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Weakly Nearly Semiprime Submodules and Some Related Concepts

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ABSTRACT

In present paper the concept of WN-semiprime submodule of unitary left R-module $\boldsymbol{G}$ over a commutative ring $\boldsymbol{R}$ with identity are introduced and studied, as a generalization of weakly semiprime submodules. Numerous basic properties of this concept are investigate. Where a proper submodule A of an R-module $\boldsymbol{G}$ is said to be WN-semiprime, if whenever $\boldsymbol{0 \neq r^n x \in A}$, for $\boldsymbol{r \in R , x \in G \text{ and } n \in Z^+}$, implies that $\boldsymbol{rx \in A + Rad(G)}$. Furthermore many characterizations of WN-semiprime submodule are introduced. Moreover, some properties of WN-semiprime submodules in some classes of modules are given.

MSC:

1. Introduction

The concept of a weakly semiprime submodule was first introduced by Tavallaee Zalfoghari in 2012, where a proper submodule A of an R-module F is called weakly semiprime, if whenever $0 \neq r^k x \in A$, for $r \in R , x \in F \text{ and } k \in Z^+$, implies that $rx \in A$[1]. Recently the concept of weakly semiprime submodule was generalization to the concept of weakly semi-2-Absorbing submodules in 2018 [2]. The concept of WE-semiprime submodule as a strong from of weakly semiprime submodule was introduced in [3]. In this note we investigate the concept WN-semiprime submodule, where a proper submodule A of an R-module G is called WN-semiprime, if whenever $0 \neq r^n x \in A$, for $r \in R , x \in G \text{ and } n \in Z^+$, implies that $rx \in A + Rad(G)$. Where $Rad(G)$ is the intersection of all maximal submodule of G, $Rad(G)$ is called the Jacobson radical of G. Also the concept of WN-semiprime submodule is a generalization of WNprime submodule, which introduced in [4]. Recall that an R-module G is faithful, if $ann_R(G) = (0)$, where $aan_R(G) = \{r \in R : rG = (0)\}$ [5]. "An R-

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module G is called multiplication if every submodule A of G is of the form $A = IG$ for some ideal I of R, in particular $A = [A:_R G]G$ [6]".

2. Basic properties of WN-semiprime submodules

In this part of this research we introduce the definition of WN-semiprime submodule and we give some basic properties, characterizations and examples.

Definition 2.1 A proper submodule A of an R-module G is called a weakly nearly semiprime (for short WN-semiprime) submodule of G, if whenever $0 \neq r^n x \in A$, for $r \in R, x \in G$ *and* $n \in Z^+$, implies that $rx \in A + Rad(G)$. And an ideal J of a ring R is called a WN-semiprime ideal of R, if J is a WN-semiprime submodule of an R-module R.

Remarks and Examples 2.2

(1). It is clear that every weakly semiprime submodule of an R-module G is a WN-semiprime submodule , but not conversely, the following example shows that.

Consider the Z-module Z_{24} and the submodule $A = \{\bar{0}, \bar{8}, \overline{16}\}$ of Z_{24}. A is a WN-semiprime submodule of Z_{24} since $Rad(Z_{24}) = \langle\bar{2}\rangle \cap \langle\bar{3}\rangle = \langle\bar{6}\rangle = \{\bar{0}, \bar{6}, \overline{12}, \overline{18}\}$ and whenever $0 \neq r^n x \in A$ for $r \in Z, x \in Z_{24}$ *and* $n \in Z^+$, implies that $rx \in A + Rad(Z_{24}) = \{\bar{0}, \bar{8}, \overline{16}\} + \{\bar{0}, \bar{6}, \overline{12}, \overline{18}\} = \langle\bar{2}\rangle = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}, \bar{8}, \overline{10}, \overline{12}, \overline{14}, \overline{16}, \overline{18}, \overline{20}, \overline{22}\}$. But A is not weakly semiprime submodule of Z_{24} since $0 \neq 2^2\, \bar{2} \in A$ for $2 \in Z, \bar{2} \in Z_{24}$, but $2\, \bar{2} \notin A$.

(2). The submodule $B = \langle\overline{12}\rangle$ of the Z-module Z_{24} is a WN-semiprime submodule of Z_{24}, since $0 \neq 2^2\, \bar{3} \in B$ for $2 \in Z, \bar{3} \in Z_{24}$, implies that $2\, \bar{3} = \bar{6} \in B + Rad(Z_{24}) = \langle\overline{12}\rangle + \langle\bar{6}\rangle = \langle\bar{6}\rangle$.

(3). It is clear that every WNprime submodule of an R-module G is a WN-semiprime submodule , but not conversely, the following example explains that.

In the Z-module Z_{24} we shows that by (2) the submodule $A = \langle\overline{12}\rangle$ is a WN-semiprime. But A is not WNprime submodule of Z_{24} for $0 \neq 3\, \bar{4} \in \langle\overline{12}\rangle$ for $3 \in Z, \bar{4} \in Z_{24}$, but $\bar{4} \notin \langle\overline{12}\rangle + Rad(Z_{24}) = \langle\overline{12}\rangle + \langle\bar{6}\rangle = \langle\bar{6}\rangle$ and $3 \notin [\langle\overline{12}\rangle + Rad(Z_{24}):_Z Z_{24}] = [\langle\bar{6}\rangle :_Z Z_{24}] = 6Z$.

The following proposition gives some equivalent characterizations of WN-semiprime submodule.

Proposition 2.3 Let G be an R-module and A be a proper submodule of G. Then the following statements are equivalent.

(1). A is a WN-semiprime submodule of G.

(2). $[A:_G \langle r^n\rangle] \subseteq [\langle 0\rangle :_G \langle r^n\rangle] \cup [A + Rad(G):_G \langle r\rangle]$.

(3). Either $[A:_G \langle r^n\rangle] \subseteq [\langle 0\rangle :_G \langle r^n\rangle]$ or $[A:_G \langle r^n\rangle] \subseteq [A + Rad(G):_G \langle r\rangle]$.

Proof (1) $\Longrightarrow$(2) Let $x \in [A:_G \langle r^n\rangle]$, implies that $r^n x \in A$, if $0 \neq r^n x \in A$ and A is a WN-semiprime it follows that $rx \in A + Rad(G)$, hence $x \in [A + Rad(G):_G \langle r\rangle]$. Thus $\in [\langle 0\rangle :_G \langle r^n\rangle] \cup [A + Rad(G):_G \langle r\rangle]$. If $r^n x = 0$, then $x \in [\langle 0\rangle :_G \langle r^n\rangle]$, it follows that $x \in [\langle 0\rangle :_G \langle r^n\rangle] \cup [A + Rad(G):_G \langle r\rangle]$. Hence $[A:_G \langle r^n\rangle] \subseteq [\langle 0\rangle :_G \langle r^n\rangle] \cup [A + Rad(G):_G \langle r\rangle]$.

(2) $\Longrightarrow$ (3) clearly.

(3) $\Longrightarrow$ (1) Suppose that $0 \neq r^n x \in A$ for $r \in R, x \in G$ *and* $n \in Z^+$, hence $x \in [A:_G \langle r^n \rangle]$ and $x \notin [\langle 0 \rangle :_G \langle r^n \rangle]$ then by hypothesis $x \in [A + Rad(G):_G \langle r \rangle]$. That is $rx \in A + Rad(G)$, and it gives that A is a WN-semiprime submodule of G .

The following proposition gives another characterization of WN-semiprime submodule of G.

Proposition 2.4 Let G be an R-module and A be a proper submodule of G. Then A is a WN-semiprime submodule of G if and only if $0 \neq \langle r \rangle^n B \subseteq A$ for $r \in R, B$ is a submodule of G and $n \in Z^+$, implies that $\langle r \rangle B \subseteq A + Rad(G)$.

Proof ($\Longrightarrow$) Suppose that $0 \neq \langle r \rangle^n B \subseteq A$ for $r \in R, B$ is a submodule of G and $n \in Z^+$that is $0 \neq \langle r^n \rangle B \subseteq A$, it follows that $B \subseteq [A:_G \langle r^n \rangle]$ with $B \nsubseteq [\langle 0 \rangle :_G \langle r^n \rangle]$. Thus by Proposition(2.3) $B \subseteq [A + Rad(G):_G \langle r \rangle]$, it follows that $\langle r \rangle B \subseteq A + Rad(G)$.

($\Longleftarrow$)Let $0 \neq r^n x \in A$ for $r \in R, x \in G$ *and* $n \in Z^+$. Implies that $0 \neq \langle r^n \rangle \langle x \rangle \subseteq A$, that is $0 \neq \langle r \rangle^n \langle x \rangle \subseteq A$, so by hypothesis $\langle r \rangle \langle x \rangle \subseteq A + Rad(G)$, implies that $r\,x \in A + Rad(G)$. Then A is a WN-semiprime submodule of G.

As a direct consequence of Proposition (2.4) we get the following corollaries

Corollary 2.5 Let G be an R-module and A be a proper submodule of G. Then A is a WN-semiprime submodule of G if and only if $0 \neq \langle r \rangle^n G \subseteq A$ for $r \in R,$ and $n \in Z^+$, implies that $\langle r \rangle G \subseteq A + Rad(G)$.

Corollary 2.6 Let G be an R-module and A be a proper submodule of G. Then A is a WN-semiprime submodule of G if and only if $0 \neq r^n B \subseteq A$ for $r \in R, B$ is a submodule of G and $n \in Z^+$, implies that $rB \subseteq A + Rad(G)$.

The following proposition gives a condition under which WN-semiprime submodules are weakly semiprime.

Proposition 2.7 Let G be an R-module and A be a WN-semiprime submodule of G with $Rad(G) \subseteq A$. Then A be a weakly semiprime submodule of G.

Proof Let $0 \neq r^n x \in A$ for $r \in R, x \in G$ *and* $n \in Z^+$. Since A a WN-semiprime submodule, then $rx \in A + Rad(G)$. But $Rad(G) \subseteq A$, it follows that $Rad(G) + A = A$, hence $rx \in A$. Hence A is a weakly semiprime submodule of G.

It is well-known that if A is a submodule of an R-module G with $Rad\left(\frac{G}{A}\right) = (0)$, then $Rad(G) \subseteq A$ [5, Theo. (9.1.4)(b)]. So we get the following corollary.

Corollary 2.8 Let G be an R-module and A be a WN-semiprime submodule of G with $Rad\left(\frac{G}{A}\right) = (0)$. Then A be a weakly semiprime submodule of G**.**

Another condition on WN-semiprime submodule of G to be weakly semiprime in the following result.

Proposition 2.9 Let G be an R-module and A be a WN-semiprime submodule of G with $Rad(G) = (0)$. Then A is a weakly semiprime submodule of G.

Proof Clear .

Recall that an R-module G is a semisimple if and only if $Rad(G) = (0)$ [5, Theo. (9.2.2)].

As a direct Application of Proposition 2.9 we get the following result.

Corollary 2.10 Let G be a semisimple R-module and A is a WN-semiprime submodule of G. Then A is a weakly semiprime submodule of G.

Recall that an R-module G is called torsion free if the set $\Im(G) = \{x \in G : rx = 0 \text{ for } some\ nonzero\ r \in R\}$ [7].

Proposition 2.11 Let G be a torsion free R-module, and A is a submodule of G with $Rad\left(\frac{G}{A}\right) = (0)$. Then A is a WN-semiprime submodule of G if and only if for any nonzero ideal I of R, $[A :_G I]$ is a WN-semiprime submodule of G.

Proof ($\Longrightarrow$) Let $0 \neq \langle r\rangle^n x \subseteq [A :_G I]$ for each $r \in R, x \in G$ and $n \in Z^+$, it follows that $\langle r\rangle^n(xI) \subseteq A$. If $0 \neq \langle r\rangle^n(xI) \subseteq A$ and A is a WN-semiprime submodule of G, then by proposition 2.4 we have $\langle r\rangle(xI) \subseteq A + Rad(G)$. But $Rad\left(\frac{G}{A}\right) = (0)$, then $Rad(G) \subseteq A$, it follows that $Rad(G) + A = A$, hence $\langle r\rangle(xI) \subseteq A$, implies that $\langle r\rangle x \subseteq [A :_G I] \subseteq [A :_G I] + Rad(G)$. Thus $\langle r\rangle x \subseteq [A :_G I] + Rad(G)$. If $(0) = \langle r\rangle^n(xI)$, so $\langle r\rangle^n x\, i = 0$ for some nonzero $i \in I$, then $\langle r\rangle^n x \subseteq \Im(G) = (0)$ contradiction .

($\Longleftarrow$) Suppose that $[A :_G I]$ is a WN-semiprime submodule of G for any nonzero ideal I of R. Put $I = R$ we get $[A :_G I] = A$ which is WN-semiprime submodule of G.

For next corollary we need to introduce the following lemma.

Lemma 2.12 [8, Lemma 4.11] Every faithful multiplication R-module is torsion free.

Corollary 2.13 Let G be a faithful multiplication R-module, and A is a proper submodule of G with $Rad\left(\frac{G}{A}\right) = (0)$. Then A is a WN-semiprime submodule of G if and only if for any nonzero ideal I of R $[A :_G I]$ is a WN-semiprime submodule of G.

Proof Follows by Lemma 2.12 and Proposition 2.11.

Recall that an R-module G is a hollow module if every submodule A of G is small, that is if $G = A + B$ for some submodule B of G, implies that $B = G$ [9].

Proposition 2.14 Let G be an R-module, and A a small submodule of G, such that $Rad(G)$is a weakly semiprime submodule of G. Then A is a WN-semiprime submodule of .

Proof Let $0 \neq r^n x \in A$ for $r \in R, x \in G\ and\ n \in Z^+$. Since A is a small submodule of G, then $A \subseteq Rad(G)$ that is $0 \neq r^n x \in Rad(G)$, but $Rad(G)$is a weakly semiprime submodule of G, it follows that $rx \in Rad(G) \subseteq A + Rad(G)$. Hence A is a WN-semiprime submodule of G.

Corollary 2.15 Let G be a hollow R-module with $Rad(\ G\ )$is a weakly semiprime submodule of G, then every proper submodule of G is a WN-semiprime submodule of G .

Recall that a submodule N of an R-module G is called coclosed if for any submodule A of G contained in N, $\frac{N}{A}$ is a small in $\frac{G}{A}$, implies that $N = A$ [9].

We need to recall the following lemma before we introduce the next proposition.

Lemma 2.16 [9, Prop. (1.2.16)] If B is a coclosed submodule of an R-module G, then $Rad(\ B\ ) = B \cap\ Rad(\ G\ )$.

Proposition 2.17 Let G be an R-module , and B, C are submodules of G with $C \nsubseteq B$. If C is a coclosed submodule of G such that $Rad\left(\frac{G}{C}\right) = (0)$ and B is a WN-semiprime submodule of G. Then $B \cap C$ is a WN-semiprime submodule of .

Proof It is clear that $B \cap C$ is a proper submodule of C. Let $0 \neq r^n x \in B \cap C$, for $r \in R, x \in C\ and\ n \in Z^+$, it follows that $0 \neq r^n x \in B$. Since B is a WN-semiprime submodule of G, implies that $rx \in B + Rad(\ G\ )$. It follows that $rx \in (\ B + Rad(\ G\ )\ ) \cap C$. Since $Rad\left(\frac{G}{C}\right) = (0)$, then $Rad(\ G\ ) \subseteq C$, hence by modular low we have $rx \in (B \cap C) + (C \cap Rad(\ G\ )\ )$. But C is a coclosed submodule of G, then by Lemma 2.16 $Rad(\ C\ ) = C \cap\ Rad(\ G\ )$, hence $rx \in (B \cap C) + Rad(\ C\ )$. That is $B \cap C$ is a WN-semiprime submodule of C.

Proposition 2.18 Let $G = G_1 \oplus G_2$ be an R-module for $G_1,\ G_2$ are R-modules, with G_2 is injective R-module and A_1 is a small submodule of G_1. Then $A_1$is a WN-semiprime submodule of G_1 if and only if $A_1 \oplus G_2$ is a WN-semiprime submodule of G.

Proof ($\Longrightarrow$) Let $(0,0) \neq r^n(x_1, x_2) \in A_1 \oplus G_2$ where $r \in R, (x_1, x_2) \in G_1 \oplus G_2$ with x_1 is nonzero in G_1 , $x_2 \in G_2\ and\ n \in Z^+$, implies that $0 \neq r^n x_1 \in A_1$. But $A_1$is a WN-semiprime submodule of G_1, then $rx_1 \in A_1 + Rad(G_1)$. SinceA_1 is a small submodule of G_1, implies that $A_1 \subseteq Rad(G_1)$, it follows that $rx_1 \in Rad(G_1)$, and since G_2 is injective, then by [10, Lemma 2.3] $Rad(G_{2)} = G_2$, implies that $rx_2 \in Rad(G_2)$, so $r\ (x_1, x_2) \in Rad(G_1) \oplus Rad(G_2) = Rad(G_1 \oplus G_2) \subseteq A_1 \oplus G_2 + Rad(G_1 \oplus G_2)$. Hence $A_1 \oplus G_2$ is a WN-semiprime submodule of G.

($\Longleftarrow$) Suppose that $A_1 \oplus G_2$ is a WN-semiprime submodule of G, and let $0 \neq r^n x_1 \in A_1$ for $r \in R$ and x_1 is nonzero element G_1 and $n \in Z^+$. It follows that for each $x_2 \in G_2$ $(0,0) \neq r^n(x_1, x_2) \in A_1 \oplus G_2$. Since $A_1 \oplus G_2$ is a WN-semiprime submodule of G, implies that so $r\ (x_1, x_2) \in (A_1 \oplus G_2) + Rad(G_1 \oplus G_2)$=$(A_1 \oplus G_2) +\ Rad(G_1) \oplus Rad(G_2)$. Since A_1 is small submodule of G_1, then $A_1 \subseteq Rad(G_1)$, and G_2 is injective, then by [8, Lemma 2.3] $Rad(G_{2)} = G_2$. Hence $r(x_1, x_2) \in (A_1 \oplus G_2) + (\ (A_1 + Rad(G_1) \oplus G_2\ ) \subseteq (\ A_1 + Rad(G_1)) \oplus G_2$ (because $A_1 \oplus G_2 \subseteq (\ A_1 + Rad(G_1) \oplus G_2\ )$. Thus $r\ x_1 \in A_1 + Rad(G_1)$. It follows that $A_1$is a WN-semiprime submodule of G_1.

Corollary 2.19 Let $G = G_1 \oplus G_2$ be an R-module where $G_1,\ G_2$ are R-modules, G_1 is a hollow module and G_2 is an injective R-module and A_1 is a proper submodule of G_1. Then $A_1$is a WN-semiprime submodule of G_1 if and only if $A_1 \oplus G_2$ is a WN-semiprime submodule of G.

We end this section by following proposition.

Proposition 2.20 Let $G = G_1 \oplus G_2$ be an R-module where G_1, G_2 are R-modules, and $A = A_1 \oplus A_2$ be a submodule of G with $A \subseteq Rad(G)$. If A is a WN-semiprime submodule of G, then A_1, A_2 are WN-semiprime submodule of G_1, G_2 respectively.

Proof let $0 \neq r^n x_1 \in A_1$ for $r \in R$ and x_1 is nonzero element G_1 and $n \in Z^+$, it follows that $(0,0) \neq r^n(x_1, 0) \in A$. Since A is a WN-semiprime submodule of G, then $r(x_1, 0) \in A + Rad(G)$. But $A \subseteq Rad(G)$, then $r(x_1, 0) \in Rad(G) = Rad(G_1 \oplus G_2) = Rad(G_1) \oplus Rad(G_2)$. It follows that $rx_1 \in Rad(G_1) \subseteq A_1 + Rad(G_1)$. Hence A_1 is a WN-semiprime submodule of G_1.

Similarly, we can prove that A_2 is a WN-semiprime submodule of G_2.

3. More properties of weakly nearly semiprime submodules

Our concern in this section is to give properties of WN-semiprime submodule in class of multiplication R-module characterize by good rings, Artinian rings and local rings before introduce the first proposition.

Recall that a ring R is a good ring if $Rad(G) = Rad(R)\, G$ for any R-module G[7].

Before we introduce the first proposition we recall the following lemma.

Lemma 3.1 [6, Coro. of Theo. 9] "Let G be a finitely generated multiplication R-module, and I, J are ideals of ring R. Then $IG \subseteq JG$ if and only if $I \subseteq J + ann_R(G)$."

Proposition 3.2 Let G be a finitely generated multiplication R-module over a good ring R and I is a WN-semiprime ideal of R with $ann_R(G) \subseteq I$. Then IG is a WN-semiprime submodule of G.

Proof Let $0 \neq r^n x \in IG$ for $r \in R, 0 \neq x \in G$ and $n \in Z^+$, that is $0 \neq r^n \langle x \rangle \subseteq IG$. Since G is a multiplication then $\langle x \rangle = JM$ for some ideal J of R, that is $0 \neq r^n JG \subseteq IG$. It follows that by Lemma 3.1, $0 \neq r^n J \subseteq I + ann_R(G)$. But $ann_R(G) \subseteq I$, implies that $ann_R(G) + I = I$, hence $0 \neq r^n J \subseteq I$. Since I is a WN-semiprime ideal of R, it follows that $rJ \subseteq I + Rad(R)$, so $rJG \subseteq IG + Rad(R)G$. But R is a good ring, then $Rad(G) = Rad(R)\, G$. Hence $r\langle x \rangle \subseteq IG + Rad(G)$, hence $rx \in IG + Rad(G)$. Hence IG is a WN-semiprime submodule of G.

It is well-known that **an** Artinian ring is a good ring [5, Coro. (9.7.3)(10)].

So, we get the following corollary.

Corollary 3.3 Let G be a finitely generated multiplication R-module over **an** Artinian ring R and I is a WN-semiprime ideal of R with $ann_R(G) \subseteq I$. Then IG is a WN-semiprime submodule of G.

It is well-known that if an R-module G over a local ring , then $Rad(G) = Rad(R)\, G$ [9, Prop. 1.2].

Proposition 3.4 Let G be a finitely generated multiplication R-module over a local ring R and I is a WN-semiprime ideal of R with $ann_R(G) \subseteq I$. Then IG is a WN-semiprime submodule of G.

Proof Let $0 \neq r^n L \subseteq IG$, for $r \in R$ and L be a nonzero submodule of G, and $n \in Z^+$. Since G is multiplication, implies that $0 \neq r^n JG \subseteq IG$for some nonzero ideal J of . Hence by Lemma 3.1, $0 \neq r^n J \subseteq I + ann_R(G)$. But $ann_R(G) \subseteq I$, implies that $ann_R(G) + I = I$, that is $0 \neq r^n J \subseteq I$. It follows by Corollary 2.6 $rJ \subseteq I + Rad(R)$, implies that $rJG \subseteq IG + Rad(R)G$. But R is a local ring, then $Rad(G) = Rad(R)\ G$. Hence $rL \subseteq IG + Rad(G)$. That is IG is a WN-semiprime submodule of G.

Proposition 3.5 Let G be a projective finitely generated multiplication R-module and I be a WN-semiprime ideal of R with $ann_R(G) \subseteq I$. Then IG be a WN-semiprime submodule of G.

Proof Let $0 \neq \langle r^n \rangle A \subseteq IG$, for $r \in R$ and A be a nonzero submodule of G, and $n \in Z^+$. Since G is multiplication, then $A = JG$ for some nonzero ideal J of R, that is $0 \neq \langle r^n \rangle JG \subseteq IG$, it follows that by Lemma 3.1, $0 \neq \langle r^n \rangle J \subseteq I + ann_R(G,$ implies that $ann_R(G) + I = I$ by hypothesis, hence is $0 \neq \langle r^n \rangle J \subseteq I$. It follows by Proposition 2.4, $\langle r \rangle J \subseteq I + Rad(R)$, it follows that $\langle r \rangle JG \subseteq IG + Rad(R)G$. But G is a projective R-module , then by [3, Prop. 17.10] $Rad(G) = Rad(R)\ G$.Thus $\langle r \rangle A \subseteq IG + Rad(G)$. That is IG is a WN-semiprime submodule of G.

Remark 3.6 If A is a WN-semiprime subodule of an R-module G, then $[A : G]$ need not to be a WN-semiprime ideal of R. The following example shows that :

Let $G = Z_8$ as a Z-module and $A = \{\bar{0}, \bar{4}\}$ be a submodule of Z_8. It is clear that $Rad(Z_8) = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}\}$. The submodule A is a WN-semiprime submodule of Z_8, since $0 \neq 2^2 \bar{1} \in A$ for $2 \in Z, \bar{1} \in Z_8$, it follows that $2\ \bar{1} \in A + Rad(Z_8) = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}\}$. But $[A : Z_8] = 4Z$ is not WN-semiprime ideal of Z, since $0 \neq 2^2 1 \in 4Z$, but $2.1 \notin 4Z + Rad(Z) = 4Z + (0) = 4Z$.

Now we look for conditions under which the result of WN-semiprime is WN-semiprime ideal.

Proposition 3.7 Let G be an R-module and B be a proper submodule of G with $Rad\left(\frac{G}{B}\right) = (0)$ and $Rad(R) \subseteq [B :_R G]$. Then B is a WN-semiprime submodule of G if and only if $[B :_R G]$ is a WN-semiprime ideal of R.

Proof $(\Longrightarrow)$ Let $0 \neq r^n s \in [B :_R G]$ for $r \in R$ and s is nonzero element of R, it follows that $0 \neq r^n sG \subseteq B$. Since B is a WN-semiprime submodule of G, then by Corollary 2.5, $rsG \subseteq B + Rad(G)$, but $Rad\left(\frac{G}{B}\right) = (0)$, then $Rad(G) \subseteq B$, it follows that $rsG \subseteq B$, hence $rs \in [B :_R G] \subseteq [B: \_R\ G] + Rad(R)$. That is $[B :_R G]$ is a WN-semiprime ideal of R.

$(\Longleftarrow)$ Let $0 \neq r^n G \subseteq B$ for r be nonzero element of R, it follows that $0 \neq r^n.1 \in [B :_R G]$, but $[B :_R G]$ is a WN-semiprime ideal of R, implies that $r.1 \in [B :_R G] + Rad(R)$. Since $Rad(R) \subseteq [B :_R G]$, it follows that that $r \in [B :_R G]$, that is $rG \subseteq B \subseteq B + Rad(G)$. Hence B is a WN-semiprime submodule of G.

Proposition 3.8 Let G be multiplication R-module over an Artinian ring R and B be a proper submodule of G. Then B is a WN-semiprime submodule of G if and only if $[B :_R G]$ is a WN-semiprime ideal of R.

Proof $(\Longrightarrow)$ Let $0 \neq r^n. I \subseteq [B :_R G]$ for $r \in R$, and I is nonzero ideal of R, $n \in Z^+$, it follows that $0 \neq r^n. IG \subseteq B$. Since B is a WN-semiprime submodule of G, then $rIG \subseteq B + Rad(G)$. But G is a multiplication R-module over an Artinian ring, then $B = [B :_R G]G$ and $Rad(G) = Rad(R)G$, it follows that $rIG \subseteq [B :_R G]G + Rad(R)G$, hence $rI \subseteq [B :_R G] + Rad(R)$. That is $[B :_R G]$ is a WN-semiprime ideal of R.

(⟸) $0 \neq r^n L \subseteq B$ for $r \in R$, and L be a submodule of G, $n \in Z^+$. Since G be multiplication R-module then $L = IG$ for some nonzero ideal I of R, that is $0 \neq r^n IG \subseteq B$, it follows that $0 \neq r^n I \subseteq [B:_R G]$. But $[B:_R G]$ is a WN-semiprime ideal of R, then $rI \subseteq [B:_R G] + Rad(R)$, implies that $rIG \subseteq [B:_R G]G + Rad(R)G$. But G is a multiplication R-module over an Artinain ring, then $rL \subseteq B + Rad(G)$. That is B is a WN-semiprime submodule of G.

Proposition 3.9 Let G be multiplication R-module over a local ring R and B be a proper submodule of G. Then B is a WN-semiprime submodule of G if and only if $[B:_R G]$ is a WN-semiprime ideal of R.

Proof It follows that as in Proposition 3.7 .

Proposition 3.10 Let G be multiplication R-module over a good ring R and B be a proper submodule of G. Then B is a WN-semiprime submodule of G if and only if $[B:_R G]$ is a WN-semiprime ideal of R.

Proof It follows that as in Proposition 3.7 .

Recall that an R-module G is a cancellation if $IG = JG$ for any ideals I, J of R, implies that $I = J$ [12].

Proposition 3.11 Let G be a multiplication faithful finitely generated projective R-module , and B be a proper submodule of G. Then the following statements are equivalent:

(1). B is a WN-semiprime submodule of G.

(**2**). $[B:_R G]$ is a WN-semiprime ideal of R.

(3). $B = IG$ for some WN-semiprime ideal I of R.

Proof (1)⟹ (2) Let $0 \neq r^n I \subseteq [B:_R G]$ for $r \in R, n \in Z^+$ and I is nonzero ideal of R, then $0 \neq r^n IG \subseteq B$. Since B is a WN-semiprime submodule of G, then by Corollary 2.6 $rIG \subseteq B + Rad(G)$. But G be a multiplication projective R-module ,then $B = [B:_R G]G$ and $Rad(G) = Rad(R)G$, it follows that $rIG \subseteq [B:_R G]G + Rad(R)G$, that is $rI \subseteq [B:_R G] + Rad(R)$. Thus $[B:_R G]$ is a WN-semiprime ideal of R.

(2)⟹(1) $0 \neq r^n A \subseteq B$ for $r \in R$, and A be a submodule of G, $n \in Z^+$. Since G be multiplication R-module then $A = JG$ for some nonzero ideal J of R, that is $0 \neq r^n JG \subseteq B$, it follows that $0 \neq r^n J \subseteq [B:_R G]$. But $[B:_R G]$ is a WN-semiprime ideal of R, implies that $rJ \subseteq [B:_R G] + Rad(R)$, it follows that $rJG \subseteq [B:_R G]G + Rad(R)G$. Since G is a multiplication projective R-module, then $rA \subseteq B + Rad(G)$. Hence B is a WN-semiprime submodule of G.

(2)⟹ (3) It is given that $[B:_R G]$ is a WN-semiprime ideal of R, and since G is a multiplication R-module ,then $B = [B:_R G]G = IG$ where $I = [B:_R G]$ is a WN-semiprime ideal of R.

(3)⟹ (2) Suppose that $B = IG$ for some WN-semiprime ideal I of R. But $B = [B:_R G]G$, hence $[B:_R G]G = IG$. Since G is a multiplication faithful finitely generated R-module then by [12, Prop. 3.1] G is a cancellation, that is $[B:_R G] = I$ which is a WN-semiprime ideal of R.

From Propositions 3.8, 3.9 and 3.10 we get the following propositions similar to Proposition 3.11.

Proposition 3.12 Let G be a multiplication faithful finitely generated R-module over an Artinian ring R, and B be a proper submodule of G. Then the following statements are equivalent:

(1). B is a WN-semiprime submodule of G.

(**2**). $[B:_R G]$ is a WN-semiprime ideal of R.

(3). $B = IG$ for some WN-semiprime ideal I of R.

Proposition 3.13 Let G be a multiplication faithful finitely generated R-module over a local ring R, and B be a proper submodule of G. Then the following statement are equivalent:

(1). B is a WN-semiprime submodule of G.

(**2**). $[B:_R G]$ is a WN-semiprime ideal of R.

(3). $B = IG$ for some WN-semiprime ideal I of R.

Proposition 3.14 Let G be a multiplication faithful finitely generated R-module over a good ring R, and B be a proper submodule of G. Then the following statement are equivalent:

(1). B is a WN-semiprime submodule of G.

(**2**). $[B:_R G]$ is a WN-semiprime ideal of R.

(3). $B = IG$ for some WN-semiprime ideal I of R.

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