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QC-Continuous and QC-Quasi-Continuous Modules

Emad Allawi Shallal
Department of Mathematics, College Of Computer Science and Mathematics
University of AL-Qadisiya, Diwaniya , Iraq
E-mail: emad_a_shallal@yahoo.com

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Abstract

In this work, we introduce the class of QC-Continuous and QC-Quasi-Continuous which are stronger than the class Continuous and Quasi-Continuous modules . where an R-module M is called QC-Continuous if M is fully Extending and has (C2) , an R-module M is called QC-Quasi-Continuous if M is fully Extending and has (C3).

Introduction

Throughout this paper all rings have an identity and modules are unitary. Let R be a ring and M be a left R-module, a submodule N of M is essential in M (denoted by $N \overset{e}{\rightarrow} M$) if $N \cap L \neq (0)$ for any non-zero submodule L of M [6]. An R-module M is uniform if every submodule of M is essential in M . Also, a submodule N of M is closed in M if it has no proper essential extensions in M, that is, if $N \overset{e}{\rightarrow} K \rightarrow M$, then $N = K$ [6]. A submodule N of an R-module M is quasi-closed (for short qc-submodule) if for each $x \in M$ with $x \notin N$, there exists a closed submodule L of M containing N such that $x \notin L$, an R-module M is called fully Extending if every qc-submodule of M is a direct summand of M [7]. An R-module M is semisimple if every submodule of M is direct summand of M [4], a submodule N of an R-module M is called stable if $f(N) \subseteq N$ for each R-homomorphism $f: N \rightarrow M$, an R-module M is fully stable if every submodule of M is stable [1], an R-module M is called Strongly Extending if every submodule of M is essential in stable direct summand of M, equivalently, every closed submodule of M is a stable direct summand [2]. For a module M consider the following conditions:

(C_1): Every closed submodule of M is a direct summand of M.

(C_1'): Every quasi-closed submodule of M is a direct summand of M.

(C_2): For each submodule $N$of M such that $N \approx K \overset{\oplus}{\rightarrow} M$, then $N \overset{\oplus}{\rightarrow} M$.

(C_3): For each submodules N and K such that $N \overset{\oplus}{\rightarrow} M$ and $K \overset{\oplus}{\rightarrow} M$ with $N \cap K = (0)$, then $N \oplus K \overset{\oplus}{\rightarrow} M$.

An R-module M is called Extending or CS if it satisfies the condition (C_1), equivalently, every submodule of M is essential in a direct summand [5]. An R-module M is called Continuous if it satisfies the conditions (C_1) and (C_2), an R-module M is called Quasi-Continuous, an R-module M is called Continuous if it satisfies the conditions (C_1) and (C_3) [8]. I.M.A.Hadi and M.A.Ahmed introduced the concept of quasi-closed submodule as a proper generalization of closed submodule [7]. We introduce and study the class of QC-Continuous and QC-Quasi-Continuous which are stronger than the class Continuous and Quasi-Continuous modules.

Definition(1):- An R-module M is called QC-Continuous module if M has (C'_1) and (C_2).

Definition(2):- An R-module M is called QC-Quasi-Continuous module if M has (C'_1) and (C_3).

Remarks and Examples(3):-

1- Every semisimple R-module is QC-Continuous and QC-Quasi-Continuous .

2- The Z-module $M = Z \oplus Z_2$ is not fully Extending [7], hence neither QC-Continuous nor QC-Quasi-Continuous.

3- The Z-module $M = Z_2 \oplus Z_4$ is not fully Extending [7], hence neither QC-Continuous nor QC-Quasi-Continuous.

4- Every Uniform R-module is QC-Quasi-Continuous. Since if M is a Uniform R-module, then M is a fully extending module [7,example 3.2]. Let N and K be two submodule of M such that $N \overset{\oplus}{\rightarrow} M$ and $K \overset{\oplus}{\rightarrow} M$ with $N \cap K = (0)$, since the only direct summand of M are M and (0), then $N \oplus K \overset{\oplus}{\rightarrow} M$, hence M has (C_2). However uniform R-module need not be QC-Continuous, for example see (5).

5- S. H. Mohamed and B. J. Muller in [8] show that if an R-module M has (C_2), then M has (C_3), therefore every QC-Continuous module is QC-Quasi-Continuous. The converse is not true in general as the following example show: The Z-module Z is QC-Quasi-Continuous but not QC-Continuous because the submodule 2Z isomorphic to Z while 2Z is not a direct summand of Z, therefore Z not QC-Continuous.

6- Every QC-Quasi-Continuous module is Quasi-Continuous, the converse is not true in general, for example, the Z-module $M = Z \oplus Z_2$ is Quasi-Continuous but not QC-Quasi-Continuous (see (1)).
7- Every Strongly Extending is fully Extending, the converse is not true in general, for example , the Z-module $M = Z_2 \oplus Z_2$ is semisimple module, hence is fully Extending (QC-Continuous ,QC-Quasi-Continuous), since $N = Z_2 \oplus (\bar{0})$ is closed submodule of M which is not stable direct submodule of M. Hence M is not Strongly Extending.
8- Every QC-Continuous module is Continuous, the converse is not true in general. Consider $M = Z_{p^\infty} \oplus Z_{p^\infty}$ as Z-module . Since $Z_{p^\infty}$is injective Z-module, then $M = Z_{p^\infty} \oplus Z_{p^\infty}$ is injective, hence M is Continuous [8], but M is not fully Extending [7], hence M is not QC-Continuous .
9- Direct sum of QC-Continuous (QC-Quasi-Continuous) module need not be QC-Continuous (QC-Quasi-Continuous), for example, the Z-module Z_2 and Z_4 are QC-Continuous (QC-Quasi-Continuous) module, but $Z_2 \oplus Z_4$ is not QC-Continuous (QC-Quasi-Continuous) module.

Proposition(4) :- Let N be a direct summand of an R-module M then:

(1) N is QC-Continuous if M is Continuous.
(2) N is QC-Quasi-Continuous if M is Quasi-Continuous .

Proof: Direct from [7,proposition 3.3] and [8,proposition 2.7].

Corollary(5) :- If $M \oplus M$ is QC-Continuous (QC-Quasi-Continuous) module, then M is QC-Continuous (QC-Quasi-Continuous) .

Recall that an R-module M is injective if for each R-monomorphism $f: A \to B$(where A and B are R-module) and for each R-homomorphism $g: A \to M$, there exists an R-homomorphism $h: B \to M$ such that $h \circ f = g$ [4]. Every injective R-module is Continuous, hence Quasi-continuous [8], but injective module need not be QC-Continuous (QC-Quasi-Continuous) , (see Remarks and Examples(3),8), an R-module M is called prime if $ann(M) = ann(N)$ for each nonzero submodule N of M.

Proposition(6) :- Every injective prime module is QC-Continuous(QC-Quasi-continuous).

Proof: Let M be an injective prime module. By [7,proposition 3.16], M is fully Extending. On the other hand, every injective module is Continuous [8], therefore M has (C_2), hence M is QC-Continuous (QC-Quasi-Continuous).

We conclude from this paper the following chart of implications:

$$\begin{array}{ccccccc} Injective\ Prime \Longrightarrow QC-Continuous & \Longrightarrow & QC-Quasi-Continuous & \Longrightarrow & fully\ Extending \\ \Downarrow & & \Downarrow & & \Downarrow \\ Injective \Longrightarrow Continuous & \Longrightarrow & Quasi-Continuous & \Longrightarrow & Extending \end{array}$$

Proposition(7) :- Let M_1 and M_2 be two isomorphic R-modules. Then M_1 is QC-Continuous R-module if and only if M_2 is QC-Continuous R-module.

Proof: Suppose M_1 is QC-Continuous R-module, then M_1 is fully Extending and by [7,example (3.2),3] M_2 is fully Extending. Let N *and* K two submodule of M_2 such that $N \approx K \overset{\oplus}{\to} M_2$, since M_1 and M_2 isomorphic, there exist R-isomorphism $f: M_2 \to M_1$, then $K \approx f(K) \overset{\oplus}{\to} M_1$ but $N \approx f(N)$, then $f(N) \approx f(K) \overset{\oplus}{\to} M_1$, since M_1 is QC-Continuous, then $f(N) \overset{\oplus}{\to} M_1$, hence $N \overset{\oplus}{\to} M_2$.

Proposition(8) :- Let M_1 and M_2 be two isomorphic R-modules. Then M_1 is QC-Quasi-Continuous R-module if and only if M_2 is QC-Quasi-Continuous R-module.

Proof: Suppose M_1 is QC-Continuous R-module, Let N and K are two direct summand M_2 with $N \cap K = (0)$, since M_1 and M_2 isomorphic, there exist R-isomorphism $f: M_2 \to M_1$, hence $f(N) \overset{\oplus}{\to} M_1$ *and* $f(K) \overset{\oplus}{\to} M_1$ with $f(N) \cap f(K) = (0)$, then $f(N) \oplus f(K) \overset{\oplus}{\to} M_1$, so $f(N+K) \overset{\oplus}{\to} M_1$, hence $N \oplus K$ is a direct summand of M_2.

An R-module M is called non-singular if $Z_M(M) = 0$, where $Z_M(M)$is the set of all $m \in M$ such that $ann(m) \overset{e}{\to} R$ [6], and M is called polyform module if for each submodule N of M and for each homomorphism $f: N \to M$,ker (f) is closed submodule in N [5].

Proposition(9) :- Let M be a polyform R-module, then:

(1) M is QC-Continuous if and only if M is Continuous.
(2) M is QC-Quasi-Continuous if and only if M is Quasi-Continuous.
(3)

Proof: If M is polyform module, then the intersection of two summands is again a summand [3,lemma 11] and by [7,theorem 3.7], M is fully extending.

Corollary(10) :- Let M be a non-singular R-module, then:

(1) M is QC-Continuous if and only if M is Continuous.
(2) M is QC-Quasi-Continuous if and only if M is Quasi-Continuous.

Proposition(11) :- For an indecomposable module M , the following are equivalent:

(1) M is strongly extending.
(2) M is uniform.
(3) M is QC-Continuous.
(4) M is QC-Quasi-Continuous.
(5) M is fully extending.
(6) M is extending.

Proof: (1)⟹(2): [2,Remark and Examples 2.1.2]

(2)⟹(3) , (3)⟹(4),(4)⟹(5) and (5)⟹(6): Clear from definitions.

(6)⟹(1): for every submodule N of M, there exist a direct summand K of M such that N essential of K but the only summand of M are M and (0), then N essential in M.

Proposition(12) :- Let M be a fully stable R-module M, the following are equivalent:

(1) M is Strongly Extending.
(2) M is fully Extending.
(3) M is QC-Continuous.
(4) M is QC-Quasi-Continuous.
(5) M is Extending.

Proof:

(1)⟹(2): [7]

(2)⟹(3) : If M is a fully Extending R-module, since every fully stable has (C_2) [1], hence M is QC-Continuous.

(3)⟹(4) and (4)⟹(5): Clear from definitions.

(5)⟹(1): [7,Corollary 3.11].

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