



Available online at [www.qu.edu.iq/journalcm](http://www.qu.edu.iq/journalcm)

JOURNAL OF AL-QADISIYAH FOR COMPUTER SCIENCE AND MATHEMATICS

ISSN:2521-3504(online) ISSN:2074-0204(print)



## On The Higher-Order Pantograph Type Delay Differential Equation Via Orthonormal Bernstein Polynomials

<sup>a</sup> *Hassan Hamad J. AL-Nasrawy* , <sup>b</sup> *Abdul Khaleq O. Al-Jubor* <sup>c</sup> and *Kasim Abbas Hussain*<sup>c</sup>

<sup>a</sup> *University Mustansiriyah, College of Science, Department of Mathematics, Baghdad-Iraq, Email : has20.aln@gmail.com*

<sup>b</sup> *University Mustansiriyah, College of Science, Department of Mathematics, Baghdad-Iraq, Email: khaleqmazeel@gmail.com*

<sup>c</sup> *University Mustansiriyah, College of Science, Department of Mathematics, Baghdad-Iraq, Email: kasimmath@gmail.com*

### ARTICLE INFO

#### Article history:

Received: 20/05/2020

Rrevised form: //

Accepted :14 /06/2020

Available online: 10/09/2020

#### Keywords:

Orthonormal Bernstein; Delay generalized pantograph equations, Operational Matrix of Derivative.

### ABSTRACT

In this study, a collocation method based on the truncated orthonormal Bernstein polynomial was adopted to obtain an approximate solution for the generalized pantograph equations with proportional delay under some initial conditions. The adopted method is based on transforming both of the generalized pantograph equations and the adopted conditions into matrix equation which corresponds to a system of linear algebraic equation. The reliability and efficiency of the proposed scheme are demonstrated by some numerical experiments and then validated by Math lab 15 package

MSC : 30C45 , 30C50

DOI : [https:// DOI 10.29304/jqcm.2020.12.3.704](https://doi.org/10.29304/jqcm.2020.12.3.704)

### 1 . Introduction

For many years, the subject of functional equations has been held a prominent place in the attention of mathematicians. Recently, this attention has been directed to a particular kind of functional equation, such as delay equation. Delay differential equation have a wide range of applications in the science such as number theory , economy, electrodynamics, physics, biology, control and engineering. Functional- differential equations with proportionate delays are usually referred as pantograph equation or generalized pantograph equation [1]. Pantograph equation allows mathematicians and engineers better modeling of a wide class of systems with anomalous dynamic behavior and give better understanding of the facets of both physical phenomena and

Corresponding author : *Hassan Hamad J. AL-Nasrawy*

Email address: *has20.aln@gmail.com*

Communicated by : *Alaa Hussein . H. Al-Ka'bi*

artificial processes. The pantograph equation is a kind of delay differential equations. That are difficult to solve analytically, they are solved by using the approximate methods.

In this study, a new orthonormal Bernstein collocation method was developed to find numerically solutions for the generalization pantograph equations with variable coefficients under the initial conditions. These solutions was obtained in terms of orthonormal Bernstein polynomial.

$$y^{(m)}(t) = \sum_{j=0}^J \sum_{k=0}^{m-1} P_{jk}(t) y^{(k)}(\lambda_{jk}x + \mu_{jk}) + g(t), t \in [0,1] \quad \dots 1$$

with an initial condition

$$y^{(l)}(0) = \lambda_i, i = 0,1, \dots, m - 1 \quad \dots 2$$

Where  $\lambda_i, \lambda_{jk}$  and  $\mu_{jk}$  are real or complex coefficients, while  $P_{jk}(t)$  and  $g(t)$  are given continuous function in the interval  $0 \leq t \leq 1$ .

However, several numbers of algorithms for solving generalization pantograph equation have been investigated. Ayşe Kurt, Salih Y. and Mehmet S.[2], investigated a new method for the solution of  $m$ th-order differential-difference equations with variable coefficients under mixed conditions. Iclall Y., Omur K. and Mehmet S.[3], adopted a numerical method to solve pantograph type functional differential equations with mixed delays under the initial conditions. Abdulnasir I., Chang P., Piau P. [4], utilized from an operational matrix for solving generalized fractional pantograph equations with an initial and boundary conditions. Salih Y., Muge A. and Mehmet S. [5], employed Hermite polynomials for approximate solution of pantograph equations with variable coefficients. I.Ahmad and A.Mukhtar[6], introduced a computational technique to solve the first order multi-pantograph differential equation.

## 2. A new Formulation of Orthonormal Bernstein Polynomials:-

Orthonormal Bernstein polynomials play an important role in different areas of mathematics, including engineering and physics.

### 2.1. The Derivative for Orthonormal Bernstein Polynomials

The Bernstein polynomials of  $n$ th degree are defined on the interval  $[0, 1]$  as [7].

$$B_{i,n}(t) = \binom{n}{i} t^i (1-t)^{n-i} \text{ for } i = 0,1,2,\dots, n$$

the set  $\{ B_{i,n}(t) \}_{i=0}^n$  in Hilbert space  $L^2[0,1]$  is complete basis polynomials .Therefore, any polynomial of degree n can be expanded in terms of linear combination of  $B_{i,n}(t)$ ,  $i = 0,1, \dots, n$  as :  $B_n(t) = \sum_{i=0}^n C_i B_{i,n}(t)$

The representation of the orthonormal Bernstein Polynomials, denoted by  $OB_{i,n}$

$$OB_{j,n}(t) = \sqrt{2(n-j)+1}(1-t)^{n-j} \sum_{k=0}^j (-1)^k \binom{2n+1-k}{j-k} \binom{j}{k} t^{j-k} \quad \dots 3$$

where  $j=0,1,\dots, n$ , Then the following sets of orthonormal polynomials  $OB_{i,n}(t)$ ,  $0 \leq i \leq n$ . For  $n = 5$ , the six orthonormal Bernstein polynomials are given as:

$$OB_{0,5}(t) = \sqrt{11} (1 - t)^5$$

$$OB_{1,5}(t) = \sqrt{9}(1 - t)^4(11t - 1)$$

$$OB_{2,5}(t) = \sqrt{7}(1 - t)^3(55t^2 - 20t + 1)$$

$$OB_{3,5}(t) = \sqrt{5}(1 - t)^2(165t^3 - 135t^2 + 27t - 1)$$

$$OB_{4,5}(t) = \sqrt{3}(1 - t)(330t^4 - 480t^3 + 210t^2 - 32t + 1)$$

$$OB_{5,5}(t) = 462t^5 - 1050t^4 + 840t^3 - 280t^2 + 35t - 1$$

## 2.2 Orthonormal Bernstein Polynomials Operational Matrix of Derivative

In this sub section , an explicit formula for orthonormal Bernstein polynomials of nth -degree operational matrix of differentiation has been obtained as follows :

$$\frac{d}{dx} OB_n(t) = M OB_n(t) \text{ wheret } t \in [0,1] \text{ and } OB_n(t) = \{OB_{0n}(t), OB_{1n}(t), \dots , OB_{nn}(t)\}^T$$

and matrix M is of order  $(n+1) \times (n+1)$ .

Also , the orthonormal Bernstein polynomials was obtained in terms of Taylor basis as follows

$$OB_n(t) = N X(t), t \in [ 0 , 1] \Rightarrow X(t) = N^{-1}OB_n(t) \quad \dots 4$$

Where

$$N_{i,j} = \sqrt{2(n-i)+1} \sum_{k=\max\{0,i+j-n\}}^{\min\{i,j\}} (-1)^{i+j-2k} \binom{n-i}{j-k} \binom{2n+1-i+k}{k} \binom{i}{i-k} \quad i, j = 0, \dots, n,$$

and

$X(t) = [1, t, t^2, \dots, t^n]^T$ . Then

$$\frac{d}{dx} OB_n(t) = N Z X(t), \quad \text{where} \quad Z = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 2 & 0 & \dots & 0 & 0 \\ 0 & 0 & 3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & n & 0 \end{bmatrix}$$

By using eq.(4) we obtain,  $\frac{d}{dx} OB_n(t) = N Z N^{-1} OB_n(t)$

hence the operational matrix of differentiation as has been obtained  $M = N Z N^{-1}$

### 3.Function Method Approximation:

In this method, the collocation method based on orthonormal Bernstein operation matrix of derivative to solve the generalized pantograph equation numerically has been adopted . the new derivation algorithm for solving eq.(1) and eq.(2). To do this, let the solution of eq.(1) is approximated orthonormal Bernstein polynomial. Hence we write

$$y_N(t) \cong \sum_{n=0}^N a_n OB_n(t) = OB(t)A \tag{5}$$

where N is order orthonormal Bernstein polynomial

And orthonormal coefficient vector A and orthonormal vector OB(t) are given by

$$A^T [a_0, a_1, \dots, a_N] \quad OB(t) = [OB_0(t), OB_1(t), \dots, OB_N(t)] \tag{6}$$

Then *k*th derivative of  $y_N(t)$  can be expressed in the matrix from by

$$y_N^{(k)}(t) = OB^{(k)}(t)A \tag{7}$$

$$OB'(t)^T = MOB(t)^T \Rightarrow OB^{(1)}(t) = OB(t)M^T$$

$$OB^{(2)}(t) = OB^{(1)}(t)M^T = OB(t)(M^T)^2$$

$$OB^{(3)}(t) = OB^{(2)}(t)M^T = OB(t)(M^T)^3$$

⋮

$$OB^{(k)}(t) = OB(t)(M^T)^k \tag{8}$$

By use eq.(8) and (7) yields

$$y_N^{(k)}(t) = OB(t)(M^T)^k A \tag{9}$$

By substituting the approximations eq.(5) into eq.(1), we get

$$OB(t)(M^T)^m A = \sum_{j=0}^J \sum_{k=0}^{m-1} P_{jk}(t) OB(\lambda_{jk}t + \mu_{jk})(M^T)^k A + g(t), \tag{10}$$

To evaluate the solution  $y_N(t)$ , we first collocate eq.(10) depend on collocation point  $t_i = \frac{i}{N-m}$ ,  $i = 0, 1, \dots, N - m$  Yield

$$OB(t_i)(M^T)^m A = \sum_{j=0}^J \sum_{k=0}^{m-1} P_{jk}(t_i) OB(\lambda_{jk}t + \mu_{jk})(M^T)^k A + g(t_i), i = 0, 1, \dots, N - m \quad \dots 11$$

Where  $OB(\lambda_{jk}t + \mu_{jk}) = [OB_0(\lambda_{jk}t + \mu_{jk}), OB_1(\lambda_{jk}t + \mu_{jk}), \dots, OB_N(\lambda_{jk}t + \mu_{jk})]$

The matrix system eq.(11) can be rewritten as follows

$$\{D(M^T)^m - \sum_{j=0}^J \sum_{k=0}^{m-1} P_{jk}E(\lambda_{jk}, \mu_{jk})(M^T)^k\} A = G \quad \dots 12$$

$$\text{Where } D = \begin{bmatrix} OB(t_0) \\ OB(t_1) \\ \vdots \\ OB(t_{N-m}) \end{bmatrix} = \begin{bmatrix} OB_0(t_0) & OB_1(t_0) & \dots & OB_N(t_0) \\ OB_0(t_1) & OB_1(t_1) & \dots & OB_N(t_1) \\ \vdots & \vdots & \ddots & \vdots \\ OB_0(t_{N-m}) & OB_1(t_{N-m}) & \dots & OB_N(t_{N-m}) \end{bmatrix}$$

$$E(\lambda_{jk}, \mu_{jk}) = \begin{bmatrix} OB(\lambda_{jk}t_0 + \mu_{jk}) \\ OB(\lambda_{jk}t_1 + \mu_{jk}) \\ \vdots \\ OB(\lambda_{jk}t_{N-m} + \mu_{jk}) \end{bmatrix} = \begin{bmatrix} OB_0(\lambda_{jk}t_0 + \mu_{jk}) & OB_1(\lambda_{jk}t_0 + \mu_{jk}) & \dots & OB_N(\lambda_{jk}t_0 + \mu_{jk}) \\ OB_0(\lambda_{jk}t_1 + \mu_{jk}) & OB_1(\lambda_{jk}t_1 + \mu_{jk}) & \dots & OB_N(\lambda_{jk}t_1 + \mu_{jk}) \\ \vdots & \vdots & \ddots & \vdots \\ OB_0(\lambda_{jk}t_{N-m} + \mu_{jk}) & OB_1(\lambda_{jk}t_{N-m} + \mu_{jk}) & \dots & OB_N(\lambda_{jk}t_{N-m} + \mu_{jk}) \end{bmatrix}$$

$$\text{and } P_{jk} = \begin{bmatrix} P_{jk}(t_0) & 0 & 0 & \dots & 0 \\ 0 & P_{jk}(t_1) & 0 & \dots & 0 \\ 0 & 0 & P_{jk}(t_2) & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & P_{jk}(t_{N-m}) \end{bmatrix}, G = \begin{bmatrix} g(t_0) \\ g(t_1) \\ g(t_2) \\ \vdots \\ g(t_{N-m}) \end{bmatrix}$$

Hence, the matrix eq.(12) corresponding to the basic eq.(1), can be written as  $W A = G$

$$\text{Where, } W = [W_{ij}] = \{D(M^T)^m - \sum_{j=0}^J \sum_{k=0}^{m-1} P_{jk}E(\lambda_{jk}, \mu_{jk})(M^T)^k\} \quad \dots 13$$

Therefore, the basic eq.(1) is changed into a system of  $(N - m + 1)$  linear algebraic equation with unknown coefficients  $a_0, a_1, \dots, a_n$  which can be written in the form :  $W A = G$ , or in augmented matrix form:

$$[W: G] = \begin{bmatrix} W_{00} & W_{01} & W_{02} & \dots & W_{0N} & \vdots & g(t_0) \\ W_{10} & W_{11} & W_{12} & \dots & W_{1N} & \vdots & g(t_1) \\ W_{20} & W_{21} & W_{22} & \dots & W_{2N} & \vdots & g(t_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ W_{(N-m)0} & W_{(N-m)1} & W_{(N-m)2} & \dots & W_{(N-m)N} & \vdots & g(t_{N-m}) \end{bmatrix} \quad \dots 14$$

Next, eq.(2), after using eq.(5) and eq.(7) at  $t = 0$ , can be written

$$OB_n(0)(M^T)^i A = [\lambda_i], i=0,1,\dots,m-1. \quad \dots 15$$

therefor the matrix from of the initial conditions can be written as

$$U_i A = [\lambda_i] \text{ or } [U_i; \lambda_i], \quad i = 0, 1, \dots, m - 1. \tag{16}$$

Where

$$U_i = OB_n(0)(M^T)^i A = [u_{i0}, u_{i1}, \dots, u_{iN}], \quad i = 0, 1, \dots, m - 1$$

Finally, from (13) and (16), the generalized pantograph eq.(1) subject to (2) reduce to the following system of algebraic equations

$$\tilde{W}A = \check{G}$$

Where

$$[\tilde{W}; \check{G}] = \begin{bmatrix} w_{00} & w_{01} & w_{02} & \dots & w_{0N} & \vdots & g(t_0) \\ w_{10} & w_{11} & w_{12} & \dots & w_{1N} & \vdots & g(t_1) \\ w_{20} & w_{21} & w_{22} & \dots & w_{2N} & \vdots & g(t_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ w_{(N-m)0} & w_{(N-m)1} & w_{(N-m)2} & \dots & w_{(N-m)N} & \vdots & g(t_{N-m}) \\ u_{00} & u_{01} & u_{02} & \dots & u_{0N} & \vdots & \lambda_0 \\ u_{10} & u_{11} & u_{12} & \dots & u_{1N} & \vdots & \lambda_1 \\ u_{20} & u_{21} & u_{22} & \dots & u_{2N} & \vdots & \lambda_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ u_{(m-1)0} & u_{(m-1)1} & u_{(m-1)2} & \dots & u_{(m-1)N} & \vdots & \lambda_{(N-m)} \end{bmatrix} \tag{17}$$

If  $rank \tilde{W} = rank [\tilde{W}; \check{G}] = N + 1$ , then we can write  $A = (\tilde{W})^{-1} \check{G}$ . Thus, the matrix  $A$  (thereby the coefficients  $a_0, a_1, \dots, a_N$ ) is uniquely determined. Also the eq.(1) with conditions eq.(2) has a unique solution. This solution is given by orthonormal Bernstein polynomial solution eq.(5). On the other hand, when  $|\tilde{W}| = 0$ , if  $rank \tilde{W} = rank [\tilde{W}; \check{G}] < N + 1$ , then we may find a particular solution. Otherwise if  $rank \tilde{W} \neq rank [\tilde{W}; \check{G}] < N + 1$ , then it is not a solution.

### 4. Illustrative Examples

In this section, several numerical examples are given to show the accuracy and the efficiency of properties of this method

**Example1:** Consider the first order pantograph equation  $y'(t) = \frac{1}{2} y(t) + \frac{1}{2} e^{t/2} y\left(\frac{t}{2}\right)$ ,  $y(0) = 1 \quad 0 \leq t \leq 1$ , with exact solution  $y(t) = e^t$ , we assume that the problem has orthonormal Bernstein polynomial solution in the form  $y_5(t) = \sum_{n=0}^5 a_n OB_n(t) = AOB_n(t)$ , where  $N = 5, P_{00}(t) = \frac{1}{2}, P_{10}(t) = \frac{1}{2} e^{\frac{t}{2}}$ ,

where  $A = [a_0 \ a_1 \ a_2 \ a_3 \ a_4 \ a_5]^T$ ,  $OB_5 = [OB_{05} \ OB_{15} \ OB_{25} \ OB_{35} \ OB_{45} \ OB_{55}]$

$$OB'_5(t) = \begin{bmatrix} -5.5000 & -0.5528 & 0 & 0 & 0 & 0 \\ 10.5026 & -4.5000 & -1.2500 & 0 & 0 & 0 \\ -8.7750 & 9.1971 & -3.50000 & -2.2819 & 0 & 0 \\ 7.4162 & -6.7082 & 8.1980 & -2.5000 & -4.1314 & 0 \\ -5.7446 & 5.1962 & -4.58260 & 8.0042 & -1.5000 & -10.1036 \\ 3.3166 & -3.0000 & 2.6458 & -2.2361 & 11.8357 & 17.5000 \end{bmatrix} \begin{bmatrix} OB(t)_{0.5} \\ OB(t)_{1.5} \\ OB(t)_{2.5} \\ OB(t)_{3.5} \\ OB(t)_{4.5} \\ OB(t)_{5.5} \end{bmatrix}$$

Where  $M$  the  $(6 \times 6)$  operation matrix of derivative according derivative  $OB_5(t)$

$$OB'_5(t)^T = MOB_5^T(t) \Rightarrow OB'_5(t) = OB_5(t)M^T$$

$$y^{(k)}(t) = OB^{(k)}(t)A$$

$$y'_5(t) = OB'_5(t)A = OB_5(t)M^T A$$

$$y^{(m)}(t) = \sum_{j=0}^J \sum_{k=0}^{m-1} P_{jk}(t)y^{(k)}(\beta_{jk}t + \mu_{jk}) + g(t)$$

$$OB_5(t)M^T A = \sum_{j=0}^J \sum_{k=0}^0 P_{jk}(t)OB_5(\beta_{jk}t + \mu_{jk})(M^T)^k A + g(t)$$

To find the solution of eq.(1). we first collocate eq.(7) at the collocation points

$$t_i = \frac{i}{N-m} = \frac{i}{5-1} = \frac{i}{4}, \text{ where } t_i = [0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1], i = 0, 1, 2, 3, 4$$

$$OB_5(t_i)M^T A = \sum_{j=0}^J \sum_{k=0}^0 P_{jk}(t_i)OB_5(\beta_{jk}t_i + \mu_{jk})(M^T)^k A + g(t_i)$$

$$OB_5(t_i)M^T A - \sum_{j=0}^J \sum_{k=0}^0 P_{jk}(t_i)OB_5(\beta_{jk}t_i + \mu_{jk})(M^T)^k A = g(t_i)$$

The matrix system can be rewritten as follows

$$\{DM^T - \sum_{j=0}^J \sum_{k=0}^0 p_{jk}E(\beta_{jk}, \mu_{jk})(M^T)^k\}A = G$$

$$D = \begin{bmatrix} 3.3166 & -3.000 & 2.6458 & -2.2361 & 1.7321 & -1.0000 \\ 0.7871 & 1.6611 & -0.6278 & -0.1376 & 0.3755 & -0.2754 \\ 0.1036 & 0.8438 & 1.5709 & -0.3464 & -0.3248 & 0.3125 \\ 0.0032 & 0.0850 & 0.7002 & 1.8059 & 0.1793 & -0.4668 \\ 0 & 0 & 0 & 0 & 0 & 6.0000 \end{bmatrix}$$

$$L_1 = DM^T = \begin{bmatrix} -16.5831 & 45.000 & 60.8523 & 64.8460 & -57.1577 & 35.0000 \\ -5.2470 & 1.5820 & 10.8827 & -11.6607 & 8.1055 & -4.1016 \\ -1.0346 & -4.6875 & 2.1497 & 10.2021 & -8.8768 & 4.3750 \\ -0.0648 & -1.2305 & -6.8185 & -0.0611 & 16.1162 & -8.4766 \\ 0 & 0 & 0 & 0 & -60.6218 & 105.00 \end{bmatrix}$$

$$\text{If } \beta_{00} = 1 \quad E(\beta_{00}, \mu) = \begin{bmatrix} 3.3166 & -3.000 & 2.6458 & -2.2361 & 1.7321 & -1.0000 \\ 0.7871 & 1.6611 & -0.6278 & -0.1376 & 0.3755 & -0.2754 \\ 0.1036 & 0.8438 & 1.5709 & -0.3464 & -0.3248 & 0.3125 \\ 0.0032 & 0.0850 & 0.7002 & 1.8059 & 0.1793 & -0.4668 \\ 0 & 0 & 0 & 0 & 0 & 6.0000 \end{bmatrix}$$

$$\text{If } \beta_{10} = \frac{1}{2} \quad E(\beta_{10}, \mu) = \begin{bmatrix} 3.3166 & -3.0000 & 2.6458 & -2.2361 & 1.7321 & -1.0000 \\ 1.7011 & 0.6595 & -1.1355 & 1.0065 & -0.7304 & 0.3984 \\ 0.7871 & 1.6611 & -0.6278 & -0.1376 & 0.3755 & -0.2754 \\ 0.3163 & 1.4305 & 0.7973 & -1.0116 & 0.6369 & -0.2912 \\ 0.1036 & 0.8438 & 1.5709 & -0.3494 & -0.3248 & 0.3125 \end{bmatrix}$$

$$P_{00} = \begin{bmatrix} 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0.5 \end{bmatrix} \quad P_{10} = \begin{bmatrix} 0.5000 & 0 & 0 & 0 & 0 \\ 0 & 0.5666 & 0 & 0 & 0 \\ 0 & 0 & 0.6420 & 0 & 0 \\ 0 & 0 & 0 & 0.7275 & 0 \\ 0 & 0 & 0 & 0 & 0.8244 \end{bmatrix}$$

$$L_2 = P_0 E(\beta_{00}, \mu) + P_1 E(\beta_{10}, \mu)$$

$$L_2 = \begin{bmatrix} 3.3166 & -3.000 & 2.6458 & -2.2361 & 1.7321 & -1.0000 \\ 1.3573 & 1.2042 & -0.9573 & 0.5015 & -0.2261 & 0.0880 \\ 0.5571 & 1.4883 & 0.3824 & -0.2630 & 0.0787 & -0.0206 \\ 0.2317 & 1.0832 & 0.9301 & 0.1670 & 0.5530 & -0.4452 \\ 0.0854 & 0.6956 & 1.2950 & 0.2880 & -0.2677 & 3.2576 \end{bmatrix}$$

$$W = L_1 - L_2 = \begin{bmatrix} -19.8997 & 48.000 & -63.498 & 67.0820 & -58.8897 & 36.0000 \\ -6.6043 & 0.3778 & 11.8400 & -12.1622 & 8.3315 & -4.1896 \\ -1.5936 & -6.1758 & 1.7673 & 10.4651 & -8.9555 & 4.3956 \\ -0.2965 & -2.3136 & -6.7487 & -0.2281 & 15.5632 & -8.0313 \\ -0.0854 & -0.6956 & -1.2950 & 0.2880 & -60.3541 & 101.7424 \end{bmatrix}$$

$$G = \begin{bmatrix} g(0) \\ g(\frac{1}{4}) \\ g(\frac{1}{2}) \\ g(\frac{3}{4}) \\ g(1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad WA = G$$

$$U_0 = OB_5(0)(M^T)^0 = [3.3166 \quad -3.000 \quad 2.6458 \quad -2.2361 \quad 1.7321 \quad -1.0000]$$

$$[U_0, \gamma_0] = [3.3166 \quad -3.000 \quad 2.6458 \quad -2.2361 \quad 1.7321 \quad -1.0000 \quad ; 1]$$

$$[\widetilde{W} : \widetilde{G}] = \begin{bmatrix} -19.8997 & 48.000 & -63.498 & 67.0820 & -58.8897 & 36.0000 & : 0 \\ -6.6043 & 0.3778 & 11.8400 & -12.1622 & 8.3315 & -4.1896 & : 0 \\ -1.5936 & -6.1758 & 1.7673 & 10.4651 & -8.9555 & 4.3956 & : 0 \\ -0.2965 & -2.3136 & -6.7487 & -0.2281 & 15.5632 & -8.0313 & : 0 \\ -0.0854 & -0.6956 & -1.2950 & 0.2880 & -60.3541 & 101.7424 & : 0 \\ 3.3166 & -3.000 & 2.6458 & -2.2361 & 1.7321 & -1.0000 & : 1 \end{bmatrix}$$



Then we can get:  $A = \tilde{W}^{-1}\check{G}$  , where  $A = \begin{bmatrix} 0.6428 \\ 0.7669 \\ 0.8471 \\ 0.8502 \\ 0.7399 \\ 0.4530 \end{bmatrix}$

t	Exact solution	Approximate N=5	Error	Approximate N=7	Error	Approximate N=9	Error
0.0	1.0000000	1.0000000	2.808e-013	1.0000000	1.571e-012	1.0000000	2.268e-012
0.1	1.1051709	1.1051674	3.560e-006	1.1051709	8.878e-009	1.1051709	7.891e-011
0.2	1.2214028	1.2213953	7.477e-006	1.2214027	1.175e-008	1.2214028	8.589e-011
0.3	1.3498588	1.3458508	8.237e-006	1.3498588	1.064e-008	1.3498588	8.130e-011
0.4	1.4918247	1.4918174	7.273e-006	1.4918247	1.229e-008	1.4918247	9.253e-011
0.5	1.6487213	1.6487141	7.147e-006	1.6487213	1.467e-008	1.6487213	1.013e-010
0.6	1.8221188	1.8221097	9.075e-006	1.8221188	1.515e-008	1.8221188	1.048e-010
0.7	2.0137527	2.0137407	1.196e-005	2.0137527	1.628e-008	2.0137527	1.273e-010
0.8	2.2255409	2.2255279	1.305e-005	2.2255409	2.035e-008	2.2255409	1.420e-010
0.9	2.4596031	2.4595926	1.048e-005	2.4596031	2.003e-008	2.4596031	1.037e-010
1.0	2.7182818	2.7182741	7.746e-005	2.7182818	1.317e-008	2.7182818	1.814e-010
MSE		7.326e-11		1.992e-016		1.182e-020	
MAE		7.818e-06		1.305e-008		1.001e-010	

Table 1

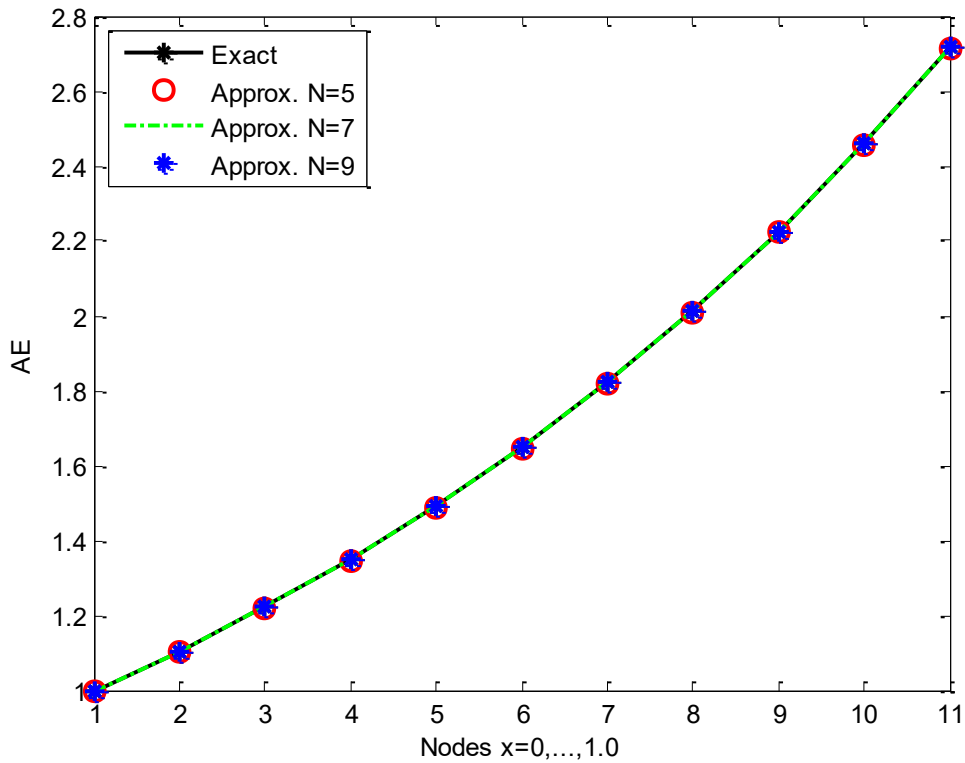


Figure 1

**Example 2:** Let us first consider the pantograph equation of second-order  $y''(t) = \frac{3}{4} y(t) + y\left(\frac{t}{2}\right) - t^2 + 2$ ,  $y(0) = 0$ ,  $y'(0) = 0$ ,  $0 < t < 1$  and the exact solution  $y(t) = t^2$ ,

we assume that the problem has orthonormal Bernstein solution is the  $y(t) = \sum_{n=0}^5 a_n OB_{n5}(t)$ , where  $N = 5$ ,  $m = 2$ ,  $p_{00} = \frac{3}{4}$ ,  $p_{01} = 1$  and  $g(t) = -t^2 + 2$ . The collocation points are computed as  $t_i = \left\{0, \frac{1}{3}, \frac{2}{3} \text{ and } 1\right\}$ , from eq. (9), the fundamental matrix equation of the problem  $\left\{D(M^T)^2 - P_{00}E(1,0)M^T - P_{10}E\left(\frac{1}{2}, 0\right)M^T\right\}A = G$ . Where

$$G = \begin{bmatrix} g(0) \\ g\left(\frac{1}{3}\right) \\ g\left(\frac{2}{3}\right) \\ g(1) \end{bmatrix} = \begin{bmatrix} 2 \\ 1.8889 \\ 1.5556 \\ 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 3.3166 & -3.000 & 2.6458 & -2.2361 & 1.7321 & -1.0000 \\ 0.4368 & 1.5802 & 0.3484 & -0.8834 & 0.7270 & -0.3951 \\ 0.0136 & 0.2346 & 1.1868 & 1.46310 & 0.7912 & 0.2099 \\ 0.0000 & 0.0000 & 0.0000 & 0.00000 & 0.0000 & 6.0000 \end{bmatrix}$$

$$P_{00} = \begin{bmatrix} 0.75 & 0 & 0 & 0 \\ 0 & 0.75 & 0 & 0 \\ 0 & 0 & 0.75 & 0 \\ 0 & 0 & 0 & 0.75 \end{bmatrix}$$

$$P_{10} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, L_1 = \begin{bmatrix} 66.33 & -300.0 & 624.40 & -849.7 & 859.1 & -560.01 \\ 19.65 & -35.56 & -26.65 & 103.4 & -104.7 & 62.221 \\ 2.457 & 15.56 & -19.21 & -76.52 & 132.4 & -62.221 \\ 0.0000 & 0.0000 & 0.0000 & 250.4 & -969.9 & 1120.0 \end{bmatrix}$$

$$L_2 = \begin{bmatrix} 5.8041 & -5.250 & 4.6301 & -3.9131 & 3.0311 & -1.7500 \\ 1.6604 & 2.3908 & -0.9721 & 0.1354 & 0.1109 & -0.1026 \\ 0.4470 & 1.7562 & 1.2385 & 0.2139 & 0.1336 & -0.2377 \\ 0.1036 & 0.4838 & 1.5709 & -0.3494 & -0.3248 & 4.8125 \end{bmatrix}$$

$$W = L_1 - L_2 = \begin{bmatrix} 60.53 & -294.8 & 619.8 & -845.8 & 856.1 & -558.2 \\ 17.99 & -37.95 & -25.68 & 103.2 & -104.8 & 62.32 \\ 2.010 & 13.8 & -20.44 & -76.74 & 132.3 & -61.981 \\ 0.1036 & -0.8438 & -1.571 & 250.8 & -969.6 & 1115.2 \end{bmatrix}$$

$$U_0 = [3.3166 \quad -3.000 \quad 2.6458 \quad -2.2361 \quad 1.7321 \quad -1.000]$$

$$U_1 = [-16.5831 \quad 45 \quad -60.8523 \quad 64.8460 \quad -57.1577 \quad 35.0]$$

$$[\bar{W} : \bar{G}] = \begin{bmatrix} 60.53 & -294.8 & 619.80 & -845.8 & 856.1 & -558.2 & : & 2.0000 \\ 17.99 & -37.95 & -25.680 & 103.2 & -104.8 & 62.32 & : & 1.8889 \\ 2.010 & 13.80 & -20.440 & -76.74 & 132.3 & -61.981 & : & 1.5556 \\ 0.1036 & -0.8438 & -1.571 & 250.8 & -969.60 & 1115.2 & : & 1.0000 \\ 3.3166 & -3.0000 & 26.458 & -2.2361 & 1.7321 & -1.0000 & : & 0.0000 \\ -16.583 & 45.000 & -60.852 & 64.846 & -57.1577 & 35.000 & : & 0.0000 \end{bmatrix}$$

$$A = \bar{W}^{-1}\bar{G}, \text{ Such that, } A = [0.0197 \quad 0.0893 \quad 0.1858 \quad 0.2529 \quad 0.2529 \quad 0.1667]$$

t	Exact solution	Approximate N=5	Error	Approximate N=7	Error	Approximat e N=9	Error
0.0	0.0	0.0000000	3.228e-013	0.0000000	7.651e-014	0.0000000	1.671e-013
0.1	0.01	0.0100000	1.999e-012	0.0100000	2.265e-012	0.0100000	2.442e-011
0.2	0.04	0.0400000	5.523e-012	0.0400000	3.648e-012	0.0400000	6.996e-011
0.3	0.09	0.0900000	1.003e-011	0.0900000	4.823e-012	0.0900000	1.134e-010
0.4	0.16	0.1600000	1.488e-011	0.1600000	6.405e-012	0.1600000	1.572e-010
0.5	0.25	0.2500000	1.919e-011	0.2500000	8.496e-012	0.2500000	2.023e-010
0.6	0.36	0.3600000	2.202e-011	0.3600000	1.079e-011	0.3600000	2.487e-010
0.7	0.49	0.4900000	2.264e-011	0.4900000	1.398e-011	0.4900000	2.947e-010
0.8	0.64	0.6400000	2.059e-011	0.6400000	1.688e-011	0.6400000	3.475e-010
0.9	0.81	0.8100000	1.610e-011	0.8100000	2.014e-011	0.8100000	4.183e-010
1.0	1.00	1.0000000	1.006e-011	1.0000000	2.185e-011	1.0000000	4.645e-010
MSE		2.279e-022		1.486e-022		6.765e-020	
MAE		1.303e-011		9.942e-012		2.128e-010	

Table 2

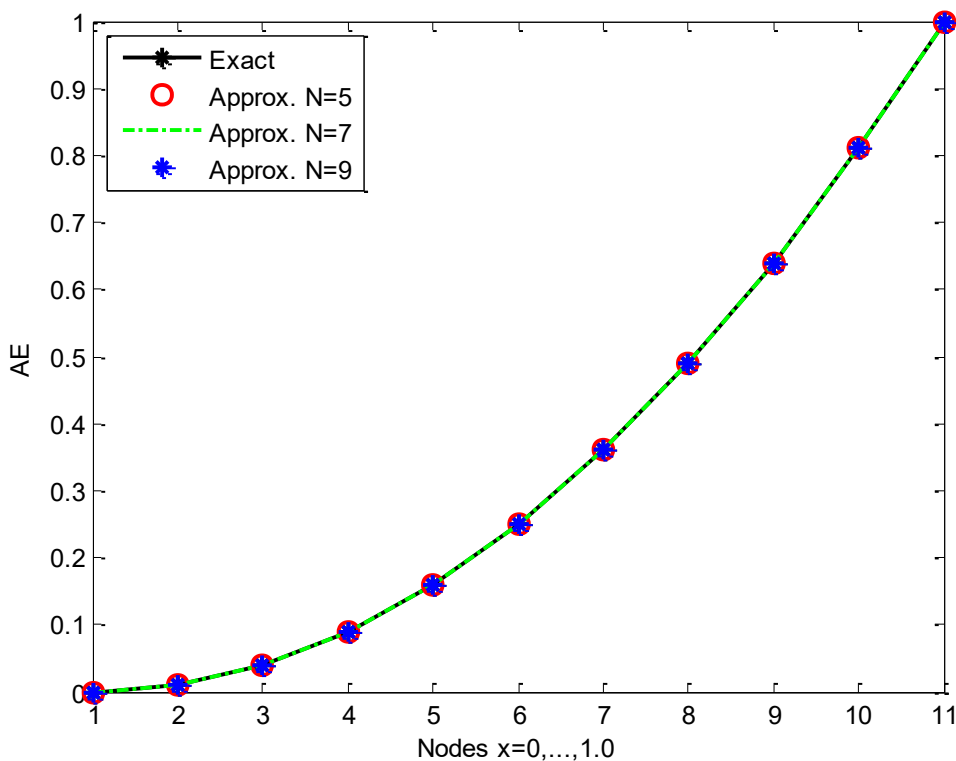


Figure 2

## **5. Conclusions**

1. Delay generalized pantograph equations with variable coefficients are usually difficult to solve analytically, therefore, approximate solutions are required. To have the best approximate solution of the mentioned equation, we take more terms from the orthonormal Bernstein expansion of functions, that is, the accuracy improves when is increased.
2. From the previous tables and figures , it has been concluded that the present method is convenient, reliable and effective. So, it can be said that the orthonormal collocation method can be a suitable method for solving numerical solutions for the delay generalized pantograph equations with variable coefficients.
3. A considerable advantage of the method is that the orthonormal Bernstein polynomial coefficients of the solution are found very easily by using computer programs with matlab15.

## **References**

- [1] J. R. Ockendon and A. B. Tayler," The dynamic of a current collocation system for a an electric locomotive", Journal mathematical and physical sciences,(1971), 447-468.
- [2] A. Kurt, S. Yalçınbaş and M. Sezer," Fibonacci collocation method for solving linear differential-difference equation", Journal mathematical and computational applications, 18(3)(2013),448-458.
- [3] Ö. Kürkçü and M. Sezer," A numerical Approach for solving differential pantograph equation with mixed delays using Dikson polynomials of the second kind",Journal of Science and Arts, 3(44)(2018),667-680.
- [4] A. Isah, C. Phang and P. Phang," collocation method based on Genocchi operational matrix for solving generalized fractional pantograph equations", International Journal of Differential Equations,2017 (2017),1-10.
- [5] S. Yalcinbas, M. Aynigul and M.Sezer," A collocation method using Hermite polynomials for approximate solution of pantograph equations", Journal Science Direct, 348(2011), 1128-1139.
- [6] I. Ahmad and A. Mukhtar," Stochastic approach for the solution of multi-pantograph differential equation arising in cell-growth model", Journal applied mathematics and computation, 261(2015),360-372.

- 
- [7] A. O. AL-JUBORY and S. H Salih," Normalization Bernstein for solving fractional linear Volterra-Integro differential equation" ,International journal of mathematical Archive , 8( 2017) , 1-7.
- [8] M. M. Ali," A new operational matrix of derivative for orthonormal Bernstein polynomial's " , Baghdad Science Journal ,11(3)(2014).