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Weak Pseudo Quasi-2- Absorbing Submodules And Related concepts

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A B S T R A C T

In this paper R stand for commutative ring with identity ,and E is a unitary left R-module. The concept of weakly quasi-2-Absorbing submodule has been generalized in this research to a Wpq-2-Absorbing submodules ,where a paper submodule H of an R-module E is a Wpq-2-Absorbiming if 0≠abcyϵH ,for a,b,cϵR, yϵE , implies that either abyϵH+Soc(E) or acyϵH+Soc(E) or bcyϵH+Soc(E) .Several characterizations ,examples and basic properties of this concept are introduced .Moreover characterizations of Wpq-2-Absorbing submodules in some types of modules are given .Keywords: Weakly quasi-2-Absorbing submodules, Wpq-2-Absorbing submodules, socel of modules ,multiplication modules ,Z-regular modules ,non-singular modules.

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1. Introduction

The concept of weakly quasi-2-Absorbing submodules was first introduced and studied by Darani and Soheihnia in 2011 ,where a proper submodule H of an R-module E is called weakly quasi-2-Absorbing if 0≠rsteϵH ,for r,s,tϵR, implies that either rseϵH or rteϵH or steϵH [5].Recontly many generalization of weakly quasi-2-Absorbing submodules are introduced see [9,6].In this research we introduced a new generalization of weakly quasi-2-Absorbing submodule which we called it Wpq-2-Absorbing submodule where a proper submodule H of an R-module E is called a Wpq-2-Absorbing if 0≠rste ϵ H for r,s,tϵR ,eϵE implies that either rse ∈H+Soc(E) or rte ϵ H+Soc(E) or steϵH+Soc(E) .Soc(E) is the socal of an R-module E defined to the intersection of all essential submodule of E [4]. And a non-zero submodule A of E is an essential if $A \cap B \neq (0)$ for all non-zero submodule B of E[4].It is clear that Soc(Z)=(0). If H is a submodule of E and J is an ideal of R , then $[H:_E J]$ is a submodule of E containing H, and $[H:_E R]$ =H and $[I:_R R]$=I [4].Recall that an R-module E is a multiplication if every submodule H of E is of the form H=JE for some ideal J of R .In particular $H=[H:_R E]E$ [2] .If H,B are submodule of a multiplication module E then $B=J_1E$,$H=J_2E$,for some ideals J_1,J_2 of R the product $BH=J_1J_2E=J_1H$. In particular $BE=J_1EE=J_1E$ =B and for any yϵE $B=J_1y$[3] .Recall that an R-module E is called a Z-regular if for any yϵE ,there exists fϵHom(E,R) such that y=f(y)y[10]. Recall that an R-module E is a non-singular if Z(E)=(0) where Z(E)={xϵE : xI=(0) for some essential I of R}[4]. Recall that an R-module E is cancellation if IE=JE for any ideals I,J of R ,implies that I=J [1].

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Main result :

In this section we introduce the definition of Wpq-2-Absorbing submodules and established same basic properties of this concept

Definition 2.1 A proper submodule H of an R-module E is said to be a weak pseudo quasi-2-Apsorbing (Breviely Wpq-2-Absorbing)submodule of E if 0≠abcy ϵH for a,b,cϵR ,yϵE ,implies that either abyϵH+Soc(E) or acyϵH+Soc(E) or bcyϵH+Soc(E) .

And an ideal J of a ring R is called a Wpq-2-Absorbing ideal of R , if J is a Wpq-2-Absorbing R-submodule of an R-module R .

There are several examples , some of which we will mention .

(1).In the Z-module Z_{24} the only essential submodules are $<\bar{2}>$,$<\bar{4}>$ and Z_{24}, itself. Thus $Soc(Z_{24})=<\bar{2}>\cap<\bar{4}>\cap Z_{24}=<\bar{4}>$.

(2). It is clear that every weakly quasi-2-Absorbing submodule H of an R-module E is a Wpq-2-Absorbing but not conversely .

The following example explain that . In the Z-module Z_{24}, and the submodule $<\overline{12}>=\{\bar{0},\overline{12}\}$, is a Wpq-2-Absorbing but not weakly quasi-2-Absorbing .Since $Soc(Z_{24})=<\bar{4}>$, and if $0\neq abc\bar{y}\in<\overline{12}>$ for $a,b,c\in Z$, $\bar{y}\in Z_{24}$ implies that either $ab\bar{y}\in<\overline{12}>+Soc(Z_{24})=<\overline{12}>+<\bar{4}>=<\bar{4}>$ or $ac\bar{y}\in<\overline{12}>+Soc(Z_{24})=<\overline{12}>+<\bar{4}>=<\bar{4}>$ or $bc\bar{y}\in<\overline{12}>+Soc(Z_{24})=<\overline{12}>+<\bar{4}>=<\bar{4}>$.

That is $0\neq 2.3.2.\bar{1}\in<\overline{12}>$, for $2,3,2\in Z$, $\bar{1}\in Z_{24}$, implies that $2.2.\bar{1}\in<\overline{12}>+Soc(Z_{24})=<\bar{4}>$. But if $0\neq 2.3.2.\bar{1}\in<\overline{12}>$,but $2.2.\bar{1}=\bar{4}\notin<\overline{12}>$ and $2.3.\bar{1}=\bar{6}\notin<\overline{12}>$ and $3.2.\bar{1}=\bar{6}\notin<\overline{12}>$.

(3).If H is a Wpq-2-Absorbing submodule of an R-module E ,then $[H:_R E]$ not necessary Wpq-2-Absorbing ideal of R .

The following example show that :

The submodule $<\overline{12}>$ of the Z-module Z_{24} is a Wpq-2-Absorbing by (2) , but $[<\overline{12}>:_Z Z_{24}]=12Z$ is not Wpq-2-Absorbing ideal of Z-module Z ,because $0\neq 3.2.2.1\in[<\overline{12}>:_Z Z_{24}]=12Z$,for 1,2,3ϵZ ,but $3.2.1=6\notin 12Z+Soc(Z)$ and $2.2.1=4\notin 12Z+Soc(Z)$.

(4).If H,K are Wpq-2-Absorbing submodule of E then $H\cap K$ need not to be Wpq-2-Absorbing submodule of E .For example it is clear that the submodules <3>,<4> of the Z-module Z are Wpq-2-Absorbing (because they are weakly quasi -2-Absorbing),but $<3>\cap<4>=<12>$ are not Wpq-2-Absorbing submodule since $2.3.2.1\in<12>$,for 2,3,1ϵZ but $2.3.1=6\notin<12>+Soc(Z)$ and $2.2.1=4\notin<12>+Soc(Z)$ and $3.2.1=6\notin<12>+Soc(Z)$

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The first result of this section characterization of Wpq-2-Absorbing submodule .

Proposition 2.2 A proper submodule H of an R-module E is a Wpq-2-Absorbing submodule E , if and only if ,$[H:_E abc]\subseteq[0:_E abc]\cup[H+Soc(E):_E ab]\cup[H+Soc(E):_E ac]\cup[H+Soc(E):_E bc]$ for every a ,b ,c ϵR .

Proof: (⇒) Let $y\epsilon[H:_E abc]$, implies that abcyϵH .If abcy=0 ,implies $y\epsilon[0:_E abc]$ .If 0≠abcyϵH and H is a Wpq-2-Absororbing submodule of E , implies that either aby ϵH +Soc(E) or acyϵH +Soc(E) , or bcyϵH+Soc(E) .That is either $y\epsilon[H+Soc(E):_E cb]$ or $y\epsilon[H+Soc(E):_E ac]$or $y\epsilon[H+Soc(E):_E bc]$ .Thus is $y\epsilon[0:_E abc]\cup[H+Soc(E):_E ab]\cup[H+Soc(E):_E ac]\cup[H+Soc(E):_E bc]$ .Therefore ,$[H:_E abc]\subseteq[0:_E abc]\cup[H+Soc(E):_E ab]\cup[H+Soc(E):_E ac]\cup[H+Soc(E):_E bc]$.

(⇐) Let 0≠abcyϵH , for a,b,c ϵR , y ϵE ,then $y\epsilon[H:_E abc]$. But 0≠abcy , it follows that $y\notin[0:_E abc]$ .Thus $y\epsilon[H+Soc(E):_E ab]\cup[H+Soc(E):_E ac]\cup[H+Soc(E):_E bc]$.Thus either aby ϵH +Soc(E) or acyϵH +Soc(E) , or bcyϵH+Soc(E) .Hence H is a Wpq-2-Absorbing submodule of E.

The next result are several characterization of Wpq-2-Absorbing submodule .

Proposition 2.3 A proper submodule H of an R-module E is a Wpq-2-Absorbing submodule E , if and only if for each a,b ϵR ,y ϵE with aby∉H+Soc(E) we have ,$[H:_R aby]\subseteq[0:_E aby]\cup[H+Soc(E):_R ay]\cup[H+Soc(E):_R by]$.

Proof: (⇒) Let $c\ \epsilon[H:_R aby]$, then abcy ϵH .If abcy=0 ,implies $c\epsilon[0:_R aby]$ .If 0≠abcyϵH and H is a Wpq-2-Absororbing submodule of E with aby∉H+Soc(E) ,implies that either acyϵH +Soc(E) , or bcyϵH+Soc(E) ,it follows that either $c\epsilon[H+Soc(E):_R ay]$ or $c\epsilon[H+Soc(E):_R by]$ .Hence $c\epsilon[0:_R aby]\cup[H+Soc(E):_R ay]\cup[H+Soc(E):_R by]$.That is $[H:_R aby]\subseteq[0:_R aby]\cup[H+Soc(E):_R ay]\cup[H+Soc(E):_R by]$.

(⇐) Let 0≠abcyϵH , for a,b,c ϵR , y ϵE ,with aby∉H+Soc(E) .Thus $c\epsilon[H:_E aby]\subseteq[0:_R aby]\cup[H+Soc(E):_R ay]\cup[H+Soc(E):_R by]$. But 0≠abcy , so $c\notin[0:_R aby]$ .Therefore either $c\ \epsilon\ [H+Soc(E):_R ay]$ or $c\ \epsilon\ [H+Soc(E):_R by]$.That is either acy ϵH +Soc(E) or bcyϵH +Soc(E) .Hence H is a Wpq-2-Absorbing submodule of E.

Proposition 2.4 A proper submodule H of an R-module E is a Wpq-2-Absorbing submodule of E if and only if , 0≠abcL ⊆H , for a,b,cϵ R, L is a submodule of E , implies that either abL⊆H +Soc(E) or ac L⊆H +Soc(E) or bcL⊆H +Soc(E).

Proof : (⇒) Let 0≠abcL⊆H . For a,b,c ϵR , L is a submodule of E . Assume that abL⊄ H+ Soc(E) and ac L⊄ H +Soc(E) and bcL⊄ H +Soc(E), it follows that there exist a nonzero elements y_1 ,y_2,y_3 ϵL such that aby_1 ∉H +Soc(E) and acy_2 ∉ H +Soc(E) and bcy_3 ∉ H +Soc(E).Now 0≠$abcy_1$ ϵH , but H is a Wpq-2-Absorbing and aby_1 ∉H +Soc(E) , it follows

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that ac y_1 ϵH +Soc(E) or bcy_1ϵ H +Soc(E). Also 0≠abcy_2 ϵH , and H is a Wpq-2-Absorbing, and acy_2 ∉H +Soc(E) , then either aby_2ϵH+Soc(E) or and bcy_2 ϵ H +Soc(E). Again 0≠abcy_3 ϵH , bcy_3 ∉H +Soc(E) , implies that either aby_3ϵH+Soc(E) or and acy_3 ϵ H +Soc(E). Now ,we have 0≠abc(y_1+y_2+y_3)ϵH and H is Wpq-2-Absorbing ,then either ab(y_1+y_2+y_3)ϵH +Soc(E) or ac(y_1+y_2+y_3)ϵH +Soc(E) or bc(y_1+y_2+y_3)ϵH +Soc(E) . If ab(y_1+y_2+y_3)=ab y_1+ aby_2+ aby_3 ϵH +Soc(E) it follows that aby_1ϵH+Soc(E) which is contradiction if ac(y_1+y_2+y_3) = acy_1+acy_2+acy_3ϵH +Soc(E) implies that acy_2ϵH+Soc(E) which is contradiction if bc(y_1+y_2 +y_3)=bc y_1+bcy_2+bcy_3ϵH +Soc(E) then bcy_3ϵH+Soc(E) which is contradiction .Thus abL⊆H +Soc(E) or ac L⊆H +Soc(E) or bcL⊆H +Soc(E).

(⇐) Obvious .

Proposition2.5 A proper submodule H of an R-module E is a Wpq-2-Absorbing submodule of E if and only if 0≠IJKA ⊆H , for I,J,K are ideals of R and A is a submodule of E , implies that either IJA⊆H +Soc(E) or IKA⊆H +Soc(E) or JKA⊆H +Soc(E).

Proof: (⇒)Let 0≠IJKA⊆H . For I,J,K are ideals of R , and A is a submodule of E . Suppose that IJA⊄ H+ Soc(E) and IKA⊄ H +Soc(E) and JK⊄ H +Soc(E), that is there exist a nonzero x_1 ,x_2,x_3 ϵL and a nonzero aϵI, bϵJ, cϵK ,such that abx_1 ∉H +Soc(E) and acx_2 ∉ H +Soc(E) and bcx_3 ∉ H +Soc(E).Now 0≠abcx_1 ϵH , and H is a Wpq-2-Absorbing and abx_1 ∉H +Soc(E) , it follows that either ac x_1 ϵH +Soc(E) or bcx_1ϵ H +Soc(E). Also 0≠abcx_2 ϵH , and H is a Wpq-2-Absorbing, and acx_2 ∉H +Soc(E) , then either abx_2ϵH+Soc(E) or bcx_2 ϵ H +Soc(E). Again 0≠abcy_3 ϵH , and H is a Wpq-2-Absorbing, bcx_3 ∉H +Soc(E) ,implies that either abx_3ϵH+Soc(E) or and acx_3 ϵ H +Soc(E). Now , 0≠abc(x_1+x_2+x_3)ϵH and H is Wpq-2-Absorbing ,then either ab(x_1+x_2+x_3)ϵH +Soc(E) or ac(x_1+x_2+x_3)ϵH +Soc(E) or bc(x_1+x_2+x_3)ϵH +Soc(E) . If ab(x_1+x_2+x_3)=ab x_1+ abx_2+ abx_3 ϵH +Soc(E) , then abx_1ϵH+ Soc(E) which is contradiction, if ac(x_1+x_2+x_3) = acx_1+acx_2+acx_3ϵH +Soc(E) implies that acx_2ϵH+Soc(E)a contradiction if bc(x_1+x_2 +x_3)=bc x_1+bcx_2+bcx_3ϵH +Soc(E) implies that bcx_3ϵH+Soc(E)a contradiction .Therefore IJA⊆H +Soc(E) or IKA⊆H +Soc(E) or JKA⊆H +Soc(E).

(⇐) Direct .

The following corollaries are direct consequence of proposition (2.4) and proposition (2.5)

Corollary 2.6 A proper submodule H of an R-module E is a Wpq-2-Absorbing submodule of E if and only if, for each a,bϵ R , and every ideal I of R and yϵ E,with (0) ≠abIy ⊆H , implies that either abyϵH +Soc(E) or aIy⊆H +Soc(E) or bIy⊆H +Soc(E).

Corollary 2.7 A proper submodule H of an R-module E is a Wpq-2-Absorbing submodule of E if and only if 0≠aIJy ⊆H for aϵ R , and every ideals I ,J of R and yϵ E, implies that either aIy ⊆ H +Soc(E) or aJy ⊆ H +Soc(E) or IJy ⊆ H +Soc(E).

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Corollary2.8 A proper submodule H of an R-module E is a Wpq-2-Absorbing submodule of E if and only if $0\neq IJKy \subseteq H$ for all ideals I ,J,K of R and y$\in$ E, implies that either IJ y $\subseteq$H +Soc(E) or IK y $\subseteq$H + Soc(E) or JK y $\subseteq$H +Soc(E).

Corollary 2.9 A proper submodule H of an R-module E is a Wpq-2-Absorbing submodule of E if and only if $0\neq abIA \subseteq H$ for a,b $\in$ R , and every ideal I of R and A is a submodule of E, implies that either abA $\subseteq$ H +Soc(E) or aIA $\subseteq$ H +Soc(E) or bIA $\subseteq$ H +Soc(E).

Corollary 2.10 A proper submodule H of an R-module E is a Wpq-2-Absorbing submodule of E if and only if, $0\neq aIJA \subseteq H$ for a$\in$ R , I ,J are ideals of R and A is a submodule of E, implies that either aIA $\subseteq$ H +Soc(E) or aJA $\subseteq$ H +Soc(E) or IJA $\subseteq$ H +Soc(E).

The following proposition is characterizations in class multiplication modules .

Proposition2.11 A proper submodule H of cyclic R-module E is a Wpq-2-Absorbing submodule E , if and only if for each a,b,c $\in$R ,y $\in$E ,$[H:_R abcy]\subseteq[0:_E abcy]\cup[H+Soc(E):_R aby]\cup[H+Soc(E):_R acy] \cup[H+Soc(E):_R bcy]$.

Proof: ($\Rightarrow$)Let d$\in$$[H:_R abcy]$, implies that abc(dy)$\in$H .If abcdy=0 ,implies that d$\in$$[0:_R abcy]$ .If $0\neq$abc(dy)$\in$H and H is a Wpq-2-Absororbing submodule of E, then either ab(dy) $\in$H +Soc(E) or ac(dy)$\in$H +Soc(E) , or bc(dy)$\in$H+Soc(E) .It follows that either d$\in$$[H+Soc(E):_R aby]$ or d$\in$$[H+Soc(E):_R acy]$ or d$\in$$[H+Soc(E):_R bcy]$ .Hence d$\in$$[0:_R abcy]\cup[H+Soc(E):_R aby] \cup [H+Soc(E):_R acy]\cup [H+Soc(E):_R bcy]$. Therefore $[H:_R abcy] \subseteq [0:_R abcy]\cup[H+Soc(E):_R acy]\cup [H+Soc(E):_R aby]\cup[H+Soc(E):_R bcy]$.

($\Leftarrow$) Let E be a cyclic R-module , then E=$<y_1>$,for some $y_1$$\in$E .Let $0\neq$abcy$\in$H , for a,b,c$\in$R , y $\in$E ,that is y=dy_1 for some d$\in$R .Thus $0\neq abc(dy_1)$ $\in$H, implies that d$\in$$[H:_E abcy_1] \subseteq [0:_R abcy_1] \cup[H+Soc(E):_R aby_1] \cup [H+Soc(E):_R acy_1] \cup [H+Soc(E):_R bcy_1]$ .Since $0\neq abcdy_1$,then d$\notin$$[0:_R abcy_1]$ .Thus d$\in$$[H+Soc(E):_R aby_1]$ or d$\in$ $[H+Soc(E):_R acy_1]$ or d$\in$ $[H+Soc(E):_R bcy_1]$ .That is either $abdy_1$ $\in$H +Soc(E) or $acdy_1$ $\in$H +Soc(E) or $bcdy_1$ $\in$H +Soc(E) .Hence either aby $\in$H +Soc(E) or acy $\in$H +Soc(E) or bcy $\in$H +Soc(E). Therefore H is a Wpq-2-Absorbing submodule of E.

The following proposition shows that Wpq-2-Absorbing submodule weakly quasi-2-Absorbing submodules are equivalent under contain condition .

Proposition2.12 A proper submodule H of an R-module E with Soc(E)$\subseteq$H is a Wpq-2-Absorbing submodule E , if and only if H is weakly quasi-2-Absorbing submodule of E.

Proof : ($\Rightarrow$) Let $0\neq$abcy $\in$H for a,b,c $\in$R ,y $\in$E .Since H is a Wpq-2-Absorbing ,then either aby$\in$H+Soc(E) or acy$\in$H+Soc(E) or bcy$\in$H+Soc(E) .But Soc(E)$\subseteq$H ,then H+Soc(E)=H .It follows that either aby$\in$H or acy$\in$H or bcy$\in$H . Hence H is a weakly-quasi-2-Absorbing submodule of E

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(⇐) Follows by example page (2) number (2) .

Proposition2.13 A proper submodule H of an R-module E with $Soc(E) \subseteq H$ is a Wpq-2-Absorbing submodule E , if and only if $[H:_E J]$ is a Wpq-2-Absorbing submodule of E for each a non-zero ideal J of R .

Proof: (⇒) Let $0 \neq abcy \in [H:_E J]$,for $a,b,c \in R$,$y \in E$,implies that $(0) \neq abc(Jy) \subseteq H$ .But H is a Wpq-2-Absorbing then by proposition (2.4) either $ab(Jy) \subseteq H+Soc(E)$ or $ac(Jy) \subseteq H+Soc(E)$ or $bc(Jy) \subseteq H+Soc(E)$. But $Soc(E) \subseteq H$ then $H+Soc(E)=H$ .Thus either $ab(Jy) \subseteq H$ or $ac(Jy) \subseteq H$ or $bc(Jy) \subseteq H$,that is either $aby \in [H:_E J] \subseteq [H:_E J] + Soc(E)$ or $acy \in [H:_E J] \subseteq [H:_E J] + Soc(E)$ or $bcy \in [H:_E J] \subseteq [H:_E J] + Soc(E)$. Therefore $[H:_E J]$ is a Wpq-2-Absorbing .

(⇐) Follows by putting J=R $[H:_E J]$.

Proposition2.14 Let H, S be Wpq-2-Absorbing submodules of E with S is not contained in H and either $Soc(E) \subseteq H$ or $Soc(E) \subseteq S$ .Then $H \cap S$ is a Wpq-2-Absorbing submodule E.

Proof: It is clear that $H \cap S$ is a proper submodule of E .Let $(0) \neq abIy \subseteq H \cap S$,for $a,b \in R$ J is an ideal of R ,implies that $(0) \neq abJy \subseteq H$,and $(0) \neq abJy \subseteq S$. But H, S are Wpq-2-Absorbing , implies that either $aby \in H + Soc(E)$ or $aJy \subseteq H+Soc(E)$ or $bJy \subseteq H + Soc(E)$, and either $aby \in S+Soc(E)$ or $aJy \subseteq S+Soc(E)$ or $bJy \subseteq S + Soc(E)$. That is either $aby \in (H + Soc(E)) \cap (S+Soc(E))$ or $aJy \subseteq (H+Soc(E)) \cap (S+Soc(E))$ or $bJy \subseteq (H + Soc(E)) \cap (S+Soc(E))$ .Suppose that $Soc(E) \subseteq S$,then $S+Soc(E)=S$ .Thus either $aby \in (H + Soc(E)) \cap S$ or $aJy \subseteq (H+Soc(E)) \cap S$ or $bJy \subseteq (H + Soc(E)) \cap S$ .Thus by modular law $(H + Soc(E)) \cap S=(H \cap S) + Soc(E)$. It follows that either $aby \in (H \cap S) + Soc(E)$ or $aJy \subseteq (H \cap S) + Soc(E)$ or $bJy \subseteq (H \cap S) + Soc(E)$ .Therefor $H \cap S$ is a Wpq-2-Absorbing submodule of E.

The following properties are characterization of Wpq-2-Absorbing submodules in the class of multiplication modules .

Proposition2.15 A proper submodule H of a multiplication R-module E is a Wpq-2-Absorbing submodule of E , if and onlyif $0 \neq L_1L_2L_3y \subseteq H$, for some submodules L_1 , L_2 ,L_3 of E , and $y \in E$ implies that either $L_1L_2y \subseteq H + Soc(E)$ or $L_1 L_3 y \subseteq H + Soc(E)$ or $L_2L_3 y \subseteq H + Soc(E)$.

Proof: (⇒) Let $0 \neq L_1L_2L_3y \subseteq H$, for some submodules L_1,L_2 ,L_3 of E, and $y \in E$ .Since E is a multiplication then $L_1 = I_1E$, $L_2 = I_2E$, $L_3 = I_3E$, for some ideals I_1 ,I_2 ,I_3 of R , that is $0 \neq I_1I_2I_3y \subseteq H$. But H is a Wpq-2-Absorbing ,then by corollary (2.8) either $I_1I_2y \subseteq H + Soc(E)$ or $I_1I_3 y \subseteq H + Soc(E)$ or $I_2I_3y \subseteq H + Soc(E)$ .That is either $L_1L_2y \subseteq H + Soc(E)$ or $L_1 L_3 y \subseteq H + Soc(E)$ or $L_2L_3y \subseteq H + Soc(E)$.

(⇐) Let $0 \neq I_1I_2I_3y \subseteq H$, for I_1 ,I_2 ,I_3 are ideals of R , and $y \in E$ .Put $L_1 = I_1E$, $L_2 = I_2E$, $L_3 = I_3E$ in by hypothesis we get $0 \neq L_1L_2L_3y \subseteq H$,implies that either $L_1L_2y \subseteq H + Soc(E)$ or $L_1 L_3 y \subseteq H$

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+Soc(E) or $L_2L_3y \subseteq H$ +Soc(E) .Thus either $I_1I_2y \subseteq H$ +Soc(E) or $I_1 I_3 y \subseteq H$ +Soc(E) or $I_2I_3y \subseteq H$ +Soc(E) .Therefore by corollary (2.8) H is a Wpq-2-Absorbing submodule of E .

Proposition2.16 A proper submodule H of a multiplication R-module E is a Wpq-2-Absorbing submodule of E , if and only if $0 \neq L_1L_2L_3L_4 \subseteq H$, for some submodules L_1 ,L_2 ,L_3 ,L_4 of E , implies that either $L_1L_2L_4 \subseteq H$ + Soc(E) or $L_1L_3L_4 \subseteq H$ + Soc(E) or $L_2L_3L_4 \subseteq H$ + Soc(E) .

Proof: (⟹) Let $0 \neq L_1L_2L_3L_4 \subseteq H$, for some submodules L_1 ,L_2 ,L_3 ,L_4 of E .But E is a multiplication then $L_1 = I_1 E$, $L_2 = I_2 E$, $L_3 = I_3E$, for some ideals I_1 ,I_2 ,I_3 of R . Thus $0 \neq I_1I_2I_3L_4 \subseteq H$ .Since H is a Wpq-2-Absorbing ,then by proposition (2.5) either $I_1I_2L_4 \subseteq H$ +Soc(E) or $I_1I_3L_4 \subseteq H$ +Soc(E) or $I_2I_3L_4 \subseteq H$ +Soc(E) .That is either $L_1L_2L_4 \subseteq H$ +Soc(E) or $L_1L_3L_4 \subseteq H$ +Soc(E) or $L_2L_3L_4 \subseteq H$ +Soc(E) .

(⟸)Trivial .

It is well-known that if E is a Z-regular R-module , then Soc(E)=Soc(R)E .[7, prop.(3.25)] .

Proposition2.17 A proper submodule H of Z-regular multiplication R-module E is a Wpq-2-Absorbing submodule E if and only if $[H:_R E]$ is a Wpq-2-Absorbing ideal of R .

Proof: (⟹)Let $0 \neq abcI \subseteq [H:_R E]$,for a,b,c∊R , I is an ideal of R .it follows that $0 \neq abc(IE) \subseteq H$. But H is a Wpq-2-Absorbing submodule of E , then by proposition(2.4) either $ab(IE) \subseteq H$+Soc(E) or $ac(IE) \subseteq H$+Soc(E) or $bc(IE) \subseteq H$+Soc(E) . Since E is a Z –regular , then Soc(E)=Soc(R)E ,and since E is multiplication , then $H=[H:_R E]E$ .Thus either $abIE \subseteq [H:_R E]E$+Soc(R)E or $acIE \subseteq [H:_R E]E$+Soc(R)E or $bcIE \subseteq [H:_R E]E$+Soc(R)E . That is , either $abI \subseteq [H:_R E]$+Soc(R) or $acI \subseteq [H:_R E]$ + Soc(R) or $bcI \subseteq [H:_R E]$+Soc(R) . Hence by proposition(2.4) $[H:_R E]$ is a Wpq-2-Absorbing ideal of R .

(⟸) Let $0 \neq aIJB \subseteq H$, for a∊R , and I,J are ideals of R, B is a submodule of E . Since E is a multiplication , then $B = J_1E$,for some ideal J_1 of R , it follows that is $0 \neq aIJJ_1 E \subseteq H$, implies that $0 \neq aIJJ_1 \subseteq [H:_R E]$. But $[H:_R E]$ is a Wpq-2-Absorbing , then by corollary (2.10) either $aIJ_1 \subseteq [H:_R E]$+ Soc(R) or $aJJ_1 \subseteq [H:_R E]$+Soc(R) or $IJJ_1 \subseteq [H:_R E]$+Soc(R). Thus either $aIJ_1E \subseteq [H:_R E]E$+Soc(R)E or $aJJ_1E \subseteq [H:_R E]E$+Soc(R)E or $IJJ_1E \subseteq [H:_R E]E$+Soc(R)E. But E is a Z-regular Soc(E)=Soc(R)E .Hence either $aIB \subseteq H$+Soc(E) or $aJB \subseteq H$+Soc(E) or $IJB \subseteq H$+Soc(E).Therefore by corollary (2.10) H is a Wpq-2-Absorbing submodule of E .

It is well-known that if an R-module E finitely generated multiplication R-module and I ,J are ideals of R . Then $IT \subseteq JT$ if and only if $I \subseteq J + ann_R(E)$.[8, coro of Theo(9)] .

Proposition2.18 Let E be a finitely generated Z-regular multiplication R-module and I is a Wpq-2-Absorbing ideal of R with $ann_R(E) \subseteq I$. Then IE is a Wpq-2-Absorbing of E.

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Proof: Let $0 \neq abJB \subseteq IE$, for a,bϵ R, J is an ideal of R and B is a submodule of E. Since E is a multiplication then B $=J_1$ E for some ideal J_1 of R. It follows that $0 \neq abJJ_1E \subseteq IE$. But E is a finitely generated , $0 \neq abJJ_1 \subseteq I+ ann_R(E)= I$ (for $ann_R(E) \subseteq I$) . Since I is a Wpq-2-Absorbing ideal R , then by corollary (2.9) either $abJ_1 \subseteq I+Soc(R)$ or $aJ\ J_2 \subseteq I+Soc(R)$ or $bJ\ J_2 \subseteq I+ Soc(R)$.That is either $0 \neq abJJ_1\ E \subseteq IE + Soc(R)E$ or $0 \neq aJ\ J_2E \subseteq IE+ Soc(R)E$ or $bJ\ J_2E \subseteq IE+Soc(R)E$. But E is a Z-regular then soc(R)E=Soc(E) ,so either $abB \subseteq IE+Soc(E)$ or $aJ\ B \subseteq IE+Soc(E)$ or $bJ\ B \subseteq I+Soc(R)$. Therefore by corollary (2.9) IE is a Wpq-2-Absorbing submodule of E .

It is well-known that ,if an R-module E is a non-singular then Soc(E) = Soc(R)E [4,Coro(1.26)] .

Proposition2.19 Let E be a non-singular multiplication R-module and H be a proper submodule of E .Then H is a Wpq-2-Absorbing submodule of E if and only if $[H:_RE]$ is a Wpq-2-Absorbing ideal of R .

Proof: Follows as in proposition (2.17) .

Proposition 2.20 Let E be a non-singular finitely generated multiplication R-module , and I is a Wpq-2-Absorbing ideal of R with $ann_R(E) \subseteq I$ .Then IE is a Wpq-2-Absorbing ideal of R .

Proof: Follows as in proposition (2.18).

Proposition2.21 Let E be a finitely generated multiplication Z-regular R-module , and H be a proper submodule of E . Then the following statements are equivalent :

1. H is a Wpq-2-Abrosorbing submodule of E .

2. $[H:_RE]$ is a Wpq-2-Abrosorbing ideal of R .

3. H = IE for some Wpq-2-Absorbing ideal I of R with $ann_R(E) \subseteq I$.

Poof: (1) $\Rightarrow$ (2) Follows by proposition (2.17) .

(2) $\Rightarrow$ (3) Let H be a submodule of E ,then $H=[H:_RE]E$,by (2) $[H:_RE]$ is a Wpq-2-Absorbing ideal, so we put $I=[H:_RE]$, that is H=IE with $ann_R(E)=[0:_RE]\subseteq[H:_RE]=I$.

(3) $\Rightarrow$ (2) Suppose that H=IE for some a some Wpq-2-Absorbing ideal I of R ,with $ann_R(E) \subseteq$ I of . But E is a multiplication , then $H= [H:_RE]E =IE$,also E is finitely generated by [10,prop.(3-1)) , E is cancellation, that is Thus $I= [H:_RE]$ which is a Wpq-2-Absorbing ideal .

Proposition2.22 Let E be a non-singular finitely generated multiplication R-module , and H be a proper submodule of E . Then the following statements are equivalent :

1.H is a Wpq-2-Abrosorbing submodule of E .

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2.[H:$_R$E] is a Wpq-2-Abrosorbing ideal of R .

3.H=IE for some Wpq-2-Absorbing ideal I of R with ann$_R$(E)$\subseteq$I .

Poof: Follows as in proposition (2.21) by using proposition (2.19) .

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المقاسات الجزئية الستحوذه على -٢- من النمط –Wpq ومفاهيم ذات العلاقه

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المستخلص:

في هذا البحث R حلقه ابداليه بمحايد و E مقاسا احاديا ايسرا مفهوم المقاسات الجزئية المستحوذه على -٢ الظاهر الضعيفة عمت في هذا البحث الى المقاسات الجزئية المستحوذه على -٢ من النمط Wpq. حيث يدعى المقاس الجزئي الفعلي H من المقاس E انه مقاسا جزئيا مستحوذا على -٢ من النمط –Wpq اذا كان 0≠abcyϵH حيث a,b,cϵR ,yϵE يؤدي الى abyϵH+Soc(E) او acy ϵH+Soc(E) او bcy ϵH +Soc(E) ,العديد من المكافئات و الامثلة و الخواص الاساسيه لهذا المفهوم قدمت ,بالاضافة الى ذالك مكافئات للمقاسات الجزئية المستحوذة على-٢ من النمط –Wpq في بعض انوع المقاسات الاخرى أعطيت .

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