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On D_b −metric and general partial b-metric spaces

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ABSTRACT

In this paper, we introduce a new type of generalize metric space, which we call D_b-metric space as a generalization of both D-metric and b-metric spaces. Then we prove some fixed point theorem in this space. Also we define a general partial b-metric space as a generalize of a partial b-metric space and study their properties. Finally, find relation between partial b-metric, D_b-metric and general partial b-metric spaces.

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1 . Introduction

There are a number of generalizations of metric space, for example Bakhtin[1] and Czerwik[2]introduced a b-metric space as a generalization of a metric space that can be found also in [3], [4], [5]

Let Y be a non-empty set and $s \geq 1$be given real number. A function $b : Y^2 \to [0,\infty)$ is said to be a b-metric if for all $\alpha,\beta,u \in Y$ the following conditions are satisfied:

b1. $b(\alpha,\beta) = 0$ *if and only if* $\alpha = \beta$;

b2. $b(\alpha,\beta) = b(\beta,\alpha)$;

b3. $b(\alpha,\beta) \leq s[b(\alpha,u) + b(u,\beta)]$.

The pair (Y,b) is called a b-metric space.

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On the other hand, the concept of D-metric space was introduce by Dahge[6]and found in [7],[8],[9]

A nonempty set Y is said to be D-metric space if there exist a function $D: Y^3 \rightarrow [0, \infty)$ satisfy the following conditions:

D1. $D(\alpha, \beta, \gamma) = 0 \Leftrightarrow \alpha = \beta = \gamma$
D2. $D(\alpha, \beta, \gamma) = D(\beta, \alpha, \gamma) = D(\gamma, \alpha, \beta) = D\ (\beta, \gamma, \alpha) = \ ...\ (Symmetry)$
D3. $D(\alpha, \beta, \gamma) \leq D(\mu\ , \beta, \gamma) + D(\alpha, \mu, \gamma) + D(\alpha, \beta, \mu)$
$\forall \alpha, \beta, \gamma$ and $u\ \in Y$, where D is D-metric on Y

And Shukla [10] and [11], [12], [13] introduce the concept of partial b-metric space as follow:

A partial b-metric on a nonempty set Y is a mapping $pb: Y^2 \rightarrow [0, \infty)$ *such that* $\forall\ \alpha, \beta, u \in Y$ and $s \geq 1$ satisfied the following conditions:

pb1. $pb(\alpha, \alpha) = pb(\alpha, \beta) = pb(\beta, \beta)$ *if and only if* $\alpha = \beta$
pb2. $pb(\alpha, \alpha) \leq pb(\alpha, \beta)$,
pb3. $pb(\alpha, \beta) = pb(\beta, \alpha)$,
pb4. $pb(\alpha, \beta) \leq s[pb(\alpha, u) + pb(u, \beta)] - pb(u, u)$.
A partial b-metric space is a pair (Y, pb) such that Y is a nonempty set and pb is a partial b-metric on Y. The number $s\ \geq 1$ is called the coefficient of (Y, pb).

2. D_b-metric spaces

Next we define a new concept namely D_b-metric space

Definition 2. 1. A nonempty set Y is said to be D_b-metric space if there exist a function $D_b: Y^3 \rightarrow [0, \infty)$ satisfy the following condition:
Db1. $D_b(\alpha, \beta, \gamma) = 0 \Leftrightarrow \alpha = \beta = \gamma$
Db2. $D_b(\alpha, \beta, \gamma) = D_b(\beta, \alpha, \gamma) = D_b(\gamma, \alpha, \beta) = D_b\ (\beta, \gamma, \alpha) = \ ...\ (Symmetry)$
Db3. $D_b(\alpha, \beta, \gamma) \leq s[D_b(\mu\ , \beta, \gamma) + D_b(\alpha, \mu, \gamma) + D_b(\alpha, \beta, \mu)]$
$\forall \alpha, \beta, \gamma\ \in Y$, The number $s\ \geq 1$ is called the coefficient of (Y, D_b)

Not that every D_b-metric space is a D-metric space with the coefficient $s = 1$
Example 2. 2. Let $Y = [0, \infty)$, $q > 1$ be a constant. Define a function on Y^3 by
1. $D_{b1}(\alpha, \beta, \gamma) = |\alpha - \beta|^q + |\beta - \gamma|^q + |\gamma - \alpha|^q$
2. $D_{b\infty}(\alpha, \beta, \gamma) = [\max\{|\alpha - \beta|, |\beta - \gamma|, |\gamma - \alpha|\}]^q$

$\forall \alpha, \beta, \gamma \in Y$, then (Y, D_{b1}) and $(Y, D_{b\infty})$ are D_b-metric spaces with the coefficient $s\ = 2^{q-1} > 1$

Solution. We prove (Y, D_{b1}) is D_b-metric space and similarity we prove the other
i. If $D_{b1}(\alpha, \beta, \gamma) = 0\ \Leftrightarrow |\alpha - \beta|^q + |\beta - \gamma|^q + |\gamma - \alpha|^q = 0$
$\Leftrightarrow |\alpha - \beta| = 0\ \Rightarrow\ \alpha = \beta, |\beta - \gamma| = 0\ \Rightarrow \beta = \gamma, |\gamma - \alpha| = 0\ \Rightarrow\ \gamma = \alpha \Leftrightarrow \alpha = \beta = \gamma$
ii. Trivial
iii. For arbitrary real numbers α, β, γ *and* u. Using convexity of the function $f\ (\alpha) = \alpha^{\ q} (\alpha > 0)$ *for* $q\ \geq\ 1$, we obtain that $(a + b)^q \leq 2^{q-1}(a^q + b^q)$, let $a = |\alpha - u|$, $b = |u - \beta|$ then we get, $0 \leq (|\alpha - u| + |u - \beta|)^q \leq 2^{q-1}(|\alpha - u|^q + |u - \beta|^q) \Rightarrow 0 \leq 2^{q-1}(|\alpha - u|^q + |u - \beta|^q)$ by substitute $|\alpha - \beta|^q$ to both side, we get, $|\alpha - \beta|^q \leq 2^{q-1}(|\alpha - u|^q + |u - \beta|^q) + |\alpha - \beta|^q \leq 2^{q-1}(|\alpha - \beta|^q + |\alpha - u|^q + |u - \beta|^q) \ldots 1$
by same way we have $|\beta - \gamma|^q \leq 2^{q-1}(|\beta - \gamma|^q + |\beta - u|^q + |u - \gamma|^q) \ldots 2$

& $|\gamma-\alpha|^q \le 2^{q-1}(|\gamma-\alpha|^q+|\gamma-u|^q+|u-\alpha|^q)$ … 3
Then by a combination 1, 2& 3 we get $|\alpha-\beta|^q+|\beta-\gamma|^q+|\gamma-\alpha|^q \le 2^{q-1}[(|u-\beta|^q+|\beta-\gamma|^q+|\gamma-u|^q)+(|\alpha-u|^q+|u-\gamma|^q+|\gamma-\alpha|^q)+(|\alpha-\beta|^q+|\beta-u|^q+|u-\alpha|^q) \Rightarrow D_{b1}(\alpha,\beta,\gamma) \le s[D_{b1}(u,\beta,\gamma)+D_{b1}(\alpha,u,\gamma)+D_{b1}(\alpha,\beta,u)]$ □

Theorem 2. 3. Let (Y,b) be a b-metric space, define real function $D_{b1}, D_{b\infty}: Y^3 \to [0,\infty)$ by

1. $D_{b1}(\alpha,\beta,\gamma) = b(\alpha,\beta)+b(\beta,\gamma)+b(\gamma,\alpha)$
2. $D_{b\infty}(\alpha,\beta,\gamma) = max\{b(\alpha,\beta), b(\beta,\gamma), b(\gamma,\alpha)\}$

Then (Y,D_{b1}), $(Y,D_{b\infty})$ are D_b −metric spaces with coefficient $s>1$

Proof. Clearly that a condition (D_b1) & (D_b2) are satisfies, we prove a condition (D_b3)

Since $b(u,\beta)\ge 0$ & $b(\alpha,u)\ge 0 \Rightarrow b(u,\beta)+b(\alpha,u)\ge 0 \Rightarrow s[b(u,\beta)+b(\alpha,u)]\ge 0$
Substitute $b(\alpha,\beta)$ to both side, we have

$0 \le b(\alpha,\beta) \le s[b(\alpha,u)+b(u,\beta)]+b(\alpha,\beta) \le s[b(\alpha,\beta)+b(\alpha,u)+b(u,\beta)]$…1
By the same way we get $b(\beta,\gamma) \le s[b(\beta,\gamma)+b(\beta,u)+b(u,\gamma)]$ … 2
& $b(\gamma,\alpha) \le s[b(\gamma,\alpha)+b(\gamma,u)+b(u,\alpha)]$ … 3,
Then by a combination 1, 2 & 3 we have $b(\alpha,\beta)+b(\beta,\gamma)+b(\gamma,\alpha) \le s[\big(b(u,\beta)+b(\beta,\gamma)+b(\gamma,u)\big)+\big(b(\alpha,u)+b(u,\gamma)+b(\gamma,\alpha)\big)+(b(\alpha,\beta)+b(\beta,u)$
$+b(u,\alpha)] \Rightarrow D_{b1}(\alpha,\beta,\gamma) \le s[D_{b1}(u,\beta,\gamma)+D_{b1}(\alpha,u,\gamma)+D_{b1}(\alpha,\beta,u)]$.
be the same way of (1) □

Remark 2. 4. The D_b-metrics given in examples2 satisfy the following properties: For every *α, β, γ, u in* X, with the coefficient $s \ge 1$

Db4. $D_b(\alpha,\beta,\beta) \le s[D(\alpha,u,u)+D_b(u,\beta,\beta)]$
Db5. $D_b(\alpha,\beta,\beta) = D_b(\alpha,\alpha,\beta)$
Db6. $D_b(\alpha,\beta,\beta) \le D_b(\alpha,\beta,\gamma)$

Proposition 2. 5. If in a D_b-metric space(Y,D_b). The conditions (D_b4), (D_b5) are satisfied then$b(\alpha,\beta) = D_b(\alpha,\beta,\beta)$ ... (1), is b-metric space.

Proof.

i. If $b(\alpha,\beta)=0 \Leftrightarrow D_b(\alpha,\beta,\beta)=0 \Leftrightarrow \alpha=\beta$
ii. Trivial
iii. $b(\alpha,\beta) = D_b(\alpha,\beta,\beta) \le s[D_b(\alpha,u,u)+D_{\boldsymbol{b}}(u,\beta,\beta)]$ by (D_b4)
$= s[b(\alpha,u)+b(u,\beta)]$. □

Theorem 2. 6. If in a D_b-metric space (Y,D_b). The condition (D_b4) holds then each of the functions b: $Y^2 \to R^+$ defined by

1. $b(\alpha,\beta) = [D_b^q(\alpha,\beta,\beta)+D_b^q(\alpha,\alpha,\beta)]^{1/q}$ *where* $1 \le q < \infty$
2. $b(\alpha,\beta) = \max\{D_b(\alpha,\beta,\beta), D_b(\alpha,\alpha,\beta)\}$ $\forall\, \alpha,\beta \in Y$,

are b-metric on Y.

Proof. We prove (1) and similarity we prove the other

i. Since $D_b(\alpha,\beta,\beta)\ge 0$ & $D_b(\alpha,\alpha,\beta)\ge 0$ *then* $b(\alpha,\beta)\ge 0$
ii. If $b(\alpha,\beta)=0$, *then* $[D_b^p(\alpha,\beta,\beta)+D_b^p(\alpha,\alpha,\beta)]^{1/p}=0 \Leftrightarrow D_b(\alpha,\beta,\beta)=0$ & $D_b(\alpha,\alpha,\beta)=0 \Leftrightarrow \alpha=\beta$
iii. Trivial

iv. $b(\alpha,\beta) = \left[D_b^p(\alpha,\beta,\beta) + D_b^p(\alpha,\alpha,\beta)\right]^{1/p} \leq \{(s[D_b(\alpha,u,u) + D_b(u,\beta,\beta)])^p + (s[D_b(\alpha,\alpha,u) + D_b(u,u,\beta)])^p\}^{1/p}$ by (D_b4)

$= s\{D_b^p(\alpha,u,u) + D_b^p(u,\beta,\beta)\}^{1/p} + s\{D_b^p(\alpha,\alpha,u) + D_b^p(u,u,\beta)\}^{1/p} = s[b(\alpha,u) + b(u,\beta)]$

Hence b is b-metric space on Y □

Definition 2. 7. Let (Y, D_b) be a D_b-metric space, then

i. A sequence $\{\alpha_n\}$ in D_b-metric space $(Y, D_b)$is converge to $\alpha \in Y$ if there exist a positive integer $m_0$such that $D_b(\alpha_n, \alpha_m, \alpha) < \epsilon \ \forall \ m, n \geq m_0$.

ii. A sequence $\{\alpha_n\}$ in D_b-metric space (Y, D_b) is said to be Cauchy if for given $\epsilon > 0$, there exists a positive integer m_0 such that $D_b(\alpha_n, \alpha_m, \alpha_l) < \epsilon \ \forall \ m, n, l \geq m_0$.

iii. A D_b-metric space (Y, D_b) is said to be complete if every Cauchy sequence in Y converges to a point α in Y.

iv. A D_b-metric space Y is said to be bounded if there exists a constant $M > 0$ such that $D_b(\alpha,\beta,\gamma) \leq M \ for \ all \ \alpha,\beta,\gamma \in Y$.

Theorem 2. 8. Let Y be a complete and bounded $D_b -$metric space and T is self-map on Y satisfying $D_b(T\alpha, T\beta, T\gamma) \leq \lambda max\{D_b(\alpha,\beta,\gamma), D_b(\beta, T\beta, T\gamma)\}$ ... (2)
For all $\alpha,\beta,\gamma \in Y, where \ 0 \leq \lambda < 1$. Then T has a unique fixed point $p \ in \ Y$.

Proof. Let $\alpha_0 \in Y$and define $\alpha_{n+1} = T\alpha_n$
if $\alpha_{n+1} = \alpha_n$ for some n. Then T has a fixed point. Assume that $\alpha_{n+1} \neq \alpha_n$ for each n. In (2), setting $\alpha = \alpha_{n-1}$, $\beta = \alpha_n$, $\gamma = \alpha_{n+p-1}$, $p \geq 2$, we have
$D_b(\alpha_n, \alpha_{n+1}, \alpha_{n+p}) \leq \lambda max\{D_b(\alpha_{n-1}, \alpha_n, \alpha_{n+p-1}), D_b(\alpha_n, \alpha_{n+1}, \alpha_{n+p})\}$ ... (3), now if
$max\{D_b(\alpha_{n-1}, \alpha_n, \alpha_{n+p-1}), D_b(\alpha_n, \alpha_{n+1}, \alpha_{n+p})\} = D_b(\alpha_n, \alpha_{n+1}, \alpha_{n+p})$ ... (4)
For some n, then from (2) we have
$D_b(\alpha_n, \alpha_{n+1}, \alpha_{n+p}) \leq \lambda D_b(\alpha_n, \alpha_{n+1}, \alpha_{n+p}) < D_b(\alpha_n, \alpha_{n+1}, \alpha_{n+p})$ ... (5)
Which is a contradiction since $D_b(\alpha_n, \alpha_{n+1}, \alpha_{n+p}) > 0$. Thus

$max\{D_b(\alpha_{n-1}, \alpha_n, \alpha_{n+p-1}), D_b(\alpha_n, \alpha_{n+1}, \alpha_{n+p})\} = D_b(\alpha_{n-1}, \alpha_n, \alpha_{n+p-1})$ ... (6), for all n. Therefore, we have $D_b(\alpha_n, \alpha_{n+1}, \alpha_{n+p}) \leq \lambda D_b(\alpha_{n-1}, \alpha_n, \alpha_{n+p-1})$ ... (7)

And so $D_b(\alpha_n, \alpha_{n+1}, \alpha_{n+p}) \leq \lambda^n \left(D_b(\alpha_0, \alpha_1, \alpha_p)\right) \ \forall n, \forall p \geq 2$ ... (8)

Let $M_p = D_b(\alpha_0, \alpha_1, \alpha_p)$ then it follows from (8) that
$D_b(\alpha_n, \alpha_{n+1}, \alpha_{n+p}) \leq \lambda^n M_p$ ... (9)
Using condition (D_b3) from definition and (9)

$D_b(\alpha_n, \alpha_{n+p}, \alpha_{n+p+t}) \leq$
$s[D_b(\alpha_{n+1}, \alpha_{n+p}, \alpha_{n+p+t}) + D_b(\alpha_n, \alpha_{n+1}, \alpha_{n+p+t}) + D_b(\alpha_n, \alpha_{n+p}, \alpha_{n+1})] \leq s\lambda^n M_p + s\lambda^n M_{p+t} + sD_b(\alpha_{n+1}, \alpha_{n+p}, \alpha_{n+p+t}) \leq$
$s\lambda^n M_p + s\lambda^n M_{p+t} + s^2[D_b(\alpha_{n+2}, \alpha_{n+p}, \alpha_{n+p+t}) + D_b(\alpha_n, \alpha_{n+2}, \alpha_{n+p+t}) + D_b(\alpha_n, \alpha_{n+p}, \alpha_{n+2})] \leq$
$s\lambda^n M_p + s\lambda^n M_{p+t} + s^2\lambda^n M_p + s^2\lambda^n M_{p+t} + s^2 D_b(\alpha_{n+2}, \alpha_{n+p}, \alpha_{n+p+t}) \leq \cdots \leq (M_p + M_{p+t})(s\lambda^n + s^2\lambda^{n+1} + \cdots + s^{p-n}\lambda^{n+p-1}) + D_b(\alpha_{n+p-1}, \alpha_{n+p}, \alpha_{n+p+t}) \leq s\lambda^n[1 + s\lambda + (s\lambda)^2 + \cdots](M_p + M_{p+t}) \leq \frac{2s\lambda^n}{1-s\lambda}(M_p + M_{p+t})$

As $\lambda \in \left[0,\frac{1}{s}\right)$ *and* $s > 1$, it follow the above inequality that $\lim_{n\to\infty} D_b\left(\alpha_n, \alpha_{n+p}, \alpha_{n+p+t}\right) = 0$, there for, $\{\alpha_n\}$ is Cauchy sequence in Y. Since Y is complete, $\{\alpha_n\}$ convergent call the limit point p. From (2) we shall show that p is fixed point of T. For any $n \in N$, *we have*

$$D_b(\alpha_n, \alpha_{n+1}, Tp) \leq \lambda max\{D_b(\alpha_{n-1}, \alpha_n, p), D_b(\alpha_n, \alpha_{n+1}, Tp)\}$$

Taking the limit as $n \to \infty$, then $D_b(p, p, Tp) \leq 0$ which implies that $p = Tp$, To prove uniqueness, assume that $w \neq p$ is also a fixed point of T. From (2),

$D_b(p,w,p) = D_b(Tp,Tw,Tp) \leq \lambda\, max\{D_b(p,w,p), D_b(w,w,p)\} = \lambda D_b(w,w,p)$ … (10),

But $D_b(w,w,p) = D_b(w,p,w) = D_b(Tw,Tp,Tw) \leq \lambda\, max\{D_b(w,p,w), D_b(p,Tp,Tw)\} = \lambda\, max\{D_b(w,p,w), D_b(p,p,w)\} = \lambda D_b(p,p,w)$ … (11)

Combining (10) and (11) yields $D_b(p,w,p) < \lambda^2 D_b(p,w,p)$, a contradiction. Therefore $p = w$.

3. General partial b-metric spaces

We begin with a new following definition

Definition 3.1. A non-empty set Y is said to be general partial b-metric space if there exists a function $D_{pb}: Y^3 \to [0,\infty)$ called D_{pb} −metric on Y, satisfy the following condition:

Dpb1. $D_{pb}(\alpha,\alpha,\alpha) = D_{pb}(\alpha,\beta,\gamma) = D_{pb}(\beta,\beta,\beta) = D_{pb}(\gamma,\gamma,\gamma) \Leftrightarrow \alpha = \beta = \gamma$
Dpb2. $D_{pb}(\alpha,\alpha,\alpha) \leq D_{pb}(\alpha,\beta,\gamma)$
Dpb3. $D_{pb}(\alpha,\beta,\gamma) = D_{pb}(\alpha,\gamma,\beta) = D_{pb}(\beta,\alpha,\gamma) = D_{pb}(\beta,\gamma,\alpha) = \ldots (Symmetry)$
Dpb4. $D_{pb}(\alpha,\beta,\gamma) \leq s[D_{pb}(\mu,\beta,\gamma) + D_{pb}(\alpha,\mu,\gamma) + D_{pb}(\alpha,\beta,\mu)] - D_{pb}(\mu,\mu,\mu)$
$\forall\, \alpha,\beta,\gamma$ *and* $\mu \in Y$, the number $s \geq 1$ is called the coefficient of $(\mathrm{Y}, D_{\mathrm{pb}})$.

Remark 3. 2. In a general partial *b*-metric space(Y, D_{pb}),

i. If $\alpha,\beta,\gamma \in Y$ and $D_{pb}(\alpha,\beta,\gamma) = 0$, then $\alpha = \beta = \gamma$, but the converse may not be true.
ii. Every *D*-metric space is a general partial *b*-metric space with the coefficient $s = 1$ and zero self-distance.
iii. Every D_b-metric space is general partial b-metric space with zero self-distance.

Example 3. 3. Let $Y = [0,\infty)$, $p > 1$ a constant and define a function D_{pb} on Y^3 by $D_{pb}(\alpha,\beta,\gamma) = [max\{\alpha,\beta,\gamma\}]^q + |\alpha-\beta|^q + |\beta-\gamma|^q + |\gamma-\alpha|^q]$, then D_{pb} is general partial b-metric space on Y. With coefficient $s = 2^{q-1}$

Proof.

i. Since $[max\{\alpha,\beta,\gamma\}]^q + |\alpha-\beta|^q + |\beta-\gamma|^q + |\gamma-\alpha|^q = [max\{\alpha,\alpha,\alpha\}]^q + |\alpha-\alpha|^q + |\alpha-\alpha|^q + |\alpha-\alpha|^q = [max\{\beta,\beta,\beta\}]^q + |\beta-\beta|^q + |\beta-\beta|^q + |\beta-\beta|^q = [max\,\{\gamma,\gamma,\gamma\}]^q + |\gamma-\gamma|^q + |\gamma-\gamma|^q + |\gamma-\gamma|^q$ if and only if $\alpha = \beta = \gamma$ then $D_{pb}(\alpha,\beta,\gamma) = D_{pb}(\alpha,\alpha,\alpha) = D_{pb}(\beta,\beta,\beta) = D_{pb}(\gamma,\gamma,\gamma)$ if and only if $\alpha = \beta = \gamma$
ii. $D_{pb}(\alpha,\alpha,\alpha) = [max\{\alpha,\alpha,\alpha\}]^q + |\alpha-\alpha|^q + |\alpha-\alpha|^q + |\alpha-\alpha|^q = \alpha^q \leq max\{\alpha,\beta,\gamma\} + |\alpha-\beta|^q + |\beta-\gamma|^q + |\gamma-\alpha|^q = D_{pb}(\alpha,\beta,\gamma)$
iii. Trivial

iv. Clearly $[max\{\alpha,\beta,\gamma\}]^q \leq [\max\{u,\beta,\gamma\}]^q + [\max\{\alpha,u,\gamma\}]^q + [\max\{\alpha,\beta,u\}]^q - [\max\{u,u,u\}]^q$ ... 1
And by the convexity of the function $f(\alpha) = \alpha^q (\alpha > 0)$ implies that $(a+b)^q \leq 2^{q-1}(a^q + b^q)$, let $a = |\alpha - u|$, $b = |u - \beta|$ then we get
$0 \leq (|\alpha - u| + |u - \beta|)^q \leq 2^{q-1}(|\alpha - u|^q + |u - \beta|^q) \Rightarrow 0 \leq 2^{q-1}(|\alpha - u|^q + |u - \beta|^q)$
by substitute $|\alpha - \beta|^q$ to both side, we get $|\alpha - \beta|^q \leq 2^{q-1}(|\alpha - u|^q + |u - \beta|^q) + |\alpha - \beta|^q \leq 2^{q-1}(|\alpha - \beta|^q + |\alpha - u|^q + |u - \beta|^q)$
By the same way we get
$|\beta - \gamma|^q \leq 2^{q-1}(|\beta - \gamma|^q + |\beta - u|^q + |u - \gamma|^q)$ & $|\gamma - \alpha|^q \leq 2^{q-1}(|\gamma - \alpha|^q + |\gamma - u|^q + |u - \alpha|^q)$, then $|\alpha - \beta|^q + |\beta - \gamma|^q + |\gamma - \alpha|^q \leq 2^{q-1}[(|\alpha - \beta|^q + |\alpha - u|^q + |u - \beta|^q) + (|\beta - \gamma|^q + |\beta - u|^q + |u - \gamma|^q) + (|\gamma - \alpha|^q + |\gamma - u|^q + |u - u|^q)$...2
From 1&2 we get
$[\max\{\alpha,\beta,\gamma\}]^q + |\alpha - \beta|^q + |\beta - \gamma|^q + |\gamma - \alpha|^q \leq 2^{q-1}[\{(\max\{u,\beta,\gamma\})^q + |u - \beta|^q + |\beta - \gamma|^q + |\gamma - u|^q\} + \{(\max\{\alpha,u,\gamma\})^q + |\alpha - u|^q + |u - \gamma|^q + |\gamma - \alpha|^q\} + \{(\max\{\alpha,\beta,u\})^q + |\alpha - \beta|^q + |\beta - u|^q + |u - \alpha|^q\}] - (\max\{u,u,u\})^q = s[D_{pb}(u,\beta,\gamma) + D_{pb}(\alpha,u,\gamma) + D_{pb}(\alpha,\beta,u)] - D_{pb}(u,u,u)$

Definition 3. 4. Let (Y, D_{pb}) be a general partial b-metric space, then

i. For $\epsilon > 0$ and $\alpha \in Y$ the open-ball with center α and radius ϵ is $B_{D_{pb}}(\alpha,\epsilon) = \{\beta \in Y \mid D_{pb}(\alpha,\beta,\beta) < D_{pb}(\alpha,\alpha,\alpha) + \epsilon\}$
ii. A sequence $\{\alpha_n\}$ in (Y, D_{pb}) is said to be converge to a point $\alpha \in Y$ if $\lim_{n,m\to\infty} D_{pb}(\alpha_n,\alpha_m,\alpha) = D_{pb}(\alpha,\alpha,\alpha)$
iii. A sequence $\{\alpha_n\}$ in (Y, D_{pb}) is said to be Cauchy sequence if $\lim_{n,m,l\to\infty} D_{pb}(\alpha_n,\alpha_m,\alpha_l)$ exists (finite)
iv. A general partial b-metric space (Y, D_{pb}) is said to be complete if every Cauchy sequence is converge to a point α in Y.
v. A mapping $F:(Y, D_{pb}) \to (Y', D_{pb}')$ is said to be continuous at α if for each open ball $B_{D_{pb}}(F(\alpha),\epsilon')$ in (Y', D'_{pb}) there exists a ball $B_{pb}(\alpha,\epsilon)$ in (Y, D_{pb}) such that $F(B_{D_{pb}}(\alpha,\epsilon)) \subseteq B_{D_{pb}}(F(\alpha),\epsilon')$.

4. Relation between D_b-metric space, pb-metric space and D_{pb}-metric space

Theorem 4. 1. Let (Y, D_{pb}) be a general partial b-metric space, then the functions $D_b^g: Y^3 \to [0,\infty)$ given by $D_b^g(\alpha,\beta,\gamma) = D_{pb}(\alpha,\alpha,\beta) + D_{pb}(\alpha,\alpha,\gamma) + D_{pb}(\beta,\beta,\alpha) + D_{pb}(\beta,\beta,\gamma) + D_{pb}(\gamma,\gamma,\alpha) + D_{pb}(\gamma,\gamma,\beta) - 2D_{pb}(\alpha,\alpha,\alpha) - 2D_{pb}(\beta,\beta,\beta) - 2D_{pb}(\gamma,\gamma,\gamma)$. (12)

is D_b-metric space on Y, with coefficient $s \geq 1$

Proof.

i. Since $D_{pb}(\alpha,\alpha,\beta) - D_{pb}(\alpha,\alpha,\alpha) \geq 0$, $D_{pb}(\alpha,\alpha,\gamma) - D_{pb}(\alpha,\alpha,\alpha) \geq 0$, $D_{pb}(\beta,\beta,\alpha) - D_{pb}(\beta,\beta,\beta) \geq 0$, $D_{pb}(\beta,\beta,\gamma) - D_{pb}(\beta,\beta,\beta) \geq 0$, $D_{pb}(\gamma,\gamma,\alpha) - D_{pb}(\gamma,\gamma,\gamma) \geq 0$ and $D_{pb}(\gamma,\gamma,\beta) - D_{pb}(\gamma,\gamma,\gamma) \geq 0$, *so* $D_b^g(\alpha,\beta,\gamma) \geq 0$.
ii. *If* $D_b^g(\alpha,\beta,\gamma) = 0$ *then*

$$D_{pb}(\alpha,\alpha,\beta) + D_{pb}(\alpha,\alpha,\gamma) + D_{pb}(\beta,\beta,\alpha) + D_{pb}(\beta,\beta,\gamma) + D_{pb}(\gamma,\gamma,\alpha) + D_{pb}(\gamma,\gamma\ \beta) - 2D_{pb}(\alpha,\alpha,\alpha) - 2D_{pb}(\beta,\beta,\beta) - 2D_{pb}(\gamma,\gamma\ \gamma) = 0$$

$Take\ D_{pb}(\alpha,\alpha,\beta)+D_{pb}(\alpha,\alpha,\gamma)-2D_{pb}(\alpha,\alpha,\alpha)=0 \Rightarrow 2D_{pb}(\alpha,\alpha,\alpha)=D_{pb}(\alpha,\alpha,\beta)+D_{pb}(\alpha,\alpha,\gamma) \ldots\ldots\ldots 1,$

$D_{pb}(\alpha,\alpha,\gamma)+D_{pb}(\beta,\beta,\alpha)-2D_{pb}(\alpha,\alpha,\alpha)=0 \Rightarrow 2D_{pb}(\alpha,\alpha,\alpha)=D_{pb}(\alpha,\alpha,\gamma)+D_{pb}(\beta,\ \beta,\ \alpha) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 2,$

$From\ 1\&2\ we\ get\ D_{pb}(\alpha,\alpha,\beta)+D_{pb}(\alpha,\alpha,\gamma)=D_{pb}(\alpha,\alpha,\gamma)+D_{pb}(\beta,\beta,\alpha)$

then $D_{pb}(\alpha,\alpha,\beta)=D_{pb}(\beta,\beta,\alpha) \ldots 3$

$Since\ D_{pb}(\alpha,\alpha,\alpha)=D_{pb}(\alpha,\alpha,\beta)+D_{pb}(\alpha,\beta,\beta)=D_{pb}(\alpha,\alpha,\beta)+D_{pb}(\alpha,\alpha,\beta)=2D_{pb}(\alpha,\alpha,\beta),\ D_{pb}(\alpha,\alpha,\alpha)=D_{pb}(\alpha,\alpha,\beta) \ldots 4$

$Now, take\ D_{pb}(\beta,\beta,\gamma)+D_{pb}(\beta,\beta,\alpha)-2D_{pb}(\beta,\beta,\beta)=0 \Rightarrow 2D_{pb}(\beta,\beta,\beta)=D_{pb}(\beta,\beta,\gamma)+D_{pb}(\beta,\beta,\alpha) \ldots 5$

$D_{pb}(\alpha,\alpha,\beta)+D_{pb}(\beta,\beta,\gamma)-2D_{pb}(\beta,\beta,\beta)=0 \Rightarrow 2D_{pb}(\beta,\beta,\beta)=D_{pb}(\alpha,\alpha,\beta)+D_{pb}(\beta,\beta,\gamma) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 6$

$From\ 5\&6\ we\ get\ D_{pb}(\beta,\ \beta,\ \alpha)+D_{pb}(\beta,\ \beta,\gamma)=D_{pb}(\alpha,\ \alpha,\ \beta)+D_{pb}(\beta,\ \beta,\gamma),$

$then\ D_{pb}(\beta,\beta,\alpha)=D_{pb}(\alpha,\alpha,\beta) \ldots 7$

$Since\ 2D_{pb}(\beta,\ \beta,\ \beta)=D_{pb}(\beta,\ \beta,\ \alpha)+D_{pb}(\alpha,\ \alpha,\ \beta)=D_{pb}(\alpha,\ \alpha,\ \beta)+D_{pb}(\alpha,\ \alpha,\beta)=\ 2D_{pb}(\alpha,\alpha,\beta \Rightarrow\ D_{pb}(\beta,\beta,\beta)=D_{pb}(\alpha,\alpha,\beta) \ldots 8$

$From\ 4\&8\ we\ get D_{pb}(\alpha,\alpha,\alpha)=D_{pb}(\alpha,\ \alpha,\ \beta)=D_{pb}(\beta,\beta,\beta),\ so\ by\ definition\ \ \alpha=\beta \ldots 9$

$and\ take\ D_{pb}(\gamma,\gamma,\alpha)+D_{pb}(\gamma,\gamma,\beta)-2D_{pb}(\gamma,\gamma,\gamma)=0 \Rightarrow 2D_{pb}(\gamma,\gamma,\gamma)=D_{pb}(\gamma,\gamma,\ \alpha)\ +D_{pb}(\gamma,\gamma,\ \beta)\ if\ \ \beta\ =\ \alpha$

$Then\ 2D_{pb}(\gamma,\gamma,\gamma)\ =D_{pb}(\gamma,\gamma,\alpha)\ +D_{pb}(\gamma,\gamma,\alpha)\ =\ 2D_{pb}(\gamma,\gamma,\alpha) \Rightarrow\ D_{pb}(\gamma,\gamma,\gamma)\ =D_{pb}(\gamma,\gamma,\alpha) \ldots 10$

and $D_{pb}(\alpha,\alpha,\beta)+D_{pb}(\alpha,\gamma,\gamma)-2D_{pb}(\alpha,\alpha,\alpha)=0 \Rightarrow 2D_{pb}(\alpha,\alpha,\alpha)=D_{pb}(\alpha,\alpha,\beta)+D_{pb}(\alpha,\gamma,\gamma) if\ \ \beta\ =\ \alpha,$

$then\ 2D_{pb}(\alpha,\alpha,\alpha)=D_{pb}(\alpha,\alpha,\alpha)+D_{pb}(\alpha,\gamma,\gamma) \Rightarrow\ D_{pb}(\alpha,\alpha,\alpha)=D_{pb}(\alpha,\gamma,\gamma) \ldots 11$

$$From\ 10\&11\ we\ get D_{pb}(\gamma,\gamma,\gamma)=\ D_{pb}(\gamma,\gamma,\alpha)=D_{pb}(\alpha,\alpha,\alpha),$$

$$so\ by\ definition\ \ \alpha=\gamma \ldots 12$$

$Then\ by\ 9\&12\ we\ get\ \alpha=\ \beta=\gamma.$

iii. $Trivial$

iv. $by\ definition\ since$

$0 \le\ D_{pb}(\mu,\mu,\beta)\ +\ D_{pb}(\mu,\mu,\gamma)\ -2D_{pb}(\mu,\mu,\mu) \le s[D_{pb}(\mu,\mu,\beta)+\ D_{pb}(\mu,\mu,\gamma)]-2D_{pb}(\mu,\mu,\mu)$

$0 \le D_{pb}(\beta,\beta,\mu)+\ D_{pb}(\beta,\beta,\mu)-2D_{pb}(\beta,\beta,\beta) \le s\left[D_{pb}(\beta,\beta,\mu)+D_{pb}(\beta,\beta,\mu)\right]-2D_{pb}(\beta,\beta,\beta)$

$0\ \le D_{pb}(\gamma,\gamma,\mu)+D_{pb}(\gamma,\gamma,\mu)-2D_{pb}(\gamma,\gamma,\gamma) \le s\left[D_{pb}(\gamma,\gamma,\mu)+D_{pb}(\gamma,\gamma,\mu)\right]-2D_{pb}(\gamma,\gamma,\gamma)$

$0 \le D_{pb}(\alpha,\alpha,\mu)+D_{pb}(\alpha,\alpha,\mu)-2D_{pb}(\alpha,\alpha,\alpha) \le s[D_{pb}(\alpha,\alpha,\mu)+D_{pb}(\alpha,\alpha,\mu)]-2D_{pb}(\alpha,\alpha,\alpha)$

$0 \le D_{pb}(\mu,\mu,\alpha)+D_{pb}(\mu,\mu,\gamma)-2D_{pb}(\mu,\mu,\mu) \le s[D_{pb}(\mu,\mu,\alpha)+D_{pb}(\mu,\mu,\gamma)]-2D_{pb}(\mu,\mu,\mu)$

$0 \le D_{pb}(\mu,\mu,\alpha) + D_{pb}(\mu,\mu,\beta) - 2D_{pb}(\mu,\mu,\mu) \le s[D_{pb}(\mu,\mu,\alpha) + D_{pb}(\mu,\mu,\beta)] - 2D_{pb}(\mu,\mu,\mu)$

When combined, it is more than and equal to zero and when adding these values $D_{pb}(\alpha,\alpha,\beta), D_{pb}(\alpha,\alpha,\gamma), D_{pb}(\beta,\beta,\alpha), D_{pb}(\beta,\beta,\gamma), D_{pb}(\gamma,\gamma,\alpha), D_{pb}(\gamma,\gamma,\beta), -2D_{pb}(\alpha,\alpha,\alpha), -2D_{pb}(\beta,\beta,\beta), -2D_{pb}(\gamma,\gamma,\gamma)$ *to two parties, we get*

$D_{pb}(\alpha,\alpha,\beta) + D_{pb}(\alpha,\alpha,\gamma) + D_{pb}(\beta,\beta,\alpha) + D_{pb}(\beta,\beta,\gamma) + D_{pb}(\gamma,\gamma,\alpha) + D_{pb}(\gamma,\gamma,\beta) - 2D_{pb}(\alpha,\alpha,\alpha) - 2D_{pb}(\beta,\beta,\beta) - 2D_{pb}(\gamma,\gamma,\gamma) \le s[D_{pb}(\alpha,\alpha,\beta) + D_{pb}(\alpha,\alpha,\gamma) + D_{pb}(\beta,\beta,\alpha) + D_{pb}(\beta,\beta,\gamma) + D_{pb}(\gamma,\gamma,\alpha) + D_{pb}(\gamma,\gamma,\beta) + D_{pb}(\mu,\mu,\beta) + D_{pb}(\mu,\mu,\gamma) + D_{pb}(\beta,\beta,\mu) + D_{pb}(\beta,\beta,\mu) + D_{pb}(\gamma,\gamma,\mu) + D_{pb}(\gamma,\gamma,\mu) + D_{pb}(\alpha,\alpha,\mu) + D_{pb}(\alpha,\alpha,\mu) + D_{pb}(\mu,\mu,\alpha) + D_{pb}(\mu,\mu,\gamma) + D_{pb}(\mu,\mu,\alpha) + D_{pb}(\mu,\mu,\beta)] - 2D_{pb}(\alpha,\alpha,\alpha) - 2D_{pb}(\beta,\beta,\beta) - 2D_{pb}(\gamma,\gamma,\gamma) - 2D_{pb}(\mu,\mu,\mu) - 2D_{pb}(\beta,\beta,\beta) - 2D_{pb}(\gamma,\gamma,\gamma) - 2D_{pb}(\alpha,\alpha,\alpha) - 2D_{pb}(\mu,\mu,\mu) - 2D_{pb}(\mu,\mu,\mu) = s[D_{pb}(\mu,\mu,\beta) + D_{pb}(\mu,\mu,\gamma) + D_{pb}(\beta,\beta,\mu) + D_{pb}(\beta,\beta,\gamma) + D_{pb}(\gamma,\gamma,\mu) + D_{pb}(\gamma,\gamma,\beta)] - 2D_{pb}(\mu,\mu,\mu) - 2D_{pb}(\beta,\beta,\beta) - 2D_{pb}(\gamma,\gamma,\gamma) + s[D_{pb}(\alpha,\alpha,\mu) + D_{pb}(\alpha,\alpha,\gamma) + D_{pb}(\mu,\mu,\alpha) + D_{bP}(\mu,\mu,\gamma) + D_{pb}(\gamma,\gamma,\alpha) + D_{pb}(\gamma,\gamma,\mu)] - 2D_{pb}(\alpha,\alpha,\alpha) - 2D_{pb}(\mu,\mu,\mu) - 2D_{pb}(\gamma,\gamma,\gamma) + s[D_{pb}(\alpha,\alpha,\beta) + D_{pb}(\alpha,\alpha,\mu) + D_{pb}(\beta,\beta,\alpha) + D_{pb}(\beta,\beta,\mu) + D_{pb}(\mu,\mu,\alpha) + D_{pb}(\mu,\mu,\beta)] - 2D_{pb}(\alpha,\alpha,\alpha) - 2D_{pb}(\beta,\beta,\beta) - 2D_{pb}(\mu,\mu,\mu)) \Rightarrow D_b^g(\alpha,\beta,\gamma) \le s[D_b^g(\mu,\beta,\gamma) + D_b^g(\alpha,\mu,\gamma) + D_b^g(\alpha,\beta,\mu)]$ □

Corollary 4. 2. Let (Y, D_{pb}) be a general partial b-metric space, the function $D_b^g: Y^3 \longrightarrow [0,\infty)$ given by

$$D_b^g(\alpha,\beta,\gamma) = D_{pb}(\alpha,\beta,\gamma) + D_{pb}(\alpha,\alpha,\beta) + D_{pb}(\alpha,\alpha,\gamma) + D_{pb}(\beta,\beta,\alpha) + D_{pb}(\beta,\beta,\gamma) + D_{pb}(\gamma,\gamma,\alpha) + D_{pb}(\gamma,\gamma,\beta) - 2D_{pb}(\alpha,\alpha,\alpha) - 2D_{pb}(\beta,\beta,\beta) - 3D_{pb}(\gamma,\gamma,\gamma) \qquad (13)$$

is D_b *−metric space.*

Proof. same way of theorem (4.1)

Lemma 4. 3. Let (Y, D_{pb}) be a general partial b-metric space if $\{D_{pb}(\alpha_m,\alpha_m,\alpha_m)\} \to \alpha$ *as* $m \to \infty$ and $\{D_b^g(\alpha_n,\alpha_m,\alpha_l)\}$ is a Cauchy sequence ,then $D_{pb}(\alpha_n,\alpha_m,\alpha_l) \to \alpha$ as $n,m,l \longrightarrow \infty$ where D_b^g define in corollary (4. 2).

Proof. Since $D_{pb}(\alpha_n,\alpha_m,\alpha_l) \to \alpha$ *as* $m \longrightarrow \infty$ then from every $\epsilon > 0$there exist $n_0 \in N$ such that

$|D_{pb}(\alpha_m,\alpha_m,\alpha_m) - a| < \frac{\epsilon}{2} \quad \forall\, m > n_0$ and $D_b^g(\alpha_n,\alpha_m,\alpha_l) < \frac{\epsilon}{2} \quad \forall\, n,m,l > n_0$

$\frac{\epsilon}{2} > D_b^g(\alpha_n,\alpha_m,\alpha_l) = D_{pb}(\alpha_n,\alpha_m,\alpha_l) + D_{pb}(\alpha_n,\alpha_n,\alpha_m) + D_{pb}(\alpha_n,\alpha_n,\alpha_l) + D_{pb}(\alpha_m,\alpha_m,\alpha_n) + D_{pb}(\alpha_m,\alpha_m,\alpha_l) + D_{pb}(\alpha_l,\alpha_l,\alpha_n) + D_{pb}(\alpha_l,\alpha_l,\alpha_m) - 2D_{pb}(\alpha_n,\alpha_n,\alpha_n) - 2D_{pb}(\alpha_m,\alpha_m,\alpha_m) - 3D_{pb}(\alpha_l,\alpha_l,\alpha_l).$

$\Rightarrow D_{pb}(\alpha_n,\alpha_m,\alpha_l) - D_{pb}(\alpha_m,\alpha_m,\alpha_m) < \frac{\epsilon}{2}$

So that $|D_{pb}(\alpha_n,\alpha_m,\alpha_l)-a| = |D_{pb}(\alpha_n,\alpha_m,\alpha_l)-D_{pb}(\alpha_m,\alpha_m,\alpha_m)+D_{pb}(\alpha_m,\alpha_m,\alpha_m)-\alpha| \leq |D_{pb}(\alpha_n,\alpha_m,\alpha_l)-D_{pb}(\alpha_m,\alpha_m,\alpha_m)|+|D_{pb}(\alpha_m,\alpha_m,\alpha_m)-\alpha| < \frac{\epsilon}{2}+\frac{\epsilon}{2} < \epsilon$

Hence $D_{pb}(\alpha_n,\alpha_m,\alpha_l) \to \alpha \; as \; n,m,l \longrightarrow \infty$

Theorem 4. 4. Let (Y,D_{pb}) be a general partial b-metric space, then

i. A sequence $\{\alpha_n\}$ is a Cauchy sequence in general partial b-metric space (Y,D_{pb}) if and only if $\{\alpha_n\}$ is a Cauchy sequence in (Y,D_b^g).

ii. A general partial metric space (Y,D_{pb}) is complete if and if (Y,D_b^g) is complete.

Where D_b^g define in corollary (4.2)

Proof . i. First we must prove that each Cauchy sequence in (Y,D_{pb}) is Cauchy in (Y,D_b^g).

Then, there exist $\alpha \in R$ such that, $\forall\epsilon > 0$ there is $n_0 \in N$ with

$|D_{pb}(\alpha_n,\alpha_m,\alpha_l)-\alpha| < \frac{\epsilon}{14}$ $\forall$ n, m, l $\geq n_0$. Hence,

$|D_b^g(\alpha_n,\alpha_m,\alpha_l)| = |D_{pb}(\alpha_n,\alpha_m,\alpha_l)+D_{pb}(\alpha_n,\alpha_n,\alpha_m)+D_{pb}(\alpha_n,\alpha_n,\alpha_l)+D_{pb}(\alpha_m,\alpha_m,\alpha_n)+D_{pb}(\alpha_m,\alpha_m,\alpha_l)+D_{pb}(\alpha_l,\alpha_l,\alpha_n)+D_{pb}(\alpha_l,\alpha_l,\alpha_m)-2D_{pb}(\alpha_n,\alpha_n,\alpha_n)-2D_{pb}(\alpha_m,\alpha_m,\alpha_m)-3D_{pb}(\alpha_l,\alpha_l,\alpha_l)| \leq |D_{pb}(\alpha_n,\alpha_m,\alpha_l)-\alpha|+|D_{pb}(\alpha_n,\alpha_n,\alpha_m)-\alpha|+|D_{pb}(\alpha_n,\alpha_n,\alpha_l)-\alpha|+|D_{pb}(\alpha_m,\alpha_m,\alpha_n)-\alpha|+|D_{pb}(\alpha_m,\alpha_m,\alpha_l)-\alpha|+|D_{pb}(\alpha_l,\alpha_l,\alpha_n)-\alpha|+|D_{pb}(\alpha_l,\alpha_l,\alpha_m)-\alpha| - 2|D_{pb}(\alpha_n,\alpha_n,\alpha_n)-\alpha|-2|D_{pb}(\alpha_m,\alpha_m,\alpha_m)-\alpha|-3|D_{pb}(\alpha_l,\alpha_l,\alpha_l)-\alpha| \frac{\epsilon}{14} < \epsilon$ $\forall$ n, m, l $\geq n_0$.

Hence $\{\alpha_n\}$ is a Cauchy sequence in (Y,D_b^g).

Conversely, now we must prove $\{\alpha_n\}$ is Cauchy sequence in (Y,D_{pb})

Since $\{\alpha_n\}$ is Cauchy sequence in (Y,D_b^g) so $\forall\epsilon > 0, \exists\, n_0 \in N$ such that

$$D_b^g(\alpha_n,\alpha_m,\alpha l) < \frac{\epsilon}{2} \quad \forall n,m,l > n_0$$

$$\begin{aligned}\frac{\epsilon}{2} &> D_b^g(\alpha_n,\alpha_m,\alpha_l)\\ &= D_{pb}(\alpha_n,\alpha_m,\alpha_l)+D_{pb}(\alpha_n,\alpha_n,\alpha_m)+D_{pb}(\alpha_n,\alpha_n,\alpha_l)+D_{pb}(\alpha_m,\alpha_m,\alpha_n)\\ &\quad + D_{pb}(\alpha_m,\alpha_m,\alpha_l)+D_{pb}(\alpha_l,\alpha_l,\alpha_n)\\ &\quad + D_{pb}(\alpha_l,\alpha_l,\alpha_m)-2D_{pb}(\alpha_n,\alpha_n,\alpha_n)-2D_{pb}(\alpha_m,\alpha_m,\alpha_m)-3D_{pb}(\alpha_l,\alpha_l,\alpha_l)\\ &\Rightarrow D_{pb}(\alpha_n,\alpha_m,\alpha_l)-D_{pb}(\alpha_n,\alpha_n,\alpha_n) \leq D_b^g(\alpha_n,\alpha_m,\alpha_l) < \frac{\epsilon}{2}\end{aligned}$$

By compensation $D_{pb}(\alpha_n,\alpha_n,\alpha_n)$ to two parties, we have

$$D_{pb}(\alpha_n,\alpha_m,\alpha_l) \leq D_b^g(\alpha_n,\alpha_m,\alpha_l)+D_{pb}(\alpha_n,\alpha_n,\alpha_n) < \frac{\epsilon}{2}+D_{pb}(\alpha_n,\alpha_n,\alpha_n)$$

And since $D_{pb}(\alpha_m,\alpha_m,\alpha_m) \leq D_{pb}(\alpha_n,\alpha_m,\alpha_l)$ *so*

$$D_{pb}(\alpha_m,\alpha_m,\alpha_m) \leq D_{pb}(\alpha_n,\alpha_m,\alpha_l) \leq D_b^g(\alpha_n,\alpha_m,\alpha l)+D_{pb}(\alpha_n,\alpha_n,\alpha_n) < \frac{\epsilon}{2}+D_{pb}(\alpha_n,\alpha_n,\alpha_n)$$

$$\Rightarrow D_{pb}(\alpha_m,\alpha_m,\alpha_m) \leq \frac{\epsilon}{2} + D_{pb}(\alpha_n,\alpha_n,\alpha_n) \quad \forall\, n\,,m > n0$$

$Let\ \alpha_n = D_{pb}(\alpha_n,\alpha_n,\alpha_n) \in R\ such\ that\ |\alpha_m - \alpha_n| < \frac{\epsilon}{2}$

$\therefore \{\alpha_n\}$ is Cauchy sequence, $\therefore \{\alpha_n\} \to \alpha$, $\therefore D_{pb}(\alpha_m,\alpha_m,\alpha_m) \to \alpha \in R$

Then by lemma (16), $D_{pb}(\alpha_m,\alpha_m,\alpha_m)$ is Cauchy sequence in (Y, D_{pb}).

ii.

If $\{\alpha_n\}$ is Cauchy sequence in (Y, D_{pb}) then it is Cauchy sequence in (Y, D_b^g) and since D_b-metric (Y, D_b^g) is complete then there exists $\alpha \in Y$ such that

$\lim_{n,m\to\infty} D_b^g(\alpha_n,\alpha_m,\alpha) = 0$, hence

$$\lim_{n,m\to\infty} \begin{bmatrix} D_{pb}(\alpha_n,\alpha_m,\alpha) + D_{pb}(\alpha_n,\alpha_n,\alpha_m) + D_{pb}(\alpha_n,\alpha_n,\alpha) + D_{pb}(\alpha_m,\alpha_m,\alpha_n) + D_{pb}(\alpha_m,\alpha_m,\alpha) \\ + D_{pb}(\alpha,\alpha,\alpha_n) + D_{pb}(\alpha,\alpha,\alpha_m) - 2D_{pb}(\alpha_n,\alpha_n,\alpha_n) \\ - 2D_{pb}(\alpha_m,\alpha_m,\alpha_m) - 3D_{pb}(\alpha,\alpha,\alpha) \end{bmatrix} = 0$$

There for $\lim_{n\ m\to\infty}[D_{pb}(\alpha_n,\alpha_m,\alpha) - D_{pb}(\alpha,\alpha,\alpha)] = 0$

$\Rightarrow \lim_{n\ m\to\infty} D_{pb}(\alpha_n,\alpha_m,\alpha) = D_{pb}(\alpha,\alpha,\alpha)$ hence (Y, D_{pb}) is converge

Thus (Y, D_{pb}) is complete.

Conversely, let $\{\alpha_n\}$ be a Cauchy sequence in (Y, D_b^g) then $\{\alpha_n\}$ is Cauchy sequence in (Y, D_{pb}) and so it is convergent to appoint $\alpha \in Y$ with $\lim_{n,m\to\infty} D_{pb}(\alpha_n,\alpha_m,\alpha) = D_{pb}(\alpha,\alpha,\alpha) \quad \forall n,m$

Then, for given $\epsilon > 0$ there exists $m_0 \in N$ such that $D_{pb}(\alpha_n,\alpha_m,\alpha) - D_{pb}(\alpha,\alpha,\alpha) < \frac{\epsilon}{11}$ and by condition (D_p2) since $D_{pb}(\alpha_n,\alpha_n,\alpha_m) \leq D_{pb}(\alpha_n,\alpha_m,\alpha) \Rightarrow D_{pb}(\alpha_n,\alpha_n,\alpha_m) - D_{pb}(\alpha,\alpha,\alpha,) \leq D_{pb}(\alpha_n,\alpha_m,\alpha) - D_{pb}(\alpha,\alpha,\alpha) < \frac{\epsilon}{11}$ by the same way we get

$D_{pb}(\alpha_m,\alpha_m,\alpha_n) - D_{pb}(\alpha,\alpha,\alpha) \leq \frac{\epsilon}{11}$, $D_{pb}(\alpha_n,\alpha,\alpha) - D_{pb}(\alpha,\alpha,\alpha) \leq \frac{\epsilon}{11}$

$D_{pb}(\alpha_m,\alpha,\alpha) - D_{pb}(\alpha,\alpha,\alpha) \leq \frac{\epsilon}{11}$, $D_{pb}(\alpha_n,\alpha_n,\alpha_n) - D_{pb}(\alpha,\alpha,\alpha) \leq \frac{\epsilon}{11}$, $D_{pb}(\alpha_n,\alpha_n,\alpha_n) - D_{pb}(\alpha,\alpha,\alpha) \leq \frac{\epsilon}{11}$, $D_{pb}(\alpha_m,\alpha_m,\alpha_m) - D_{pb}(\alpha,\alpha,\alpha) \leq \frac{\epsilon}{11}$, $D_{pb}(\alpha_m,\alpha_m,\alpha_m) - D_{pb}(\alpha,\alpha,\alpha) \leq \frac{\epsilon}{11}$, there for

$\left|D_b^g(\alpha_n,\alpha_m,\alpha)\right| = \left|D_{pb}(\alpha_n,\alpha_m,\alpha) + D_{pb}(\alpha_n,\alpha_n,\alpha_m) + D_{pb}(\alpha_n,\alpha_n,\alpha) + D_{pb}(\alpha_m,\alpha_m,\alpha_n) + D_{pb}(\alpha_m,\alpha_m,\alpha) + D_{pb}(\alpha,\alpha,\alpha_n) + D_{pb}(\alpha,\alpha,\alpha_m) - 2D_{pb}(\alpha_n,\alpha_n,\alpha_n) - 2D_{pb}(\alpha_m,\alpha_m,\alpha_m) - 3D_{pb}(\alpha,\alpha,\alpha)\right| \leq \left|D_{pb}(\alpha_n,\alpha_m,\alpha) - D_{pb}(\alpha,\alpha,\alpha)\right| + \left|D_{pb}(\alpha_n,\alpha_n,\alpha_m) - D_{pb}(\alpha,\alpha,\alpha)\right| + \left|D_{pb}(\alpha_n,\alpha_n,\alpha) - D_{pb}(\alpha,\alpha,\alpha)\right| + \left|D_{pb}(\alpha_m,\alpha_m,\alpha_n) - D_{pb}(\alpha,\alpha,\alpha)\right| + \left|D_{pb}(\alpha_m,\alpha_m,\alpha) - D_{pb}(\alpha,\alpha,\alpha)\right| + \left|D_{pb}(\alpha,\alpha,\alpha_n) - D_{pb}(\alpha,\alpha,\alpha)\right| + \left|D_{pb}(\alpha,\alpha,\alpha_m) - D_{pb}(\alpha,\alpha,\alpha)\right| + 2\left|D_{pb}(\alpha,\alpha,\alpha) - D_{pb}(\alpha_n,\alpha_n,\alpha_n)\right| + 2\left|D_{pb}(\alpha,\alpha,\alpha) - D_{pb}(\alpha_m,\alpha_m,\alpha_m)\right| < \frac{11\epsilon}{11} < \epsilon \Rightarrow D_b^g(\alpha_n,\alpha_m,\alpha) < \epsilon$

Hence $D_b^g(\alpha_n, \alpha_m, \alpha)$ is converge, thus (Y, D_b^g) is complete. □

Proposition 4. 5. If $\{\alpha_n\}$ is Cauchy sequence in (Y, D_b^g) then $\lim_{n,m\to\infty} D_{pb}(\alpha_n, \alpha_n, \alpha_m) = \lim_{n\to\infty} D_{pb}(\alpha_n, \alpha_n, \alpha_n)$ *in* (Y, D_{pb})

Proof. Since $\{\alpha_n\}$ is Cauchy sequence in (Y, D_b^g) then $\lim_{n,m,l\to\infty} D_b^g(\alpha_n, \alpha_m, \alpha_l) = 0$

and $D_b^g(\alpha_n, \alpha_m, \alpha_l) = D_{pb}(\alpha_n, \alpha_n, \alpha_m) + D_{pb}(\alpha_n, \alpha_n, \alpha_l) + D_{pb}(\alpha_m, \alpha_m, \alpha_n) + D_{pb}(\alpha_m, \alpha_m, \alpha_l) + D_{pb}(\alpha_l, \alpha_l, \alpha_n) + D_{pb}(\alpha_l, \alpha_l, \alpha_m) - D_{pb}(\alpha_n, \alpha_n, \alpha_n) - 2D_{pb}(\alpha_m, \alpha_m, \alpha_m) - 2D_{pb}(\alpha_l, \alpha_l, \alpha_l)$.

and $D_{pb}(\alpha_n, \alpha_n, \alpha_m) + D_{pb}(\alpha_n, \alpha_n, \alpha_n) \leq D_b^g(\alpha_n, \alpha_m, \alpha_l)$ then

$lim_{n,m\to\infty} D_{pb}(\alpha_n, \alpha_n, \alpha_m) - D_{pb}(\alpha_n, \alpha_n, \alpha_n) \to 0$ …1, similarity

$lim_{n,m\to\infty} D_{pb}(\alpha_n, \alpha_n, \alpha_m) - D_{pb}(\alpha_m, \alpha_m, \alpha_m) \to 0$ …2

Since $D_{pb}(\alpha_m, \alpha_m, \alpha_m) - D_{pb}(\alpha_n, \alpha_n, \alpha_n) = D_{pb}(\alpha_m, \alpha_m, \alpha_m) + D_{pb}(\alpha_n, \alpha_n, \alpha_m) - D_{pb}(\alpha_n, \alpha_n, \alpha_m) - D_{pb}(\alpha_n, \alpha_n, \alpha_n)$ then $\lim_{n,m\to\infty}[D_{pb}(\alpha_m, \alpha_m, \alpha_m) - D_{pb}(\alpha_n, \alpha_n, \alpha_n)] = lim_{n,m\to\infty}[D_{pb}(\alpha_m, \alpha_m, \alpha_m) - D_{pb}(\alpha_n, \alpha_n, \alpha_m)] + \lim_{n,m\to\infty}[D_{pb}(\alpha_n, \alpha_n, \alpha_m) - D_{pb}(\alpha_n, \alpha_n, \alpha_n)] \longrightarrow 0$

So that $lim_{n,m\to\infty}[D_{pb}(\alpha_m, \alpha_m, \alpha_m) - D_{pb}(\alpha_n, \alpha_n, \alpha_n)] \longrightarrow 0$

Let $\alpha_n = D_{pb}(\alpha_n, \alpha_n, \alpha_n)$

$\therefore |\alpha_m - \alpha_n| \to 0$ as $n, m \longrightarrow \infty$

Hence $\{\alpha_n\}$ is a Cauchy sequence in R, there for $\{D_{pb}(\alpha_n, \alpha_n, \alpha_n)\}$ converge to α.

Also, $\lim_{n\to\infty} D_{pb}(\alpha_n, \alpha_n, \alpha_m) = \lim_{n,m\to\infty}[D_{pb}(\alpha_n, \alpha_n, \alpha_m) + D_{pb}(\alpha_n, \alpha_n, \alpha_n) - D_{pb}(\alpha_n, \alpha_n, \alpha_n)]$

Then $[D_{pb}(\alpha_n, \alpha_n, \alpha_m) - D_{pb}(\alpha_n, \alpha_n, \alpha_n)] \longrightarrow 0$ so $\lim_{n,m\to\infty} D_{pb}(\alpha_n, \alpha_m, \alpha_m) = \alpha$

Thus $\lim_{n,m\to\infty} D_{pb}(\alpha_n, \alpha_n, \alpha_m) = \lim_{n\to\infty} D_{pb}(\alpha_n, \alpha_n, \alpha_n)$.

Theorem 4. 6. If (Y, pb) is partial b-metric space then

$$D_{pb}(\alpha, \beta, \gamma) = pb(\alpha, \beta) + pb(\alpha, \gamma) + pb(\beta, \gamma) - pb(\alpha, \alpha) - pb(\beta, \beta) - pb(\gamma, \gamma)$$

is general partial b-metric space. (13)

i. *Since* $pb(\alpha, \beta) - pb(\alpha, \alpha) \geq 0$, $pb(\alpha, \gamma) - pb(\gamma, \gamma) \geq 0$, $pb(\beta, \gamma) - pb(\beta, \beta) \geq 0$ then $D_{pb}(\alpha, \beta, \gamma) \geq 0$
ii. *Let* $D_{pb}(\alpha, \beta, \gamma) = D_{pb}(\alpha, \alpha, \alpha) = D_{pb}(\beta, \beta, \beta) = D_{pb}(\gamma, \gamma, \gamma)$
since $D_{pb}(\alpha, \alpha, \alpha) = D_{pb}(\beta, \beta, \beta) = D_{pb}(\gamma, \gamma, \gamma) = 0 \Rightarrow D_{pb}(\alpha, \beta, \gamma) = 0 \Rightarrow$
$pb(\alpha, \beta) + pb(\beta, \gamma) + pb(\alpha, \gamma) - pb(\alpha, \alpha) - pb(\beta, \beta) - pb(\gamma, \gamma) = 0 \Rightarrow$
$pb(\alpha, \beta) - pb(\alpha, \alpha) = 0 \Rightarrow pb(\alpha, \beta) = pb(\alpha, \alpha)$ … 1,
$\Rightarrow pb(\alpha, \gamma) - pb(\gamma, \gamma) = 0 \Rightarrow pb(\alpha, \gamma) = pb(\gamma, \gamma)$ … 2,

$and\ pb(\beta,\gamma) - pb(\beta,\beta) = 0 \Rightarrow pb(\beta,\gamma) = pb(\beta,\beta) \ldots 3.$
$From\ 1\ pb(\alpha,\alpha) = pb(\alpha,\beta)$
$and\ by\ definition\ pb(\alpha,\alpha) = pb(\alpha,\beta) \le s[pb(\alpha,\gamma) + pb(\gamma,\beta)] - pb(\gamma,\gamma)$
$Since\ pb(\alpha,\gamma) = pb(\gamma,\gamma)\ \&\ pb(\beta,\gamma) = pb(\beta,\beta)\ we\ get\ pb(\alpha,\alpha) = pb(\beta,\beta)$
$From\ 2\ pb(\beta,\beta) = pb(\beta,\gamma)$
$and\ by\ definition\ pb(\beta,\beta) = pb(\beta,\gamma) \le s[pb(\beta,\alpha) + pb(\alpha,\gamma)] - pb(\alpha,\alpha)$
$by\ 1\&2\ we\ get\ pb(\beta,\beta) = pb(\gamma,\gamma),\ From\ 3\ pb(\gamma,\gamma) = pb(\alpha,\gamma)$
and by definition $pb(\gamma,\gamma) = pb(\alpha,\gamma) \le s[pb(\alpha,\beta) + pb(\beta,\gamma)] - pb(\beta,\beta)$
$by\ 1\&3\ we\ get\ pb(\gamma,\gamma) = pb(\alpha,\alpha)$
$Hence\ pb(\alpha,\alpha) = pb(\beta,\beta) = pb(\gamma,\gamma)$
$Thus\ pb(\alpha,\alpha) = pb(\alpha,\beta) = pb(\beta,\beta)\ by\ definition\ \alpha = \beta, and\ pb(\beta,\beta) = pb(\beta,\gamma) = pb(\gamma,\gamma)\ \beta y\ definition\ \beta = \gamma\ then\ we\ get\ \alpha = \beta = \gamma.$

iii. $Trivial$

iv. $Since\ 0 \le pb(\mu,\gamma) - pb(\gamma,\gamma) \le spb(\mu,\gamma) - pb(\gamma,\gamma), 0 \le pb(\mu,\beta) - pb(\mu,\mu) \le spb(\mu,\beta) - pb(\mu,\mu), 0 \le pb(\alpha,\mu) - pb(\alpha,\alpha) \le spb(\alpha,\mu) - pb(\alpha,\alpha), 0 \le pb(\mu,\gamma) - pb(\mu,\mu) \le spb(\mu,\gamma) - pb(\mu,\mu), 0 \le pb(\beta,\mu) - pb(\beta,\beta) \le spb(\beta,\mu) - pb(\beta,\beta), 0 \le pb(\alpha,\mu) - pb(\mu,\mu) \le spb(\alpha,\mu) - pb(\mu,\mu)$

When combined, it is more than and equal to zero and when adding these values $pb(\alpha,\beta), pb(\beta,\gamma), pb(\alpha,\gamma), -pb(\alpha,\alpha), -pb(\beta,\beta), -pb(\gamma,\gamma)$ to both said, we get

$$bp(\alpha,\beta) + pb(\beta,\gamma) + pb(\alpha,\gamma) - pb(\alpha,\alpha) - pb(\beta,\beta) - pb(\gamma,\gamma) \le s[pb(\alpha,\beta) + pb(\beta,\gamma) + pb(\alpha,\gamma) + pb(\mu,\gamma) + pb(\alpha,\mu) + pb(\beta,\mu) + pb(\mu,\beta) + pb(\alpha,\mu) + pb(\mu,\gamma)] - pb(\mu,\mu) - pb(\mu,\mu) - pb(\mu,\mu) - pb(\alpha,\alpha) - pb(\beta,\beta) - pb(\gamma,\gamma) - pb(\alpha,\alpha) - pb(\gamma,\gamma) - pb(\beta,\beta).$$

$$\Rightarrow D_{pb}(\alpha,\beta,\gamma) \le \{s[pb(\mu,\beta) + pb(\mu,\gamma) + pb(\gamma,\beta)] - pb(\mu,\mu) - pb(\beta,\beta) - pb(\gamma,\gamma)\} + \{s[pb(\alpha,\mu) + pb(\alpha,\gamma) + pb(\mu,\gamma)] - pb(\alpha,\alpha) - pb(\mu,\mu) - pb(\gamma,\gamma)\} + \{s[pb(\alpha,\beta) + pb(\alpha,\mu) + pb(\beta,\mu)] - pb(\alpha,\alpha) - pb(\beta,\beta) - pb(\mu,\mu)\}$$

$$D_{pb}(\alpha,\beta,\gamma) \le s[D_{bP}(\mu,\beta,\gamma) + D_{pb}(\alpha,\mu,\gamma) + D_{pb}(\alpha,\beta,\mu)] - D_{pb}(\mu,\mu,\mu)$$ □

Proposition 4. 7. Let (Y, D_{pb}) be a general partial b-metric space and

$$D_{pb}(\alpha,\beta,\beta) \le s[D_{pb}(\alpha,\gamma,\gamma) + D_{pb}(\gamma,\beta,\beta)] - D_{pb}(\gamma,\gamma,\gamma) \qquad (14)$$

holds then the function $pb: Y^2 \to [0,\infty)$ which is define by $pb(\alpha,\beta) = D_{pb}(\alpha,\beta,\beta)$ is a partial b-metric on Y .

Proof.

i. $Since\ pb(\alpha,\beta) = pb(\alpha,\alpha) = pb(\beta,\beta) \Leftrightarrow D_{pb}(\alpha,\beta,\beta) = D_{pb}(\alpha,\alpha,\alpha) = D_{pb}(\beta,\beta,\beta) \Leftrightarrow \alpha = \beta.$

ii. $Since\ D_{pb}(\alpha,\alpha,\alpha) \le D_{pb}(\alpha,\beta,\beta)\ then\ pb(\alpha,\alpha) \le pb(\alpha,\beta)\ \forall\ \alpha, \beta \in Y$

iii. Trivial

iv. $pb(\alpha,\beta) = D_{pb}(\alpha,\beta,\beta) \le s[D_{pb}(\alpha,\gamma,\gamma) + D_{pb}(\gamma,\beta,\beta)] - D_{pb}(\gamma,\gamma,\gamma)$ from(14) $= s[pb(\alpha,\gamma) + pb(\gamma,\beta)] - pb(\gamma,\gamma).$ □

5. Conclusion

The key messages of this study are the following: define a new two concept of metric space generalization namely D_b −metric and general partial b-metric.

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