The Principle Indecomposable Projective Characters of The Symmetric Groups $S_{22}$ Modulo $p = 19$

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**ABSTRACT**

In this paper we invention all the principle indecomposable spin (projective) characters of the symmetric group $S_n$ when $n = 22$ and the characteristic of the field is 19. The principle indecomposable spin characters of the symmetric group $S_n$ also can get it by determine all irreducible spin characters for $S_{22}, p = 19,$ and all irreducible modular spin characters for $S_{22}, p = 19,$ the sum of multiplication of these characters it represents the principle indecomposable spin (projective) characters of the symmetric group $S_{22}.$ For their more can get on the irreducible spin characters for $S_{22}, p = 19$ by fixing all bar partitions for $S_{22},$ also we can get on all the irreducible modular spin characters for $S_{22}, p = 19$ by $(r, r')$-inducing method, finally induce the principal indecomposable characters (P.i.s) from $S_{21}$ (see creek*) give us principal indecomposable characters (P.i.s) or principal characters (P.s) of $S_{22}.$

**MSC**: 15C15, 15C20, 15C25

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1. Introduction

The principle indecomposable spin (projective) characters is represent the sum of multiplication for irreducible projective characters and irreducible modular projective characters [1]. In General characters is known modular or ordinary corresponding to the characteristic of the field is prime or zero, respectively [2]. Every finite group has covering group [3], then $S_n$ has as this group. The characters of the covering group which are identical the characters of $S_n$ are called modular or ordinary characters of $S_n,$ the survival characters are called projective (spin) of $S_n$ [4]. The irreducible projective characters are described by the bar partitions of $n,$ also these characters are recognize double or associate corresponding to the result $n - m$ is even or odd, respectively where $m$ is the number of parts for the bar partition of
n[3]. A lot of researchers activate in this field like A. K.Yaseen, S. A.Taban, A. H.Jassimand M. M.Jawad[5],[7]. In this paper the principle indecomposablespin(projective) characters of $S_{22}$ modulo $p = 19$ have been calculated by using $(r, \bar{r})$-inducing method, we induce the principal indecomposable characters of $S_{21}$ (see creek*) to have the principal indecomposable characters or principal characters of $S_{22}$.

2. Rudiments

1- The spin characters of $S_n$ can be written as a linear combination, with non-negative integer coefficients, of the irreducible spin characters [8].
2- Projective characters $\langle \omega \rangle = \langle \omega_1, ..., \omega_u \rangle$ for $S_\infty$ have degree which is equal to the number of parts $[9].$
3- The values of associate characters $\langle \omega \rangle$, $\langle \omega \rangle'$ are same on the class except on the class corresponding to $\omega$ they have values $\pm i \frac{2^n-1}{2} \sqrt{\frac{\omega_1...\omega_u}{2}} [9].$
4- The inducing from group or restriction from the subgroup of the projective characters are also projective characters [1].
5- If $z$ is even and $p \nmid (z)$, then $\langle z \rangle$ and $\langle z \rangle'$ are distinct irreducible modular spin characters I.m.s of grade $2 \frac{[\frac{z-1}{2}]}{2}$ which are denoted by $\omega(z)$ and $\omega(z)$ [9].
6- Let $p$ be an odd prime and let $\mu$, $\omega$ be a bar partition of $z$ which are not $p$-bar core. Then $\langle \mu \rangle$ (and $\langle \mu \rangle'$ if $\mu$ is odd) and $\langle \omega \rangle$ (and $\langle \omega \rangle'$ if $\omega$ is odd) are in the same $p$-block $\leftrightarrow (\mu) = (\omega)$ (where $\mu$, $\omega$ are representativesp-bar core for $\mu$, $\omega$ respectively). If $\mu$ is a bar partition of $z$ and $\mu = (\mu)$, then $\langle \mu \rangle$ (and $\langle \mu \rangle'$ if $\sigma$ is odd) forms a $p$-block of defect 0 [4].
7- Let $p$ be an odd prime and $\omega = (\omega_1, ..., \omega_u)$ be a bar partition of $z$, then all irreducible modular spin characters I.m.s in the block B are double(associate), if $(z - u - u_0)$ is even(odd), where $u_0$ the number of parts of $\omega$ divisible by $p$ [4].
8- Each block of defect zero contains exactly one irreducible ordinary characters $\tau$, one irreducible modular characterse and one principal indecomposable character $\vartheta$ such that $\tau = \epsilon = \vartheta$ [10].
9- If $C$ is a principal character for an odd prime $p$ and all the entries in $C$ are divisible by positive integer $q$, then $(1/q)C$ is a principal character [1].

3- The projective block of $S_{22}$

The number of the principle indecomposablespin(projective) characters of $S_{22}$, $p = 19$ is 130 which is equal to the $(19,\alpha)$-regular classes [10].

From preliminaries (6), there are 103 blocks of $S_{22}$, $p = 19$, theses blocks are $j_3, ..., j_{105}$ of defect zero except the block $j_1$, $j_2$ of defect one.

The blocks of defect zero $j_3, ..., j_{105}$ includes

$(1,3,5,6,7), (1,3,5,6,7)', (2,3,4,6,7), (2,3,4,6,7)', (4,5,6,7)^*, (1,2,3,4,5,6,7)^*, (1,2,3,5,6,8), (1,2,5,6,8), (3,5,6,8)^*, (2,3,4,5,8),$
$(2,3,4,5,8)', (1,3,4,6,8), (1,3,4,6,8)', (2,5,7,8)', (3,4,7,8)^*, (1,2,4,7,8), (1,2,4,7,8)', (1,6,7,8)^*, (1,3,4,5,9), (1,3,4,5,9)',

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\( \langle 1,2,4,6,9 \rangle, \langle 1,2,4,6,9 \rangle^\prime, \langle 3,4,6,9 \rangle^*, \langle 2,5,6,9 \rangle^*, \langle 1,2,3,7,9 \rangle, \langle 1,2,3,7,9 \rangle^\prime, \langle 2,4,7,9 \rangle^*\langle 1,5,7,9 \rangle^*\langle 6,7,9 \rangle^*, \langle 2,3,8,9 \rangle^*, \langle 1,4,8,9 \rangle^*, \langle 5,8,9 \rangle^*, \langle 1,2,4,5,10 \rangle, \langle 1,2,4,5,10 \rangle^\prime, \langle 3,4,5,10 \rangle^*, \langle 1,2,3,6,10 \rangle, \langle 1,2,3,6,10 \rangle^\prime, \langle 2,4,6,10 \rangle^*, \langle 1,5,6,10 \rangle^*, \langle 2,3,7,10 \rangle^*, \langle 1,4,7,10 \rangle^*, \langle 5,7,10 \rangle, \langle 5,7,10 \rangle^\prime, \langle 1,3,8,10 \rangle^*, \langle 4,8,10 \rangle, \langle 4,8,10 \rangle^\prime, \langle 1,2,3,5,11 \rangle, \langle 1,2,3,5,11 \rangle^\prime, \langle 2,4,5,11 \rangle^*, \langle 2,3,6,11 \rangle^*, \langle 1,4,6,11 \rangle^*, \langle 5,6,11 \rangle, \langle 5,6,11 \rangle^\prime, \langle 4,7,11 \rangle, \langle 4,7,11 \rangle^\prime, \langle 2,9,11 \rangle, \langle 2,9,11 \rangle^\prime, \langle 1,10,11 \rangle, \langle 1,10,11 \rangle^\prime, \langle 1,2,3,4,12 \rangle, \langle 1,2,3,4,12 \rangle^\prime, \langle 2,3,5,12 \rangle^*, \langle 1,4,5,12 \rangle^*, \langle 1,3,6,12 \rangle^*, \langle 4,6,12 \rangle, \langle 4,6,12 \rangle^\prime, \langle 2,8,12 \rangle, \langle 2,8,12 \rangle^\prime, \langle 1,9,12 \rangle, \langle 1,9,12 \rangle^\prime, \langle 10,12 \rangle^*, \langle 2,3,4,13 \rangle^*, \langle 1,3,5,13 \rangle^*, \langle 4,5,13 \rangle, \langle 4,5,13 \rangle^\prime, \langle 2,7,13 \rangle, \langle 2,7,13 \rangle^\prime, \langle 1,8,13 \rangle, \langle 1,8,13 \rangle^\prime, \langle 9,13 \rangle^*, \langle 1,3,4,14 \rangle^*, \langle 2,6,14 \rangle, \langle 2,6,14 \rangle^\prime, \langle 1,7,14 \rangle, \langle 1,7,14 \rangle^\prime, \langle 8,14 \rangle^*, \langle 2,5,15 \rangle, \langle 2,5,15 \rangle^\prime, \langle 1,6,15 \rangle, \langle 1,6,15 \rangle^\prime, \langle 7,15 \rangle^*, \langle 2,4,16 \rangle, \langle 2,4,16 \rangle^\prime, \langle 1,5,16 \rangle, \langle 1,5,16 \rangle^\prime, \langle 6,16 \rangle^*, \langle 1,4,17 \rangle, \langle 1,4,17 \rangle^\prime, \langle 5,17 \rangle^*, \langle 1,3,18 \rangle, \langle 1,3,18 \rangle^\prime, \langle 4,18 \rangle^* \rangle.

respectively, these characters are principle indecomposable spin characters (preliminaries 6).

The block \( j_2 \) contains the projective characters \( \langle 1,21 \rangle^*, \langle 2,20 \rangle^*, \langle 1,2,19 \rangle, \langle 1,2,19 \rangle^\prime, \langle 1,2,16 \rangle^*, \langle 1,2,4,15 \rangle^*, \langle 1,2,6,13 \rangle^*, \langle 1,2,7,12 \rangle^*, \langle 1,2,8,11 \rangle^*, \langle 1,2,9,10 \rangle^* \rangle.

The principle block \( j_1 \) contains the remaining projective characters.

4. The principle indecomposable spin(projective) characters for the block \( j_2 \) of defect one

From preliminaries \( (7,3) \) allirreducible modular spin characters I.m.s. for the block \( j_2 \) are double and \( \langle \omega \rangle = \langle \omega \rangle^\prime \) on \( (19,\omega) \)-regular classes respectively.

**Theorem (4.1):** The required characters of \( S_{22} \) are \( x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9 \).

**Proof:** Through technique and the method \( (r, \tilde{r}) \)-inducing of principle indecomposable spin characters P.i.s. of \( S_{21}, p = 19 \) (see creek *) to \( S_{22} \) we have

\[
\begin{align*}
    y_1 & \uparrow^{(2,18)} S_{22} = x_1, \\
    y_2 & \uparrow^{(1,0)} S_{22} = x_2, \\
    y_3 & \uparrow^{(17,3)} S_{22} = x_3, \\
    y_4 & \uparrow^{(15,5)} S_{22} = x_4, \\
    y_5 & \uparrow^{(14,6)} S_{22} = x_5, \\
    y_6 & \uparrow^{(12,8)} S_{22} = x_6, \\
    y_7 & \uparrow^{(12,8)} S_{22} = x_7, \\
    y_8 & \uparrow^{(13,7)} S_{22} = x_8, \\
    y_9 & \uparrow^{(12,8)} S_{22} = x_9.
\end{align*}
\]

**Issue (4.1.1):** \( x_3 \) is not subtracted from \( x_1 \)

Suppose \( x_3 \) is subtracted from \( x_1 \), then \( x_3 \cdot x_1 = \langle 1,21 \rangle^* + 2 \langle 2,20 \rangle^* \), and we have \( 2 \langle 1,21 \rangle^* - \langle 2,20 \rangle^* + \langle 1,2,19 \rangle \rangle \).

Is m.s for \( S_{22} \), but \( 2 \langle 1,21 \rangle - \langle 2,20 \rangle^* + \langle 1,2,19 \rangle \rangle_{S_{21}} = \langle 1,2,18 \rangle^* - \langle 21 \rangle^* \) is not m.s. for \( S_{21} \), so, \( x_3 \) is not subtracted from \( x_1 \).

**Issue (4.1.2):** \( \frac{1}{2} \) \( x_2 \) is not P.i.s for \( S_{22} \)

Suppose \( \frac{1}{2} \) \( x_2 \) is P.i.s for \( S_{22} \), then \( \langle 1,21 \rangle^* - \langle 2,20 \rangle^* + \langle 1,2,19 \rangle \rangle \).

Is m.s for \( S_{22} \), but \( \langle 1,21 \rangle - \langle 2,20 \rangle^* + \langle 1,2,19 \rangle \rangle_{S_{21}} = \langle 1,2,18 \rangle^* - \langle 21 \rangle^* \) is not m.s. for \( S_{21} \), so, \( \frac{1}{2} \) \( x_2 \) is not P.i.s. for \( S_{22} \).

**Issue (4.1.3):** \( x_3 \) is not subtracted from \( x_2 \)
Suppose $x_3$ is subtracted from $x_2$, then $x_2 - x_3 = 2 \langle 2,20 \rangle^* \langle 1,2,19 \rangle + \langle 1,2,19 \rangle$ and we have $2 \langle 1,21 \rangle^* - \langle 2,20 \rangle^* + \langle 1,2,19 \rangle$
Is m.s for $S_{22}$, but $(2 \langle 1,21 \rangle^* - \langle 2,20 \rangle^* + \langle 1,2,19 \rangle) \downarrow S_{21} = \langle 1,2,18 \rangle^* - \langle 21 \rangle^*$ is not m.s for $S_{21}$, so $x_3$ is not subtracted from $x_2$.
So, the characters required for this block are given as:
$x_1 = (1,2,19)^* + 2 \langle 2,20 \rangle^* + (1,2,19)^* + (1,2,19)^*$, $x_4 = (1,2,4,15)^* + (1,2,5,14)^*$,
$x_2 = 2 \langle 2,20 \rangle^* + 2 \langle 1,2,19 \rangle + 2 \langle 1,2,19 \rangle^*$, $x_5 = (1,2,4,15)^* + (1,2,5,14)^*$,
$x_3 = (1,2,19)^* + (1,2,19)^*$, $x_6 = (1,2,5,14)^* + (1,2,6,13)^*$,
$x_7 = (1,2,6,13)^* + (1,2,7,12)^*$, $x_8 = (1,2,7,12)^* + (1,2,8,11)^*$, $x_9 = (1,2,8,11)^* + (1,2,9,10)^*$.

5-The principle indecomposable spin characters for the principle block $j_1$ of defect one

From preliminaries (7,3) all irreducible modular spin characters l.m.s. for the block $j_1$ are associate and $\langle \omega \rangle \neq \langle \omega \rangle$ on $(19,\omega)$-regular classes respectively.

**Theorem (5.1):** The required characters for this block are $x_{10}, x_{11}, \ldots, x_{27}$

**Proof:** Through technique and the method $(r, \overline{r})$-inducing of P.i.s. of $S_{21}, p = 19$ (see creek *) to $S_{22}, \langle \omega \rangle \neq \langle \omega \rangle$ on $(19,\omega)$-regular classes and rudiments 5 we have:

$y_1 \uparrow (1,2,18)^* S_{22} = x_{10}, y_1 \uparrow (1,0)^* S_{22} = x_{11}, y_2 \uparrow (1,0)^* S_{22} = x_{12}, y_3 \uparrow (1,0)^* S_{22} = x_{13}, y_3 \uparrow (1,0)^* S_{22} = x_{14}$

$y_3 \uparrow (1,7,3)^* S_{22} = x_{15}, y_4 \uparrow (1,7,3)^* S_{22} = x_{16}, y_4 \uparrow (1,7,3)^* S_{22} = x_{17}, y_5 \uparrow (1,7,3)^* S_{22} = x_{18}, y_5 \uparrow (1,7,3)^* S_{22} = x_{19}$

$y_6 \uparrow (1,7,3)^* S_{22} = x_{20}, y_6 \uparrow (1,7,3)^* S_{22} = x_{21}, y_7 \uparrow (1,7,3)^* S_{22} = x_{22}, y_7 \uparrow (1,7,3)^* S_{22} = x_{23}, y_8 \uparrow (1,7,3)^* S_{22} = x_{24}$

$y_8 \uparrow (1,7,3)^* S_{22} = x_{25}, y_9 \uparrow (1,7,3)^* S_{22} = x_{26}, y_9 \uparrow (1,7,3)^* S_{22} = x_{27}$

So, the characters required for this block are:

$x_{10} = (1,2,18)^* + (3,19)^* + (3,19)^* + (1,3,18)^* + (3,19)^* + (1,3,18)^* + (3,19)^*$,

$x_{11} = (2,3,17)^* + (3,19)^* + (1,3,18)^* + (2,3,17)^* + (3,19)^* + (3,19)^* + (1,3,18)^* + (3,19)^*$,

$x_{12} = (3,19)^* + (3,19)^* + (3,19)^* + (3,19)^* + (3,19)^* + (3,19)^* + (3,19)^* + (3,19)^*$,

$x_{13} = (3,19)^* + (3,19)^* + (3,19)^* + (3,19)^* + (3,19)^* + (3,19)^* + (3,19)^* + (3,19)^*$,

$x_{14} = (3,19)^* + (3,19)^* + (3,19)^* + (3,19)^* + (3,19)^* + (3,19)^* + (3,19)^* + (3,19)^*$,


$x_{17} = (3,19)^* + (3,19)^* + (3,19)^* + (3,19)^* + (3,19)^* + (3,19)^* + (3,19)^* + (3,19)^*$,

$x_{18} = (3,19)^* + (3,19)^* + (3,19)^* + (3,19)^* + (3,19)^* + (3,19)^* + (3,19)^* + (3,19)^*$,

$x_{19} = (3,19)^* + (3,19)^* + (3,19)^* + (3,19)^* + (3,19)^* + (3,19)^* + (3,19)^* + (3,19)^*$,

$x_{20} = (3,19)^* + (3,19)^* + (3,19)^* + (3,19)^* + (3,19)^* + (3,19)^* + (3,19)^* + (3,19)^*$,


$x_{22} = (3,19)^* + (3,19)^* + (3,19)^* + (3,19)^* + (3,19)^* + (3,19)^* + (3,19)^* + (3,19)^*$,


The grade of the projective characters

<table>
<thead>
<tr>
<th>The grade of the projective characters</th>
<th>The projective characters</th>
<th>$H_{21,19}^{1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1024</td>
<td>(21)*</td>
<td>1</td>
</tr>
<tr>
<td>193536</td>
<td>(19,2)</td>
<td>1</td>
</tr>
<tr>
<td>193536</td>
<td>(19,2)′</td>
<td>1</td>
</tr>
<tr>
<td>487424</td>
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<td>1</td>
</tr>
<tr>
<td>62899200</td>
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<td>1</td>
</tr>
<tr>
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<td>(15,4,2)*</td>
<td>1</td>
</tr>
<tr>
<td>684343296</td>
<td>(14,5,2)*</td>
<td>1</td>
</tr>
<tr>
<td>1316044800</td>
<td>(13,6,2)*</td>
<td>1</td>
</tr>
<tr>
<td>1809561600</td>
<td>(12,7,2)*</td>
<td>1</td>
</tr>
<tr>
<td>1663334400</td>
<td>(11,8,2)*</td>
<td>1</td>
</tr>
<tr>
<td>684343296</td>
<td>(10,9,2)*</td>
<td>1</td>
</tr>
</tbody>
</table>

\[\sum y_i\]

**Conclusion and Future work**

We characterize all principal indecomposable spin characters P.i.s. for $S_{22}$ modulo $p = 19$ by method $(r, \bar{r})$-inducing. Our proof strongly depends on theorems (4.1) and (5.1). In future work, we can find the principle indecomposable spin (projective) characters for the symmetric group $S_{23}$ modulo $p = 19$.

**References**


