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Generalized Soft Mappings between Soft Closure Spaces

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ABSTRACT

In the closure spaces, Boonpok and Khampakdee [1] were introduced the notion of generalized closed sets and applied this notion to defined and studied generalized continuous functions. In this paper, the concept of soft generalized closed sets in soft closure spaces is considered to introduce weaker forms of soft continuous and soft closed mappings which are named as, g-soft continuous and g-soft closed mappings. There are several exciting results are gotten.

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1-Introduction

Čech [2] was the first to present and study the definition of closure spaces. Later, after Molodtsov introduced the notion of soft sets in 1999 [3], several researchers began to introduce and study the notion of closure spaces in the soft setting. Gowri and Jegadeesan [4] proposed and discussed Čech soft closure spaces. Furthermore, in a fuzzy soft situation, Majeed [5] existing the fundamental construction of Čech fuzzy soft closure spaces. The notion of soft closure spaces and their fundamental structures was also introduced very

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recently by us in [6, 7]. Our notion of soft closure spaces is an extension of the notion of closure spaces that are introduced in [2].

Levine [8] introduced the concept of generalized closed sets in topological space. He applies certain essential properties of closed sets to a larger set's family. For example, compactness, normality, and completeness are inherited through generalized closed sets in a uniform space. Balachandran et al. [9] introduced the notion of generalized continuous maps and studied some of their properties. Kannan [10] introduced in soft topological spaces the notion of soft generalized, closed sets. Also, definitions of generalized closed sets and continuous maps have been applied to closure spaces in [1]. And, in [11] we present the concepts of soft generalized closed sets and $\mathcal{T}_{\frac{1}{2}}$-soft closure spaces. In the present paper, the concept of soft gc-sets are using to introduce a new type of soft mappings, these are generalized soft continuous mappings and generalized soft closed mappings. Some properties of these soft mappings are studied and discussed.

2- Preliminaries

In this section, some basic concepts and results described and discussed via various authors of soft set theory in the literature are listed.

Definition 2.1 [3] Let $\mathcal{M}$ be a universal set, R be a set of parameters, and $P(\mathcal{M})$ denote the power set of $\mathcal{M}$. A mapping $\mathcal{F}: R \to P(\mathcal{M})$ is called a soft set over $\mathcal{M}$. The family of all soft sets over $\mathcal{M}$ denotes $\mathcal{SS}(\mathcal{M}, R)$.

Definition 2.2 Let $\mathcal{F}_R$ and G_R be two soft sets over $\mathcal{M}$. Then,

1- $\mathcal{F}_R$ is said to be a null soft set [12], denoted $\widetilde{\Phi}_R$, is defined as $\mathcal{F}(R) = \emptyset$ (empty set) for all $r \in R$.
2- $\mathcal{F}_R$ is said to be an absolute soft set [12], denoted $\widetilde{\mathcal{M}}$, is defined as $\mathcal{F}(R) = \mathcal{M}$ for all $r \in R$.
3- $\mathcal{F}_R$ is called a soft subset of G_R [13], denoted $\mathcal{F}_R \sqsubseteq G_R$, if $\mathcal{F}(r) \subseteq G(r)$ for all $r \in R$. $\mathcal{F}_R$ equals G_R, denoted $\mathcal{F}_R = G_R$ if $\mathcal{F}_R \sqsubseteq G_R$ and $G_R \sqsubseteq \mathcal{F}_R$.
4- The union [12] of $\mathcal{F}_R$ and G_R is denoted $\mathcal{F}_R \sqcup G_R = \mathcal{H}_R$ and defined as $\mathcal{H}(r) = \mathcal{F}(r) \cup G(r)$ for all $r \in R$.
5- The intersection [12] of $\mathcal{F}_R$ and G_R is denoted $\mathcal{F}_R \sqcap G_R = \mathcal{H}_R$ and defined as $\mathcal{H}(r) = \mathcal{F}(r) \cap G(r)$ for all $r \in R$.
6- The difference [14] between $\mathcal{F}_R$ and G_R over $\mathcal{M}$, denoted $\mathcal{F}_R - G_R = \mathcal{H}_R$ and defined as $\mathcal{H}(r) = \mathcal{F}(r) - G(r)$ for all $r \in R$.
7- The relative complement [14] of a $\mathcal{F}_R$ is denoted $\mathcal{F}_R^c$, such that $\mathcal{F}^c: R \longrightarrow P(\mathcal{M})$ defined as $\mathcal{F}^c(r) = \mathcal{M} - \mathcal{F}(r)$, for all $r \in R$. Clearly, $\mathcal{F}_R^c = \widetilde{\mathcal{M}} - \mathcal{F}_R$.

Definition 2.3 [15] The soft set $\mathcal{F}_R \in \mathcal{SS}(\mathcal{M}, R)$ is called soft point in $\mathcal{M}$, denoted x_r, if for the element $r \in R$, $\mathcal{F}(r) = \{x\}$ and $\mathcal{F}(r') = \emptyset$ for every $r' \in R - \{r\}$.

Definition 2.4 [15] The soft point x_r is said to be in the soft set G_R, indicate $x_r \widetilde{\in} G_R$, if we have $\{x\} \subseteq G(r)$ for the parameter $r \in R$.

Definition 2.5 [16] Let $\mathcal{SS}(\mathcal{M}, R)$ and $\mathcal{SS}(\mathcal{N}, \mathcal{K})$ be families of soft sets, $\psi: \mathcal{M} \longrightarrow \mathcal{N}$ and $\ell: R \longrightarrow \mathcal{K}$ be mappings. The mapping $\psi_\ell: \mathcal{SS}(\mathcal{M}, R) \longrightarrow \mathcal{SS}(\mathcal{N}, \mathcal{K})$ defined as:

1- If $\mathcal{F}_R \in \mathcal{SS}(\mathcal{M}, R)$, then the image of $\mathcal{F}_R$ under ψ_ℓ , written as $\psi_\ell(\mathcal{F}_R) = \left(\psi_\ell(\mathcal{F}_R), \ell(R)\right)$, is a soft set in $\mathcal{SS}(\mathcal{N}, \mathcal{K})$ such that:

$$\psi_\ell(\mathcal{F}_R)(k) = \begin{cases} \psi(\cup_{r \in \ell^{-1}(k) \cap R} \mathcal{F}(r)), & if\ r \in \ell^{-1}(k) \cap R \neq \emptyset; \\ \emptyset, & otherwise. \end{cases}$$

2- If $G_\mathcal{K} \in \mathcal{SS}(\mathcal{N}, \mathcal{K})$, then the pre-image of $G_\mathcal{K}$ under ψ_ℓ, written as $\psi_\ell^{-1}(G_\mathcal{K}) = (\psi_\ell^{-1}(G_\mathcal{K}), \ell^{-1}(\mathcal{K}))$ is a soft set in $\mathcal{SS}(\mathcal{M}, R)$ such that

$$\psi_\ell^{-1}(G_\mathcal{K})(r) = \begin{cases} \psi^{-1}\left(G\left(\ell(r)\right)\right), & if\ \ell(r) \in \mathcal{K}; \\ \emptyset, & otherwise. \end{cases}$$

Definition 2.6 [17] Let $\mathcal{F}_R \in \mathcal{SS}(\mathcal{M}, R)$ and $G_\mathcal{K} \in \mathcal{SS}(\mathcal{N}, \mathcal{K})$. The Cartesian product $\mathcal{F}_R \times G_\mathcal{K}$ is defined as $(\mathcal{F} \times G)_{R \times \mathcal{K}}$ where

$$(\mathcal{F} \times G)_{R \times \mathcal{K}}(r, k) = \mathcal{F}(r) \times G(k), \text{ for all } (r, k) \in R \times \mathcal{K}.$$

Through this description, the $\mathcal{F}_R \times G_\mathcal{K}$ the soft set is a soft set over $\mathcal{M} \times \mathcal{N}$ and its universe parameter is $R \times \mathcal{K}$.

The projection pairs $p_\mathcal{M}: \mathcal{M} \times \mathcal{N} \longrightarrow \mathcal{M}$, $q_R: R \times \mathcal{K} \longrightarrow R$ and $p_\mathcal{N}: \mathcal{M} \times \mathcal{N} \longrightarrow \mathcal{N}$, $q_\mathcal{K}: R \times \mathcal{K} \longrightarrow \mathcal{K}$ determine morphisms respectively $(p_\mathcal{M}, q_R)$ from $\mathcal{M} \times \mathcal{N}$ to $\mathcal{M}$ and $(p_\mathcal{N}, q_\mathcal{K})$ from $\mathcal{M} \times \mathcal{N}$ to $\mathcal{N}$, where

$(p_\mathcal{M}, q_R)(\mathcal{F}_R \times G_\mathcal{K}) = p_\mathcal{M}(\mathcal{F} \times G)_{q_R(R \times \mathcal{K})}$ and

$(p_\mathcal{N}, q_\mathcal{K})(\mathcal{F}_R \times G_\mathcal{K}) = p_\mathcal{N}(\mathcal{F} \times G)_{q_\mathcal{K}(R \times \mathcal{K})}$ [18].

Definition 2.7 [6] An operator $\tilde{u}: \mathcal{SS}(\mathcal{M}, R) \longrightarrow \mathcal{SS}(\mathcal{M}, R)$ is referred to as a soft closure operator (soft- co, for short) on $\mathcal{M}$, if for all $\mathcal{F}_R, G_R \in \mathcal{SS}(\mathcal{M}, R)$ the following hold:

$(\mathcal{C}1)$ $\widetilde{\Phi}_R = \tilde{u}(\widetilde{\Phi}_R)$;

$(\mathcal{C}2)$ $\mathcal{F}_R \sqsubseteq \tilde{u}(\mathcal{F}_R)$;

$(\mathcal{C}3)$ $\mathcal{F}_R \sqsubseteq G_R \Longrightarrow \tilde{u}(\mathcal{F}_R) \sqsubseteq \tilde{u}(G_R)$.

The triple $(\mathcal{M}, \tilde{u}, R)$ is called a soft closure space (soft-cs, for short). It is said that soft subset $\mathcal{F}_R$ over $\mathcal{M}$ is a soft closed set (soft c-set, for short), if $\mathcal{F}_R = \tilde{u}(\mathcal{F}_R)$. The relative complement $\widetilde{\mathcal{M}} - \mathcal{F}_R$ of any soft c-set over $\mathcal{M}$ is said to be a soft open set (soft o- set, for short).

Definition 2.8 [11] A soft set $\mathcal{F}_R$ is called a soft generalized closed set (soft gc-set, for short) in a soft-cs $(\mathcal{M}, \tilde{u}, R)$, if $\tilde{u}(\mathcal{F}_R) \sqsubseteq G_R$ whenever $\mathcal{F}_R \sqsubseteq G_R$ and G_R is a soft o- set in $\widetilde{\mathcal{M}}$. A soft set $\mathcal{F}_R$ is called a soft generalized open set (soft go-set, for short), if the relative complement $\widetilde{\mathcal{M}} - \mathcal{F}_R$ is a soft gc-set in $(\mathcal{M}, \tilde{u}, R)$.

Proposition 2.9 [11] Every soft c-set in a soft-cs $(\mathcal{M}, \tilde{u}, R)$ is a soft gc-set.

Definition 2.10 [11] A soft-cs $(\mathcal{M}, \tilde{u}, R)$ is said to be a $\mathcal{T}_{\frac{1}{2}}$-soft-$cs$, if every soft gc-set is a soft c-set in $\mathcal{M}$.

Definition 2.11 [7] Let $(\mathcal{M}, \tilde{u}, R)$ and $(\mathcal{N}, \tilde{v}, \mathcal{K})$ be soft-cs's. A soft mapping $\psi_\ell : (\mathcal{M}, \tilde{u}, R) \to (\mathcal{N}, \tilde{v}, \mathcal{K})$ is said to be soft continuous, if $\psi_\ell(\tilde{u}(\mathcal{F}_R)) \sqsubseteq \tilde{v}\ (\psi_\ell(\mathcal{F}_R))$ for every soft set $\mathcal{F}_R \in \mathcal{SS}(\mathcal{M}, R)$.

Proposition 2.12 [7] If $\psi_\ell : (\mathcal{M}, \tilde{u}, R) \to (\mathcal{N}, \tilde{v}, \mathcal{K})$ is a soft continuous mapping, then $\psi_\ell^{-1}(G_\mathcal{K})$ is a soft c-set of $(\mathcal{M}, \tilde{u}, R)$ for every soft c-set $G_\mathcal{K}$ of $(\mathcal{N}, \tilde{v}, \mathcal{K})$.

Definition 2.13 [7] Let $(\mathcal{M}, \tilde{u}, R)$ and $(\mathcal{N}, \tilde{v}, \mathcal{K})$ be soft-cs's. A soft mapping $\psi_\ell : (\mathcal{M}, \tilde{u}, R) \to (\mathcal{N}, \tilde{v}, \mathcal{K})$ is said to be soft closed, if $\psi_\ell(\mathcal{F}_R)$ is a soft c-set of $(\mathcal{N}, \tilde{v}, \mathcal{K})$ whenever $\mathcal{F}_R$ is a soft c-set of $(\mathcal{M}, \tilde{u}, R)$.

Theorem 2.14 [6] Let $\{(\mathcal{M}_\alpha, \tilde{u}_\alpha, R_\alpha) : \alpha \in \mathcal{I}\}$ be a family of soft-cs's. Define a soft operator $\otimes\tilde{u} : \mathcal{SS}(\prod_{\alpha\in\mathcal{I}} \mathcal{M}_\alpha, \prod_{\alpha\in\mathcal{I}} R_\alpha) \longrightarrow \mathcal{SS}(\prod_{\alpha\in\mathcal{I}} \mathcal{M}_\alpha, \prod_{\alpha\in\mathcal{I}} R_\alpha)$, where $\prod_{\alpha\in\mathcal{I}} \mathcal{M}_\alpha$ and $\prod_{\alpha\in\mathcal{I}} R_\alpha$ denotes to the Cartesian product of the sets $\mathcal{M}_\alpha$ and R_α, $\alpha \in \mathcal{I}$, respectively as follows:

$\otimes\tilde{u}(\mathcal{F}_{\prod_{\alpha\in\mathcal{I}} R_\alpha}) = \prod_{\alpha\in\mathcal{I}} \tilde{u}_\alpha\ ((p_{\mathcal{M}_\alpha}, q_{R_\alpha})(\mathcal{F}_{\prod_{\alpha\in\mathcal{I}} R_\alpha}))$, $\forall\, \mathcal{F}_{\prod_{\alpha\in\mathcal{I}} R_\alpha} \in \mathcal{SS}(\prod_{\alpha\in\mathcal{I}} \mathcal{M}_\alpha, \prod_{\alpha\in\mathcal{I}} R_\alpha)$. Then, the operator $\otimes\tilde{u}$ is a soft closure operator on $\prod_{\alpha\in\mathcal{I}} \mathcal{M}_\alpha$. The triple $(\prod_{\alpha\in\mathcal{I}} \mathcal{M}_\alpha, \otimes\tilde{u}, \prod_{\alpha\in\mathcal{I}} R_\alpha)$ is said to be the product soft-cs of the family $\{(\mathcal{M}_\alpha, \tilde{u}_\alpha, R_\alpha) : \alpha \in \mathcal{I}\}$.

Theorem 2.15 [7] Let $\{(\mathcal{M}_\alpha, \tilde{u}_\alpha, R_\alpha) : \alpha \in \mathcal{I}\}$ be a family of soft-cs's and let $v \in \mathcal{I}$. Then, $\mathcal{F}_{R_v}$ is a closed set of $(\mathcal{M}_v, \tilde{u}_v, R_v)$ if and only if $\mathcal{F}_{R_v} \times \widetilde{\prod_{\substack{\alpha\neq v\\ \alpha\in\mathcal{I}}} \mathcal{M}_\alpha}$ is a soft c-set in $(\prod_{\alpha\in\mathcal{I}} \mathcal{M}_\alpha, \otimes\tilde{u}, \prod_{\alpha\in\mathcal{I}} R_\alpha)$.

Theorem 2.16 [11] Let $\{(\mathcal{M}_\alpha, \tilde{u}_\alpha, R_\alpha): \alpha \in \mathcal{J}\}$ be a family of soft-cs's, let $\nu \in \mathcal{J}$ and let $\mathcal{F}_{R_\nu} \sqsubseteq \widetilde{\mathcal{M}}_\nu$. Then, $\mathcal{F}_{R_\nu}$ is soft gc-set of $(\mathcal{M}_\nu, \tilde{u}_\nu, R_\nu)$ if and only if $\mathcal{F}_{R_\nu} \times \widetilde{\prod_{\substack{\alpha \neq \nu \\ \alpha \in \mathcal{J}}} \mathcal{M}_\alpha}$ is a soft gc-set of $(\prod_{\alpha \in \mathcal{J}} \mathcal{M}_\alpha, \otimes\tilde{u}, \prod_{\alpha \in \mathcal{J}} R_\alpha)$.

3. Generalized Soft Continuous Mappings

In this section, we introduce the notion of generalized soft continuous mappings and discussed its relationship with soft continuous mappings. Also, several properties of this new notion are introduced.

Definition 3.1 Let $(\mathcal{M}, \tilde{u}, R)$ and $(\mathcal{N}, \tilde{v}, \mathcal{K})$ be soft- cs's. A soft mapping $\psi_\ell: (\mathcal{M}, \tilde{u}, R) \to (\mathcal{N}, \tilde{v}, \mathcal{K})$ is called a generalized soft continuous (g-soft continuous, for short), if $\psi_\ell^{-1}(\mathcal{F}_\mathcal{K})$ is soft gc-set of $(\mathcal{M}, \tilde{u}, R)$ for every soft c-set $\mathcal{F}_\mathcal{K}$ of $(\mathcal{N}, \tilde{v}, \mathcal{K})$.

The next proposition gives a characterization of Definition 3.1.

Proposition 3.2 A soft mapping $\psi_\ell: (\mathcal{M}, \tilde{u}, R) \to (\mathcal{N}, \tilde{v}, \mathcal{K})$ is g-soft continuous if and only if $\psi_\ell^{-1}(G_\mathcal{K})$ soft go- set of $(\mathcal{M}, \tilde{u}, R)$ for every soft o-set $G_\mathcal{K}$ of $(\mathcal{N}, \tilde{v}, \mathcal{K})$.

Proof. Suppose ψ_ℓ is a g-soft continuous mapping and let $G_\mathcal{K}$ be a soft o-set of $(\mathcal{N}, \tilde{v}, \mathcal{K})$. Since $G_\mathcal{K}$ is a soft o-set, then $\widetilde{\mathcal{N}} - G_\mathcal{K}$ is a soft c-set and from ψ_ℓ is g-soft continuous mapping, we have $\psi_\ell^{-1}(\widetilde{\mathcal{N}} - G_\mathcal{K})$ is a gc- soft set. Since $\psi_\ell^{-1}(\widetilde{\mathcal{N}} - G_\mathcal{K}) = \widetilde{\mathcal{M}} - \psi_\ell^{-1}(G_\mathcal{K})$. Therefore, $\psi_\ell^{-1}(G_\mathcal{K})$ is a soft go-set of $(\mathcal{M}, \tilde{u}, R)$.

Conversely, let $\mathcal{F}_\mathcal{K}$ be a soft c-set of $\mathcal{N}$. Then $\widetilde{\mathcal{N}} - \mathcal{F}_\mathcal{K}$ is a soft o- set of $\mathcal{N}$. From hypothesis, $\psi_\ell^{-1}(\widetilde{\mathcal{N}} - \mathcal{F}_\mathcal{K})$ is a soft go-set of $\mathcal{M}$. Hence $\psi_\ell^{-1}(\mathcal{F}_\mathcal{K})$ is a soft gc-set. Thus, ψ_ℓ is a g- soft continuous mapping. □

The next proposition gives the relationship between the concept of soft continuous and g-soft continuous mappings.

Proposition 3.3 Let $\psi_\ell: (\mathcal{M}, \tilde{u}, R) \to (\mathcal{N}, \tilde{v}, \mathcal{K})$ be a soft mapping. If ψ_ℓ is a soft continuous, then ψ_ℓ is a g-soft continuous mapping.

Proof. Let $\mathcal{F}_\mathcal{K}$ be a soft c-set of $(\mathcal{N}, \tilde{v}, \mathcal{K})$. Since ψ_ℓ is a soft continuous, then from Proposition 2.12, $\psi_\ell^{-1}(\mathcal{F}_\mathcal{K})$ is a soft c-set of $(\mathcal{M}, \tilde{u}, R)$. This implies $\psi_\ell^{-1}(\mathcal{F}_\mathcal{K})$ is soft gc- set of $(\mathcal{M}, \tilde{u}, R)$. Hence, ψ_ℓ *is* g-soft continuous mapping. □

In the following example, we explain the converse of Proposition 3.3 maybe not holds.

Example 3.4 Let $\mathcal{M} = \{a, b, c\}, R = \{r_1, r_2\}$ and $\mathcal{N} = \{x, y, z\}$, $\mathcal{K} = \{k_1, k_2\}$. Let $\tilde{u}: \mathcal{SS}(\mathcal{M}, R) \longrightarrow \mathcal{SS}(\mathcal{M}, R)$ and $\tilde{v}: \mathcal{SS}(\mathcal{N}, \mathcal{K}) \longrightarrow \mathcal{SS}(\mathcal{N}, \mathcal{K})$ be soft-co's defined as follows:

$$\tilde{u}(\mathcal{F}_R) = \begin{cases} \widetilde{\Phi}_R & if\ \mathcal{F}_R = \widetilde{\Phi}_R, \\ \{(r_1,\{b\})\} & if\ \mathcal{F}_R = \{(r_1,\{b\})\}, \\ \{(r_2,\{b,c\})\} & if\ \mathcal{F}_R \sqsubseteq \{(r_2,\{b,c\})\}, \\ \widetilde{\mathcal{M}} & otherwise. \end{cases}$$

$$\tilde{v}(G_{\mathcal{K}}) = \begin{cases} \widetilde{\Phi}_{\mathcal{K}} & if\ G_{\mathcal{K}} = \widetilde{\Phi}_{\mathcal{K}}, \\ \{(k_1,\{x,y\})\} & if\ G_{\mathcal{K}} = \{(k_1,\{x\})\}, \\ \{(k_1,\{y\})\} & if\ G_{\mathcal{K}} = \{(k_1,\{y\})\}, \\ \widetilde{\mathcal{N}} & otherwise. \end{cases}$$

Clearly, $(\mathcal{M},\tilde{u},R)$ and $(\mathcal{N},\tilde{v},\mathcal{K})$ are soft-cs's. Then, the soft mapping $\psi_\ell:(\mathcal{M},\tilde{u},R) \to (\mathcal{N},\tilde{v},\mathcal{K})$ defined as $\psi(a) = x, \psi(b) = \psi(c) = y$ and $\ell(r_1) = k_2$, $\ell(r_2) = k_1$ is $\mathcal{g}$-soft continuous mapping. However, it is not a soft continuous mapping. Since there exists a soft set $\mathcal{F}_R = \{(r_2,\{a\})\} \in \mathcal{SS}(\mathcal{M},R)$ such that $\psi_\ell\big(\tilde{u}(\mathcal{F}_R)\big) = \widetilde{\mathcal{N}} \not\sqsubseteq \{(k_1,\{x,y\})\} = \tilde{v}\big(\psi_\ell(\mathcal{F}_R)\big)$.

Proposition 3.5 Let $(\mathcal{M},\tilde{u},R)$, $(\mathcal{N},\tilde{v},\mathcal{K})$ and $(\mathcal{Z},\widetilde{w},\mathcal{Q})$ be soft-cs's. If $\psi_\ell:(\mathcal{M},\tilde{u},R) \to (\mathcal{N},\tilde{v},\mathcal{K})$ is $\mathcal{g}$-soft continuous and $\varphi_q:(\mathcal{N},\tilde{v},\mathcal{K}) \to (\mathcal{Z},\widetilde{w},\mathcal{Q})$ is soft continuous, then $\varphi_q o \psi_\ell$ is a $\mathcal{g}$-soft continuous.

Proof. Let H_Q be a soft c-set of $(\mathcal{Z},\widetilde{w},\mathcal{Q})$. We must prove $\left(\varphi_q o \psi_\ell\right)^{-1}(H_Q)$ is soft $\mathcal{g}c$-set of $(\mathcal{M},\tilde{u},R)$. Since φ_q is soft continuous, then $\varphi_q^{-1}(H_Q)$ is soft c-set of $(\mathcal{N},\tilde{v},\mathcal{K})$. Since ψ_ℓ is a $\mathcal{g}$-soft continuous, then $\psi_\ell^{-1}(\varphi_q^{-1}(H_Q))$ is soft $\mathcal{g}c$-set of $(\mathcal{M},\tilde{u},R)$. That means $\left(\varphi_q o \psi_\ell\right)^{-1}(H_Q)$ is soft $\mathcal{g}c$-set of $(\mathcal{M},\tilde{u},R)$. Hence, $\varphi_q o \psi_\ell$ is a $\mathcal{g}$-soft continuous mapping. □

Proposition 3.6 Let $(\mathcal{M},\tilde{u},R)$ and $(\mathcal{Z},\widetilde{w},\mathcal{Q})$ be soft-cs's and $(\mathcal{N},\tilde{v},\mathcal{K})$ be a $\mathcal{T}_{\frac{1}{2}}$-soft-cs. If $\psi_\ell:(\mathcal{M},\tilde{u},R) \to (\mathcal{N},\tilde{v},\mathcal{K})$ is a soft continuous and $\varphi_q:(\mathcal{N},\tilde{v},\mathcal{K}) \to (\mathcal{Z},\widetilde{w},\mathcal{Q})$ is a $\mathcal{g}$-soft continuous, then $\varphi_q o \psi_\ell$ is a $\mathcal{g}$- soft continuous.

Proof. Let $\mathcal{H}_Q$ be a soft c-set of $(\mathcal{Z},\widetilde{w},\mathcal{Q})$. Since φ_q is $\mathcal{g}$-soft continuous, then $\varphi_q^{-1}(H_Q)$ is soft $\mathcal{g}c$-set of $(\mathcal{N},\tilde{v},\mathcal{K})$. Since $(\mathcal{N},\tilde{v},\mathcal{K})$ is $\mathcal{T}_{\frac{1}{2}}$-soft- cs, then $\varphi_q^{-1}(H_Q)$ is a soft c-set of $(\mathcal{N},\tilde{v},\mathcal{K})$. Since ψ_ℓ is soft continuous, then $\psi_\ell^{-1}\left(\varphi_q^{-1}(H_Q)\right) = \left(\varphi_q o \psi_\ell\right)^{-1}(H_Q)$ is a soft c-set of $(\mathcal{M},\tilde{u},R)$. Since every soft c-set is soft $\mathcal{g}c$-set, then $\left(\varphi_q o \psi_\ell\right)^{-1}(H_Q)$ is a soft $\mathcal{g}c$-set of $(\mathcal{M},\tilde{u},R)$. Thus, $\varphi_q o \psi_\ell$ is a $\mathcal{g}$- soft continuous. □

Proposition 3.7 Let $(\mathcal{M},\tilde{u},R)$ and $(\mathcal{Z},\widetilde{w},\mathcal{Q})$ be soft- cs's and $(\mathcal{N},\tilde{v},\mathcal{K})$ be a $\mathcal{T}_{\frac{1}{2}}$-soft- cs. If $\psi_\ell:(\mathcal{M},\tilde{u},R) \to (\mathcal{N},\tilde{v},\mathcal{K})$ and $\varphi_q:(\mathcal{N},\tilde{v},\mathcal{K}) \to (\mathcal{Z},\widetilde{w},\mathcal{Q})$ are $\mathcal{g}$-soft continuous mapping, then $\varphi_q o \psi_\ell$ is a $\mathcal{g}$-soft continuous mapping.

Proof. Let $H_{\mathcal{Q}}$ be a soft c-set of $(\mathcal{Z}, \widetilde{w}, \mathcal{Q})$. Since φ_{q} is g-soft continuous, then $\varphi_{q}^{-1}(H_{\mathcal{Q}})$ is soft gc-set of $(\mathcal{N}, \tilde{v}, \mathcal{K})$. But $(\mathcal{N}, \tilde{v}, \mathcal{K})$ is $\mathcal{T}_{\frac{1}{2}}$-soft-cs this implies $\varphi_{q}^{-1}(H_{\mathcal{Q}})$ is a soft c-set. Since ψ_{ℓ} is g-soft continuous, then $\psi_{\ell}^{-1}\left(\varphi_{q}^{-1}(H_{\mathcal{Q}})\right) = \left(\varphi_{q} o \psi_{\ell}\right)^{-1}(H_{\mathcal{Q}})$ is a soft gc-set. Hence, the result. □

Theorem 3.8 Let $\{(\mathcal{M}_{\alpha}, \tilde{u}_{\alpha}, R_{\alpha}) : \alpha \in \mathcal{I}\}$ and $\{(\mathcal{N}_{\alpha}, \widetilde{\omega}_{\alpha}, \mathcal{K}_{\alpha}) : \alpha \in \mathcal{I}\}$ be families of soft-cs's. For each $\alpha \in \mathcal{I}$, let $(\psi_{\ell})_{\alpha} : (\mathcal{M}_{\alpha}, \tilde{u}_{\alpha}, R_{\alpha}) \longrightarrow (\mathcal{N}_{\alpha}, \widetilde{\omega}_{\alpha}, \mathcal{K}_{\alpha})$ be a soft mapping and $\psi_{\ell} : (\prod_{\alpha\in\mathcal{I}} \mathcal{M}_{\alpha}, \otimes\tilde{u}, \prod_{\alpha\in\mathcal{I}} R_{\alpha}) \longrightarrow (\prod_{\alpha\in\mathcal{I}} \mathcal{N}_{\alpha}, \otimes\widetilde{\omega}, \prod_{\alpha\in\mathcal{I}} \mathcal{K}_{\alpha})$ be the soft mapping defined as $\psi_{\ell}((x_{\alpha_{r_\alpha}})_{\alpha\in\mathcal{I}}) = ((\psi_{\ell})_{\alpha}(x_{\alpha_{r_\alpha}}))_{\alpha\in\mathcal{I}}$. If ψ_{ℓ} is g-soft continuous, then $(\psi_{\ell})_{\alpha}$ is g-soft continuous for each $\alpha \in \mathcal{I}$.

Proof. Let $v \in \mathcal{I}$ and let $\mathcal{F}_{\mathcal{K}_v}$ be a soft c-set of $(\mathcal{N}_v, \widetilde{\omega}_v, \mathcal{K}_v)$. Then, from Theorem 2.15, $\mathcal{F}_{\mathcal{K}_v} \times \widetilde{\prod_{\substack{\alpha\neq v\\ \alpha\in\mathcal{I}}} \mathcal{N}_{\alpha}}$ is a soft c-set of $(\prod_{\alpha\in\mathcal{I}} \mathcal{N}_{\alpha},, \otimes\widetilde{\omega}, \prod_{\alpha\in\mathcal{I}} \mathcal{K}_{\alpha})$. Since ψ_{ℓ} is g-soft continuous, then $\psi_{\ell}^{-1}\left(\mathcal{F}_{\mathcal{K}_v} \times \widetilde{\prod_{\substack{\alpha\neq v\\ \alpha\in\mathcal{I}}} \mathcal{N}_{\alpha}}\right) = (\psi_{\ell})_{v}^{-1}(\mathcal{F}_{\mathcal{K}_v}) \times \widetilde{\prod_{\substack{\alpha\neq v\\ \alpha\in\mathcal{I}}} \mathcal{M}_{\alpha}}$ is soft gc-set of $(\prod_{\alpha\in\mathcal{I}} \mathcal{M}_{\alpha}, \otimes\tilde{u}, \prod_{\alpha\in\mathcal{I}} R_{\alpha})$. From Theorem 2.16, $(\psi_{\ell})_{v}^{-1}(\mathcal{F}_{\mathcal{K}_v})$ is soft gc-set of $(\mathcal{M}_v, \tilde{u}_v, R_v)$. Therefore, $(\psi_{\ell})_v$ is g-soft continuous mapping. □

4- Generalized Soft Closed Mappings

In this section, we introduce the notion of generalized soft closed mappings and investigated some of their properties.

Definition 4.1 Let $(\mathcal{M}, \tilde{u}, R)$ and $(\mathcal{N}, \tilde{v}, \mathcal{K})$ be soft-cs's. A soft mapping $\psi_{\ell} : (\mathcal{M}, \tilde{u}, R) \rightarrow (\mathcal{N}, \tilde{v}, \mathcal{K})$ is called a generalized soft closed (g-soft closed, for short), if $\psi_{\ell}(\mathcal{F}_R)$ is soft gc-set of $(\mathcal{N}, \tilde{v}, \mathcal{K})$ for every soft c-set $\mathcal{F}_R$ of $(\mathcal{M}, \tilde{u}, R)$.

Proposition 4.2 Every soft closed mapping is g-soft closed mapping.

Proof Let ψ_{ℓ} be a soft closed mapping from a soft- cs $(\mathcal{M}, \tilde{u}, R)$ into a soft- cs $(\mathcal{N}, \tilde{v}, \mathcal{K})$. Let $\mathcal{F}_R$ be a soft c-set of $(\mathcal{M}, \tilde{u}, R)$. Since ψ_{ℓ} be a soft closed mapping, then $\psi_{\ell}(\mathcal{F}_R)$ is soft c-set of $(\mathcal{N}, \tilde{v}, \mathcal{K})$. This implies $\psi_{\ell}(\mathcal{F}_R)$ is soft gc-set of $(\mathcal{N}, \tilde{v}, \mathcal{K})$. Hence, ψ_{ℓ} is g-soft closed mapping. □

The converse of Proposition 4.2 is not true as the following example shows.

Example 4.3 Let $\mathcal{M} = \{a, b, c\}, R = \{r_1, r_2\}$ and $\mathcal{N} = \{x, y, z\}$, $\mathcal{K} = \{k_1, k_2\}$. Let $\tilde{u} : \mathcal{SS}(\mathcal{M}, R) \longrightarrow \mathcal{SS}(\mathcal{M}, R)$ and $\tilde{v} : \mathcal{SS}(\mathcal{N}, \mathcal{K}) \longrightarrow \mathcal{SS}(\mathcal{N}, \mathcal{K})$ be soft-co's defined as follows:

$$\tilde{u}(\mathcal{F}_R) = \begin{cases} \widetilde{\Phi}_R & if\ \mathcal{F}_R = \widetilde{\Phi}_R, \\ \{(r_1,\{a\}),(r_2,\{a\})\} & if\ \mathcal{F}_R \sqsubseteq \{(r_1,\{a\}),(r_2,\{a\})\}, \\ \widetilde{\mathcal{M}} & otherwise. \end{cases}$$

$$\tilde{v}(G_{\mathcal{K}}) = \begin{cases} \widetilde{\Phi}_{\mathcal{K}} & if\, G_{\mathcal{K}} = \widetilde{\Phi}_{\mathcal{K}}, \\ \{(k_1,\{x\})\} & if\ G_{\mathcal{K}} = \{(k_1,\{x\})\}, \\ \widetilde{\mathcal{N}} & otherwise. \end{cases}$$

Clearly, $(\mathcal{M},\tilde{u},R)$ and $(\mathcal{N},\tilde{v},\mathcal{K})$ are soft-cs's. Then, the soft mapping $\psi_\ell:(\mathcal{M},\tilde{u},R) \to (\mathcal{N},\tilde{v},\mathcal{K})$ defined as $\psi(a)=x, \psi(b)=z, \psi(c)=y$ and $\ell(r_1)=k_1$, $\ell(r_2)=k_2$ is g-soft closed mapping. But ψ_ℓ is not soft closed mapping, because there exists a soft c-set $\mathcal{F}_R = \{(r_1,\{a\}),(r_2,\{a\})\} \in \mathcal{SS}(\mathcal{M},R)$ such that $\psi_\ell(\mathcal{F}_R) = \{(k_1,\{x\}),(k_2,\{x\})\}$ is not soft c-set of $(\mathcal{N},\tilde{v},\mathcal{K})$.

The following theorem gives a characterization for the definition of g-soft closed mapping.

Theorem 4.4 Let $(\mathcal{M},\tilde{u},R)$ and $(\mathcal{N},\tilde{v},\mathcal{K})$ be soft-cs's. A soft mapping $\psi_\ell:(\mathcal{M},\tilde{u},R) \to (\mathcal{N},\tilde{v},\mathcal{K})$ is g-soft closed if and only if for each soft set $\mathcal{F}_{\mathcal{K}} \in \mathcal{SS}(\mathcal{N},\mathcal{K})$ and each soft σ-set G_R of $(\mathcal{M},\tilde{u},R)$ with $\psi_\ell^{-1}(\mathcal{F}_{\mathcal{K}}) \sqsubseteq G_R$, there is a soft $g\sigma$-set $H_{\mathcal{K}}$ of $(\mathcal{N},\tilde{v},\mathcal{K})$ such that $\mathcal{F}_{\mathcal{K}} \sqsubseteq H_{\mathcal{K}}$ and $\psi_\ell^{-1}(H_{\mathcal{K}}) \sqsubseteq G_R$.

Proof. Suppose ψ_ℓ is g-soft closed mapping. Let G_R be a soft σ-set of $\mathcal{M}$ such that $\psi_\ell^{-1}(\mathcal{F}_{\mathcal{K}}) \sqsubseteq G_R$. Then $\widetilde{\mathcal{M}} - G_R$ is soft c-set of $\mathcal{M}$ and $\psi_\ell(\widetilde{\mathcal{M}} - G_R)$ is soft gc-set of $(\mathcal{N},\tilde{v},\mathcal{K})$. Let $H_{\mathcal{K}} = \widetilde{\mathcal{N}} - \psi_\ell(\widetilde{\mathcal{M}} - G_R)$. Then $H_{\mathcal{K}}$ is a soft $g\sigma$-set of $(\mathcal{N},\tilde{v},\mathcal{K})$ and $\psi_\ell^{-1}(H_{\mathcal{K}}) = \psi_\ell^{-1}\left(\widetilde{\mathcal{N}} - \psi_\ell(\widetilde{\mathcal{M}} - G_R)\right) = \widetilde{\mathcal{M}} - \psi_\ell^{-1}\left(\psi_\ell(\widetilde{\mathcal{M}} - G_R)\right) \sqsubseteq \widetilde{\mathcal{M}} - \left(\widetilde{\mathcal{M}} - G_R\right) = G_R$. Therefore, $H_{\mathcal{K}}$ is a soft $g\sigma$-set, $\mathcal{F}_{\mathcal{K}} \sqsubseteq H_{\mathcal{K}}$ and $\psi_\ell^{-1}(H_{\mathcal{K}}) \sqsubseteq G_R$.

Conversely, suppose that $\mathcal{F}_R$ is a soft c-set of $(\mathcal{M},\tilde{u},R)$. To prove $\psi_\ell(\mathcal{F}_R)$ is a soft gc-set. Then, $\psi_\ell^{-1}\left(\widetilde{\mathcal{N}} - \psi_\ell(\mathcal{F}_R)\right) \sqsubseteq \widetilde{\mathcal{M}} - \mathcal{F}_R$ and $\widetilde{\mathcal{M}} - \mathcal{F}_R$ is a soft σ-set of $\mathcal{M}$. From hypotheses, there exists a soft $g\sigma$-set $H_{\mathcal{K}}$ of $(\mathcal{N},\tilde{v},\mathcal{K})$ such that $\widetilde{\mathcal{N}} - \psi_\ell(\mathcal{F}_R) \sqsubseteq H_{\mathcal{K}}$ and $\psi_\ell^{-1}(H_{\mathcal{K}}) \sqsubseteq \widetilde{\mathcal{M}} - \mathcal{F}_R$. Therefore, $\mathcal{F}_R \sqsubseteq \widetilde{\mathcal{M}} - \psi_\ell^{-1}(H_{\mathcal{K}})$. Hence, $\widetilde{\mathcal{N}} - H_{\mathcal{K}} \sqsubseteq \psi_\ell(\mathcal{F}_R) \sqsubseteq \psi_\ell(\widetilde{\mathcal{M}} - \psi_\ell^{-1}(H_{\mathcal{K}})) \sqsubseteq \widetilde{\mathcal{N}} - H_{\mathcal{K}}$ this implies $\psi_\ell(\mathcal{F}_R) = \widetilde{\mathcal{N}} - H_{\mathcal{K}}$. Thus, $\psi_\ell(\mathcal{F}_R)$ is a soft gc-set. □

Proposition 4.5 Let $(\mathcal{M},\tilde{u},R)$, $(\mathcal{N},\tilde{v},\mathcal{K})$ and $(\mathcal{Z},\widetilde{w},\mathcal{Q})$ be soft-cs's. If $\psi_\ell:(\mathcal{M},\tilde{u},R) \to (\mathcal{N},\tilde{v},\mathcal{K})$ and $\varphi_q:(\mathcal{N},\tilde{v},\mathcal{K}) \to (\mathcal{Z},\widetilde{w},\mathcal{Q})$ are soft mappings. Then

1. If ψ_ℓ is soft closed mapping and φ_q is g-soft closed mapping, then $\varphi_q o \psi_\ell$ is g-soft closed mapping.

2. If $\varphi_{q} o\psi_{\ell}$ is$g$-soft closed mapping and ψ_{ℓ} is soft continuous and surjective, then φ_{q} g-soft closed mapping.
3. If $\varphi_{q} o\psi_{\ell}$ is soft closed mapping and φ_{q} is g-soft continuous and injective, then ψ_{ℓ} is g-soft closed mapping.

Proof.

1. Let $\mathcal{F}_R$ be a soft c-set of $(\mathcal{M},\tilde{u},R)$, since ψ_{ℓ} is soft closed, then $\psi_{\ell}(\mathcal{F}_R)$ is soft c-set of $(\mathcal{N},\tilde{v},\mathcal{K})$. Since φ_{q} is g-soft continuous, then $\varphi_{q}(\psi_{\ell}(\mathcal{F}_R))$ is soft gc-set of $(\mathcal{Z},\widetilde{w},\mathcal{Q})$. This implies $\varphi_{q} o\psi_{\ell}(\mathcal{F}_R)$ is a soft gc-set. Therefore, $\varphi_{q} o\psi_{\ell}$ is g-soft closed mapping.
2. Let $G_{\mathcal{K}}$ be a soft c-set of $(\mathcal{N},\tilde{v},\mathcal{K})$. Since ψ_{ℓ} is a soft continuous, then $\psi_{\ell}^{-1}(G_{\mathcal{K}})$ is soft closed in $(\mathcal{M},\tilde{u},R)$. Since $\varphi_{q} o\psi_{\ell}$ is g -soft closed mapping, then $\varphi_{q} o\psi_{\ell}\left(\psi_{\ell}^{-1}(G_{\mathcal{K}})\right)=\varphi_{q}(\psi_{\ell}\left(\psi_{\ell}^{-1}(G_{\mathcal{K}})\right))$ is soft gc -set in $(\mathcal{Z},\widetilde{w},\mathcal{Q})$. But ψ_{ℓ} is surjection, then $\varphi_{q} o\psi_{\ell}\left(\psi_{\ell}^{-1}(G_{\mathcal{K}})\right)=\varphi_{q}\left(\psi_{\ell}\left(\psi_{\ell}^{-1}(G_{\mathcal{K}})\right)\right)=\varphi_{q}(G_{\mathcal{K}})$. Consequently, $\varphi_{q}(G_{\mathcal{K}})$ is soft gc-set in $(\mathcal{Z},\widetilde{w},\mathcal{Q})$. Hence, φ_{q} is g-soft closed mapping.
3. Let $\mathcal{F}_R$ soft c-set in $(\mathcal{M},\tilde{u},R)$, to prove $\psi_{\ell}(\mathcal{F}_R)$ soft gc-set in $(\mathcal{N},\tilde{v},\mathcal{K})$. Since $\varphi_{q} o\psi_{\ell}$ soft closed mapping, then $(\varphi_{q} o\psi_{\ell})(\mathcal{F}_R)$ is soft c-set in $(\mathcal{Z},\widetilde{w},\mathcal{Q})$ and since φ_{q} is g-soft continuous, then $\varphi_{q}^{-1}((\varphi_{q} o\psi_{\ell})(\mathcal{F}_R))$ is soft gc-set of $(\mathcal{N},\tilde{v},\mathcal{K})$. That means $\varphi_{q}^{-1}(\varphi_{q}(\psi_{\ell}(\mathcal{F}_R))$ is a soft gc-set of $(\mathcal{N},\tilde{v},\mathcal{K})$. Since φ_{q} is one to one, then $\psi_{\ell}(\mathcal{F}_R)$ is soft gc-set in $(\mathcal{N},\tilde{v},\mathcal{K})$. Hence, ψ_{ℓ} is g-soft closed mapping. □

5-Conclusion

In the present work, in soft closure spaces, we have introduced the notions of generalized soft continuous mappings and generalized soft closed mappings and presented their associated properties. In their description, these soft mappings depend on the notion of soft generalized closed sets in soft closure spaces. To clarify some outcomes in our work, some examples are added.

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