JQCM University Of AL-Qadisiyah

Available online at www.qu.edu.iq/journalcm
JOURNAL OF AL-QADISIYAH FOR COMPUTER SCIENCE AND MATHEMATICS
ISSN:2521-3504(online) ISSN:2074-0204(print)

Weakly Approximaitly –Prime Submodules And Related concepts

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ARTICLE INFO

Article history:
Received: 20 /04/2020
Rrevised form: //
Accepted :14/05/2020
Available online: 17/05/2020
Keywords:

Weakly prime submodules, WAPP-prime submodules Socal of submodules,multiplication modules.

ABSTRACT

Let R be a commutative ring with identity and P a unital left R-module . The purpose of this paper is to introduce and study the concept of WAPP-prime submodules as a generalization of weakly prime submodules , where a proper submodule L of an R-module P is called a WAPP-prime , if whenever 0≠re ϵ L for rϵR , eϵP , implies that either e∈ $L+Soc(P)$ or r∈[$L+Soc(P)$:$_R$ P] . Several examples , characterizations and basic properties of this concept are given .Moreover many characterizations of WAPP-prime submodules in class of multiplication modules are introduced .

MSC:

DOI : https://DOI 10.29304/jqcm.2020.12.2.690

1. Introduction

Throughout this article all ring are commutative rings with identity and all modules are Unital left R-modules . Weakly prime submodules were first introduced in 2004 as a generalization of prime submodules , where a paper submodule L of an R-module P is called weakly prime , if whenever 0≠re ϵ L r∈R , e∈P , implies that either e∈ L or r∈[L:$_R$ P] , where [L:$_R$ P]={ r∈R: rP⊆L} [1] , and a proper submodule L of an R-module P is called prime , if whenever re ϵ L rϵR , eϵP , implies that either e∈ L or r∈[L:$_R$ P] [2] . The concept of weakly prime submodule studied extensively in [3 ,4 , 5 ,6] . Several generalization of weakly prime submodule were introduced such as weakly primary , weakly quasi primary and weakly semi prime submodules see [7 ,8 , 9] . In this paper we introduce and study a new generalization of weakly prime submodule called WAPP-prime submodule . Recall that a nonzero submodule B of an R-module P is called essential if B∩ $C\neq(0)$ for every nonzero submodule C of V[10] . The socale of an R-module P denoted by Soc(P) is defined to be the intersection of all essential submodules of P [11] . It is well known if a submodule N of P is essential then Soc(P)=Soc(N) [10 , P. 29] . Multiplication module plays importaint role in this work , where an R-module P is called multiplication if every submodule L of P is the form L=JP for some ideal J of R . Equivalently P is a multiplication if L=[L:$_R$ P]P [12] . Recalled that for any submodules L, K of a multiplication R-module P with L=IP and K=JK for some ideals I and J of R , the product LK=IPJP=IJP , that is LK=IK . In particular LP=IPP=IP=L , also for every e∈P , L=Ie [13] . Recall that an R-module P is Z-regular if for each e∈P there exists f∈ P^* =Hom$_R$(P,R) such that e=f(e)e [14] . This paper divide into two parts . We introduce in part one the definition of WAPP-prime submodules and give examples , characterizations and some properties of this concept to part two deals with introducing many characterizations of WAPP-prime submodules in class of multiplication modules

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Main result :

We introduce in this part of the paper the definition of WAPP-prime submodule and give some basic properties ,also characterizations of this concept .

Definition 2.1

A proper submodule L of an R-module P is called weakly approximaitly prime (Briefly WAPP-prime) submodule of P , if whenever $0 \neq re \in L$ for $r \in R$, $e \in P$, implies that either $e \in L+Soc(P)$ or $r \in [L+Soc(P):_R P]$.

And an ideal J of a ring R is a WAPP-prime ideal of R if and only if J is a WAPP-prime submodule of an R-module R .

There are several examples ,some of which we will mention .

1. In the Z-module Z_{24} ,the proper submodules are $<\bar{2}>,<\bar{3}>,<\bar{4}>,<\bar{6}>,<\bar{8}>$ and $<\overline{12}>$. The only essential submodules of Z_{24} are $<\bar{2}>,<\bar{4}>$ So, $Soc(Z_{24})= <\bar{2}> \cap < \bar{4} > = <\bar{4}>$.
2. It is clear that the submodules $<\bar{2}>,<\bar{3}>$ of the Z-module Z_{24} , are weakly prime , but $<\bar{4}>$, $<\bar{6}>,<\bar{8}>$ and $<\overline{12}>$ are not weakly prime submodules .
3. It is clear that the submodules $<\bar{6}>,<\bar{8}>$ of the Z-module Z_{24} are WAPP-prime , but the submodules $<\bar{4}>$ and $<\overline{12}>$ are not WAPP-prime submodules of Z_{24} .
4. It is clear that every weakly prime submodule of an R- module P is a WAPP-prime but not conversely .

the following example explains that .
In the Z-module $Z_{24}$the submodule $<\bar{6}>$ is a WAPP-prime submodule of Z_{24} by (3), but $<\bar{6}>$ is not weakly prime by (2) .

5. The submodules $<\bar{2}>,<\overline{3}>$of the Z-module Z_{24} are WAPP-prime by (4)
6. If L is a WAPP-prime submodule of P , then $[L:_R P]$ need not to be a WAPP-prime ideal of R . The following example shows that .

In the Z-module Z_{24} the submodule $<\bar{8}>$ is a WAPP-prime by (3) , but $[<\bar{8}>:_Z Z_{24}]$ =8Z is not a WAPP-prime ideal of Z because $0 \neq 2.4 \in 8Z$, for $2,4 \in Z$ but $4 \notin 8Z+Soc(Z)$ and $2 \notin [8Z+Soc(Z) :_Z Z]$ since 8Z+Soc(Z) =8Z (since Soc(Z)=(0))

7. Let A , B be submodules of an R-module P with $A \subseteq B$. If B is a WAPP-prime submodule of P , then A need not to be a WAPP-prime submodule of P .

For example the submodules $<\bar{4}>$and $<\bar{2}>$ of Z-module Z_{24} $<\bar{4}> \subseteq <\bar{2}>$, we see that $<\bar{2}>$ is a WAPP-prime submodule by (5) and $<\bar{4}>$ is not WAPP-prime submodule by (3) .

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8. Let A , B be submodules of an R-module P with $A \subseteq B$. If A is a WAPP-prime submodule of P ,then B need not to be a WAPP-prime submodule of P .

For example the submodules $<\bar{8}>$and $<\bar{4}>$ of the Z-module Z_{24} , $<\bar{8}> \subseteq <\bar{4}>$, we see that $<\bar{8}>$ is a WAPP-prime submodule of Z_{24} by (3), but $<\bar{4}>$ is not WAPP-prime submodule of Z_{24} , also by (3)

9. The intersection of two WAPP-prime submodules of P need not to be WAPP-prime submodule of P .

The following example shows that :
Let N=<2> ,K=<3> are two WAPP-prime submodules of the Z –modules Z (because they are weakly prime). But $<2> \cap <3> = <6>$ is not a WAPP-prime submodule of Z ,since $0 \neq 2.3 \in <6>$,for $2,3 \in Z$ but $2 \notin <6>+Soc(Z)$ and $3 \notin <6>+Soc(Z)$.

The following proposition are characterizations of WAPP-prime submodules .

Proposition 2.2 Let P be an R-module , and L a proper submodule of P . Then L is a WAPP-prime submodule of P if and only if $[L:_R e] \subseteq [L+Soc(P):_R P] \cup [0:_R e]$ for all $e \in P$ and $e \notin L+Soc(P)$.

Proof (⇒) Suppose that L is a WAPP-prime submodule of P , and let $a \in [L:_R e]$ and $e \notin L+Soc(P)$, then $ae \in L$. If ae=0 , then $a \in [0:_R e]$, thus $a \in [L+Soc(P):_R P] \cup [0:_R e]$. If $0 \neq ae \in L$, and L is a WAPP-prime of P and $e \notin L+Soc(P)$, implies that $a \in [L+Soc(P):_R P]$. Thus $a \in [L+Soc(P):_R P] \cup [0:_R e]$. Therefore $[L:_R e] \subseteq [L+Soc(P):_R P] \cup [0:_R e]$.

(⇐) Suppose that $[L:_R e] \subseteq [L+Soc(P):_R P] \cup [0:_R e]$ for all $e \in P$ and $e \notin L+Soc(P)$, and let $0 \neq ae \in L$, implies that $a \in [L:_R e]$, it follows by hypothesis $a \in [L+Soc(P):_R P] \cup [0:_R e]$. But $0 \neq ae$, hence $a \notin [0:_R e]$, therefore $a \in [L+Soc(P):_R P]$. Thus L is a WAPP-prime submodule of P.

Proposition 2.3 Let P be an R-module , and L a proper submodule of P . Then L is a WAPP-prime submodule of P if and only if whenever $(0) \neq aK \subseteq L$ for $a \in R$, K is a submodule of P , implies that either $K \subseteq L+Soc(P)$ or $a \in [L+Soc(P):_R P]$.

Proof (⇒) Let $(0) \neq aK \subseteq L$ for $a \in R$, K is a submodule of P with $K \not\subset L+Soc(P)$. That is there exists $0 \neq e \in K$ such that $e \notin L+Soc(P)$. Now , since $(0)) \neq aK \subseteq L$, then $0 \neq ae \in L$. But L is a WAPP-prime submodule of P and $e \notin L+Soc(P)$. It follows that $a \in [L+Soc(P):_R P]$.

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(⇐)Let $0 \neq ae \in L$ for $a \epsilon R$, $e \epsilon P$, that is $0 \neq a<e> \subseteq L$, hence by hypothesis either $<e> \subseteq L+Soc(P)$ or $a \epsilon [L+Soc(P):_R P]$. That is either $e \epsilon L+Soc(P)$ or $a \epsilon [L+Soc(P):_R P]$. Hence L is a WAPP-prime submodule of P .

Proposition 2.4 Let P be an R-module , and L a proper submodule of P . Then L is a WAPP-prime submodule of P if and only if whenever $(0) \neq JK \subseteq L$ for J is an ideal of R and K is a submodule of P , implies that either $K \subseteq L+Soc(P)$ or $J \subseteq [L+Soc(P):_R P]$.

Proof (⇒) Let $(0) \neq JK \subseteq L$ with $K \not\subset L+Soc(P)$, that is there exists $0 \neq e \in K$ and $e \notin L+Soc(P)$. To prove that $J \subseteq [L+Soc(P):_R P]$. Let $r \epsilon J$, if $0 \neq re \epsilon L$ and L is a WAPP-prime submodule , gives $r \epsilon [L+Soc(P):_R P]$, it follows that $J \subseteq [L+Soc(P):_R P]$. Assume that $re=0$, and first suppose that $rK \neq (0)$, that is $0 \neq rd \epsilon L$ for some $d \epsilon K$. If $d \notin L \subseteq L+Soc(P)$, and L is a WAPP-prime , then $r \in [L+Soc(P):_R P]$, hence $J \subseteq [L+Soc(P):_R P]$. If $d \epsilon L \subseteq L+Soc(P)$, then $0 \neq rd=r(d+e) \epsilon L$ and L is a WAPP-prime submodule , so either $d+e \epsilon L+Soc(P)$ or $r \epsilon [L+Soc(P):_R P]$. Thus $J \subseteq [L+Soc(P):_R P]$. So ,we can assume that $rk=0$. Suppose that $Je \neq (0)$, that is $0 \neq se \epsilon L$ for some $s \epsilon J$ and L is a WAPP-prime submodule of P gives $s \epsilon [L+Soc(P):_R P]$ .As $0 \neq se=(r+s)e \epsilon L$ and L is a WAPP-prime submodule of P ,we get $r+s \in [L+Soc(P):_R P]$, it follows that $r \epsilon [L+Soc(P):_R P]$, so $J \subseteq [L+Soc(P):_R P]$. Therefore we can assume that $Je=(0)$. Since $JK \neq (0)$,then there exists $d_1 \epsilon K$, $b \in J$ such that $0 \neq bd_1$,and $0 \neq bd_1 =b(d_1 +e) \epsilon L$, so we have two cases :

- If $b \epsilon [L+Soc(P):_R P]$ and $d_1 +e \notin L+Soc(P)$. Since $0 \neq (r+b)(d_1 +e)= bd_1 \epsilon L$ and L is a WAPP-prime , it follows that $(r+b) \epsilon [L+Soc(P):_R P]$ so $r \epsilon [L+Soc(P):_R P]$. Hence $J \subseteq [L+Soc(P):_R P]$.
- If $b \notin [L+Soc(P):_R P]$ and $d_1 +e \epsilon L+Soc(P)$. As ,$0 \neq bd_1 \epsilon L$, we have $d_1 \in L+Soc(P)$, so $e \epsilon L+Soc(P)$ which is a contradiction . Thus $J \subseteq [L+Soc(P):_R P]$.

(⇐)Let $0 \neq re \epsilon L$ for $r \epsilon R$, $e \epsilon P$, implies that $0 \neq <r><e> \subseteq L$, thus by hypothesis either $<e> \subseteq L+Soc(P)$ or $<r> \subseteq [L+Soc(P):_R P]$. That is either $e \epsilon L+Soc(P)$ or $r \epsilon [L+Soc(P):_R P]$ .Therefor L is a WAPP-prime submodule of P .

As a direct application of Proposition (2.5) , we have the following corollary .

Corollary 2.5 Let P be an R-module , and L a proper submodule of P . Then L is a WAPP-prime submodule of P if and only if whenever $(0) \neq Je \subseteq L$ for J is an ideal of R and $e \epsilon P$, implies that either $e \in L+Soc(P)$ or $J \subseteq [L+Soc(P):_R P]$.

The following propositions are basic properties of WAPP-prime submodules.

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Proposition 2.6 Let P be an R-module , and N,L submodules of P such that L⊆N and Nis a WAPP-prime submodule of P . Then $\frac{N}{L}$ is a WAPP-prime submodule of $\frac{P}{L}$.

Proof : Let 0≠r(e+L)=re+L$\epsilon\frac{N}{L}$ for rϵR , e+L$\epsilon\frac{P}{L}$, eϵP . Then reϵN. If re=0,then r(e+L)=0 which is a contradiction . Thus 0≠reϵN and N is a WAPP-prime submodule of P , implies that either eϵN+Soc(P) or rϵ[N+Soc(P):$_R$ P], that is either eϵN+Soc(P) or rP⊆ N+Soc(P) . It follows that e+Lϵ $\frac{N+Soc(P)}{L}$ or r $\frac{P}{L}$ ⊆ $\frac{N+Soc(P)}{L}$,that is either e+Lϵ $\frac{N}{L}$ + $\frac{N+Soc(P)}{L}$ ⊆$\frac{N}{L}$ +Soc($\frac{p}{L}$) or r $\frac{P}{L}$ ⊆ $\frac{N}{L}$ + $\frac{N+Soc(P)}{L}$ ⊆ $\frac{N}{L}$ +Soc($\frac{p}{L}$) . Hence $\frac{N}{L}$ is a WAPP-prime submodule of $\frac{P}{L}$.

Recall that an R-module P is semi simple , if every submodule of P is a direct summand of P[11].Equivalently P is a semi simple if and only if Soc(P)=p [7].

It is well known that an R-module P is semi simple if and only if Soc($\frac{P}{L}$)=$\frac{Soc(P)+L}{L}$ for all submodule L of P [11 , Ex.(12)c].

Now , we give the converse of proposition (2.7)

Proposition 2.7 Let P be a semi simple R-module , and N,L submodules of P such that L⊆N and Nis a proper submodule of P . If L and $\frac{N}{L}$ are WAPP-prime submodules of P and $\frac{P}{L}$ respectively ,then N is a WAPP-prime submodule of *P* .

Proof : Let 0≠reϵN for rϵR ,eϵP , so 0≠r(e+L)=re+L$\epsilon\frac{N}{L}$. If 0≠reϵL and L is a WAPP-prime submodule of P , then either eϵL+Soc(P)⊆ N+Soc(P) or rP⊆ L+Soc(P)⊆ N+Soc(P) . Hence N is a WAPP-prime submodule of P . So , we may assume that re∉L. It follows that 0≠r(e+L) ϵ $\frac{N}{L}$, but $\frac{N}{L}$ a WAPP-prime submodule of $\frac{P}{L}$, implies that either e+Lϵ $\frac{N}{L}$ +Soc($\frac{p}{L}$) or r $\frac{P}{L}$ ⊆ $\frac{N}{L}$ +Soc($\frac{p}{L}$). Since P is a semi simple , then Soc($\frac{p}{L}$) = $\frac{L+Soc(P)}{L}$,hence either e+Lϵ $\frac{N}{L}$ + $\frac{L+Soc(P)}{L}$ or r $\frac{P}{L}$ ⊆ $\frac{N}{L}$ + $\frac{L+Soc(P)}{L}$. Since L⊆N ,it follows that L+Soc(P)⊆N+Soc(P). Thus , $\frac{N}{L}$ + $\frac{L+Soc(P)}{L}$ ⊆$\frac{N}{L}$ + $\frac{N+Soc(P)}{L}$ and since $\frac{N}{L}$ ⊆ $\frac{N+Soc(P)}{L}$, implies that $\frac{N}{L}$ + $\frac{N+Soc(P)}{L}$ = $\frac{N+Soc(P)}{L}$ .Thus either e+Lϵ $\frac{N+Soc(P)}{L}$ or r $\frac{P}{L}$ ⊆ $\frac{N+Soc(P)}{L}$. It follows that either eϵN+Soc(P) or rP⊆ N+Soc(P) . Hence N is a WAPP-prime submodule of P .

Recall that an R-module P is compressible if it can be embedded in any of it is nonzero submodules [10] .

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Proposition 2.8 Let P be an R- module , and N a proper submodule of P such that $\frac{P}{N}$ is a compressible R-module . Then N is a WAPP-prime of P .

Proof: Suppose that K is a submodule of P such that N⊆K , and let (0)≠rK⊆N for rϵR with K⊄N+Soc(P) . Thus $\frac{K}{N}$ is a submodule of $\frac{P}{N}$ .But $\frac{P}{N}$ is compressible , then there exists a monomorphism f: $\frac{P}{N} \to \frac{K}{N}$, that is rf ($\frac{P}{N}$)=(0) , implies that f ($r\ \frac{P}{N}$)=(0) . It follows that r ($\frac{P}{N}$)=(0) .That is rP⊆N⊆N+Soc(P) . Hence r∈ [N+Soc(P):$_R$ P] . Therefore N is a WAPP-prime submodule of P .

Proposition 2.9 Let P be an R-module , and N,K submodules of P with N⊆K and K is an essential in P . If N is a WAPP-prime submodule of P . Then N is a WAPP-prime submodule of K .

Proof : Suppose that N is a WAPP-prime submodule of P , and (0)≠rB⊆N for rϵR and B is a submodule of K , that is B is a submodule of P . But Nis a WAPP-prime submodule of P , then either B ⊆N+Soc(P) or rϵ[N+Soc(P):$_R$ P] . Since K is an essential submodule of P , then Soc(K)=Soc(P) .Thus either B⊆N+Soc(K) or rϵ[N+Soc(K):$_R$ P]⊆ [N+Soc(K):$_R$ K] . Therefore N is a WAPP-prime submodule of K

It is well known that if L is a submodule of an R-module P , then Soc(L)=L∩ $Soc(p)$ [11 , Lemma(2.3.15)]

Proposition 2.10 Let P be an R-module , and N,K be a submodules of P with K⊄N and Soc(P)⊆K . If N is a WAPP-prime submodule of P . Then N∩ K is a WAPP-prime submodule of K .

Proof: It is clear that N∩ K is a proper submodule of K . Let (0)≠IA⊆ N∩ K for I is an ideal of R and A is a submodule of K , that is A is a submodule of P .Thus IA⊆N and IA⊆ K. But N is a WAPP-prime submodule of P then either A⊆N+Soc(P) or IP⊆N+Soc(P) . Thus either A⊆ (N+Soc(P))∩ K or IP⊆(N+Soc(P))∩ K . Since Soc(P)⊆K then by modular law either A⊆ (N∩ K)+(K ∩Soc(P)) or IP⊆(N∩ K)+(K ∩ Soc(P)). Hence either A⊆ (N∩ K)+ Soc(K) or IP⊆(N∩ K)+ Soc(K). Therefore by Proposition (2.5) N∩ K is a WAPP-prime submodule of K .

Proposition 2.11 Let P be an R-module , and N,K are WAPP-prime submodules of P with K is not contained in N and either Soc(P)⊆N or Soc(P)⊆K . Then N∩ K is a WAPP-prime submodule of P .

Proof: It is clear that N∩ K is a proper submodule of K , and K is a proper submodule of P, it follows that N∩ K is a proper submodule of P . Let 0≠ JA⊆N

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$\cap K$ for J is an ideal of R and A is a submodule of P . Then 0≠ JA⊆N and 0≠ JA⊆K . But N and K are WAPP-prime submodules of P , then either A⊆N+Soc(P) or JP⊆N+Soc(P) and either A⊆K+Soc(P) or JP⊆K+Soc(P) . Thus either A⊆ (N+Soc(P))∩ (K + Soc(P)) or JP⊆(N+Soc(P))∩ (K + Soc(P)) .If Soc(p)⊆K then K+Soc(P)=K , and if follows that either A⊆ (N+Soc(P))∩ K or JP⊆(N+Soc(P))∩ K , and so by modular law , we have either A⊆ (N∩ K)+Soc(P) or JP⊆(N∩ K)+Soc(P) . If Soc(P)⊆N , in the same way we get either A⊆ (N∩ K)+Soc(P) or JP⊆(N∩ K)+Soc(P) . Therefore N∩ K is a WAPP-prime submodule of P .

Proposition 2.12 Let P be an R-module , and N a submodule of P with [N+Soc(P):$_R$P] is a maximal ideal of R . Then N is a WAPP-prime submodule of P .

Proof: Let 0≠ae϶ N for aϵR , eϵP with a∉[N+Soc(P):$_R$P] . Since [N+Soc(P):$_R$P] is a maximal ideal of R , then R=<a> +[N+Soc(P):$_R$P] .That is 1=ar +s for some rϵR , sϵ[N+Soc(P):$_R$P] . That is e =are +se ϵN+Soc(P) .Hence N is a WAPP-prime submodule of P .

Proposition 2.13 Let P be an R-module , and J a maximal ideal of R with JP+Soc(P)is a proper submodule of P . Then JP is a WAPP-prime submodule of P.

Proof: Since JP⊆JP+Soc(P) , it follows that J⊆[JP+Soc(P):$_R$ P] , that is there exists rϵ[JP+Soc(P):$_R$ P] and r∉J . But J is a maximal ideal of R and r∉J then R=J+<r>, it follows that 1=b+rs for some sϵR , bϵJ , that is e=be +sae for each eϵP . Thus eϵJP +Soc(P) for each eϵP ,that is P⊆JP+Soc(P) , hence JP+Soc(P)=P which is a contradiction . Thus rϵJ and it follows that [JP+Soc(P):$_R$ P]⊆J hence [JP+Soc(P):$_R$ P]=J which is a maximal ideal of R ,it follows by Proposition (2.13) JP is a WAPP-prime submodule of P .

Proposition 2.14 Let P be an R-module and L a proper submodule of P with [L+ Soc(P):$_R$ P]= [L+Soc(P):$_R$ K] for each submodule K of P and L+Soc(P)⊆K . Then L is a WAPP-prime submodule of P .

Proof: Let (0)≠Ie⊆L for eϵP , I is an ideal of R with e∉L+Soc(P) . That is K=L+Soc(P) +<e> . It is clear that L+Soc(P)⊆K , then eϵK . Now since (0)≠Ie⊆L and eϵK ,implies that I⊆[L:$_R$ K] .Since L⊆L+Soc(P) , then [L:$_R$ K]⊆ [L+Soc(P):$_R$ K] . But it is given that [L+Soc(P):$_R$ K]= [L+Soc(P):$_R$ P] ,implies that [L:$_R$ K]⊆ [L+Soc(P):$_R$ P] ,that is I⊆[L+ Soc(P):$_R$ P] . Therefore by Corollary (2.6) L is a WAPP-prime submodule of P .

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Proposition 2.15 Let P be an R-module with Soc(P) a weakly prime submodule of P . If L is a proper submodule of P with $L\subseteq Soc(P)$, then L is a WAPP-prime of P .

Proof: Let $0\neq JA\subseteq L$ for J is an ideal of R and A a submodule of P . Since $L\subseteq Soc(P)$ then $(0)\neq JA\Rightarrow Soc(P)$. But Soc(P) is a weakly prime submodule of P , then either $A\subseteq Soc(P)\subseteq L+Soc(P)$ or $JP\subseteq Soc(P)\subseteq L+Soc(P)$. Therefor L is a WAPP-prime submodule of P .

We end this section by the following proposition .

Proposition 2.16 Let P be an R-module and L a submodule of P with $Soc(P)\subseteq L$. Then L is a WAPP-prime submodule of P if and only if $[L:_P I]$ is a WAPP-prime submodule of P, for every nonzero ideal I of R .

Proof Let $(0)\neq Je\subseteq [L:_P I]$ for $e\in P$ and J is an ideal of R , that is $(0)\neq J(Ie)\subseteq L$ .Since L is a WAPP-prime submodule of P , then by Proposition (2.5) either $Ie\subseteq L+Soc(P)$ or $JP\subseteq L+Soc(P)$. But $Soc(P)\subseteq L$, implies that $L+Soc(P)=L$. Hence either $Ie\subseteq L$ or $JP\subseteq L$, implies that either $e\in [L:_P I]$ or $JP\subseteq L\subseteq[L:_P I]$. That is either $e\in[L:_P I]+Soc(P)$ or $JP\subseteq[L:_P I]+Soc(P)$ .Thus by Corollary (2.6) $[L:_P I]$ is a WAPP-prime submodule of P.

($\Leftarrow$) Follows by taking I=R .

2. Characterizations in class of multiplication modules .

In this section we give many characterizations of WAPP-prime submodules in the class of multiplication modules .

The first characterization of WAPP-prime submodules was introduced in the next proposition .

Proposition 3.1 Let P be a multiplication R-module , and L a proper submodule of P. Then L is a WAPP-prime submodule of P if and only if , whenever $(0)\neq AB\subseteq L$ for A,B are submodules of P , implies that either $A\subseteq L+Soc(P)$ or $B\subseteq L+Soc(P)$.

Proof($\Rightarrow$) $0\neq AB\subseteq L$ for A,B are submodules of P . Since P is multiplication , then A=IP , B=JP , for some ideals I,J of R . That is $0\neq I(JP)\subseteq L$, but L is a WAPP-prime submodule of P , then by Proposition (2.5)) either $JP\subseteq L+Soc(P)$ or $IP\subseteq L+Soc(P)$. It follows that either $B\subseteq L+Soc(P)$ or $A\subseteq L+Soc(P)$.

($\Leftarrow$) Let $0\neq I_1 C\subseteq L$ for C is a submodule of P and I_1 is an ideal of R . Since P is a multiplication , then $C=I_2P$ for some ideal I_2 of R , that is $0\neq I_1 I_2P\subseteq L$. Put $B=I_1P$,then $0\neq B C\subseteq L$, then by hypothesis either $B\subseteq L+Soc(P)$ or $C\subseteq L+Soc(P)$. That is either $C\subseteq L+Soc(P)$ or $I_1P\subseteq L+Soc(P)$. Hence by Proposition (2.5) L is a WAPP-prime submodule of P .

The following corollary is a direct consequence of Proposition (3.1) .

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Corollary(3.2) Let P be a multiplication R-module , and L a proper submodule of P. Then L is a WAPP-prime submodule of P if and only if , whenever $(0)\neq Ae\subseteq L$ for A is a submodule of P and $e\epsilon P$, implies that either $e\epsilon L+Soc(P)$ or $A\subseteq L+Soc(P)$.

It is well known that in a Z-regular R-module P, Soc(P)=Soc(R)P [15 **,**Prop. (3.25)] .

Proposition 3.3 Let P be a multiplication Z-regular R-module , and L a proper submodule of P. Then L is a WAPP-prime submodule of P if and only if $[L:_R P]$ is a WAPP-prime ideal of R .

Proof(⇒) Let $0\neq IJ\subseteq[L:_R P]$ for I,J are ideals of R ,then $0\neq IJP\subseteq L$. Since P is a multiplication , then IJP=AB , by taking A=IP ,B=JP are submodule of P , that is $0\neq AB\subseteq L$. Since L is a WAPP-prime submodule of P, and P is a multiplication , then by Proposition (3.1) either $A\subseteq L+Soc(P)$ or $B\subseteq L+Soc(P)$. Again since P is a multiplication , then $L=[L:_R P]P$, and since P is a Z-regular then Soc(P)=Soc(R)P .Hence either $IP\subseteq[L:_R P]P+Soc(R)P$ or $JP\subseteq[L:_R P]P+Soc(R)P$ .That is either $I\subseteq[L:_R P]+Soc(R)$ or $J\subseteq[L:_R P]+Soc(R)$ .It follows that either $J\subseteq[L:_R P]+Soc(R)$ or $I\subseteq[[L:_R P]+Soc(R):_R R]$. Thus by Proposition(2.5) $[L:_R P]$ is a WAPP-prime ideal of R .

(⇐) Let $(0)\neq I_1C\subseteq L$ for I_1 is an ideal of R and C is a submodule of P .Since P is a multiplication , then $C=I_2P$ for some ideal I_2 of R , it follows that $0\neq I_1 I_2P\subseteq L$, implies that $0\neq I_1 I_2\subseteq[L:_R P]$ .But $[L:_R P]$ is a WAPP-prime ideal of R , then by proposition (2.5) either $I_2\subseteq[L:_R P]+Soc(R)$ or $I_1\subseteq[[L:_R P]+Soc(R):_R R]=[L:_R P]+Soc(R)$. That is either $I_2P\subseteq[L:_R P]P+Soc(R)P$ or $I_1P\subseteq[L:_R P]P+Soc(R)P$ .Since P is a Z-regular R-module , then Soc(P)=Soc(R)P . Thus either $C\subseteq L+Soc(P)$ or $IP\subseteq L+Soc(P)$. It follows that either $C\subseteq L+Soc(P)$ or $I\subseteq[L+Soc(P):_R P]$. Hence by Proposition (2.5) L is a WAPP-prime submodule of P .

It is well known that in projective R-module P,Soc(P)=Soc(R)P[15 , Prop(3-24)].

Proposition 3.4 Let P be a projective multiplication R-module , and L a proper submodule of P. Then L is a WAPP-prime submodule of P if and only if $[L:_R P]$ is a WAPP-prime ideal of R .

Proof:(⇒) Let $0\neq rI\subseteq[L:_R P]$ for $r\epsilon R$ and I ideal of R .It follows that $0\neq rIP\subseteq L$ since L is a WAPP-prime submodule of P , then by Proposition (2.4) either $IP\subseteq L+Soc(P)$ or $rP\subseteq L+Soc(P)$. But L is a multiplication projective R-module , then either $IP\subseteq[L:_R P]P+Soc(R)P$ or $rP\subseteq[L:_R P]P+Soc(R)P$ .That is either $I\subseteq[L:_R P]+Soc(R)$ or $r\epsilon[L:_R P]+Soc(R)=[[L:_R P]+Soc(R):_R R]$. Therefore by proposition (2.4) $[L:_R P]$ is a WAPP-prime ideal of R .

(⇐) Let $(0)\neq rB\subseteq L$ for $r\epsilon R$ and B is a submodule of P .Since P is a multiplication , then B=JP for some ideal J of R , that is $0\neq rJP\subseteq L$, it follows that $0\neq rJ\subseteq[L:_R P]$ .But $[L:_R P]$ is a WAPP-prime ideal of R , then by Proposition (2.4) either $J\subseteq[L:_R P]+Soc(R)$ or $r\epsilon[[L:_R P]+Soc(R):_R R]=[L:_R P]+Soc(R)$.

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That is either IP $\subseteq$[L:$_R$ P] P +Soc(R)P or rP$\subseteq$[L:$_R$ P] P +Soc(R)P .But P is a projective R-module , then either B$\subseteq$L+Soc(P) or rP$\subseteq$L+Soc(P) .That is either B$\subseteq$L+Soc(P) or rϵ[L+Soc(P):$_R$ P] . Therefor by Proposition (2.4) L is a WAPP-prime submodule of P .

It is well known if an R-module P is a finitely generated multiplication , and I,J are ideals of R .Then IP$\subseteq$JP if and only if I$\subseteq$J+ann$_R$(P)[16] .

Proposition 3.5 Let P be a finitely generated multiplication Z-regular R-module , and I a WAPP-prime ideal of R with ann$_R$(P)$\subseteq$I .Then IP is a WAPP-prime submodule of P .

Proof: Let (0)$\neq$LK$\subseteq$IP for L,K are submodules of P . Since P is multiplication ,then L=I_1P, K=I_2P for some ideals I_1,I_2 of R ,that is (0)$\neq I_1I_2$P $\subseteq$IP .Since P is a finitely generated multiplication ,then (0)$\neq I_1I_2$ $\subseteq$I+ann$_R$(P).But ann$_R$(P)$\subseteq$I ,implies that I+ann$_R$(P)=I ,that is (0)$\neq I_1I_2$ $\subseteq$I .But I is a WAPP-prime ideal of R then by proposition (2.5) either $I_2 \subseteq$ I+Soc(R) or $I_1 \subseteq$ [I+Soc(R):$_R$R]= I+Soc(R) .Thus either I_2P $\subseteq$ IP+Soc(R)P or I_1P $\subseteq$ IP+Soc(R)P ,implies that either K $\subseteq$ IP+Soc(P) or L $\subseteq$ IP+Soc(P). Therefore IP is a WAPP-prime submodule of P by Proposition (3.1) .

Proposition 3.6 Let P be a finitely generated multiplication projective R-module , and I a WAPP-prime ideal of R with ann$_R$(P)$\subseteq$I .Then IP is a WAPP-prime submodule of P .

Proof: Let (0)$\neq$JB$\subseteq$IP for J is an ideal of R ,and B a submodule of P . Since P is a multiplication ,then (0)$\neq$JI_1P $\subseteq$IP for some ideal $I_1$of R .But P a finitely generated ,then (0)$\neq$JI_1 $\subseteq$I+ann$_R$(P) ,that is (0)$\neq$JI_1 $\subseteq$I (for ann$_R\subseteq$I) .But I is a WAPP-prime ideal of R then by Proposition (2.5) either $I_1 \subseteq$ I+Soc(R) or J $\subseteq$ [I+Soc(R):$_R$R]= I+Soc(R) .Thus either I_1P $\subseteq$ IP+Soc(R)P or JP $\subseteq$ IP+Soc(R)P .But P is a projective then either B $\subseteq$ IP+Soc(P) or J $\subseteq$[IP+Soc(P):$_R$P]. Therefore by Proposition (2.5) IP is a WAPP-prime submodule of P .

It is well known that cyclic R-module is multiplication [12]. Also ,every cyclic R-module a finitely generated[11].We have the following corollaries as direct consequence of Proposition (3.1),Corollary (3.2) and Proposition (3.3,3.4,3.5,3.6) .

Corollary3.7 Let P be a cyclic R-module , and L a proper submodule of P .Then L is a WAPP-prime submodule if and only if whenever (0)$\neq$AB$\subseteq$L ,for A,B are submodules of P, implies that either A$\subseteq$L+Soc(P) or B $\subseteq$L+Soc(P) .

Corollary3.8 Let P be a cyclic R-module , and L a proper submodule of P .Then L is a WAPP-prime submodule if and only if whenever (0)$\neq$Ae$\subseteq$L ,for A is a submodule of P ,eϵP , implies that either eϵL+Soc(P) or A $\subseteq$L+Soc(P) .

Corollary3.9 Let P be a cyclic Z-regular R-module , and L a WAPP-prime submodule of P. Then [L:$_R$P] is a WAPP-prime ideal of R .

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Corollary3.10 Let P be a cyclic projective R-module , and L a WAPP-prime submodule of P if and only if $[L:_R P]$ is a WAPP-prime ideal of R .

Corollary3.11 Let P be a cyclic Z-regular R-module , and I a WAPP-prime ideal of R with $ann_R(P)\subseteq I$ .Then IP is a WAPP-prime submodule of P .

Corollary3.12 Let P be a cyclic projective R-module , and I a WAPP-prime ideal of R with $ann_R(P)\subseteq I$ .Then IP is a WAPP-prime submodule of P .

Proposition 3.13 Let P be a finitely generated multiplication Z-regular R-module , and L a proper submodule of P with $ann_R(P)\subseteq[L:_R P]$,then the following statements are equivalent :

1. L is a WAPP-prime submodule of P.
2. $[L:_R P]$ is a WAPP-prime ideal of R .
3. L=IP for some WAPP-prime ideal I of R with $ann_R(P)\subseteq I$.

Proof (1)⇔(2) Follows by Proposition (3.3)

(2)⇒(3) Since P is a multiplication module then $L= [L:_R P]P$ where $[L:_R P]$ is a WAPP-prime ideal of R with $ann_R(P)= [0:_R P]\subseteq [L:_R P]$ .Put $I=[L:_R P]$ .Thus L=IP where I is a WAPP-prime ideal of R .

(3)⇒(2)Suppose L=IP for some WAPP-prime ideal I of R with $ann_R(P)\subseteq I$ .But P is a multiplication it follows that $L=[L:_R P]P=IP$ .Since P is a finitely generated then by [16,Prop(3-9)] P is a weak cancellation that is $[L:_R P]+ann_R(P)=I+ ann_R(P)$ .But $ann_R(P) \subseteq[L:_R P]$ and $ann_R(P)\subseteq I$, it follows that $[L:_R P]+ann_R(P)= [L:_R P]$ and $ann_R(P)+I=I$ .Therefor $[L:_R P] =I$ but I is a WAPP-prime ideal of R ,then $[L:_R P]$ is a WAPP-prime ideal of R .

Proposition 3.14 Let P be a finitely generated multiplication projective R-module , and L a proper submodule of P with $ann_R(P)\subseteq[L:_R P]$,then the following statements are equivalent :

1. L is a WAPP-prime submodule of P.
2. $[L:_R P]$ is a WAPP-prime ideal of R .
3. L=IP for some WAPP-prime ideal I of R with $ann_R(P)\subseteq I$.

Proof (1)⇔(2) Follows by Proposition (3.4)

(2)⇔(3)The same as in Proposition (3.13)

The following corollaries are direct consequence of propositions(3.12 and 3.14).

Corollary 3.15 Let P be a cyclic Z-regular R-module , and L a proper submodule of P with $ann_R(P)\subseteq[L:_R P]$ .Then the following statements are equivalent :

1.L is a WAPP-prime submodule of P.

2. $[L:_R P]$ is a WAPP-prime ideal of R .

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3.L=IP for some WAPP-prime ideal I of R with $ann_R(P) \subseteq I$.

Corollary 3.16 Let P be a cyclic projective R-module , and L a proper submodule of P with $ann_R(P) \subseteq [L:_R P]$ .Then the following statements are equivalent :

1. L is a WAPP-prime submodule of P.
2. $[L:_R P]$ is a WAPP-prime ideal of R .
3. L=IP for some WAPP-prime ideal I of R with $ann_R(P) \subseteq I$.

We end this section by the following proposition .

Proposition 3.17 Let P be a multiplication R-module and L a proper submodule of P with $[L+Soc(P):_R P]$ is a prime ideal of R , and $L+Soc(P) \subseteq K$ for each submodule K of P .Then P is a WAPP-prime submodule of P.

Proof :Let $0 \neq ae \in L$ for $a \in R$,$e \in P$ with $e \notin L+Soc(P)$,it follows that $L+Soc(P) \subseteq L+Soc(P)+\langle e\rangle = K$ .Since P is a multiplication then$[K:_R P] \not\subset [L+Soc(P):_R P]$, then there exists $r \in [K:_R P]$ and $r \notin [L+Soc(P):_R P]$. That is $rP \subseteq K$ and $rP \not\subset L+Soc(P)$ .Thus $rP \subseteq K$,implies that $arP \subseteq a(L+Soc(P)+\langle e\rangle) \subseteq L+Soc(P)$. That is $ar \in [L+Soc(P):_R P]$ .But $[L+Soc(P):_R P]$ is a prime ideal of R and $r \notin [L+Soc(P):_R P]$ then $a \in [L+Soc(P):_R P]$. Thus L is a WAPP- prime submodule of P .

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المقاسات الجزئية الاولية من النمط-WAPPومفاهيم ذات العلاقة

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المستخلص:

لتكن R حلقه ابدالية بمحايد و P مقاسا احاديا ايسر ,الغرض من هذا البحث هو لتقديم و دراسة مفهوم المقاسات الجزئية الاوليه من النمط –WAPP كأعمام لمفهوم المقاسات الجزئية الضعيفة حيث يدعى المقاس الجزئي الفعلي L من المقاس P انه مقاس جزئي اولي من نمط –WAPP اذا كان $0\neq re\epsilon L$ حيث ان $e\epsilon P$,$r\epsilon R$ فانه يؤدي الى انه اما $e\epsilon L+Soc(P)$ او $r\epsilon[L+Soc(P) :_R P]$, العديد من الامثله و المكافئات والصفات الاساسيه لهذا المفهوم قد اعطيت اضافه لذلك مكافئات لمفهوم المقاسات الجزئيه من النمط – WAPP في صف المقاسات الضربيه قدمت .

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