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Some Properties of Algebra Fuzzy Metric Space

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ABSTRACT

Our goal in the present paper is to introduce new type of fuzzy metric space called algebra fuzzy metric space after that some examples is introduced to illustrate this notion. Then basic properties of algebra fuzzy metric space is proved.

MSC : 30C45 , 30C50

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1 . Introduction

The fuzzy topological structure of a fuzzy normed space was studied by Sadeqi and Kia in 2009 [1]. Kider introduced a fuzzy normed space in 2011 [2]. Also he proved this fuzzy normed space has a completion in [3]. Again Kider introduced a new fuzzy normed space in 2012 [4]. The properties of fuzzy continuous mapping which is defined on a fuzzy normed spaces was studied byNadaban in 2015 [5].

Kider and Kadhum in 2017 [6] introduce the fuzzy norm for a fuzzy bounded operator on a fuzzy normed space and proved its basic properties then other properties was proved by Kadhum in 2017

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[7]. Ali in 2018 [8] proved basic properties of complete fuzzy normed algebra. Kider and Ali in 2018 [9] introduce the notion of fuzzy absolute value and study properties of finite dimensional fuzzy normed space.

The concept of general fuzzy normed space were presented by Kider and Gheeab in 2019 [10] [11] also they proved basic properties of this space and the general fuzzy normed space GFB(V, U). Kider and Kadhum in 2019 [12] introduce the notion fuzzy compact linear operator and proved its basic properties. For more information about fuzzy metric spaces also see [13, 14].

In present paper first we introduce the notion algebra fuzzy metric space which is a new type of a fuzzy metric spaces and a generalization of ordinary metric space after that two examples is solved to show that the existence of such type of fuzzy metric. Then we continuo this study by introducing open fuzzy ball, fuzzy open set, fuzzy convergence of sequences at this point basic properties of algebra fuzzy metric space is proved. Finely the notion fuzzy continuous function and uniform fuzzy continuous function between two algebra fuzzy metric spaces is introduced also basic properties of theses notions is proved.

2. Basic Properties of Algebra fuzzy metric space

Definition 2.1[15].

Let $\circledcirc:[0, 1] \times [0, 1] \to [0, 1]$ be a binary operation function then $\circledcirc$ is said to be continuous t-conorm (or simply t-conorm) if it satisfies the following conditions $s, r, z, w \in [0, 1]$

(i) $s \circledcirc r = r \circledcirc s$

(ii) $s \circledcirc [r \circledcirc z] = [s \circledcirc r] \circledcirc z$

(iii) $\circledcirc$ is continuous function

(iv) $s \circledcirc 0 = 0$

(v) $(s \circledcirc r) \leq (z \circledcirc w)$ whenever $s \leq z$ and $r \leq w$.

Lemma 2.2[15].

If $\circledcirc$ is a continuous t-conorm on [0, 1] then

(i) $1 \circledcirc 1 = 1$

(ii) $0 \circledcirc 1 = 1 \circledcirc 0 = 1$

(iii) $0 \circledcirc 0 = 0$

(iv) $a \circledcirc a \geq a$ for all $a \in [0, 1]$.

Remark 2.3[15].

If $\circledcirc$ is a continuous t-conrm then

(i) For any $a, b \in (0, 1)$ with $a > b$ we have $d \in (0, 1)$ whenever $a > d \circledcirc b$

(ii) For any $a \in (0, 1)$ there exists $c \in (0, 1)$ such that $c \circledcirc c \leq a$.

Example 2.4[15].

The algebra product a ⊙ b = a + b – ab is a continuous t-conorm for all a, b ∈ [0, 1].

Definition 2.5[15].

Assume that S≠ ∅, a fuzzy set $\widetilde{D}$ in S is represented by $\widetilde{D} = \{(s, \mu_{\widetilde{D}}(s)): s \in S, 0 \leq \mu_{\widetilde{D}}(s) \leq 1\}$

where $\mu_{\widetilde{D}}(x)$: S→ [0,1] is a membership function.

The following definition the main definition

Definition 2.6:

A triple (S, m, ⊙) is said to be the algebra fuzzy metric space if S≠ ∅, ⊙ is a continuous t-conorm and m: S×S→[0, 1] satisfying the following conditions:

(A_1)$0 \leq m(s, r) \leq 1$;

(A_2) m(s, r) = 0 if and only if s = r;

(A_3) m(s, r) = m(r, s) ;

(A_4) m(s, t) ≤ m(s, r) ⊙ m(r, t)

For all s, r, t ∈ S then the triple (S, m, ⊙) is said to be the algebra fuzzy metric space

Example 2.7:

If (S, d) is a metric space and t ⊙ r = t + r – tr for all t, r ∈ [0, 1]. Put $m_d(s, u) = \frac{d(s,u)}{1+d(s,u)}$ for all s, u ∈ S. Then (S, m_d, ⊙) is algebra fuzzy metric space. m_d is known as the algebra fuzzy metric comes from d.

Proof:

We will show that m_d satisfy all the conditions of definition 2.6

(A_1) It is clear that $0 \leq m_d(s, u) \leq 1$ for all s, u ∈ S.

(A_2) $m_d(s, u) = 0$ if and only if $\frac{d(s,u)}{1+d(s,u)}$ =0 if and only if d(s, u)=0 if and only if s = u.

(A_3) $m_d(s, u) = \frac{d(s,u)}{1+d(s,u)} = \frac{d(u,s)}{1+d(s,u)} = m_d(u, s)$ for all s, u ∈ S.

(A_4) Now we will show that $m_d(s, v) \leq m_d(s, u) ⊙ m_d(u, v)$ for all s, u, v ∈S. Notice that

$m_d(s, u) ⊙ m_d(u, v)= m_d(s, u) + m_d(u, v) - m_d(s, u)\, m_d(u, v)$

$$= \frac{d(s,u)}{[1+d(s,u)]} + \frac{d(u,v)}{[1+d(u,v)]} - \frac{d(s,u)}{[1+d(s,u)]}\frac{d(u,v)}{[1+d(u,v)]}$$

$$= \frac{d(u,v)[1+d(s,u)]+d(s,u)[1+d(u,v)]}{[1+d(s,u)][1+d(u,v)]} - \frac{d(s,u)d(u,v)}{[1+d(s,u)][1+d(u,v)]}$$

$$= \frac{d(u,v)[1+d(s,u)]+d(s,u)[1+d(u,v)]-d(s,u)d(u,v)}{[1+d(s,u)][1+d(u,v)]}$$

$$\geq \frac{d(s,v)}{1+d(s,v)} = m_d(\mathrm{s, v})$$

Hence (S, m_d, ⊚) is algebra fuzzy metric space.

The proof of the next example is clear and hence is omitted.

Example 2.8:

If S≠ ∅ put $m_D(\mathrm{s, u}) = \begin{cases} 0 & if\ s = u \\ 1 & if\ s \neq u \end{cases}$

By simple calculation we see that (S, m_D, ⊚) is algebra fuzzy metric space known as the discrete.

Definition 2.9:

If (S, m, ⊚) is algebra fuzzy metric space then FB(s, j) ={u∈S: m(s, u) <j} is known as an open fuzzy ball with center s∈S and radius j ∈(0, 1). Similarly closed fuzzy ball is defined by FB[s, j] ={u∈S: m(s, u) ≤j}.

Definition 2.10:

If (S, m, ⊚) is algebra fuzzy metric space and W⊆S is known as fuzzy open if FB(w, j) ⊆W for any arbitrary w∈W and for some j ∈(0, 1). Also D⊆S is known as fuzzy closed if D^C is fuzzy open then the fuzzy closure of D, $\overline{\mathrm{D}}$ is defined to be the smallest fuzzy closed set contains D.

Definition 2,11:

If (S, m, ⊚) is algebra fuzzy metric space then D⊆S is known as fuzzy dance in S whenever $\overline{\mathrm{D}}$=S.

Theorem 2.12:

If FB(s, j) is open fuzzy ball in algebra fuzzy metric space (S, m, ⊚) then it is a fuzzy open set.

Proof:

Let FB(s, j) be open fuzzy ball open where s∈S and j∈ (0, 1) . Let u∈FB(s, j) so m(s, u) < j , let t = m(s, u) so t< j , then there is i ∈ (0, 1) such that t ∗ i < j by Rremark 2.3 (ii). Now assume the open fuzzy ball FB(u, i), we will show that FB(u, i) ⊆ FB(s, j). Let z∈FB(u, i) so m(u, z) <i. Hence m(s, z) ≤ m(s, u) ⊚ m(u, z) by definition 2.6 (A4) or m(s, z) ≤ t ⊚ i < j. so z∈ FB(s, j) that is FB(u, i) ⊆ FB(s, j).Therefore FB(s, j) is a fuzzy open set.

Definition 2.13 [1].

Let X be any nonempty set a collection of subset of τ of is said to be a fuzzy topology on X if

(i)X and φ belongs to τ.

(ii)if $A_1, A_2, \ldots, A_n \in \tau$ then $\cap_{i=1}^{n} A_i \in \tau$.

(iii)if $\{A_i: i\in I\} \in \tau$ then $\cup_{i\in I} A_i \in \tau$.

Theorem 2.14:

Every algebra fuzzy metric space is a fuzzy topological space.

Proof:

If (S, m, ⊙) is algebra fuzzy metric space. Put τ_m= {W⊂S: w∈W if and only if we have j∈ (0, 1) with FB(w, j) ⊂W }. Now we will prove τ_m is a fuzzy topology on S.

(i)Clearly ϕ and S belong to τ_M since ϕ and S are fuzzy open.

Let $W_1, W_2, ..., W_n \in \tau_m$ and put $V = \cap_{i=1}^{n} W_i$.We shall show that $V \in \tau_m$. Let v∈V then v∈W_i for each $1 \leq i \leq n$. Hence there exists $0 \leq r_i \leq 1$ such that FB(v, r_i) ⊂ W_i since W_i is fuzzy open for each i=1, 2, ...,n. Put r = min{ r_i : $1 \leq i \leq n$ } thus $r \leq r_i$ for all $1 \leq i \leq n$. So FB(v, r) ⊆ W_i for all $1 \leq i \leq n$ .Therefore FB(v, r) ⊆ $\cap_{i=1}^{n} W_i = V$, thus V∈τ_m.

(iii) Suppose that {W_i: i∈I }∈ τ_m and put $W = \cup_{i\in I} W_i$. We shall show that W∈ τ_m.

Let w∈W then w∈$\cup_{i\in I} W_i$ so w∈ W_i for some i∈I since $W_i \in \tau_M$ there exists 0< r < 1 such that FB(w, r) ⊂ W_i.Thus FB(w, r) ⊂ $W_i \subseteq \cup_{i\in I} W_i = W$ that is W∈ τ_m.

Hence (S, τ_m) is a fuzzy topological space. τ_m is known as the fuzzy topology induced by m.

Theorem 2.15:

Every algebra fuzzy metric space is a Hausdorff space.

Proof :

Suppose that (S, m, ⊙) is algebra fuzzy metric space. Let s, u ∈ S with s≠u. Then 0<m(s, u) <1. Put m(s, u) = r, for some r∈(0, 1). Using by Remark 2.3 we can find r_1 ∈(0, 1) with $r_1 \odot r_1 < r$. Now assume the open fuzzy balls FB(s, r_1) and FB(u, r_1). It is Clear that FB(s, r_1) ∩ FB(u, r_1) = ∅ since if there is z∈FB(x, r_1) ∩ FB(y, r_1) implies

$r = m(s, u) \leq m(s, z) \odot m(z, u) \leq r_1 \odot r_1 < r$ that is impossible. Thus (S, m, ⊙) is a Hausdorff space.

Definition 2.16:

In algebra fuzzy metric space (S, m, ⊙) a sequence (s_n) is said to be fuzzy converge to a point s in S(or simply $s_n \to s$) if for each r ∈(0, 1) then we can find N with m(s_n, s) < r , for each n≥N.

The proof of the next result is clear and hence is omitted.

Proposition 2.17:

In algebra fuzzy metric space (S, m, ⊙) $s_n \to s$ if and only if m(s_n, s) →0.

Definition 2.18:

In algebra fuzzy metric space (S, m, ⊙) a sequence (s_n) is fuzzy Cauchy if for each r∈(0, 1) then we can find N such that m(s_n, s_m)< r, for each m, n ≤ N.

Definition 2.19:

An algebra fuzzy metric space (S, m, ⊚) is known as fuzzy complete if (s_n) is fuzzy Cauchy sequence then s_n →s ∈ S.

Theorem 2.20:

In algebra fuzzy metric space (S, m, ⊚) if s_n →s ∈ S is fuzzy Cauchy.

Proof:

Suppose that (s_n) in S and s_n →s ∈ S then for any r ∈(0, 1) we can find N with m(s_n , s) < r , for all n≥N . Using Remark 2.3 we can find t ∈(0, 1) with r ⊚ r < t. Now

m(s_n, s_m) ≤ m(s_n, s) ⊚ m(s, s_m))< r ⊚ r < t for each m, n ≤ N.

Hence (x_n) is fuzzy Cauchy sequence.

Theorem 2.21:

In algebra fuzzy metric space (S, M, ⊚) if (s_n) ∈ S with s_n → s and (d_n) ∈ S with m(s_n, d_n) → 0 as n→ ∞. Then d_n → s.

Proof:

Since s_n → s so m(s_n, s) → 0 as n→ ∞. Now

m(d_n, s) ≤ m(d_n, s_n) ⊚ M(s_n, s) → 0 as n→ ∞. Hence d_n → s.

Definition 2.22:

Anon empty set D in algebra fuzzy metric space (S, m, ⊚) is known as fuzzy bounded whenever we can find s∈(0, 1) with D ⊂ FB(d, s) for some d ∈S. Also a sequence (d_n) in algebra fuzzy metric space (S, m, ⊚) is fuzzy bounded if we can find s∈(0, 1) with (d_n) ∈ FB(d, s) for some d ∈S.

Lemma 2.23:

In algebra fuzzy metric space (S, m, ⊚) If (s_n) ∈S with s_n →s ∈S. Then (s_n) is fuzzy bounded.

Proof:

Assume that (s_n) ∈S with s_n →s ∈S we can find r∈(0, 1), N with m(s_n, s) < r , for each n≥N. Put a = max{ m(s_1, s). m(s_2, s), ..., m(s_N, s)}then we can find t∈(0, 1) with a ⊚ r < t. Now since m(s_n, s) ≤ m(s_n , s_N) ⊚ m(s_N , s) < a ⊚ r < t. Hence (s_n) is fuzzy bounded.

Lemma 2.24:

In algebra fuzzy metric space (S, m, ⊚) if (s_n) ∈ S with s_n →s ∈S and s_n → d ∈S as n→ ∞. Then s=d.

Proof:

Now assume that s_n → s and s_n → d as n→ ∞ then m(s, d) ≤ m(s, s_n) ⊚ m(s_n, d)

Taking limit to both sides as n→ ∞ we have m(s, d)=0 which implies that s=d.

Lemma 2.25:

In algebra fuzzy metric space (S, m, ⊙) if (s_n) ∈S and (d_n) ∈S s_n →s ∈S and d_n →d ∈S. Then $m(s_n, d_n)$ fuzzy converges to m(s, d) where a ⊙ b = a + b – ab for all a, b ∈ [0, 1].

Proof:

Suppose that (s_n) and (d_n) are fuzzy converges to s and d in S. Then

$m(s_n, d_n) \leq m(s_n, s) \odot m(s, d) \odot m(d, d_n)$

$=[m(s_n, s) + m(s, d) - m(s_n, s)\, m(s, d)] \odot m(d, d_n)$

$=m(s_n, s) + m(s, d) - m(s_n, s)\, m(s, d) + m(d, d_n) - [m(s_n, s) + m(s, d) - m(s_n, s)\, m(s, d)]\, m(d, d_n)$

$= m(s_n, s) + m(s, d) - m(s_n, s)\, m(s, d) + m(d, d_n) - m(s_n, s)\, m(d, d_n) - m(s, d)\, m(d, d_n) + m(s_n, s)\, m(s, d)\, m(d, d_n)$

Now

$m(s_n, d_n) - m(s, d) \leq m(s_n, s) - m(s_n, s)\, m(s, d) + m(d, d_n) - m(s_n, s)\, m(d, d_n) - m(s, d)\, m(d, d_n) + m(s_n, s)\, m(s, d)\, m(d, d_n)$…….(1)

Similarly

$m(s, d) \leq m(s, s_n) \odot m(s_n, d_n) \odot m(d_n, d)$

$=[m(s, s_n) + m(s_n, d_n) - m(s, s_n) m(s_n, d_n)] \odot m(d_n, d)$

$= m(s, s_n) + m(s_n, d_n) - m(s, s_n) m(s_n, d_n) + m(d_n, d) - [m(s, s_n) + m(s_n, d_n) - m(s, s_n) m(s_n, d_n)]\, m(d_n, d)$

$= m(s, s_n) + m(s_n, d_n) - m(s, s_n) m(s_n, d_n) + m(d_n, d) - m(s, s_n)\, m(d_n, d) - m(s_n, d_n)\, m(d_n, d) + m(s, s_n) m(s_n, d_n)\, m(d_n, d)$

Now

$m(s_n, d_n) - m(s, d) \geq m(s, s_n) m(s_n, d_n) - m(s, s_n) - m(d_n, d) + m(s, s_n)\, m(d_n, d) + m(s_n, d_n)\, m(d_n, d) - m(s, s_n) m(s_n, d_n)\, m(d_n, d)$…..(2)

From 1 and 2 we have

$|m(s_n, d_n) - m(s, d)| \leq m(s_n, s) - m(s_n, s)\, m(s, d) + m(d, d_n) - m(s_n, s)\, m(d, d_n) - m(s, d)\, m(d, d_n) + m(s_n, s)\, m(s, d)\, m(d, d_n)$

But as n→ ∞ $m(s_n, s) \to 0$ and $m(d, d_n) \to 0$ so $|m(s_n, d_n) - m(s, d)| \to 0$

Hence $m(s_n, d_n)$ fuzzy converges to m(s, d).

Theorem 2.26:

In algebra fuzzy metric space (S, m, ⊙) when D⊂S then d∈$\overline{D}$ if and only if there is (d_n) ∈ D with $d_n \to$ d.

Proof:

Suppose that d∈ $\overline{D}$, if d∈D then choose the sequence of that type is (d, d,…,d,…) .

If d∉D, it is a limit point of D. Hence we construct the sequence (d_n)∈D by $m(d_n, d) < \frac{1}{n}$ for each n = 1, 2, 3,…..

The fuzzy ball FB(d, $\frac{1}{n}$) contains d_n ∈D and $d_n \to d$ because $\lim_{n\to\infty} m(d_n, d) = 0$.

Conversely if (d_n) in D and $d_n \to d$ then d ∈D or every fuzzy ball of a contain points $d_n \neq d$, so that d is a point of accumulation of D, hence d∈ $\overline{D}$ by the definition of the closure.

Theorem 2.27:

In algebra fuzzy metric space (S, m, ⊚) when D⊂S then $\overline{D}$=S if and only if for any s∈S there is d∈D with m(s, d) < r for some r∈(0, 1).

Proof:

Suppose that D is fuzzy dense in S and s∈S so s∈ $\overline{D}$ and by Theorem 2.26 there is a sequence (d_n) ∈D such that $d_n \to s$ that is for any r∈(0, 1) we can find N with $m(d_n, s)$ <r for all n ≥ N. Take $d = d_N$, so m(d, s) <r.

Conversely to prove D is fuzzy dense in S we have to show that S ⊆ $\overline{D}$. Let s∈S then there is d_k ∈D such that $m(d_k, s) < \frac{1}{k}$. Now take 0 < r < 1 such that $\frac{1}{k} < r$ for each k ≥ N for some positive number N. Hence we have a sequence (d_k) ∈D such that $n(d_k, s) < \frac{1}{k} < r$ for all k ≥ N that is $d_k \to$ s so s ∈ $\overline{D}$.

Proposition 2.28:

Let (S, m_d, ⊚) be the algebra fuzzy metric space induced by d where (S, d) is a metric space and Let (s_n) ∈S. Then $s_n \to$ s∈S in (S, d) if and only if $s_n \to$ s∈S in (S, m_d, ⊚).

Proof:

we know that $m_d(s, y) = \frac{d(s,y)}{1+d(s,y)}$, let $s_n \to$ s∈S then for any $\varepsilon > 0$ there is N with $d(s_n, s) < \varepsilon$ for all n≥N. Now $m_d(s_n, s) = \frac{d(s_n,s)}{1+d(s_n,s)} < \frac{\varepsilon}{1+\varepsilon}$ put $\frac{\varepsilon}{1+\varepsilon}$=r then r ∈(0, 1) and $m_d(s_n, s)$ <r for all n≥N that is (s_n) fuzzy converges to s in (S, m_d, ⊚).

Conversely let (s_n) fuzzy converges to x in $(S, m_d, \circledcirc)$ then for any given r>0 there is N with

$m_d(s_n, s) < r$ so $\frac{d(s_n,s)}{1+d(s_n,s)} < r$ or

$d(s_n, s) < r[1+ d(s_n, s)] = r + r\, d(s_n, s)$ or $d(s_n, x) - r\, d(s_n, s) < r$ or

$d(s_n, s)[1-r] < r$ or $d(s_n, s) < \frac{r}{(1-r)}$. Put $\frac{r}{(1-r)} = \varepsilon$ then $\varepsilon > 0$ then $d(s_n, s) < \varepsilon$

for all $n \geq N$ that is (s_n) converges to $s \in S$ in (S, d).

Proposition 2.29:

Let $(S, m_d, \circledcirc)$ be the algebra fuzzy metric space induced by d where (S, d) is a metric space and Let $(s_n) \in S$. Then (s_n) is a Cauchy sequence in (S, d) if and only if (s_n) is a fuzzy Cauchy sequence in $(S, m_d, \circledcirc)$.

Proof:

Suppose that (s_n) is a Cauchy sequence in (S, d) then for any $\varepsilon > 0$ we can find N such that $d(s_n, s_m) < \varepsilon$ for all $n, m \geq N$. Now

$m_d(s_n, s_m) = \frac{d(s_n,s_m)}{1+d(s_n,s_m)} < \frac{\varepsilon}{1+\varepsilon}$ put $\frac{\varepsilon}{1+\varepsilon} = r$ then $0<r<1$ and $m_d(s_n, s) < r$ for all $n \geq N$ that is (s_n) fuzzy Cauchy in $(S, m_d, \circledcirc)$.

Conversely let (s_n) fuzzy Cauchy sequence in $(S, m_d, \circledcirc)$ then for any r>0 we can find N with

$m_d(s_n, s_m) < r$ so $\frac{d(s_n,s_m)}{1+d(s_n, s_m)} < r$ or $d(s_n, s_m) < r[1+ d(s_n, s_m)] = r + r\, d(s_n, s_m)$ or

$d(s_n, s_m) - r\, d(s_n, s_m) < r$ or $d(s_n, s_m)[1-r] < r$ or $d(s_n, s_m) < \frac{r}{(1-r)}$. Put $\frac{r}{(1-r)} = \varepsilon$ so $\varepsilon > 0$ then

$d(s_n, s_m) < \varepsilon$ for all $n, m \geq N$. Hence (s_n) Cauchy sequence in (S, d).

Theorem 2.30:

Let D be a dense subset of algebra fuzzy metric space $(S, m, \circledcirc)$. If every Cauchy sequence of point of D converges in S then S is fuzzy complete.

Proof:

Let (s_n) be a Cauchy sequence in S, since D is fuzzy dense then for every $s_n \in S$ there is $a_n \in D$ such that $m(s_n, a_n) < s$ for some $0 < s < 1$ by Theorem 2.27. Now by Remark 2.3 we can find $t \in (0,1)$ with $s \circledcirc s < t$. But (s_n) is fuzzy Cauchy so (a_n) is fuzzy Cauchy thus $a_n \to s \in S$

by assumption. Now $m(s_n, s) \leq m(s_n, a_n) \circledcirc N(a_n, s) \leq s \circledcirc s < t$.

Hence $s_n \to s$.

3. Fuzzy continuous and uniform fuzzy continuous function

Definition 3.1:

If $(S, m_S, \circledcirc)$ and $(V, m_V, \circledcirc)$ are two algebra fuzzy metric spaces and $W \subseteq S$. Then a function $T:S \to V$ is called fuzzy continuous at $w \in W$. If for every $0 < r < 1$, we can find some $0 < t < 1$, with $m_V[T(w), T(s)] < r$ as $s \in W$ and $m_S(w, s) < t$.

Also f is said to be fuzzy continuous on W if it is fuzzy continuous at every point of W.

Theorem 3.2:

If $(S, m_S, \circledcirc)$ and $(V, m_V, \circledcirc)$ are two algebra fuzzy metric spaces and $W \subseteq S$. Then a function $T:S \to V$ is fuzzy continuous at $w \in W$ if and only if whenever $w_n \to w$ in W then $T(w_n) \to T(w)$ in V.

Proof:

Let $T:W \to V$ be fuzzy continuous at $w \in W$ and let (w_n) be a sequence in W fuzzy converge to w. Let $0 < t < 1$ be given. By fuzzy continuity of T at w we can find $0 < r < 1$ with $s \in W$ and $m_S(s, w) < r$, implies $m_V[T(s), T(w)] < t$. Since $w_n \to w$ then we can find N such that $n \geq N$ implies $m_S(w_n, w) < r$. Therefore $n \geq N$ implies $m_V[T(w_n), T(w)) < t$. Thus $T(w_n) \to T(w)$.

Conversely assume that $w_n \to w$ in W has the property that $T(w_n) \to T(w)$ in V. To prove that T is fuzzy continuous at w. Assume that f is not fuzzy continuous at w that for every r, $0 < r < 1$ there exists $s \in W$ with $m_S(s, w) < r$ but $m_V[T(s), T(w)] \geq t$. For every $n \in N$. Now for each n we can find $w_n \in W$ such that $m_S(w_n, w) < \frac{1}{n}$ but $m_V[T(w_n), T(w)] \geq t$. That is $w_n \to w$ but $(T(w_n))$ does not fuzzy converges to T(w). This is a contradiction to the assumption. Therefore, the supposition that T is not fuzzy continuous at w must be false.

Proposition 3.3:

The function T of algebra fuzzy metric space $(S, m_S, \circledcirc)$ into algebra fuzzy metric space $(V, m_V, \circledcirc)$ is fuzzy continuous at a point $s \in S$ if and only if for every $0 < t < 1$, there exists $0 < r < 1$ such that , $FB(s, r) \subseteq T^{-1}[FB(T(s), t)]$ where FB(s, r) denotes the open fuzzy ball of radius r with center s.

Proof:

The function T:S→V is fuzzy continuous at s∈S if and only if for every $0 < t < 1$, there exists $0 < r < 1$ such that $m_V[T(s), T(w)] < t$ for all w satisfying $m_S(s, w) < r$ that is w∈ FB(s, r) implies T(s)∈ FB(T(s), t) or T[FB(s, r)] ⊆ FB(T(s), t). This is equivalent to the condition FB(s, r) ⊆ T^{-1}[FB(T(s), t)].

Theorem 3.4:

The function T:S→V is fuzzy continuous on S if and only if $T^{-1}(D)$ is fuzzy open in S for all fuzzy open subset D of V where (S, m_S, ⊚) and (V, m_V, ⊚) are algebra fuzzy metric spaces.

Proof:

Let T be fuzzy continuous on S and let D be a fuzzy open subset of V. If $T^{-1}(D) = \emptyset$ or $T^{-1}(D) = S$ then the proof is completed since ∅ and S are fuzzy open. Now let $T^{-1}(D) \neq \emptyset$ and $T^{-1}(D) \neq S$. Assume that s ∈ $T^{-1}(D)$. Then T(s)∈D. Since D is fuzzy open we can find $0 < t < 1$ with FB(T(s), t) ⊆ D. Since T is fuzzy continuous at s, by Proposition 3.3 for this t we can find $0 < r < 1$ with FB(s, r) ⊆ T^{-1} [FB(T(s), t]) ⊆ $T^{-1}(G)$.Thus $T^{-1}(D)$ is fuzzy open in S.

Suppose conversely, that $T^{-1}(D)$ is fuzzy open in S for all fuzzy open subsets D of V. Let s∈S for each $0 < t < 1$ since the fuzzy ball FB(T(s), t) is fuzzy open set and so T^{-1}[FB(f(s), t)] is fuzzy open in S. Whenever s∈ T^{-1} [FB(T(x), t)] it implied that we can find $0 < r < 1$ with FB(s, r) ⊆ T^{-1}[FB(T(s), t)]. Hence T is fuzzy continuous of S by Proposition 3.3.

Theorem 3.5:

The function T:S→V is fuzzy continuous on S if and only if $T^{-1}(B)$ is fuzzy closed in S for all fuzzy closed subset B of V where (S, m_S, ⊚) and (V, m_V, ⊚) are algebra fuzzy metric spaces.

Proof:

If B is a fuzzy closed subset of V then V−B is fuzzy open in V so that $T^{-1}(V-B)$ is fuzzy open in S by Theorem 3.4. But $T^{-1}(V-B) = V - T^{-1}(B)$ so $T^{-1}(B)$ is fuzzy closed in X.

For the converse assume that that $T^{-1}(B)$ is fuzzy closed in S for all fuzzy closed subset B of V. But ∅ and S are fuzzy closed sets. Then V− $T^{-1}(B)$ is fuzzy open in V and $T^{-1}(V-B) = V - f^{-1}(B)$ is fuzzy open in S. Since every fuzzy open subset of Y is of the type V−B, where B is suitable fuzzy closed set. It follows by using Theorem 3.4 that T is fuzzy continuous.

Theorem 3.6:

Let (S, m_S, ⊚), (V, m_V, ⊚) and (W, m_W, ⊚) be algebra fuzzy metric spaces and let T:X→Y and H:Y→Z are fuzzy continuous functions. Then T∘H is a fuzzy continuous function from S into W.

Proof:

If D is fuzzy open subset of W then by Theorem 3.4 $H^{-1}(D)$ is a fuzzy open subset of V and using theorem 3.4 again we have $T^{-1}(H^{-1}(D))$ is a fuzzy open subset of S. But $(H \circ T)^{-1}(D) = T^{-1}(H^{-1}(D))$ by using the same theorem again that T∘H is fuzzy continuous.

Theorem 3.7:

If $(S, m_S,*)$ and $(V, m_V,*)$ are algebra fuzzy metric spaces and T:S→V then (1)⟺(2)⟺(3) where

(1) T is fuzzy continuous on S.

(2) For all subsets D of V we have $\overline{T^{-1}(D)} \subseteq T^{-1}(\overline{D})$.

(3) For all subsets E of S we have $T(\overline{E}) \subseteq \overline{T(E)}$.

Proof: (1)⇒(2):

If D ⊆V then $\overline{D}$ is a fuzzy closed subset of V so $T^{-1}(\overline{D})$ is fuzzy closed in S. Also $T^{-1}(D) \subseteq T^{-1}(\overline{D})$. Hence $\overline{T^{-1}(D)} \subseteq T^{-1}(\overline{D})$ since $\overline{T^{-1}(D)}$ is the smallest fuzzy closed set containing $T^{-1}(D)$.

Proof: (2)⇒(3):

If E ⊆S then when $B = f(E)$, we have $E \subseteq f^{-1}(B)$ and $\overline{E} \subseteq \overline{T^{-1}(B)} \subseteq T^{-1}(\overline{B})$. Thus $T(\overline{E}) \subseteq T(T^{-1}(\overline{B})) = \overline{B} = \overline{T(E)}$.

Proof: (3)⇒(1):

If D is fuzzy closed set in V and set $T^{-1}(D) = F$ then by Theorem 3.4 it is sufficient to show that F is fuzzy closed in S, that is, $F= \overline{F}$. Now $T(\overline{F}) \subseteq \overline{T(T^{-1}(D))} \subseteq \overline{D} = D$ so that $\overline{F} \subseteq T^{-1}(T(\overline{F})) \subseteq T^{-1}(D) = F$.

Theorem 3.8:

If $(S, m_S, \circledcirc)$, $(V, m_V, \circledcirc)$ are two algebra fuzzy metric spaces and T:S→V and H:S→V are two fuzzy continuous functions. Then the set $\{s \in S: m_V[T(s), H(s)] = 0\}$ is fuzzy closed subset of S.

Proof:

Put $D=\{S \in S: m_V[T(s), H(s)] = 0\}$. Then $S-D=\{s \in S: 0 < m_V[T(s), H(s)] < 1\}$. If S−D is empty, then it is fuzzy open. Thus assume that S−D is nonempty and if s∈S−D. Then $m_V[T(s), H(s)] < 1$. Put

$m_V[T(s), H(s)] = t$, for some $0 < t < 1$. By using the fuzzy continuity of T and H we can find $0 < q < 1$ with $m_S(s, a) < q$ implies that $m_V[T(s), T(a)] < t$ and $m_V[H(s), H(a)] < t$. Hence there exists r for some $0 < r < 1$. By Remark 2.3 such that $t \circledcirc t \circledcirc t < r$. Now

$$m_V(T(a), H(a)) \leq m_V(T(a), T(s)) \circledcirc m_V(T(s), H(s)) \circledcirc m_V(H(s), H(a))$$

$$\leq t \circledcirc t \circledcirc t < r, \text{ for all a satisfying } m_x(s, a) < q.$$

Thus for each $a \in FB(s, q)$, $m_V(T(a), H(a)) < 1$, i.e, $f(a) \neq g(a)$.

So $FB(s, q) \subseteq S-D$. Hence $S-D$ is fuzzy open. Thus F is fuzzy closed.

Corollary 3.9:

Assume that $(S, m_S, \circledcirc)$, $(V, m_V, \circledcirc)$ are two algebra fuzzy metric spaces and $T:S\to V$ and $H:S\to V$ are two fuzzy continuous functions. If the set $D=\{s\in S: m_V[T(s), H(s)] = 0\}$ is fuzzy dense in S then $f = g$.

Proof:

Since D is fuzzy dense in S it is fuzzy closed. we have $S = \overline{D} = D$ that is $T(s) = H(s)$ for all $s \in S$ thus $T = H$.

Definition 3.10:

If $(S, m_S, \circledcirc)$ and $(V, M_V, \circledcirc)$ are two algebra fuzzy metric spaces then function $T:S\to V$ is known as uniformly fuzzy continuous on S, if for every $0 < t < 1$, we can find r, $0 < r < 1$ (depending on t alone) with $m_Y[T(s_1), T(s_2)) < t$ whenever $m_x(s_1, s_2) < r$.

Theorem 3.11:

Assume that $(S, m_S, \circledcirc)$, $(V, M_V, \circledcirc)$ are two algebra fuzzy metric spaces and $T:S\to V$ is a function. Then so is $(T(s_n))$ in V If (s_n) is a fuzzy Cauchy sequence in S.

Proof:

Using our assumption T is uniformly fuzzy continuous thus for any t, $0 < t<1$ we can find r, $0 < r<1$ with $m_V[T(s), T(u)]< t$ such that $m_x(s, u) <r$ for all $s, u\in S$. But (s_n) is fuzzy Cauchy, so for any $0 <r< 1$ we can find N with $m_S(s_n, s_m) <r$ for all $m, n \geq N$. Now it is concluded

that $m_V[T(s_n), T(s_m)] < t$ for all $n, m \geq N$. Hence $(T(s_n))$ is a fuzzy Cauchy in V.

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