Available online at www.qu.edu.iq/journalcm
JOURNAL OF AL-QADISIYAH FOR COMPUTER SCIENCE AND MATHEMATICS
ISSN:2521-3504(online) ISSN:2074-0204(print)

Some Results for Fractional Derivative Associated with Fuzzy Differential Subordinations

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ARTICLE INFO

Article history:
Received: 05 /01/2020
Rrevised form: 00 /00/0000
Accepted : 06 /02/2020
Available online: 15 /11/2020

Keywords: Fuzzy differential subordination, Fuzzy best dominant, Fractional derivative, Differential operator.

Big data, IT companies, Energy Management, Confusion.

ABSTRACT

In the present article, by making use of fuzzy differential subordination, we establish some interesting results of fractional derivative related to differential operator defined in the open unit disk.

MSC : 30C45,30A20.

DOI : https://doi.org/10.29304/jqcm.2020.12.3.708

1 . Introduction

Let the notation $\mathcal{H}(\mathfrak{U})$ stands for the family of holomorphic functions in the unit disk $\mathfrak{U} = \{z \in \mathbb{C} : |z| < 1\}$. For $n \in \mathbb{N}$ and $a \in \mathbb{C}$, we indicate by

$$\mathcal{H}[a, n] = \{f \in \mathcal{H}(\mathfrak{U}) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots , \ z \in \mathfrak{U}\}$$

and

$$\mathcal{A}_n = \{f \in \mathcal{H}(\mathfrak{U}) : f(z) = z + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \cdots , \ z \in \mathfrak{U}\},$$

with $\mathcal{A}_1 = \mathcal{A}$.

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Communicated by : ***Alaa Hussein Hamadi***

Definition 1.1 [11]. "Let X be a non-empty set. An application $F : X \to [0,1]$ is called fuzzy subset. An alternate definition, more precise, would be the following:

$$\text{A pair } (S, F_S), \text{ where } F_S : X \to [0,1] \text{ and } supp(S, F_S) = \{x \in X : 0 < F_S(x) \le 1\}$$

is called fuzzy subset. The function F_S is called membership function of the fuzzy subset (S, F_S)."

Definition 1.2 [3]. Let two fuzzy subsets of X, (M, F_M) and (N, F_N). "We say that the fuzzy subsets M and N are equal if and only if $F_M(x) = F_N(x), x \in X$ and we denote this by $(M, F_M) = (N, F_N)$. The fuzzy subset (M, F_M) is contained in the fuzzy subset (N, F_N) if and only if $F_M(x) \le F_N(x), x \in X$ and we denote the inclusion relation by $(M, F_M) \subseteq (N, F_N)$".

Assume that D be a set in $\mathbb{C}$ and f, g holomorphic functions. We indicate by

$$f(D) = supp\big(f(D), F_{f(D)}\big) = \big\{f(z) : 0 < F_{f(D)}\big(f(z)\big) \le 1, z \in D\big\}$$

and

$$g(D) = supp\big(g(D), F_{g(D)}\big) = \big\{g(z) : 0 < F_{g(D)}\big(g(z)\big) \le 1, z \in D\big\}.$$

Definition 1.3 [3]. Suppose that D be a set in $\mathbb{C}$, $z_0 \in D$ be a fixed point and let the functions $f, g \in \mathcal{H}(D)$. The function f is named a fuzzy subordinate to g and write $f \prec_F g$ or $f(z) \prec_F g(z)$ if satisfies the following:

1) $f(z_0) = g(z_0)$,
2) $F_{f(D)}\big(f(z)\big) \le F_{g(D)}\big(g(z)\big), z \in D$.

Definition 1.4 [4]. Let h be univalent in $\mathfrak{U}$ and $\psi : \mathbb{C}^3 \times \mathfrak{U} \longrightarrow \mathbb{C}$. If $\mathcal{P}$ is holomorphic in $\mathfrak{U}$ satisfies the fuzzy differential subordination:

$$F_{\psi(\mathbb{C}^3 \times \mathfrak{U})}\big(\psi(\mathcal{P}(z), z\mathcal{P}'(z), z^2\mathcal{P}''(z); z)\big) \le F_{h(\mathfrak{U})}\big(h(z)\big), \tag{1.1}$$

i.e.

$$\psi(\mathcal{P}(z), z\mathcal{P}'(z), z^2\mathcal{P}''(z); z) \prec_F h(z), z \in \mathfrak{U},$$

"then $\mathcal{P}$ is called a fuzzy solution of the fuzzy differential subordination. The univalent function q is called a fuzzy dominant of the fuzzy solutions of the fuzzy differential subordination, or more simple a fuzzy dominant, if $\mathcal{P}(z) \prec_F q(z), z \in \mathfrak{U}$ for all $\mathcal{P}$ satisfying (1.1). A fuzzy dominant $\tilde{q}$ that satisfies $\tilde{q}(z) \prec_F q(z), z \in \mathfrak{U}$ for all fuzzy dominant q of (1.1) is said to be the fuzzy best dominant of (1.1)".

For $f_i \in \mathcal{A}$ $(i = 1,2)$ defined by

$$f_i(z) = z + \sum_{n=2}^{\infty} a_{n,i} z^n \ \ (i = 1,2),$$

the convolution product of f_1 and f_2 is defined by

$$(f_1 * f_2)(z) = z + \sum_{n=2}^{\infty} a_{n,1} a_{n,2} z^n = (f_2 * f_1)(z)$$

Definition 1.5 [6]. The operator $I^{n,m}_{\alpha,\beta,\ell,d}$ is defined by $I^{n,m}_{\alpha,\beta,\ell,d} : \mathcal{A} \longrightarrow \mathcal{A}$

$$I^{n,m}_{\alpha,\beta,\ell,d} f(z) = \mathcal{M}^{m}_{\alpha,\beta,\ell,d}(z) * R^n f(z), \ \ z \in \mathfrak{U},$$

where

$$\mathcal{M}^{m}_{\alpha,\beta,\ell,d}(z) = z + \sum_{n=2}^{\infty} \left[\frac{\ell\big(1 + (\alpha + \beta)(n-1)\big) + d}{\ell\big(1 + \beta(n-1)\big) + d}\right]^m z^n ,$$

and $R^k f(z)$ indicates the Ruscheweyh derivative operator [6] given by

$$R^k f(z) = z + \sum_{n=2}^{\infty} C(k,n) a_n z^n ,$$

where $C(k,n) = \binom{n+k-1}{k}, k, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\},\ \beta \geq \alpha > 0\ , \ell > 0$ and $\ell + d > 0$.

Simple computations show that

$$I_{\alpha,\beta,\ell,d}^{k,m} f(z) = z + \sum_{n=2}^{\infty} \left[\frac{\ell\big(1 + (\alpha+\beta)(n-1)\big) + d}{\ell\big(1 + \beta(n-1)\big) + d} \right]^m \binom{n+k-1}{k} a_n z^n.$$

Definition 1.6 [8]. Let $0 \leq \lambda < 1$, the fractional derivative of order λ of a function f is defined by

$$D_z^{\lambda} f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(t)}{(z-t)^{\lambda}} dt,$$

"where the function f is holomorphic in a simply-connected region of the z-plane containing the origin and the multiplicity of $(z-t)^{-\lambda}$ is removed by requiring $\log(z-t)$ to be real, when $\mathrm{Re}(z-t) > 0$".

Now, in the light of Definition 1.5 and Definition 1.6, we conclude that

$$D_z^{\lambda} I_{\alpha,\beta,\ell,d}^{k,m} f(z) = \frac{1}{\Gamma(2-\lambda)} z^{1-\lambda} + \sum_{n=2}^{\infty} \frac{n\Gamma(k+n)}{\Gamma(n-\lambda+1)\Gamma(k+1)} \left[\frac{\ell\big(1 + (\alpha+\beta)(n-1)\big) + d}{\ell\big(1 + \beta(n-1)\big) + d} \right]^m a_n z^{n-\lambda}. \qquad (1.2)$$

By making use of (1.2), it is evident that

$$z\left(D_z^{\lambda} I_{\alpha,\beta,\ell,d}^{k,m} f(z)\right)' = \frac{\ell\big(1+\beta(n-1)\big)+d}{\ell\alpha} D_z^{\lambda} I_{\alpha,\beta,\ell,d}^{k,m+1} f(z) - \frac{\ell(1+\beta(n-1)-(1-\lambda)\alpha)+d}{\ell\alpha} D_z^{\lambda} I_{\alpha,\beta,\ell,d}^{k,m} f(z). \qquad (1.3)$$

We will need the following lemmas in investigating "our main results".

Lemma 1.1 [5]. Suppose that the convex function h satisfies $h(0) = a$, let $\mu \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ such that $\mathrm{Re}(\mu) \geq 0$. If $\mathcal{P} \in \mathcal{H}[a,n]$ with $\mathcal{P}(0) = a$ and $\psi: \mathbb{C}^2 \times \mathfrak{U} \to \mathbb{C}$, $\psi\big(\mathcal{P}(z), z\mathcal{P}'(z)\big) = \mathcal{P}(z) + \frac{1}{\mu} z\mathcal{P}'(z)$ is holomorphic in $\mathfrak{U}$, then

$$F_{\psi(\mathbb{C}^2\times\mathfrak{U})}\left[\mathcal{P}(z) + \frac{1}{\mu} z\mathcal{P}'(z)\right] \leq F_{h(\mathfrak{U})} h(z),$$

implies

$$F_{\mathcal{P}(\mathfrak{U})} \mathcal{P}(z) \leq F_{q(\mathfrak{U})} q(z) \leq F_{h(\mathfrak{U})} h(z),\ \ z \in \mathfrak{U},$$

i.e.

$$\mathcal{P}(z) \prec_F q(z) \prec_F h(z),$$

where

$$q(z) = \frac{\mu}{n z^{\frac{\mu}{n}}} \int_0^z h(t)\, t^{\frac{\mu}{n}-1} dt$$

is convex and is the fuzzy best dominant.

Lemma 1.2 [5]. Suppose that q be a convex function in $\mathfrak{U}$, let $h(z) = q(z) + n\nu z q'(z), \nu > 0$ and $n \in \mathbb{N}$. If $\mathcal{P} \in \mathcal{H}[q(0), n]$ and $\psi: \mathbb{C}^2 \times \mathfrak{U} \to \mathbb{C}$, $\psi\big(\mathcal{P}(z), z\mathcal{P}'(z)\big) = \mathcal{P}(z) + \nu z\mathcal{P}'(z)$ is holomorphic in $\mathfrak{U}$, then

$$F_{\psi(\mathbb{C}^2\times\mathfrak{U})}[\mathcal{P}(z)+\nu z\mathcal{P}'(z)]\le F_{h(\mathfrak{U})}h(z),$$

implies

$$F_{\mathcal{P}(\mathfrak{U})}\mathcal{P}(z)\le F_{q(\mathfrak{U})}q(z),\ \ z\in\mathfrak{U},$$

i.e.

$$\mathcal{P}(z)\prec_F q(z)$$

and q is the fuzzy best dominant.

"Recently, Oros and Oros [4,5], Magesh [2], Haydar [1] and Wanas and Majeed [9,10] have obtained fuzzy differential subordination results for certain classes of holomorphic functions".

2. Main Results

Theorem 2.1. Suppose that convex function h satisfies $h(0)=1$. Let $f\in\mathcal{A}$ and $G(\gamma,k,\alpha,\beta,\daleth;z)$ is holomorphic in $\mathfrak{U}$, where

$$G(\daleth,k,m,n,\alpha,\beta,\ell,d;z)=\frac{(\daleth+1)\Gamma(1-\daleth)}{\ell\alpha z^{1-\daleth}}\Big(\big[\ell\big(1+\beta(n-1)\big)+d\big]D_z^{\daleth}I_{\alpha,\beta,\ell,d}^{k,m+1}f(z)$$

$$-[\ell(1+\beta(n-1)-(1-\daleth)\alpha)+d]D_z^{\daleth}I_{\alpha,\beta,\ell,d}^{k,m}f(z)\Big)+\frac{\Gamma(1-\daleth)}{z^{-1-\daleth}}\left(D_z^{\daleth}I_{\alpha,\beta,\ell,d}^{k,m}f(z)\right)''. \quad (2.1)$$

If

$$F_{\psi(\mathbb{C}^2\times\mathfrak{U})}[G(\daleth,k,m,n,\alpha,\beta,\ell,d;z)]\le F_{h(\mathfrak{U})}h(z), \quad (2.2)$$

then

$$F_{\left(D_z^{\daleth}I_{\alpha,\beta,\ell,d}^{k,m}f\right)'(\mathfrak{U})}\left(\frac{\Gamma(1-\daleth)\left(D_z^{\daleth}I_{\alpha,\beta,\ell,d}^{k,m}f(z)\right)'}{z^{-\daleth}}\right)\le F_{q(\mathfrak{U})}q(z)\le F_{h(\mathfrak{U})}h(z),$$

i.e.

$$\frac{\Gamma(1-\daleth)\left(D_z^{\daleth}I_{\alpha,\beta,\ell,d}^{k,m}f(z)\right)'}{z^{-\daleth}}\prec_F q(z)\prec_F h(z),$$

where $q(z)=\frac{1}{z}\int_0^z h(t)\,dt$ is convex and is the fuzzy best dominant.

Proof. Assume that

$$\mathcal{P}(z)=\frac{\Gamma(1-\daleth)\left(D_z^{\daleth}I_{\alpha,\beta,\ell,d}^{k,m}f(z)\right)'}{z^{-\daleth}}. \quad (2.3)$$

Then $\mathcal{P}\in\mathcal{H}[1,1]$ and $\mathcal{P}(0)=1$. Therefore, in view of (1.3) and (2.3), we have

$$\mathcal{P}(z)+z\mathcal{P}'(z)=1+\sum_{n=2}^{\infty}\frac{n^2(n-\daleth)\Gamma(1-\daleth)\Gamma(k+n)}{\Gamma(n-\daleth+1)\Gamma(k+1)}\left[\frac{\ell\big(1+(\alpha+\beta)(n-1)\big)+d}{\ell\big(1+\beta(n-1)\big)+d}\right]^m a_n z^{n-1}$$

$$=\frac{(\daleth+1)\Gamma(1-\daleth)\big[\ell\big(1+\beta(n-1)\big)+d\big]}{\ell\alpha z^{1-\daleth}}\times$$

$$\times\left[\frac{1}{\Gamma(2-\lambda)}z^{1-\lambda}+\sum_{n=2}^{\infty}\frac{n\Gamma(k+n)}{\Gamma(n-\lambda+1)\Gamma(k+1)}\left[\frac{\ell\big(1+(\alpha+\beta)(n-1)\big)+d}{\ell\big(1+\beta(n-1)\big)+d}\right]^{m+1}a_n z^{n-\lambda}\right]$$

$$-\frac{(\lambda+1)[\ell(1+\beta(n-1)-(1-\lambda)\alpha)+d]+\ell\alpha\lambda(1-\lambda)}{\ell\alpha(1-\lambda)}$$

$$-\sum_{n=2}^{\infty}\frac{n(\lambda+1)\Gamma(1-\lambda)\Gamma(k+n)[\ell(1+\beta(n-1)-(1-\lambda)\alpha)+d]}{\ell\alpha\Gamma(n-\lambda+1)\Gamma(k+1)}\left[\frac{\ell\big(1+(\alpha+\beta)(n-1)\big)+d}{\ell\big(1+\beta(n-1)\big)+d}\right]^{m}a_n z^{n-1}$$

$$+\sum_{n=2}^{\infty}\frac{n(n-\lambda)(n-\lambda-1)\Gamma(1-\lambda)\Gamma(k+n)}{\Gamma(n-\lambda+1)\Gamma(k+1)}\left[\frac{\ell\big(1+(\alpha+\beta)(n-1)\big)+d}{\ell\big(1+\beta(n-1)\big)+d}\right]^{m}a_n z^{n-1}=G(\lambda,k,m,n,\alpha,\beta,\ell,d;z), \quad (2.4)$$

where $G(\lambda,k,m,n,\alpha,\beta,\ell,d;z)$ is defined as (2.1).

According to (2.2) and (2.4), we deduce that

$$F_{\psi(\mathbb{C}^2\times\mathfrak{U})}[\mathcal{P}(z)+z\mathcal{P}'(z)]\le F_{h(\mathfrak{U})}h(z).$$

Thus applying Lemma 1.1 with $\mu=1$, we obtain

$$F_{\mathcal{P}(\mathfrak{U})}\mathcal{P}(z)\le F_{q(\mathfrak{U})}q(z)\le F_{h(\mathfrak{U})}h(z).$$

From (2.3), we find that

$$F_{\left(D_z^{\lambda}I_{\alpha,\beta,\ell,d}^{k,m}f\right)'(\mathfrak{U})}\left(\frac{\Gamma(1-\lambda)\left(D_z^{\lambda}I_{\alpha,\beta,\ell,d}^{k,m}f(z)\right)'}{z^{-\lambda}}\right)\le F_{q(\mathfrak{U})}q(z)\le F_{h(\mathfrak{U})}h(z),$$

i.e.

$$\frac{\Gamma(1-\lambda)\left(D_z^{\lambda}I_{\alpha,\beta,\ell,d}^{k,m}f(z)\right)'}{z^{-\lambda}}\prec_F q(z)\prec_F h(z),$$

where $q(z)=\frac{1}{z}\int_0^z h(t)\,dt$ is convex and is "the fuzzy best dominant".

"Putting $m=k=0$ and $h(z)=\frac{1+(2\rho-1)z}{1+z}$ $(0\le\rho<1)$ in Theorem 2.1, we obtain the following corollary":

Corollary 2.1. Let $f\in\mathcal{A}$ and

$$\frac{(\lambda+1)\Gamma(1-\lambda)}{\ell\alpha z^{1-\lambda}}\left(D_z^{\lambda}zf'(z)+\lambda D_z^{\lambda}f(z)\right)+\frac{\Gamma(1-\lambda)}{z^{-1-\lambda}}\left(D_z^{\lambda}f(z)\right)''$$

is holomorphic in $\mathfrak{U}$. If

$$\frac{(\lambda+1)\Gamma(1-\lambda)}{\ell\alpha z^{1-\lambda}}\left(D_z^{\lambda}zf'(z)+\lambda D_z^{\lambda}f(z)\right)+\frac{\Gamma(1-\lambda)}{z^{-1-\lambda}}\left(D_z^{\lambda}f(z)\right)''\prec_F\frac{1+(2\rho-1)z}{1+z},$$

then

$$\frac{\Gamma(1-\lambda)\left(D_z^{\lambda}f(z)\right)'}{z^{-\lambda}}\prec_F q(z)\prec_F\frac{1+(2\rho-1)z}{1+z},$$

where $q(z)=2\rho-1+\frac{2(1-\rho)}{z}\ln(1+z)$ is convex and is the "fuzzy best dominant".

Theorem 2.2. Suppose that the convex function h satisfies $h(0) = 1$. Let $f \in \mathcal{A}$ and $\frac{\Gamma(1-\lambda)\left(D_z^\lambda I_{\alpha,\beta,\ell,d}^{k,m} f(z)\right)'}{z^{-\lambda}}$ is holomorphic in $\mathfrak{U}$. If

$$F_{\psi(\mathbb{C}^2\times\mathfrak{U})}\left[\frac{\Gamma(1-\lambda)\left(D_z^\lambda I_{\alpha,\beta,\ell,d}^{k,m} f(z)\right)'}{z^{-\lambda}}\right] \leq F_{h(\mathfrak{U})}h(z), \tag{2.5}$$

then

$$F_{\left(D_z^\lambda I_{\alpha,\beta,\ell,d}^{k,m} f\right)(\mathfrak{U})}\left(\frac{\Gamma(2-\lambda)D_z^\lambda I_{\alpha,\beta,\ell,d}^{k,m} f(z)}{z^{1-\lambda}}\right) \leq F_{q(\mathfrak{U})}q(z) \leq F_{h(\mathfrak{U})}h(z),$$

i.e.

$$\frac{\Gamma(2-\lambda)D_z^\lambda I_{\alpha,\beta,\ell,d}^{k,m} f(z)}{z^{1-\lambda}} \prec_F q(z) \prec_F h(z),$$

where $q(z) = \frac{1}{z}\int_0^z h(t)\,dt$ is convex and is "the fuzzy best dominant".

Proof. Assume that

$$\mathcal{P}(z) = \frac{\Gamma(2-\lambda)D_z^\lambda I_{\alpha,\beta,\ell,d}^{k,m} f(z)}{z^{1-\lambda}}. \tag{2.6}$$

It is clear that $\mathcal{P} \in \mathcal{H}[1,1]$ and $\mathcal{P}(0) = 1$.

We find

$$\mathcal{P}(z) + \frac{1}{1-\lambda}z\mathcal{P}'(z) = \frac{\Gamma(1-\lambda)\left(D_z^\lambda I_{\alpha,\beta,\ell,d}^{k,m} f(z)\right)'}{z^{-\lambda}}. \tag{2.7}$$

In view of (2.7), the fuzzy differential subordination (2.5) becomes

$$F_{\psi(\mathbb{C}^2\times\mathfrak{U})}\left[\mathcal{P}(z) + \frac{1}{1-\lambda}z\mathcal{P}'(z)\right] \leq F_{h(\mathfrak{U})}h(z).$$

Thus applying Lemma 1.1 with $\mu = 1-\lambda$, we obtain

$$F_{\mathcal{P}(\mathfrak{U})}\mathcal{P}(z) \leq F_{q(\mathfrak{U})}q(z) \leq F_{h(\mathfrak{U})}h(z).$$

From (2.6), we get

$$F_{\left(D_z^\lambda I_{\alpha,\beta,\ell,d}^{k,m} f\right)(\mathfrak{U})}\left(\frac{\Gamma(2-\lambda)D_z^\lambda I_{\alpha,\beta,\ell,d}^{k,m} f(z)}{z^{1-\lambda}}\right) \leq F_{q(\mathfrak{U})}q(z) \leq F_{h(\mathfrak{U})}h(z),$$

i.e.

$$\frac{\Gamma(2-\lambda)D_z^\lambda I_{\alpha,\beta,\ell,d}^{k,m} f(z)}{z^{1-\lambda}} \prec_F q(z) \prec_F h(z),$$

where $q(z) = \frac{1}{z}\int_0^z h(t)\,dt$ is convex and is "the fuzzy best dominant".

Putting $m = k = 0$ and $h(z) = e^{bz}$, $|b| \leq 1$ in Theorem 2.2, we obtain the following corollary :

Corollary 2.2. If $f \in \mathcal{A}$, $\frac{\Gamma(1-\lambda)\left(D_z^\lambda f(z)\right)'}{z^{-\lambda}}$ is holomorphic in $\mathfrak{U}$ and

$$\frac{\Gamma(1-\lambda)\left(D_z^{\lambda} f(z)\right)'}{z^{-\lambda}} \prec_F e^{bz},$$

then

$$\frac{\Gamma(2-\lambda) D_z^{\lambda} f(z)}{z^{1-\lambda}} \prec_F q(z) \prec_F e^{bz},$$

where $q(z) = \frac{e^{bz}-1}{bz}$ is convex and is "the fuzzy best dominant".

Theorem 2.3. Suppose that q be a convex function in $\mathfrak{U}$ such that $q(0) = 1$, $h(z) = q(z) + \frac{\ell\alpha}{\ell(1+\beta(n-1))+d} z q'(z)$. Let $f \in \mathcal{A}$ and $\frac{\Gamma(1-\lambda)\left(D_z^{\lambda} I_{\alpha,\beta,\ell,d}^{k,m+1} f(z)\right)'}{z^{-\lambda}}$ is holomorphic in $\mathfrak{U}$. If

$$F_{\psi(\mathbb{C}^2\times\mathfrak{U})}\left[\frac{\Gamma(1-\lambda)\left(D_z^{\lambda} I_{\alpha,\beta,\ell,d}^{k,m+1} f(z)\right)'}{z^{-\lambda}}\right] \leq F_{h(\mathfrak{U})} h(z), \tag{2.8}$$

then

$$F_{\left(D_z^{\lambda} I_{\alpha,\beta,\ell,d}^{k,m} f\right)'(\mathfrak{U})}\left(\frac{\Gamma(1-\lambda)\left(D_z^{\lambda} I_{\alpha,\beta,\ell,d}^{k,m} f(z)\right)'}{z^{-\lambda}}\right) \leq F_{q(\mathfrak{U})} q(z),$$

i.e.

$$\frac{\Gamma(1-\lambda)\left(D_z^{\lambda} I_{\alpha,\beta,\ell,d}^{k,m} f(z)\right)'}{z^{-\lambda}} \prec_F q(z)$$

and q is "fuzzy best dominant".

Proof. Assume that

$$\mathcal{P}(z) = \frac{\Gamma(1-\lambda)\left(D_z^{\lambda} I_{\alpha,\beta,\ell,d}^{k,m} f(z)\right)'}{z^{-\lambda}}. \tag{2.9}$$

It is clear that $p \in \mathcal{H}[1,1]$.

Differentiating both sides of (2.9) with respect to z, we have

$$\mathcal{P}(z) + \frac{\ell\alpha}{\ell(1+\beta(n-1))+d} z\mathcal{P}'(z) = \frac{\ell\alpha\Gamma(1-\lambda)\left(D_z^{\lambda} I_{\alpha,\beta,\ell,d}^{k,m} f(z)\right)''}{[\ell(1+\beta(n-1))+d] z^{-1-\lambda}}$$

$$+\frac{\Gamma(1-\lambda)[\ell(1+\alpha\lambda+\beta(n-1))+d]\left(D_z^{\lambda} I_{\alpha,\beta,\ell,d}^{k,m} f(z)\right)'}{[\ell(1+\beta(n-1))+d] z^{-\lambda}}. \tag{2.10}$$

Using (1.3) and differentiating with respect to z, we obtain

$$\left(D_z^{\lambda} I_{\alpha,\beta,\ell,d}^{k,m+1} f(z)\right)' = \frac{\ell\alpha z\left(D_z^{\lambda} I_{\alpha,\beta,\ell,d}^{k,m} f(z)\right)''}{[\ell(1+\beta(n-1))+d]} + \frac{[\ell(1+\alpha\lambda+\beta(n-1))+d]\left(D_z^{\lambda} I_{\alpha,\beta,\ell,d}^{k,m} f(z)\right)'}{[\ell(1+\beta(n-1))+d]}.$$

So

$$\frac{\Gamma(1-\lambda)\left(D_z^{\lambda} I_{\alpha,\beta,\ell,d}^{k,m+1} f(z)\right)'}{z^{-\lambda}} = \frac{\ell\alpha\Gamma(1-\lambda)\left(D_z^{\lambda} I_{\alpha,\beta,\ell,d}^{k,m} f(z)\right)''}{\left[\ell\big(1+\beta(n-1)\big)+d\right] z^{-1-\lambda}}$$

$$+\frac{\Gamma(1-\lambda)\left[\ell\big(1+\alpha\lambda+\beta(n-1)\big)+d\right]\left(D_z^{\lambda} I_{\alpha,\beta,\ell,d}^{k,m} f(z)\right)'}{\left[\ell\big(1+\beta(n-1)\big)+d\right] z^{-\lambda}}. \tag{2.11}$$

In the light of (2.10) and (2.11), (2.8) becomes

$$F_{\psi(\mathbb{C}^2\times\mathfrak{A})}\left[\mathcal{P}(z)+\frac{\ell\alpha}{\ell\big(1+\beta(n-1)\big)+d} z\mathcal{P}'(z)\right] \le F_{h(\mathfrak{A})} h(z).$$

Thus applying Lemma 1.2 with $\nu=\frac{\ell\alpha}{\ell(1+\beta(n-1))+d}$, we obtain

$$F_{\left(D_z^{\lambda} I_{\alpha,\beta,\ell,d}^{k,m} f\right)'(\mathfrak{A})}\left(\frac{\Gamma(1-\lambda)\left(D_z^{\lambda} I_{\alpha,\beta,\ell,d}^{k,m} f(z)\right)'}{z^{-\lambda}}\right) \le F_{q(\mathfrak{A})} q(z),$$

i.e.

$$\frac{\Gamma(1-\lambda)\left(D_z^{\lambda} I_{\alpha,\beta,\ell,d}^{k,m} f(z)\right)'}{z^{-\lambda}} \prec_F q(z)$$

and q is fuzzy "best dominant".

Theorem 2.4. Suppose that q be a convex function in $\mathfrak{A}$ such that $q(0)=1$, $h(z)=q(z)+zq'(z)$. Let $f\in\mathcal{A}$ and $\left(\frac{z\, D_z^{\lambda} I_{\alpha,\beta,\ell,d}^{k,m+1} f(z)}{D_z^{\lambda} I_{\alpha,\beta,\ell,d}^{k,m} f(z)}\right)'$ is holomorphic in $\mathfrak{A}$. If

$$F_{\psi(\mathbb{C}^2\times\mathfrak{A})}\left[\left(\frac{z\, D_z^{\lambda} I_{\alpha,\beta,\ell,d}^{k,m+1} f(z)}{D_z^{\lambda} I_{\alpha,\beta,\ell,d}^{k,m} f(z)}\right)'\right] \le F_{h(\mathfrak{A})} h(z), \tag{2.12}$$

then

$$F_{\left(\frac{D_z^{\lambda} I_{\alpha,\beta,\ell,d}^{k,m+1} f}{D_z^{\lambda} I_{\alpha,\beta,\ell,d}^{k,m} f}\right)(\mathfrak{A})}\left(\frac{D_z^{\lambda} I_{\alpha,\beta,\ell,d}^{k,m+1} f(z)}{D_z^{\lambda} I_{\alpha,\beta,\ell,d}^{k,m} f(z)}\right) \le F_{q(\mathfrak{A})} q(z),$$

i.e.

$$\frac{D_z^{\lambda} I_{\alpha,\beta,\ell,d}^{k,m+1} f(z)}{D_z^{\lambda} I_{\alpha,\beta,\ell,d}^{k,m} f(z)} \prec_F q(z)$$

and q is "fuzzy best dominant".

Proof. Assume that

$$\mathcal{P}(z)=\frac{D_z^{\lambda} I_{\alpha,\beta,\ell,d}^{k,m+1} f(z)}{D_z^{\lambda} I_{\alpha,\beta,\ell,d}^{k,m} f(z)}. \tag{2.13}$$

Therefore, we note that $\mathcal{P}\in\mathcal{H}[1,1]$.

Differentiating both sides of (2.13) with respect to z, it yields

$$\mathcal{P}'(z)=\frac{\left(D_z^{\lambda}I_{\alpha,\beta,\ell,d}^{k,m+1}f(z)\right)'}{D_z^{\lambda}I_{\alpha,\beta,\ell,d}^{k,m}f(z)}-\mathcal{P}(z)\frac{\left(D_z^{\lambda}I_{\alpha,\beta,\ell,d}^{k,m}f(z)\right)'}{D_z^{\lambda}I_{\alpha,\beta,\ell,d}^{k,m}f(z)}.$$

Then

$$\mathcal{P}(z)+z\mathcal{P}'(z)=\frac{D_z^{\lambda}I_{\alpha,\beta,\ell,d}^{k,m}f(z)\left(z\left(D_z^{\lambda}I_{\alpha,\beta,\ell,d}^{k,m+1}f(z)\right)'+D_z^{\lambda}I_{\alpha,\beta,\ell,d}^{k,m+1}f(z)\right)-zD_z^{\lambda}I_{\alpha,\beta,\ell,d}^{k,m+1}f(z)\left(D_z^{\lambda}I_{\alpha,\beta,\ell,d}^{k,m}f(z)\right)'}{\left(D_z^{\lambda}I_{\alpha,\beta,\ell,d}^{k,m}f(z)\right)^2}.$$

$$=\left(\frac{z\,D_z^{\lambda}I_{\alpha,\beta,\ell,d}^{k,m+1}f(z)}{D_z^{\lambda}I_{\alpha,\beta,\ell,d}^{k,m}f(z)}\right)'. \tag{2.14}$$

Utilizing (2.14) in (2.12), we can get

$$F_{\psi(\mathbb{C}^2\times\mathfrak{U})}[\mathcal{P}(z)+z\mathcal{P}'(z)]\le F_{h(\mathfrak{U})}h(z).$$

Thus applying Lemma 1.2 with $\nu=1$, we obtain

$$F_{\left(\frac{D_z^{\lambda}I_{\alpha,\beta,\ell,d}^{k,m+1}f}{D_z^{\lambda}I_{\alpha,\beta,\ell,d}^{k,m}f}\right)(\mathfrak{U})}\left(\frac{D_z^{\lambda}I_{\alpha,\beta,\ell,d}^{k,m+1}f(z)}{D_z^{\lambda}I_{\alpha,\beta,\ell,d}^{k,m}f(z)}\right)\le F_{q(\mathfrak{U})}q(z),$$

i.e.

$$\frac{D_z^{\lambda}I_{\alpha,\beta,\ell,d}^{k,m+1}f(z)}{D_z^{\lambda}I_{\alpha,\beta,\ell,d}^{k,m}f(z)}\prec_F q(z)$$

and q is fuzzy best dominant.

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