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Page 314 -325

VARIATIONAL FORMULATION OF NONLINEAR ORDINARY DELAY DIFFERENTIAL EQUATIONS

Zaineb Hamed. SH
Department of Mathematics, Collage of Science, Al-Mustansiriyah University, Baghdad, Iraq.

Fadhel. F. S
Department of Mathematics, Collage of Science, Al-Nahrain University, Baghdad, Iraq.

ABSTRACT

In this paper, the variational formulation of nonlinear problems is considered to find the approximate solution of an important type of differential equations which is the nonlinear ordinary delay differential equations and illustrated by an example.

1- INTRODUCTION

The variational calculus gives a method for finding the maximal and minimal values of functionals. Problems that consist of finding the maxima or the minima of a functional are called variational problems, [Elsgolc, 1962]. As an example, the solution of any problem (such as partial differential equations, ordinary differential equations, integral equations, etc.) is equivalent to the problem of minimizing a functional that corresponding to this problem, [Magri, 1974].

The basic analysis of the subject of calculus of variation depends mainly on minimizing some functional with a suitable condition to be satisfied. However, the problem of evaluating this functional has some difficulties and therefore, much attention on Tonti's approach is given for every nonlinear operator equation of the form N(u) = 0, [Tonti E., 1984].

In addition, delay differential equations occur as in the works of L. Euler (in the second half of the eighteenth century), but systematically the study of such equations was first considered in the twentieth century, to meet the demands of applied science, in particular of control theory, [Driver, R. D., 1977]. The significance of these equations lies in their ability to describe processes with after effect. The importance of these equations in various branches of technology, economics, biology and medical sciences have been recognized recently and has caused mathematicians to study with increasing interest.

2- BASIC CONCEPTS IN CALCULUS OF VARIATION

Following are some of the fundamental concepts related to this paper considering the subject of calculus of variation and its inverse problem for evaluating the related functional of certain problem, these concepts may be summarized as follows:

A functional F(u, v) is said to be bilinear form if it is linear in both of its arguments (i.e., in u and v) which is denoted by (u, v) and may be defined as:

$$< u, v >= \int_0^T u(x)v(x)dx, \quad 0 \le x \le T \qquad \ldots(1)$$

(see [Marie, N., 2001] for other types of bilinear forms).

Also, the bilinear form <u, v> is said to be symmetric if <u, v> = <v, u>, $\forall$ n $\in$ U, v $\in$ V, where U, is a linear normed space and it is called non degenerate if $<u, \overline{v}> = 0$ implies $\overline{v} = 0$, $\forall$ n $\in$ U (and vise versa follows from the symmetry of <u, v>).

The follows definitions seem to be necessary:

Definition (2.1), [Magri, 1974]:

A linear operator L is said to be symmetric with respect to the chosen bilinear form <., .> if L satisfies:

$$< Lu_1, u_2 >=< Lu_2, u_1 >, \ \forall \ u_1, u_2 \in D(L).$$

and in addition, the linear operator L is said to be invertible (A is called the inverse of L) if there exists a bounded linear operator A such that LA = AL = I, when I is the identity operator.

Definition (2.2), [Taylor, 1961]:

Let $L : D(L) \subseteq U \longrightarrow R(L) \subseteq V$ be a linear operator. The operator L* is called the adjoint operator of L if:

$<v, Lu> = <L^*v, u>, \forall u, v \in U.$

Definition (2.3), [Ogata, 1967]:

An operator L is said to be positive definite if the following two conditions are satisfied:

a- $< u, Lu > > 0, \ \forall \ u \in D(L)$ and $u \ne 0$.

b- $< u, Lu >= 0$ if and only if $u = 0$.

Definition (2.4), [Tonti, 1984]:

Let $N : D(N) \subseteq U \longrightarrow R(N) \subseteq V$ be a nonlinear operator. Then the operator N' defined by:

$$N'_u(u:\phi) \equiv N'_u(\phi) = \frac{d}{d\varepsilon}[N(u + \varepsilon\phi)] \;\Big|_{\varepsilon=0}, \; \forall u \in D(N), \varepsilon \in \square$$

is called the Gateaux derivative of N at u, where ϕ is an arbitrary element in D(N).

Also it is symbolically referred to the gradient of functional as the potential operator.

3- ORDINARY DELAY DIFFERENTIAL EQUATIONS

An n-th order ordinary delay differential equation may take the following general form:

$$f(t; y(t), y(t - \tau_1(t)), \ldots \quad \ldots \quad \ldots$$

$$y^{(n)}(t - \tau_k(t)) = g(t) \qquad \ldots(2)$$

where F is a given function and $\tau_1(t)$, $\tau_2(t)$, …, $\tau_k(t)$ are given real valued functions called "time delays" [Bellman and Cooke, 1963]. If in eq.(2) we set $\tau_1(t) = \tau_2(t) = \ldots = \tau_k(t) = 0$, then we have an n-th order ordinary differential equation. Also,

Other literatures writes eq.(2) in the following form:

$$F(t, x(t), x(k_1(t)), \ldots, x(k_2(t)), x'(t), x'(k_1(t)), \ldots, x'(k_n(t)), \ldots, x^{(n)}(t), x^{(n)}(k_1(t))\ldots, x^{(n)}(k_n(t)) = g(t) \qquad \ldots(3)$$

where $k_1(t)$, $k_2(t)$, …, $k_n(t)$ are real valued functions not all of them equals to one. Also we can consider the ODE as a special case delay differential equation with either $k_1 = k_2 = \ldots = k_n = 1$. Hence, as a conclusion, one can assume that the theory of delay differential equations is a generalization the theory of ordinary differential equations.

For simplicity, one can write a first order delay differential equation with constant coefficients and with one delay as follows, [Abbas, H. J., 1994]:

$$a_0y'(t) + a_1y'(t - \tau(t)) + b_0y(t) + b_1y(t - \tau(t)) = g(t) \qquad \ldots(4)$$

where g(t) is a given continuous function, $\tau(t)$ is a positive constant and a_0, a_1, b_0 and b_1 are constants.

Equation (4) can be classified into three kinds: the first kind (retarded) occurs when ($a_0 \neq 0$ and $a_1 = 0$), [Stepan, G., 1989], i.e., the delay comes in x only, and the differential equation takes the form:

$$a_0 y'(t) + + b_0 y(t) + b_1 y(t - \tau(t)) = g(t) \qquad \ldots(5)$$

The second kind (neutral) is obtained when ($a_1 \neq 0$ and $b_0 = 0$), i.e., the delay comes in y′, and the differential equation takes the form:

$$a_0 y'(t) + a_1 y'(t - \tau(t)) + b_1 y(t - \tau(t)) = g(t) \qquad \ldots(6)$$

The third kind (mixed) is obtained when ($a_1 \neq 0$ and $b_0 \neq 0$), i.e., the delay comes in x and y′, and the differential equation also takes the form:

$$a_0 y'(t) + a_1 y'(t - \tau(t)) + b_0 y(t) + b_1 y(t - \tau(t)) = g(t) \qquad \ldots(7)$$

Remark (3.1):

The main different between delay and ordinary differential equations is the kind of initial conditions that should be used in delay differential equations which are different from differential equations so that one should specify in delay differential equations an initial function on some interval of length τ say $[y_0 - \tau, y_0]$ and then try to find the solution of eq.(4) for all $y \geq y_0$, [Ladde, G.S.,1987].

4- TONTI'S APPROACH, [TONTI, 1984]

Tonti in 1984 give the variational formulation for every linear or nonlinear equation, with ordinary or partial derivatives, of any order (odd or even).This equation may be of an integral or integro-differential type, or it may even be a system of differential or integral equations, etc.

Now, in order to find the variational formulation corresponding to Tonti's approach we follow the following steps:

Step (1):

Find an integral operator k of the form $ku(x) = \int_a^b k(x,t)u(t)dt$ that transform the given problem into another problem, with the following two conditions must be satisfied:

(1) The integral operator must be invertible to ensure that the new problem has the same solution as the original one.

(2) The operator of the new problem must be a potential operator with respect to the bilinear form.

Step (2):

Find a functional F, such that the operator of the new problem is the gradient of F.

Now the procedure that Tonti had used to find the variational formulation for every nonlinear problem is given in the following theorem:

Theorem (4.1), [Tonti, 1984]:

Consider the nonlinear problem:

$$N(u) = 0 \qquad \ldots(8)$$

where $N : D(N) \subset U \longrightarrow R(N) \subset V$ is a nonlinear operator such that:

1- The solution of this problem exists and unique.

2- D(N) is simply connected.

3- $N'_u(u,.)$ exists.

4-D(N'_u) is dense in U.

5- $N'_u * (u,.)$ is invertible for every $u \in D(N)$.

Then for every operator K that satisfy the following conditions:

6- K is linear with $D(k) \supset R(N)$ and $R(k) \subset D(N'^n_u)$.

7- K is invertible.

8- K is symmetric.

Then the operator $\overline{N}$ defined by:

$$\overline{N}(u) = N^*_u(u, kN(u)) \qquad \ldots(9)$$

has the following properties.

a- Its domain coincides with that of N.

b-The problems N(u) = 0 and $\overline{N}$ (u) = 0 have the same solution;

c- It is a potential operator.

From properties (b) and (c) it follows that the solution of problem (8) is the critical point of the functional:

$$\overline{F}[u] = \frac{1}{2} < N(u), KN(u) >$$

Whose gradient is the operator $\overline{N}$. The functional vanishes when the solution is reached. Moreover if:

9- k is positive definite, then

d- $\overline{F}[u]$ has its minimum value at the critical point.

5- VARIATIONAL FORMULATION FOR NONLINEAR ORDINARY DELAY DIFFERENTIAL EQUATIONS

Now, we use Tontie's approach to find the variational formulation of nonlinear ordinary delay differential equations, which is as follows:

Now, consider the nonlinear delay differential equation which single constant delay τ, as:

$$\frac{d}{dt}u(t) = F(t, u(t), u(t-\tau)) \quad , \quad t \geq t_0 \qquad \ldots(10)$$

with initial condition

$$u(t) = g(t) \, , \quad t_0 - \tau \leq t \leq t_0 \qquad \ldots(11)$$

where f is a nonlinear function and to solve this problem for all $t \geq t_0$ using the method of stops in connection with Tontie's approach given by theorem (4.1).

In eq.(1) we have $n \in U = C^1[0,T]$ and $f \in V = C[0,T]$, and as a first step we must check if the given operator satisfies the conditions of theorem (4.1).

Now, let:

$$N(u) = \left\{ \frac{d}{dt}u(t) - f(t, u(t), u(t-\tau), u(t) = g(t), t_0 - \tau \leq t \leq t_0, u(t_0) = g(t_0), u \in C^1[0,T] \right\}$$

$$= \left\{ \frac{d}{dt}u(t) - f(t, u(t)), g(t-\tau), u(t_0) = g(t_0) \right\}$$

and hence:

$$= \left\{ \frac{d}{dt}u(t) - f(t, u(t), u(t_0) = g(t_0), u \in C^1[0,T] \right\}$$

$$= N'_u\phi = \frac{d}{d\varepsilon}[N(u+\varepsilon\phi)] \Big|_{\varepsilon=0}$$

$$= \frac{d}{d\varepsilon}\left[\frac{d}{dt}\{(u+\varepsilon\phi)(t) - f(t, (u+\varepsilon\phi)(t), (u+\varepsilon\phi)(t-\tau)\} \right]_{\varepsilon=0}$$

and for using the method of steps for solving delay differential equations and for the first time step $t_0 \le t \le t_0 + \tau$, we have:

$$u(t-\tau) = g(t), \quad t_0 \le t \le t_0 + \tau$$

and we may let $(u+\varepsilon\phi)(t-\tau) = g(t)$, with error of order O(ε). Hence:

$$N'_u\phi = \frac{d}{d\varepsilon}\left[\frac{d}{dt}\left[(u+\varepsilon\phi)(t) - f(t,(u+\varepsilon\phi)(t), g(t))\right]\right]$$

$$= \frac{d}{d\in}\left[\frac{d}{dt}\left[(u+\varepsilon\phi)(t) - f(t, u+\in\phi)(t)\right]\right]_{\varepsilon=0}$$

$$= \frac{d}{dt}\phi(t) - \frac{\partial F}{\partial u}\phi(t)$$

Hence:

$$N'_u\phi = \left\{\frac{d}{dt}\phi(t) - \frac{\partial F}{\partial u}\phi(t), \phi(0) = 0, \phi \in C^1[0,T]\right\}$$

Now, in order to find the ad joint Gateaux derivative, we must satisfy:

$$\int_0^T \psi N'_u\phi\, dt = \int_0^T \phi N'^*_u\psi\, dt, \ \forall\ \phi, \psi \in U$$

or equivalently:

$$\int_0^T \psi N'_u\phi\ dt = \int_0^T \psi\left(\frac{d}{dt}\phi(t) - \frac{\partial F}{\partial u}\phi(t)\right) dt \qquad \ldots(12)$$

$$\int_0^T \psi(t)\frac{d}{dt}\phi(t)\, dt - \int_0^T \psi(t)\frac{\partial F}{\partial u}\phi(t)\, dt = 0$$

and using the method of integration by parts to the both integrals in the right hand side of the last equation, we get:

$$N'^*_u\psi = -\frac{d}{dt}\psi(t) - \frac{\partial F}{\partial u}\psi(t)$$

Hence:

$$N'^*_u\psi = \left\{-\frac{d}{dt}\psi(t) - \frac{\partial F}{\partial u}\psi(t), \psi(t) = 0, \psi \in C^1[0,T]\right\}$$

Then, the ad joint homogeneous problem is given by

$$-\frac{d}{dt}\psi(t)-\frac{\partial f}{\partial u}\psi(t)=0,\psi(T)=0$$

which is a linear equitation. With variable coefficients and can by solved to have the null solution, and thus the operator $N_u'^*$ is invertible. Therefore conditions (1)-(6) of theorem (3.1) are satisfied.

Also, choose the integral operator to be:

$$KV=\int_0^T e^{st}\phi(t)\phi(s)V(s)\,ds$$

where ϕ satisfy the homogeneous initial conditions for $D(N_u'^*)$, which is$\phi(T) = 0$ and $u(0) = 0$ is already satisfied with the operator N, hence $\phi(0) = 0$. Therefore, letting $\phi(t) = t^2 - Tt$, yields:

$$KV=\int_0^T e^{st}(t^2-Tt)(s^2-Ts)V(s)ds$$

and hence from theorem (3.1), we have the functional:

$$\overline{F}(u)=\frac{1}{2}\langle N(u),KN(u)\rangle$$

$$=\frac{1}{2}\int_0^T\left(\frac{du}{dt}-F(t,u)\right)\int_0^T e^{st}(t^2-Tt)(s^2-Ts)\left(\frac{du}{ds}-F(s,u(s)\right)dsdt \qquad \ldots(13)$$

which can be minimized using the direct Ritz method to find the critical points of eq.(13) which are equivalent to the solution of the nonlinear ordinary delay differential equation given by eq.(10) with initial condition (11).

As an illustration consider the following example:

Example (5.1):

Consider the nonlinear delay differential equation:

$$\frac{du}{dt}=u^2(t)+u(t-1),t\in[0,1],u(t)\in U=C^1[-1,1]$$

with initial condition:

$$u(t)=g(t)=t \quad ,-1\le t\le 0$$

which can be solved for the first time step [0, 1] and by using the method of steps, we have:

$$\frac{du(t)}{dt} = u^2(t) + u(t-1)$$

$$= u^2(t) + g(t-1)$$

$$= u^2(t) + t - 1$$

$$= F(t, u(t))$$

Hence, using the variational formulation (13), we have the functional:

$$\bar{F}(u) = \frac{1}{2}\int_0^1 \left(\frac{du}{dt} - F(t,u) \int_0^1 e^{ts}(t^2-t)(s^2-s)\left(\frac{du}{ds} - F(s,u(s)) \right) dsdt \right)$$

$$= \frac{1}{2}\int_0^1 \left(\frac{du}{dt} - u^2(t) - t + 1 \right) \int_0^1 e^{ts}(t^2-t)(s^2-s)\left(\frac{du}{ds} - u^2(s) - s + 1 \right) dsdt \quad \ldots(14)$$

and by using the direct Ritz method, let:

$$u(t) = a_1 + a_2 t + a_3 t^2$$

and since u(0), then $a_1 = 0$ and hence:

$$u(t) = a_2 t + a_3 t^2 \quad \ldots(15)$$

Thus, substitute (115) in (14), yields to:

$$\bar{F}(a_2, a_3) = \frac{1}{2}\int_0^1 \left[a_2 + 2a_3 t - (a_2 t + a_3 t^2)^2 - t + 1 \right] \int_0^1 [e^{ts}(t^2-t)(s^2-s)(a_2 + 2a_3 s -$$

$$(a_2 s + a_3 s^2)^2 - s + 1)]\ dsdt$$

which may be minimized and find the critical points a_2^*, a_3^* for the function which are found to be:

$$a_2^* = -0.997,\ a_3^* = 0.607$$

Therefore:

$$u(t) = -0.997t + 0.607t^2 \quad \ldots(16)$$

which is the approximate solution of the problem.

A comparison between the approximate solution and Euler numerical method is given in figure (1).

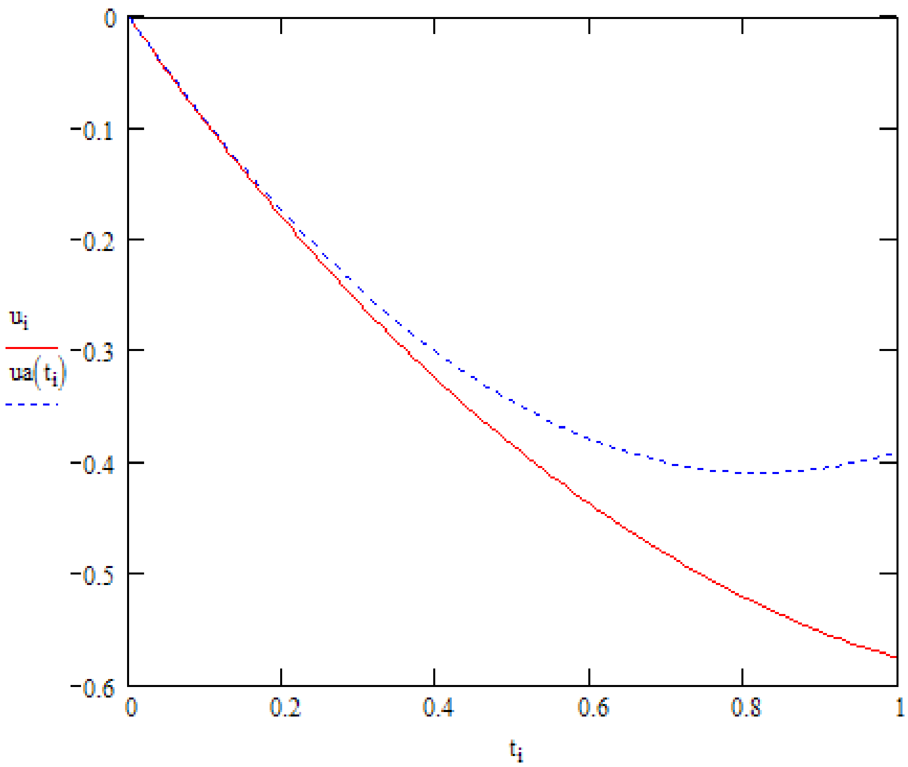

Figure (1) Approximate and Numerical Solutions of Example (5.1).

From the results, one can see the accuracy of the results, which may occur due to the application of Euler's method which is of the first order and/or due to the nonlinearity of the delay differential equations.

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الصياغة التغايرية للمعادلات التفاضلية التباطؤية الغير خطية

زينب شهاب حامد* د.فاضل صبحي فاضل**
*قسم الرياضيات ,كلية العلوم , الجامعة المستنصرية , بغداد -العراق
**قسم الرياضيات , كلية العلوم , جامعة النهرين , بغداد -العراق

المستخلص:

في هذا البحث تم ايجاد الصياغة التغايرية للمعادلات التفاضلية التباطؤية الغير خطية ومن ثم ايجاد الحل التقريبي لها موضحين ذلك بمثال توضيحي.