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Some Results on Weak and Strong Fuzzy Convergence for Fuzzy Normed Spaces

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Abstract

In this paper we study the fuzzy norm and the fuzzy normed space, then we define the fuzzy normed space and study the notion of weak and strong fuzzy convergence of sequences in fuzzy normed spaces. After that we prove some basic results of fuzzy convergent in these spaces.

Keywords : Fuzzy norm, fuzzy continuity, fuzzy boundedness , weak and strong fuzzy convergence.

Mathematics Subject Classification: 46S40 .

1. Introduction

The notion of fuzzy norm on a linear space was introduced by Katsaras [1] in 1984. Later on many other Mathematicians like Felbin [2] in 1992, Cheng and Mordeson [3] in 1994, Bag and Samanta [4] in 2003 etc, have given different definitions of fuzzy normed spaces. In this paper we define fuzzy continuity and fuzzy boundedness of functions in fuzzy normed spaces also we define strong and weak fuzzy convergence of sequences in fuzzy normed space and discuss the relation between them. Finaly we prove some new results on fuzzy convergent in these spaces.

2. Preliminaries

In this section some fundamental definitions are given which are used in this paper.

Definition (2.1) : [5] A binary operation $\ast : [0,1] \times [0,1] \to [0,1]$ is called a t-norm if $\ast$ is satisfies the following conditions:

(i) $\ast$ is commutative and associative,
(ii) $a \ast 1 = a$ for all $a \in [0,1]$,
(iii) $a \ast b \leq c \ast d$ whenever $a \leq c$ and $b \leq d$ for all $(a, b, c, d \in [0,1])$.

If $*$ is continuous then it is called continuous t-norm.

Definition (2.2) : [6] The 3-tuple $(X, N,*)$ is said to be a fuzzy normed space if X is a vector space, $*$ be a continuous t-norm and N is a fuzzy set on $X \times (0, \infty)$ satisfying the following conditions for every $x, y \in X$ and $t, s > 0$:

(N.1) $N(x, t) > 0$,
(N.2) $N(x, t) = 1 \iff x = 0$,
(N.3) $N(\alpha x, t) = N\left(x, \frac{t}{|\alpha|}\right)$ for all $\alpha \neq 0$,
(N.4) $N(x, t) * N(y, s) \leq N(x + y, t + s)$,
(N.5) $N(x, .): (0, \infty) \to [0,1]$ is continuous,
(N.6) $\lim_{t\to\infty} N(x, t) = 1$.

Definition (2.3) : [4] Let $(X, N_1 ,*)$ and $(Y, N_2 ,*)$ be two fuzzy normed spaces and $f : X \to Y$ be a function:

(1) f is called weakly fuzzy continuous at $x_0 \in X$ if for given $\varepsilon > 0$ and $\alpha \in (0,1)$, there exists some $\delta \in \mathbb{Z}^+$ such that for all $x \in X$,

$$N_1(x - x_0 , \delta) \geq \alpha \ \text{ implies } \ N_2(f(x) - f(x_0) , \varepsilon) \geq \alpha.$$

(2) f is called strongly fuzzy continuous at $x_0 \in X$ if given $\varepsilon > 0$, there exists some $\delta \in \mathbb{Z}^+$ such that for all $x \in X$,

$$N_2(f(x) - f(x_0) , \varepsilon) \geq N_1(x - x_0 , \delta).$$

(3) Let f be linear function. f is called weakly fuzzy bounded on X if for every $\alpha \in (0,1)$, there exists some $m_\alpha > 0$ such that for all $x \in X$,

$$N_1\left(x , \frac{t}{m_\alpha}\right) \geq \alpha \ \text{ implies } \ N_2(f(x), t) \geq \alpha, \forall t > 0.$$

(4) Let f be linear function. f is called strongly fuzzy bounded on X if for every $\alpha \in (0,1)$, there exists some $M > 0$ such that for all $x \in X$,

$$N_2(f(x), t) \geq N_1\left(x , \frac{t}{M}\right), \ \forall t > 0.$$

As in classical theory, the following is easy to prove.

Theorem (2.4) : [4] Let $(X, N_1 ,*)$ and $(X, N_2 ,*)$ be two fuzzy normed spaces and $f : X \to Y$ be a linear function. Then f is strongly $(weakly)$ fuzzy continuous if and only if strongly $(weakly)$ fuzzy bounded.

3. Main results

Theorem (3.1) : Let $(X, N,*)$ be a fuzzy normed space, we further assume that,

(N.7) $\alpha * \alpha = \alpha \quad \forall \alpha \in [0,1]$
(N.8) $N(x, t) > 0 \ \forall t > 0 \Longrightarrow x = 0$.

Define $\|x\|_\alpha = \inf\{t > 0 : N(x, t) \geq \alpha\}$. Then$\{ \|x\|_\alpha : \alpha \in (0,1)\}$ is an ascending family of norms on X. We call these norms as α-norms on X corresponding to fuzzy norm N on X.

Proof : Let $\alpha \in (0,1)$. To prove $\|x\|_\alpha$ is a norm on X. It is easy to see that (N.1), (N.2), (N.3), (N.5) and (N.6) are true. We now prove (N.4) :

$$\begin{aligned}\|x\|_\alpha + \|y\|_\alpha &= \inf\{ s > 0 : N(x,s) \geq \alpha \} + \inf\{ t > 0 : N(y,t) \geq \alpha \} \\ &= \inf\{ s + t > 0 : N(x,s) \geq \alpha , N(y,t) \geq \alpha \} \\ &= \inf\{ s + t > 0 : N(x,s) * N(y,t) \geq \alpha * \alpha = \alpha \} \\ &\geq \inf\{ s + t > 0 : N(x + y , s + t) \geq \alpha \} \\ &= \|x + y\|_\alpha,\end{aligned}$$

which proves (N.4).

Let $0 < \alpha_1 < \alpha_2 < 1$.

$\|x\|_{\alpha_1} = \inf\{t > 0 : N(x,t) \geq \alpha_1\}$ and $\|x\|_{\alpha_2} = \inf\{ t > 0 : N(x,t) \geq \alpha_2 \}$.

Since $\alpha_1 < \alpha_2$, $\{ t > 0 : N(x,t) \geq \alpha_2 \} \subset \{ t > 0 : N(x,t) \geq \alpha_1 \}$

$\Rightarrow \inf\{ t > 0 : N(x,t) \geq \alpha_2 \} \geq \inf\{ t > 0 : N(x,t) \geq \alpha_1 \}$

$\Rightarrow \|x\|_{\alpha_2} \geq \|x\|_{\alpha_1}$.

Thus, we see that $\{ \|x\|_\alpha : \alpha \in (0,1)\}$ is an ascending family of norms on X.

Definition (3.2) : [7] Let $(X, N,*)$ be a fuzzy normed space.

(a) A sequence $\{x_n\}$ in X is said to be fuzzy converges to x in X if for each $\varepsilon \in (0,1)$ and each $t > 0$, there exists $n_0 \in \mathbb{Z}^+$ such that $N(x_n - x , t) > 1 - \varepsilon$ for all $n \geq n_0$ (or equivalently $\lim_{n\to\infty} N(x_n - x , t) = 1$) and x is called the limit of the sequence $\{x_n\}$.

(b) A sequence $\{x_n\}$ in X is said to be fuzzy Cauchy if for each $\varepsilon \in (0,1)$ and each $t > 0$, there exists $n_0 \in \mathbb{Z}^+$ such that $N(x_n - x_m , t) > 1 - \varepsilon$ for all $n, m \geq n_0$ (or equivalently $\lim_{n,m\to\infty} N(x_n - x_m , t) = 1$) and x is called the limit of the sequence $\{x_n\}$.

(c) A fuzzy normed space in which every fuzzy Cauchy sequence is a fuzzy convergent is said to be complete.

Definition (3.3) : Let $(X, N,*)$ be a fuzzy normed space. The sequence $\{x_n\}$ is said to be:

(i) weakly fuzzy convergent to $x \in X$ if and only if, for every $\varepsilon > 0$ and $\alpha \in (0,1)$, there exists some $n_0 \in \mathbb{Z}^+$ such that $N(x_n - x , \varepsilon) \geq 1 - \alpha$ for all $n \geq n_0$. In this case we write $x_n \xrightarrow{wf} x$.

(ii) strongly fuzzy convergent to $x \in X$ if and only if , for every $\alpha \in (0,1)$, there exists some $n_0 \in \mathbb{Z}^+$ such that $N(x_n - x , t) \geq 1 - \alpha$ for all $t > 0$. In this case we write $x_n \xrightarrow{sf} x$.

Theorem (3.4) : If a sequence $\{x_n\}$ is sf-convergent then it is wf- convergent to the same limit, but not conversely. Therefore sf-convergence implies wf-convergence. For converse, we have the following example.

Example (3.5) : Let $X = \mathbb{C}$ and consider the fuzzy norm

$$N(x,t) = \begin{cases} \dfrac{t-|x|}{t+|x|} & t > |x| \\ 0 & t \le |x| \end{cases}$$

on X . We can find α-norms of N since it satisfies (N.6) condition. Thus $N(x\,,t) \ge \alpha \Leftrightarrow \frac{t-|x|}{t+|x|} \ge \alpha \Leftrightarrow \frac{1+\alpha}{1-\alpha} |x| \le t$.

This show that $\|x\|_\alpha = \inf\{\, t > 0 : N(x,t) \ge \alpha \,\} = \frac{1+\alpha}{1-\alpha}|x|$.

We now show that the sequence $\{\, x_n \,\} = \{\frac{1}{n}\}$ is wf-convergent but not sf-convergent. Since each $\|.\|_\alpha$ is equivalent to $|\ .\ |$, obviously, $\{\, x_n \,\}$ is wf- convergent to 0. However, this convergence is not uniform in α. Indeed; for given $\varepsilon > 0$,

$$\|x_n\|_\alpha = \frac{1+\alpha}{1-\alpha}|x_n| < \varepsilon \Leftrightarrow \frac{1+\alpha}{(1-\alpha)\varepsilon} < n.$$

We cannot find desired n_0 since $\frac{1+\alpha}{(1-\alpha)\varepsilon} \to \infty$ as $\alpha \to 1$.

Theorem (3.6) : Let $\{x_n\}$ and $\{y_n\}$ are sequences in a fuzzy normed space $(X, N, *)$ and for all $\alpha_1 \in (0,1)$ there exists $\alpha \in (0,1)$ such that $\alpha * \alpha \ge \alpha_1$

(1) The weak limit x of $\{x_n\}$ is unique.

(2) If $x_n \xrightarrow{wf} x$ then $cx_n \xrightarrow{wf} cx$ for all $c \in F/\{0\}$.

(3) If $x_n \xrightarrow{wf} x$, $y_n \xrightarrow{wf} y$, then $x_n + y_n \xrightarrow{wf} x + y$.

Proof :

(1) Let $\{x_n\}$ be a sequence in X such that $x_n \xrightarrow{wf} x$ and $x_n \xrightarrow{wf} y$ as $n \to \infty$. Then for all $\varepsilon\,, \varepsilon_1 > 0$ such that $\lim_{n\to\infty} N(x_n - x\,, \varepsilon_1) = 1, \lim_{n\to\infty} N(x_n - y\,, \varepsilon - \varepsilon_1) = 1$,
$N(x-y\,,\varepsilon) \ge N(x_n - x\,, \varepsilon_1\,) * N(x_n - y\,, \varepsilon - \varepsilon_1\,)$.
Taking limit as $n \to \infty$: $N(x-y\,,\varepsilon) \ge 1 * 1 = 1$.
But $N(x-y\,,\varepsilon) \le 1 \Longrightarrow N(x-y\,,\varepsilon) = 1$.
Then by axiom (N.2) $x - y = 0 \Longrightarrow x = y$.

(2) Since $x_n \xrightarrow{wf} x$ then for every $\varepsilon > 0$ and $\alpha \in (0,1)$ there exists $n_0 \in \mathbb{Z}^+$ such that $N(x_n - x\,, \varepsilon) \ge 1 - \alpha$ for all $n \ge n_0$.
Put $\varepsilon = \frac{\varepsilon_1}{|c|}$ such that $\varepsilon_1 > 0$, $c \in F/\{0\}$.
$N(cx_n - cx\,, \varepsilon_1\,) = N\left(x_n - x\,, \frac{\varepsilon_1}{|c|}\right) = N(x_n - x\,, \varepsilon) \ge 1 - \alpha$.
Then $cx_n \xrightarrow{wf} cx$.

(3) For each $\alpha_1 \in (0,1)$ there exists $\alpha \in (0,1)$ such that
$(1-\alpha) * (1-\alpha) \ge (1-\alpha_1)$.

Since $x_n \overset{wf}{\rightarrow} x$ then for every $\varepsilon > 0$ and $\alpha \in (0,1)$ there exists $n_1 \in \mathbb{Z}^+$, $N\left(x_n - x, \frac{\varepsilon}{2}\right) \geq 1 - \alpha$ for all $n \geq n_1$.

Since $y_n \overset{wf}{\rightarrow} y$ then for every $\varepsilon > 0$ and $\alpha \in (0,1)$ there exists $n_2 \in \mathbb{Z}^+$ Such that $N\left(y_n - y, \frac{\varepsilon}{2}\right) \geq 1 - \alpha$ for all $n \geq n_2$.

Take $n_0 = \min\{ n_1, n_2 \}$ and for all $\varepsilon > 0$, there exists $n_0 \in \mathbb{Z}^+$ such that:
$N\big((x_n + y_n) - (x + y), \varepsilon\big) = N\big((x_n - x) + (y_n - y), \varepsilon\big) \geq$
$N\left(x_n - x, \frac{\varepsilon}{2}\right) * N\left(y_n - y, \frac{\varepsilon}{2}\right) > (1 - \alpha) * (1 - \alpha) \geq (1 - \alpha_1)$
for all $n \geq n_0$. Then $x_n + y_n \overset{wf}{\rightarrow} x + y$.

Proposition (3.7) : [8] Let $(X, N, *)$ be a fuzzy normed space satisfying (N.8) and $\{x_n\}$ be a sequence in X. Then $\lim_{n\to\infty} N(x_n - x, t) = 1$ if and only if $\lim_{n\to\infty} \|x_n - x\|_\alpha = 0$ for all $\alpha \in (0,1)$.

Theorem (3.8) : Let $\{x_n\}$ be a sequence in fuzzy normed space $(X, N, *)$ satisfying (N.8). Then:

(1) $x_n \overset{wf}{\rightarrow} x$ if and only if for each $\alpha \in (0,1)$, $\lim_{n\to\infty} \|x_n - x\|_\alpha = 0$.
(2) $x_n \overset{wf}{\rightarrow} x$ if and only if $\lim_{n\to\infty} \|x_n - x\|_\alpha = 0$ uniformly in α where $\|.\|_\alpha$ are α-normes of N.

Proof :

(1) Let $x_n \overset{wf}{\rightarrow} x \Longrightarrow \forall\ \alpha \in (0,1)$ and $\varepsilon > 0$, there exists some $K \in \mathbb{Z}^+$ such that $N(x_n - x, \varepsilon) \geq 1 - \alpha$ for all $n \geq K$.
Since $\|x_n - x\|_\alpha = \inf\{ t > 0 : N(x_n - x, t) \geq \alpha \}$ by (N.8)
$\lim_{n\to\infty} N(x_n - x, t) = 1$ and by proposition (3.7) we obtain
$\lim_{n\to\infty} \|x_n - x\|_\alpha = 0$.

(2) Let $x_n \overset{sf}{\rightarrow} x \Longrightarrow \forall\ \alpha \in (0,1)$, there exists some $K \in \mathbb{Z}^+$ such that
$N(x_n - x, t) \geq 1 - \alpha$ for all $t > 0$.
Since $\|x_n - x\|_\alpha = \inf\{ t > 0 : N(x_n - x, t) \geq \alpha \}$ by (N.8)
$\lim_{n\to\infty} N(x_n - x, t) = 1$ and by proposition (3.7) we obtain
$\lim_{n\to\infty} \|x_n - x\|_\alpha = 0$.

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بعض النتائج عن التقارب الضبابي القوي والضعيف في الفضاءات المعيارية الضبابية

نوري فرحان المياحي　　　　هند وليم طوير
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المستخلص :

في هذا البحث درسنا المعيار الضبابي والفضاء المعياري الضبابي ثم عرفنا الفضاء المعياري الضبابي ودرسنا مفهوم التقارب الضبابي القوي والضعيف للمتتابعات في الفضاءات المعيارية الضبابية بعد ذلك برهنا بعض النتائج الأساسية للتقارب الضبابي في تلك الفضاءات.