar{\bar{x}}$ and $y\bar{\bar{y}}_1 > y_1\bar{\bar{y}}$

Proof: As proof in Lemma 4.

Figures 7-a and 7-b, with fixed parameters as: $r = 0.6, \alpha_0 = 0.44,$
$\alpha_1 = 0.309, \alpha_4 = 0.479, \alpha_5 = 0.292, \mu = 0.406, k = 0.095, d_1 = 0.126, c = 0.202$

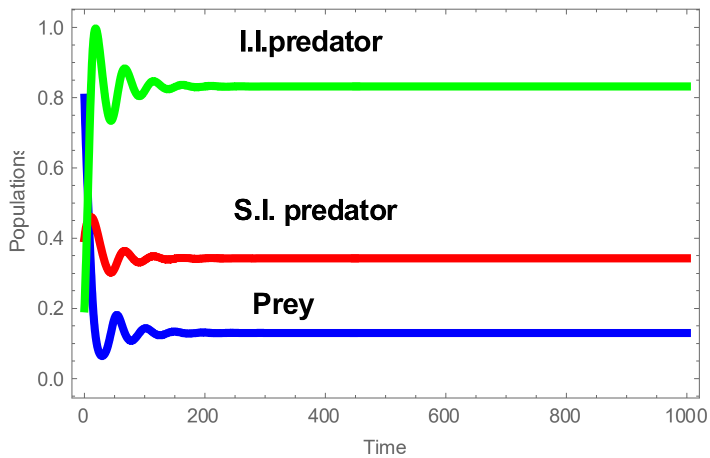

Figure 7-a: Global stability of subsystem (5) without harvesting

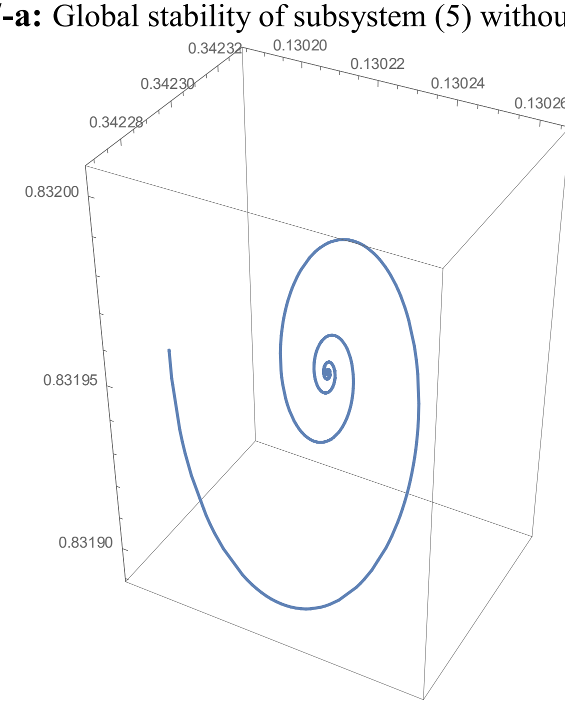

Figure 7-b: Global stability of subsystem (5) without harvesting

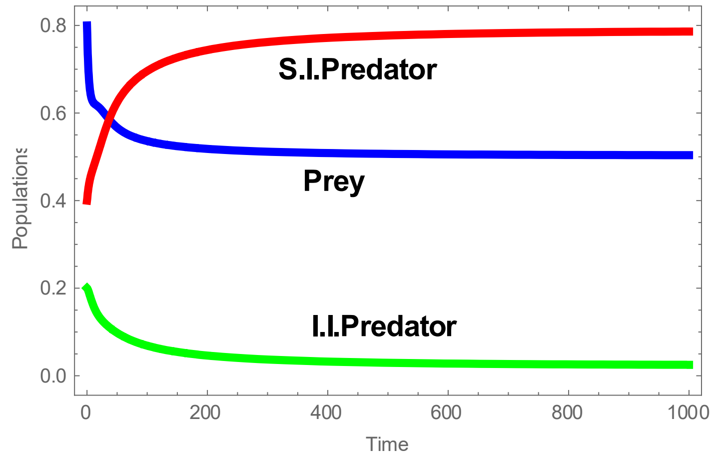

Figure 8: global stability of subsystem (5) with harvesting

5. Existence of Equilibrium Points and Stability of Original System

- Always vanishing equilibrium point $E_0(0,0,0,0)$ exist.
- Boundary point $E_1(1,0,0,0)$.
- Finally, positive equilibrium point $E_2(x^*,y^*,y_1^*,z^*)$, where

$$x^*=\frac{\left(\alpha_2\frac{y^*z^*}{1+\mu y^*}+\frac{cy^*y_1^*}{y^*+y_1^*}-ky_1^*+d_1y^*\right)}{\alpha_1y^*-\mu\left(\alpha_2\frac{y^*z^*}{1+\mu y^*}+\frac{cy^*y_1^*}{y^*+y_1^*}-ky_1^*+d_1y^*\right)},$$

$$y^*=\frac{y_1\left(k+q-\alpha_5\frac{x^*}{1+\mu x^*}\right)}{c-\left(k+q-\alpha_5\frac{x^*}{1+\mu x^*}\right)},y_1^*=\frac{1+\mu x^*}{\alpha_4}\left[r(1-x^*)-\alpha_0\frac{y^*}{1+\mu x^*}\right]\text{and}$$

$$z^*=\frac{\alpha_3}{\alpha_2d_2}\left(-\alpha_1\frac{x^*}{1+\mu x^*}+\frac{cy_1^*}{y^*+y_1^*}-\frac{ky_1^*}{y^*}+d_1\right)y^*\text{ with conditions}$$

$$\alpha_1y^*>\mu\left(\alpha_2\frac{y^*z^*}{1+\mu y^*}+\frac{cy^*y_1^*}{y^*+y_1^*}-ky_1^*+d_1y^*\right),c>\left(k+q-\alpha_5\frac{x^*}{1+\mu x^*}\right),$$

$$\frac{cy_1^*}{y^*+y_1^*}+d_1>\frac{ky_1^*}{y^*}+\alpha_1\frac{x^*}{1+\mu x^*}$$

$$r(1-x^*)>\alpha_0\frac{y^*}{1+\mu x^*}$$

The Jacobian matrix is

$$J=\begin{bmatrix} r-2rx-\alpha_0\frac{y}{(1+\mu x)^2}-\alpha_4\frac{y_1}{(1+\mu x)^2} & -\alpha_0\frac{x}{1+\mu x} & -\alpha_4\frac{x}{1+\mu x} & 0 \\ \alpha_1\frac{y}{(1+\mu x)^2} & \alpha_1\frac{x}{1+\mu x}-\alpha_2\frac{z}{(1+\mu y)^2}-\frac{cy_1^2}{(y+y_1)^2}-d_1 & -\frac{cy^2}{(y+y_1)^2}-k & -\alpha_2\frac{y}{1+\mu y} \\ \alpha_5\frac{y_1}{(1+\mu x)^2} & \frac{cy_1^2}{(y+y_1)^2} & -\frac{cy^2}{(y+y_1)^2}+k-q & 0 \\ 0 & \alpha_3\frac{z}{(1+\mu y)^2} & 0 & \alpha_3\frac{y}{1+\mu y}-d_2 \end{bmatrix}$$

$$E=\begin{bmatrix} r-2rx^*-\frac{r(1-x^*)}{1+\mu x^*}-\lambda & -\alpha_0\frac{x^*}{1+\mu x^*} & \alpha_4\frac{x^*}{(1+\mu y_1^2)^2} & 0 \\ \alpha_1\frac{y^*}{(1+\mu x^*)^2} & \alpha_1\frac{x^*}{1+\mu x^*}-\alpha_2\frac{z^*}{(1+\mu y^*)^2}-\frac{cy_1^*}{(y+y_1^*)^2}-d_1-\lambda & -\frac{cy^{*2}}{(y^*+y_1^*)^2} & -\alpha_2\frac{y^*}{1+\mu y^*} \\ \alpha_5\frac{y_1^*}{1+\mu y_1^*} & \frac{cy_1^*}{(y^*+y_1^*)^2} & -\frac{cy^*}{(y^*+y_1^*)^2}+k-q-\lambda & 0 \\ 0 & \alpha_3\frac{z^*}{(1+\mu y^*)^2} & 0 & -\lambda \end{bmatrix}$$

$$\begin{vmatrix} J_{11}-\lambda & J_{12} & J_{13} & 0 \\ J_{21} & J_{22}-\lambda & J_{23} & J_{24} \\ J_{31} & J_{32} & J_{33}-\lambda & 0 \\ 0 & J_{42} & 0 & -\lambda \end{vmatrix}=0$$

The stability of these points as follows:

Then the characteristic equation of $J(E_2)$ is given by:

$$\lambda^4+A\lambda^3+B\lambda^2+C\lambda+D=0$$

$$A=-J_{11}-J_{22}-J_{33}$$

$$B=J_{11}J_{22}+J_{11}J_{33}-J_{23}J_{32}+J_{22}J_{33}-J_{12}J_{21}-J_{13}J_{31}-J_{24}J_{42}$$

$$C=-J_{11}J_{22}J_{33}+J_{11}J_{24}J_{42}+J_{11}J_{23}J_{32}+J_{12}J_{21}J_{33}-J_{13}J_{21}J_{32}+J_{13}J_{22}J_{31}+J_{24}J_{33}J_{42}-J_{12}J_{23}J_{31}$$

$$D=-J_{11}J_{24}J_{33}J_{42}+J_{13}J_{24}J_{31}J_{42}$$

$$\therefore \Delta=ABC-C^2-A^2D$$

Also, by Routh Hurwitz theorem this is stable if $A>0, C>0, D>0$

Lemma 14: Assume that the positive equilibrium point $E(x^*,y^*,y^*_1,z^*)$ of system (1) is globally provided that the following conditions hold $xy^*>yx^*, xy_1^*>y_1x^*, y^*z>yz^*$ and $yy_1^*>y_1y^*$

Proof: As proof in Lemma 4.

6. Numerical Simulation

In this section, we want illustration some results by employ Mathematica Programing in the system (1). To discuss the effect of cure rate from the disease and harvesting on the behavior of the solution. Above all, by looking at a number of papers and taking advantage of the conditions discussed in this paper, we have installed parameters as follows: $r = 0.88, \alpha_0 = 0.662$, $\alpha_1 = 0.558, \alpha_2 = 0.51, \alpha_3 = 0.413, \alpha_4 = 0.576, \alpha_5 = 0.192, c = 0,247. k = 0$, $d_1 = 0.109, \mathrm{d}_2 = 0.077$ .Two cases that will be discussed: First case, when the model is the kind of *SI*. In such model, susceptible intermediate predator (S.I. Predator) become infected intermediate predator (I.I. Predator) and not able to become susceptible again. Then we employ the harvest to see its impact on behavior. Figure (9) the behavior of solution of system (1) as *SI* model without harvesting. Figure (10) the behavior of solution with harvesting. Note in these two cases how employ the harvesting to disease control.

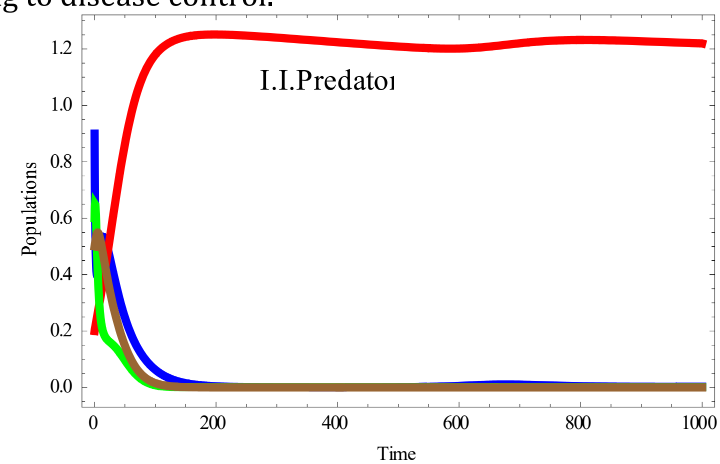

Figure 9: *SI* model without harvesting **(q=0)**

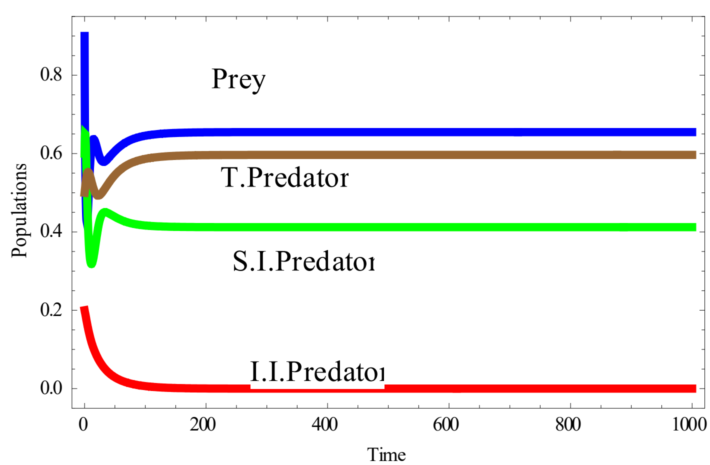

Figure 10: *SI* model with harvesting **(q=0.067)**

The second case, the model is the kind *SIS*. In such model susceptible intermediate predator become infected intermediate predator and become susceptible again. Figure (11) behavior of solution of system (1) as *SIS* model without harvesting, while figure (12) employ the harvesting to disease control. Also, we note the effect of harvesting on disease to not become as epidemic.

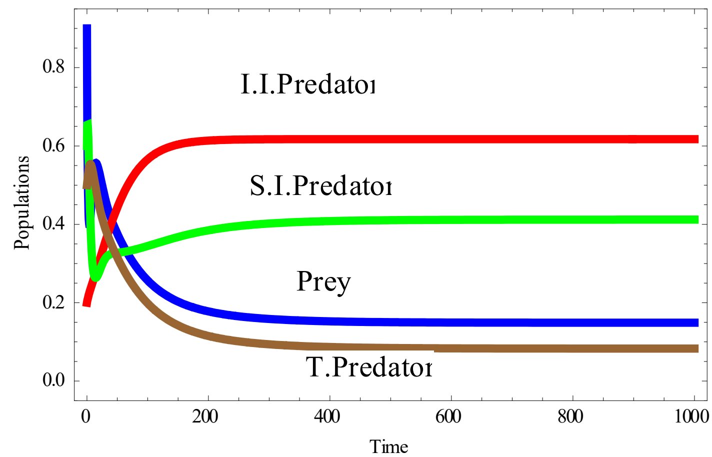

Figure. 11: SIS model without harvesting**(k=0.026,q=0)**

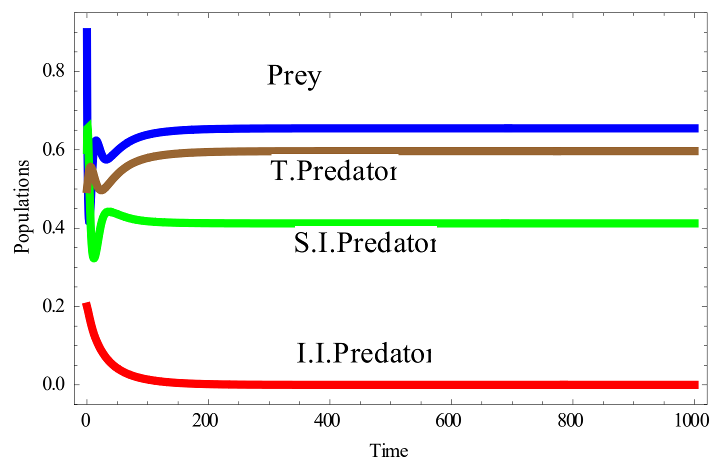

Figure.12: SIS model with harvesting **(k=0.026,q=0.067)**

7. Consolation

In this research, we studied the dynamic system of the food chain model when the intermediate predator is at risk of disease. All system solutions have proven to be positive and limited, including subsystems. Equilibrium points were found in the subsystems and the original system and the necessary and sufficient conditions for their existence were found in addition to conditions of local and global stability. It turns out that the stability conditions of the subsystems are necessary for the stability of the original system. The disease is controlled and prevented from becoming as epidemic by numerical simulation.

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