Journal of AL-Qadisiyah for computer science and mathematics
Vol.7 No.2 Year 2015

Page 132-145

On Asymptotic Behavior of Metric Dynamical Systems

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Recived :11\3\2015 **Revised : 24\5\2015** **Accepted : 16\6\2015**

Abstract

In this paper we shall study the asymptotic behavior of metric dynamical systems when the time domain is any locally compact topological group. We investigate some new properties of ergodic, mixing, and weakly mixing metric dynamical system.

Key words metric dynamical system, ergodic, mixing and weakly mixing.

Mathematics Subject Classification: 37A25 .

Introduction

Theergodicity, mixing , and weakly mixing [1] are very important tools in the study of metric dynamical system and to describe the behavior of many phenomena's in biology, chemistry, biochemistry, physics,…etc. when the time goes to infinity. The mixing, weakly mixing and exact are define in case of the time domain is an infinite. In this paper we define these concepts when the time domain is any locally compact topological group. That is, we shall study the asymptoticbehaveor of metric dynamical systems when the time domain is any locally compact topological group. We investigate some new properties ofergodic, mixing and weakly mixing metric dynamical system.

There are many studies of the asymptotic behavior of metric dynamical systems with respect to many authors. In the following we shall list some of these studies.H. Furstenberg , Y. Katznelson and D. Ornstein in 1982[2]they give an exposition, as widely accessible as possible, ofthe ergodic theoretic proof of the (Let $A \subseteq \mathbb{Z}$ be a subset of the integers of positive upper density,then Acontains arbitrarily long arithmetic progressions).John Earman and MiklosRedei in 1996[3] they argue that, contrary to some analyses in the philosophy of science liter-ature, ergodic theory falls short in explaining the success of classical equilibrium sta-tistical mechanics. H. R. Biswas and M. S. Islam in 2012 [4] study one dimensional linear and non-linear maps and its dynamical behavior. They study measure theoretic-cal dynamical behavior of the maps.

P. J. Mitkowski, W.Mitkowski in 2012[5] discuss basic notions of the ergodic theory approach to chaos. Throughout this paper G be any locally compact group [6,7] and it is considered as a measurable space with Haar measure[6-9] A sequence $\{g_n\}$ in a topological (semi-)group G is said to be diverge [6,7] (written $g_n \to \infty$ as $n \to \infty$) if for any compact subset K of G there exists $n_0 \in \mathbb{N}$ such that $g_n \notin K$ for $n \geq n_0$.The triple $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space[8,9]. The paper is organized as follows. In Sec.1 we state the definition of MDS and give some examples. In sec.2 the study of methods for constructing new MDS's from given one. In sec.3 we study the ergodicity of metric dynamical systems in terms of general locally compact topological groups.In Sec.4 In this section we study mixingandweakly mixing of MDS and give some properties of such systems.

1.Metric Dynamical Systems(Definitions and Examples)

Inthissectionwe state the definition of metric dynamical system(shortly, MDS)in terms of locally compact group and give some examples.

Definition1.1 [10,11]The 5-tuple $(G, \Omega, \mathcal{F}, \mathbb{P}, \theta)$ is called ***metric dynamical system***(Shortly MDS) if $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and

(i) $\theta: G \times \Omega \to \Omega$ is $(\mathcal{B}(G) \otimes \mathcal{F}, \mathcal{F})$ $-$measurable,

(ii) $\theta(e, \omega) = Id_{\Omega}$,

(iii) $\theta(g * h, \omega) = \theta(g, \theta(h, \omega))$ and

(iv) $\mathbb{P}(\theta(g)F) = \mathbb{P}(F)$, for every $F \in \mathcal{F}$ and every $\in G$.

Note that we write $\theta: G \times \Omega \to \Omega$ either in the form$\theta(g, \omega)$ (as a function of two variable or in the form$\theta(g)\omega$.

Example 1.2 [11]Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with

$\Omega \coloneqq \{\omega \in C(\mathbb{R}, l^2 \times \mathbb{R}) : \omega(0) = 0\}$.

$\mathcal{F}$is the Borel σ $-$algebra generated by the compact-open topology on $C(\mathbb{R}, l^2 \times \mathbb{R})$, $\mathbb{P}$ is the Wiener measure on $(\Omega, \mathcal{F})$. Define$\theta: \mathbb{R} \times \Omega \to \Omega$by

$\theta(t)\omega(\cdot) = \omega(\cdot + t) - \omega(t)$, $t \in \mathbb{R}$.

Then $(\mathbb{R}, \Omega, \mathcal{F}, \mathbb{P}, \theta)$ is anMDS.

Example 1.3[11](Ordinary Differential Equations). An MDS can be also generated by ordinary differential equations (ODE). Let us consider a system ofODEs in $\mathbb{R}^n$:

$\frac{dx_i}{dt} = f_i(x_1, x_2, \dots, x_n)$, $i = 1, \dots, n$.(1)

Assume that the Cauchy problem for this system is well-posed. We define$\theta: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ by$\theta(t, x) \coloneqq x(t)$, where $x(t)$ is the solution of (1) with$x(0) = x$. Assume that a nonnegative smooth function $\rho(x_1, x_2, \dots, x_n)$ satisfiesthe stationary Liouville equation

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \big(\rho(x_1, x_2, \dots, x_n), f_i(x_1, x_2, \dots, x_n)\big) = 0 \quad (2)$$

and possesses the property$\int_{\mathbb{R}^n} \rho(x) dx = 1$. Then $\rho(x)$is a density of a probability measure on $\mathbb{R}^n$. By Liouville's theorem

$$\int_{\mathbb{R}^n} f(\theta(t)x)\rho(x)dx = \int_{\mathbb{R}^n} f(x)\rho(x)dx$$

for any bounded continuous function $f(x)$on $\mathbb{R}^n$ and therefore in this situation an MDS arises with $\Omega \coloneqq \mathbb{R}^n$, $\mathcal{F} \coloneqq \mathcal{B}(\mathbb{R})$and $\mathbb{P}(dx) = \rho(x)dx$. Here$\mathcal{B}(\mathbb{R})$ is the Borel σ −algebra of sets in $\mathbb{R}$**.**

2. Constriction of Metric Dynamical Systems.

We begin now the study of methods for constructing new MDS's from given one. This will lead in later sections in terms of simpler, more familiar ones**.**

The following definition is a development of the concept of "fiber preserving action" [12].

Definition 2.1 The action of the metric dynamical system $\Phi = (G, \Omega, \mathcal{F}, \mathbb{P}, \theta)$ is said to be a ***measurablefiber preserving action*** with respect to surjective measurable mapping ξ of a measurable space Ω onto the measurable space Ξ if the following condition satisfies : if $\xi(\omega_1) = \xi(\omega_2)$, then $\xi(\theta(g)\omega_1) = \xi(\theta(g)\omega_2)$ for every $g \in \alpha^{-1}(h)$ and $\omega_1, \omega_2 \in \Omega$.

Definition 2.2[12] Let $(G, \Omega, \mathcal{F}, \mathbb{P}, \theta)$be a metric dynamical system , H be a topological group and $\alpha: G \to H$ be a surjective map. Then $\alpha^{-1}(h)$ is said to be ***acts constantly*** on every point of Ω if $\theta(g_1, \omega) = \theta(g_2, \omega)$ for every $g_1, g_2 \in \alpha^{-1}(h)$ and $\omega \in \Omega$.

In the following definition we make a simple modification on Definition 1.2 in [1].

Definition2.3 Let $(G_1,\Omega_1,\mathcal{F}_1,\mathbb{P}_1,\theta_1)$ and $(G_2,\Omega_2,\mathcal{F}_2,\mathbb{P}_2,\theta_2)$be two MDS'sif there existbijectivemeasurable transformation $\xi:\Omega_1\to\Omega_2$ and group isomorphism$\alpha: G_1\to G_2$such that for all $B_2\in\mathcal{F}_2$ and $g_2\in G_2$ we have

(1)$\mathbb{P}_1\left(\xi^{-1}(B_2)\right)=\mathbb{P}_2(B_2)$,

(2)$\mathbb{P}_1[\theta_1\left(\alpha^{-1}(g_2),\xi^{-1}(B_2)\right)\Delta\xi^{-1}(\theta_2(g_2,B_2))]=0$.

then we say that $(G_1,\Omega_1,\mathcal{F}_1,\mathbb{P}_1,\theta_1)$ and $(G_2,\Omega_2,\mathcal{F}_2,\mathbb{P}_2,\theta_2)$, are equivalent via (ξ,α) and we write $(G_1,\Omega_1,\mathcal{F}_1,\mathbb{P}_1,\theta_1)\cong_{(\xi,\alpha)}(G_2,\Omega_2,\mathcal{F}_2,\mathbb{P}_2,\theta_2)$.

Theorem 2.4[13]Let $(G,\Omega,\mathcal{F},\mathbb{P},\theta)$ be a metric dynamical system,H be a topological group ,Ξ be measurable space, $\alpha: G\to H$ be surjective open group homomorphism , $\xi:\Omega\to\Xi$ be surjective measurable function. If $\alpha^{-1}(h)$ is acts constantly on every point of X and θ is fiber preserving action with respect to ξ, , then $(\Xi,\Sigma,\mathbb{Q},\sigma)$ is metric random dynamical system where

(1) $(\Xi,\Sigma,\mathbb{Q})$ is probability space with $\mathbb{Q}:\Sigma\to[0,1]$ defined by $\mathbb{Q}(B):=\mathbb{P}(\xi^{-1}(B))\forall B\in\Sigma$.

(2) The action $\sigma: H\times\Xi\to\Xi$ is defined by$\sigma(h,\varpi):=\xi(\theta(g,\omega)),\forall(h,\varpi)\in H\times\Xi$.

In the following theorem we study the direct product $\Phi_1\otimes\Phi_2$ of MDS's Φ_1 and Φ_2. We have already used a special case of this construction, namely$\Phi\otimes\Phi=\Phi^2$.

Theorem2.5[13] If $\Phi_1=(G_1,\Omega_1,\mathcal{F}_1,\mathbb{P}_1,\theta_1),\Phi_2(G_2,\Omega_2,\mathcal{F}_2,\mathbb{P}_2,\theta_2)$ are two MDS's, then their direct product$\Phi_1\otimes\Phi_2$ is a MDS.

Theorem2.6 Let $(G,\Omega,\mathcal{F},\mathbb{P},\theta)$ be anMDS. If $\Omega^*\subset\Omega$ with $\mathbb{P}(\Omega^*)>0$, then so is $(G,\Omega^*,\mathcal{F}^*,\mathbb{P}^*,\theta^*)$ where $\mathcal{F}^*:=\mathcal{F}\cap\Omega^*$, $\mathbb{P}^*:=\mathbb{P}\,/\,\mathbb{P}(\Omega^*)$ and $\theta^*:=\theta|_{G\times\Omega^*}$.

Proof The collection $\mathcal{F}^* \coloneqq \mathcal{F} \cap \Omega^*$ is $\sigma -$algebra on Ω^*.It is easy to see that $\mathbb{P}^*$ is a probability on Ω^* and so $(\Omega^*, \mathcal{F}^*, \mathbb{P}^*)$ is probability space. Now, $\theta^*(e, \omega) = \theta(e, \omega) = \omega$, for every $\omega \in \Omega^*$. If $g, h \in G$, then

$\theta^*(gh, \omega) = \theta(gh, \omega) = \theta\big(g, \theta(h, \omega)\big) = \theta^*(g, \theta^*(h, \omega))$.

If $A^* \in \mathcal{F}^*$, then there exists $A \in \mathcal{F}$ such that $A^* = A \cap \Omega^*$. Since $\Omega^* \in \mathcal{F}$, then so is A^*. Thus $\theta^*(g, A^*) = \theta(g, A^*)$ and consequently $\mathbb{P}^*\big(\theta^*(g, A^*)\big) = \mathbb{P}^*(A^*)$.This means that θ^*preserve-measure. Since θ is $(\mathcal{B}(G) \otimes \mathcal{F}, \mathcal{F}) -$measurable, then$\theta^*$ is $(\mathcal{B}(G) \otimes \mathcal{F}^*, \mathcal{F}^*) -$measurable.

3. Ergodicity of Metric Dynamical Systems

In this section, we study the ergodicity of metric dynamical systems in terms of general locally compact topological groups. First we shall state the definition of invariant set and give some new properties of such set.

Definition 3.1 [1] Let $(G, \Omega, \mathcal{F}, \mathbb{P}, \theta)$ be a MDS. A set $F \in \mathcal{F}$ is said to be ***θ −invariantunder***$\mathbb{P}$ if $\mathbb{P}(\theta(g)F \vartriangle F) = 0$, for all $g \in G$.

Theorem 3.2 Let $(G_1, \Omega_1, \mathcal{F}_1, \mathbb{P}_1, \theta_1)$ and $(G_2, \Omega_2, \mathcal{F}_2, \mathbb{P}_2, \theta_2)$ be equivalent MDS's via (ψ, α) . If $F_1 \in \mathcal{F}_1$ is $\theta_1 -$invariant under the $\mathbb{P}_1$, then $F_2 \coloneqq \psi(F_1) \in \mathcal{F}_2$ is $\theta_2 -$invariant under the $\mathbb{P}_2$.

Proof First, note that since ψ is bijective and measurable, then $F_2 \coloneqq \psi(F_1) \in \mathcal{F}_2$. To show that F_2 is $\theta_2 -$invariant under the $\mathbb{P}_2$. Let $g_2 \in G_2$. Since α is group isomorphism, then there exists $g_1 \in G_1$ such that $g_1 = \alpha^{-1}(g_2)$ or equivalently$g_2 = \alpha(g_1)$. Now

$$\mathbb{P}_2[\theta_2(g_2)F_2 \vartriangle F_2] = \mathbb{P}_1[\psi^{-1}(\theta_2(g_2)F_2 \vartriangle F_2)]$$

$$= \mathbb{P}_1[\psi^{-1}(\theta_2(g_2)F_2) \vartriangle \psi^{-1}(F_2)]$$

$= \mathbb{P}_1[\psi^{-1}(\theta_2(g_2)F_2) \vartriangle \theta_1(\alpha^{-1}(g_2))\psi^{-1}(F_2)]$.

By Definition2.3, we get $\mathbb{P}_2[\theta_2(g_2)F_2 \vartriangle F_2] = 0$ and this complete the proof.

Theorem 3.3 Let $(G_1, \Omega_1, \mathcal{F}_1, \mathbb{P}_1, \theta_1)$ and $(G_2, \Omega_2, \mathcal{F}_2, \mathbb{P}_2, \theta_2)$ be two MDS's. Then $F_1 \times F_2 \in \mathcal{F}_1 \otimes \mathcal{F}_2$ is $\theta_1 \times \theta_2$ −invariant under $\mathbb{P}_1 \otimes \mathbb{P}_2$ if, and only if either $F_1 \in \mathcal{F}_1$ is θ_1 −invariant under $\mathbb{P}_1$ or $F_2 \in \mathcal{F}_2$ is θ_2 −invariant under $\mathbb{P}_2$.

Proof Let $(g_1, g_2) \in G_1 \times G_2$, then$g_1 \in G_1$ and $g_2 \in G_2$. Now

$$\begin{aligned} &\mathbb{P}_1 \otimes \mathbb{P}_2\left[\theta_1 \times \theta_2\big((g_1, g_2)\big)F_1 \times F_2 \vartriangle F_1 \times F_2\right] \\ &= \mathbb{P}_1 \otimes \mathbb{P}_2[\theta_1(g_1)F_1 \times \theta_2(g_2)F_2 \vartriangle F_1 \times F_2] \\ &= \mathbb{P}_1 \otimes \mathbb{P}_2[\theta_1(g_1)F_1 \times \theta_2(g_2)F_2 \vartriangle F_1 \times F_2] \\ &= \mathbb{P}_1 \otimes \mathbb{P}_2[\theta_1(g_1)F_1 \vartriangle F_1 \times \theta_2(g_2)F_2 \vartriangle F_2] \\ &= \mathbb{P}_1[\theta_1(g_1)F_1 \vartriangle F_1] \times \mathbb{P}_2[\theta_2(g_2)F_2 \vartriangle F_2]. \end{aligned}$$

From this we get our result.

Definition 3.4[1] An MDS $(G, \Omega, \mathcal{F}, \mathbb{P}, \theta)$ is said to be ***ergodic*** under $\mathbb{P}$ if for any θ −invariant set $F \in \mathcal{F}$ under $\mathbb{P}$ we have either $\mathbb{P}(F) = 0$ or $\mathbb{P}(F) = 1$. That is, if $\mathbb{P}(\theta(g)F \vartriangle F) = 0$, implies $\mathbb{P}(F) = 0$ or $\mathbb{P}(F) = 1$, for all $F \in \mathcal{F}$ and all $g \in S$, where S is a syndetic subset of G.

Theorem 3.5 Let $(G_1, \Omega_1, \mathcal{F}_1, \mathbb{P}_1, \theta_1)$ and $(G_2, \Omega_2, \mathcal{F}_2, \mathbb{P}_2, \theta_2)$ be equivalent MDS's via (ψ, α) . If $(G_1, \Omega_1, \mathcal{F}_1, \mathbb{P}_1, \theta_1)$ is ergodic, then so is $(G_2, \Omega_2, \mathcal{F}_2, \mathbb{P}_2, \theta_2)$.

Proof Suppose that $(G_1, \Omega_1, \mathcal{F}_1, \mathbb{P}_1, \theta_1)$ is ergodic. Let $F_2 \in \mathcal{F}_2$ be a θ_2 −invariant under $\mathbb{P}_2$. Then $\psi^{-1}(F_2) \in \mathcal{F}_1$ is θ_1 −invariant under $\mathbb{P}_1$. By hypothesis either $\mathbb{P}_1\big(\psi^{-1}(F_2)\big) = 0$ or 1. But by Definition 3.4, then we have $\mathbb{P}_2(F_2) = 0$ or 1. This complete the proof.

Definition 3.6[1] An MDS $(G, \Omega, \mathcal{F}, \mathbb{P}, \theta)$ is said to be ***uniquely ergodic*** under $\mathbb{P}$ if there is exactly one θ −invarint probability measure $\mathbb{P}$ on Ω.

Theorem3.7 Let $(G, \Omega, \mathcal{F}, \mathbb{P}, \theta) \cong_{(\alpha,\Phi)} (H, \Xi, \mathcal{G}, \mathbb{Q}, \sigma)$. If $(G, \Omega, \mathcal{F}, \mathbb{P}, \theta)$ is uniquely ergodic, then so is $(H, \Xi, \mathcal{G}, \mathbb{Q}, \sigma)$.

Proof. Suppose that $(G, \Omega, \mathcal{F}, \mathbb{P}, \theta)$ is uniquely ergodic. To show that $(H, \Xi, \mathcal{G}, \mathbb{Q}, \sigma)$ is uniquely ergodic. By hypothesis there is exactly one θ −invarint probability measure $\mathbb{P}$ on Ω. That is $\mathbb{P}(\theta(g)A) = \mathbb{P}(A)$ for every $A \in \mathcal{F}$. If $B \in \mathcal{G}$,then $\Phi^{-1}(B) \in \mathcal{F}$ since Φ is measurable. Now,

$\mathbb{Q}(\sigma(h)B) = \mathbb{Q}(\Phi(\theta(\alpha^{-1}(h))\Phi^{-1}(B))) = \mathbb{P}(\Phi^{-1}(B)) = \mathbb{Q}(B)$.

Finally, weneedto show that $\mathbb{Q}$ is unique. Let $\mathbb{Q}_1$ be any σ −invarint probability measure on Ξ such that $\mathbb{Q}_1 \neq \mathbb{Q}_0$. Then there exists $B \in \mathcal{G}$ such that $\mathbb{Q}_1(B) \neq \mathbb{Q}_0(B)$.Then $\mathbb{Q} \circ \Phi^{-1}(B) \neq \mathbb{Q} \circ \Phi^{-1}(B)$, i.e.,$\mathbb{Q}(\Phi^{-1}(B)) \neq \mathbb{Q}(\Phi^{-1}(B))$.But that is a contradiction. Therefore $(H, \Xi, \mathcal{G}, \mathbb{Q}, \sigma)$ is uniquely ergodic.

<u>4.Mixingand Weakly Mixing Metric Dynamical Systems</u>

In this section we introduce a new definition of mixing metric dynamical systemswhere the time domain is a general locally compact group and give some properties of such systems. The following two definitionsare generalizing of Definition 1.5 and Definition 1.6 in [1] respectively.

Definition4.1 An MDS$(G, \Omega, \mathcal{F}, \theta, \mathbb{P})$is said to be mixing if for each $A, B \in \mathcal{F}$, there exists a divergent sequence $\{g_n\}$ in G such that

$$\lim_{k\to\infty} \mathbb{P}(A \cap \theta({g_k}^{-1})B) = \mathbb{P}(A)\mathbb{P}(B). \qquad (1)$$

We shall say that a set J of positive integers has density zero [1]if the number of elements in $J \cap \{1,2, \dots, n\}$ divided by n tends to 0 as $n \to \infty$.

Definition 4.2 An MDS $(G, \Omega, \mathcal{F}, \theta, \mathbb{P})$ is said to be weakly mixing if for each $A, B \in \mathcal{F}$, there exists a divergent sequence $\{g_n\}$ in G such that

$\lim_{n\to\infty, n\notin J} \mathbb{P}(A \cap \theta({g_n}^{-1})B) = \mathbb{P}(A)\mathbb{P}(B)$ (2)

where J is a set of density zero, which may vary for different choices of A and B.

Remark 4.3 The concept of mixing (weakly mixing) MDS given in [1] is a special case of Definition 4.1 (Definition 4.2) above. For we can take $\{g_n\} = \{n\}$ if $G = \mathbb{R}$ or $\mathbb{Z}$.

The following theorem shows that weak mixing lies logically between mixing and ergodicity.(Here we make simple modification on Proposition 1.5 in [1])

Lemma4.4[1] For a bounded sequence $\{a_n\}$ let us write $a \coloneqq * - \lim_{n\to\infty} a_n$ provided that $a \coloneqq \lim_{n\to\infty, n\notin J} a_n$, where J has density zero. Then, in general,

$a \coloneqq * - \lim_{n\to\infty} a_n$ iff $\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1}|a_k - a| = 0.$ (3)

Theorem4.5 Let $(G, \Omega, \mathcal{F}, \mathbb{P}, \theta)$ be anMDS. Then the following are equivalent:

(a) $(G, \Omega, \mathcal{F}, \mathbb{P}, \theta)$ is weakly mixing;

(b) $\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1}|\mathbb{P}(A \cap \theta({g_k}^{-1})B) - \mathbb{P}(A)\mathbb{P}(B)| = 0$ $(A, B \in \mathcal{F})$;

(c) $\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1}[\mathbb{P}(A \cap \theta({g_k}^{-1})B)]^2 = \mathbb{P}(A)^2\mathbb{P}(B)^2$ $(A, B \in \mathcal{F})$;

(d) The MDS$(G \times G, \Omega \times \Omega, \mathcal{F}\otimes\mathcal{F}, \mathbb{P}\otimes\mathbb{P}, \theta \times \theta)$is ergodic.

(e) $(G \times G, \Omega \times \Omega, \mathcal{F}\otimes\mathcal{F}, \mathbb{P}\otimes\mathbb{P}, \theta \times \theta)$ is weakly mixing.

Proof $(a) \Leftrightarrow (b)$. Suppose (a). Then$\lim_{n\to\infty, n\notin J} \mathbb{P}(A \cap \theta({g_n}^{-1})B) = \mathbb{P}(A)\mathbb{P}(B)$,for all $A, B \in \mathcal{F}$,$g \in G$. Set $\{a_n\} \coloneqq \{\mathbb{P}(A \cap \theta({g_n}^{-1})B\}$ and $a \coloneqq \mathbb{P}(A)\mathbb{P}(B)$ in Lemma 4.4 we get $(a) \Leftrightarrow (b)$. Also$* - \lim_{n\to\infty} \mathbb{P}(A \cap \theta({g_n}^{-1})B) = \mathbb{P}(A)\mathbb{P}(B)$if and only if$* - \lim_{n\to\infty}|\mathbb{P}(A \cap \theta({g_n}^{-1})B - \mathbb{P}(A)\mathbb{P}(B)| = 0$if and only if

$* - \lim_{n\to\infty}|\mathbb{P}(A \cap \theta({g_n}^{-1})B - \mathbb{P}(A)\mathbb{P}(B)|^2 = 0$if and only if

$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1}|\mathbb{P}(A \cap \theta({g_n}^{-1})B - \mathbb{P}(A)\mathbb{P}(B)|^2 = 0.$

Now if $(G,\Omega,\mathcal{F},\mathbb{P},\theta)$ is ergodic, then $\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}|\mathbb{P}(A\cap\theta({g_n}^{-1})B-\mathbb{P}(A)\mathbb{P}(B)|^2$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}[\mathbb{P}(A\cap\theta({g_n}^{-1})B)^2-2\mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(A\cap\theta({g_n}^{-1})B]+\mathbb{P}(A)^2\mathbb{P}(B)^2$$

$=\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\mathbb{P}(A\cap\theta({g_n}^{-1})B)^2-\mathbb{P}(A)^2\mathbb{P}(B)^2.$

so that in this case (b) and (c) are equivalent. However, either (b) or (c) implies that $(G,\Omega,\mathcal{F},\mathbb{P},\theta)$ is ergodic. Thus (b) and (c) are equivalent. To show that (2) holds for all $A,B\in\mathcal{F}\times\mathcal{F}$, with θ replaced by $\theta\times\theta$, it is sufficient to show that it holds for measurable rectangles. Condition (2) then becomes

$*-\lim_{n\to\infty}\mathbb{P}(A\cap\theta({g_n}^{-1})B\,\mathbb{P}(C\cap\theta({g_n}^{-1})D)=\mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C)\mathbb{P}(D),\ A,B,C,D\in\mathcal{F}.$

Since the union of two sets of density zero has density zero, the last equality follows from (2). That is, (a) implies (e). Since (d) obviously follows from (e), it only remains to show that (d) implies (c). If $(G\times G,\Omega\times\Omega,\mathcal{F}\otimes\mathcal{F},\mathbb{P}\otimes\mathbb{P},\theta\times\theta)$ is ergodic and $A,B\in\mathcal{F}$, then

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\mathbb{P}(A\cap\theta({g_n}^{-1})B)^2$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\mathbb{P}\otimes\mathbb{P}((A\times A)\cap\theta\times\theta({g_n}^{-1})(B\times B))^2$$

$=\mathbb{P}\otimes\mathbb{P}(A\times A)\mathbb{P}\otimes\mathbb{P}(B\times B)=\mathbb{P}(A)^2\mathbb{P}(B)^2,$

as was to be shown.

Theorem 4.6 If an MDS $(G,\Omega,\mathcal{F},\theta,\mathbb{P})$ is $\{g_n\}$ −mixing, then it is ergodic.

Proof

Suppose that $(G,\Omega,\mathcal{F},\theta,\mathbb{P})$ is a $\{g_n\}$ −mixing MDS. Let $F\in\mathcal{F}$ be a θ −invariant set. By hypothesis, there exists a divergent sequence $\{g_n\}$ in G such that

$\lim_{k\to\infty}\mathbb{P}(F\cap\theta({g_k}^{-1})F)=\mathbb{P}(F)\mathbb{P}(\theta({g_k}^{-1})F).$

But $\theta({g_k}^{-1})F = F$, then$\lim_{k\to\infty} \mathbb{P}(F) = (\mathbb{P}(F))^2$,or $(\mathbb{P}(A))^2 = \mathbb{P}(A)$. Then either $\mathbb{P}(A) = 0$ or $\mathbb{P}(A) = 1$ and this complete the proof.

Theorem 4.7 Let $(G_1, \Omega_1, \mathcal{F}_1, \mathbb{P}_1, \theta_1) \cong_{(\xi,\alpha)} (G_2, \Omega_2, \mathcal{F}_2, \mathbb{P}_2, \theta_2)$. If $(G_1, \Omega_1, \mathcal{F}_1, \mathbb{P}_1, \theta_1)$ is (weakly) $\{g_k^1\}$ −mixing, then $(G_2, \Omega_2, \mathcal{F}_2, \mathbb{P}_2, \theta_2)$ is (weakly) $\{g_k^2\}$–mixing where $g_k^2 \coloneqq \alpha \circ g_k^1$.

Proof We shall proof for "mixing" and "weakly mixing" follow similarly. Let $A_2, B_2 \in \mathcal{F}_2$, then $A_1 \coloneqq \xi^{-1}(A_2), B_1 \coloneqq \xi^{-1}(B_2) \in \mathcal{F}_1$. By hypothesis, there exists a divergent sequence$\{g_k^1\}$ inG_1 such that $\lim_{k\to\infty} \mathbb{P}_1(A_1 \cap \theta_1(g_k^1)^{-1}B_1) = \mathbb{P}_1(A_1)\mathbb{P}_1(B_1)$.Set$g_k^2 \coloneqq \alpha \circ g_k^1$, then g_k^2 is divergent sequence in G_2. Thus

$\lim_{k\to\infty} \mathbb{P}_1(A_1 \cap \theta_1(\alpha^{-1} \circ g_k^2)^{-1}B_1) = \mathbb{P}_1(A_1)\mathbb{P}_1(B_1)$.

$\Rightarrow \lim_{k\to\infty} \mathbb{P}_1(A_1 \cap \theta_1(\alpha^{-1} \circ g_k^2)^{-1}\xi^{-1} \circ \xi(B_1)) = \mathbb{P}_1(A_1)\mathbb{P}_1(B_1)$.

$\Rightarrow \lim_{k\to\infty} \mathbb{P}_1(A_1 \cap \xi^{-1}\theta_2((g_k^2)^{-1})\xi(B_1)) = \mathbb{P}_1(A_1)\mathbb{P}_1(B_1)$.

$\Rightarrow \lim\limits_{k\to\infty} \mathbb{P}_2(A_2 \cap \theta_2((g_k^2)^{-1})B_2) = \mathbb{P}_2(A_2)\mathbb{P}_2(B_2)$

This means that $(G_2, \Omega_2, \mathcal{F}_2, \mathbb{P}_2, \theta_2)$ is $\{g_k^2\}$–mixing and the proof is completed.

Theorem4.8 If $(G_1, \Omega_1, \mathcal{F}_1, \mathbb{P}_1, \theta_1)$ is (weakly) $\{g_k^1\}$ −mixing and $(G_2, \Omega_2, \mathcal{F}_2, \mathbb{P}_2, \theta_2)$ is(weakly)$\{g_k^2\}$ −mixingthen their product is (weakly)$\{g_k^1\} \times \{g_k^2\}$ −mixing.

Proof

We shall proof for "mixing" and "weakly mixing" follow similarly. First, note that if $(G_1, \Omega_1, \mathcal{F}_1, \mathbb{P}_1, \theta_1), (G_2, \Omega_2, \mathcal{F}_2, \mathbb{P}_2, \theta_2)$ are two MDS's, then by Theorem 2.5 their product is also MDS. Suppose that $(G_1, \Omega_1, \mathcal{F}_1, \mathbb{P}_1, \theta_1)$ is $\{g_k^1\}$ −mixing and $(G_2, \Omega_2, \mathcal{F}_2, \mathbb{P}_2, \theta_2)$ is $\{g_k^2\}$ −mixing. Set

$$\mathcal{S} \coloneqq \{A \times B : A \in \mathcal{F}_1, B \in \mathcal{F}_2\}$$

For any $E_1 \coloneqq A_1 \times B_1$, $E_2 \coloneqq A_2 \times B_2 \in \mathcal{S}$. By hypothesis there exists a divergent sequence $\{g_k^i\}$ in G_i such that

$\lim_{k\to\infty} \mathbb{P}_i(A_i \cap \theta_i\left(g_k^i\right)^{-1} B_i) = \mathbb{P}_i(A_i)\mathbb{P}_i(B_i), i = 1,2.$

Put $\{g_k\} \coloneqq \{g_k^1\} \times \{g_k^2\} = \{(g_k^1, g_k^2)\}$. Then $\{g_k\}$ is divergent sequence in $G_1 \times G_2$. Now

$$\lim_{k\to\infty} \mathbb{P}(E_1 \cap \theta(g_k)^{-1} E_2)$$
$$= \lim_{k\to\infty} \mathbb{P}_1 \otimes \mathbb{P}_2((A_1 \times B_1) \cap \theta_1 \times \theta_2((g_k^1, g_k^2)^{-1})(A_2 \times B_2))$$
$$= \lim_{k\to\infty} \mathbb{P}_1 \otimes \mathbb{P}_2([A_1 \times B_1] \cap [\theta_1(g_k^1)^{-1} A_2 \times \theta_2(g_k^2)^{-1} B_2])$$
$$= \lim_{k\to\infty} \mathbb{P}_1 \otimes \mathbb{P}_2((A_1 \cap \theta_1(g_k^1)^{-1} A_2) \times (B_1 \cap \theta_2(g_k^2)^{-1} B_2)$$
$$= \lim_{k\to\infty} \mathbb{P}_1((A_1 \cap \theta_1(g_k^1)^{-1} A_2))\mathbb{P}_2(B_1 \cap \theta_2(g_k^2)^{-1} B_2)$$
$$= \lim_{k\to\infty} \mathbb{P}_1(A_1 \cap \theta_1(g_k^1)^{-1} A_2) \lim_{k\to\infty} \mathbb{P}_2(B_1 \cap \theta_2(g_k^2)^{-1} B_2)$$
$$= \mathbb{P}_1(A_1)\mathbb{P}_1(A_2)\, \mathbb{P}_2(B_1)\mathbb{P}_2(B_2)$$
$$= [\mathbb{P}_1(A_1)\mathbb{P}_2(B_1)][\mathbb{P}_1(A_2)\, \mathbb{P}_2(B_2)]$$
$$= \mathbb{P}_1 \otimes \mathbb{P}_2(A_1 \times B_1)\mathbb{P}_1 \otimes \mathbb{P}_2(A_2 \times B_2)$$
$$= \mathbb{P}(E_1)\mathbb{P}(E_2).$$

Since $\mathcal{S}$ is semi-algebra which generates $\mathcal{F}_1 \otimes \mathcal{F}_2$, any element of $\mathcal{F}_1 \otimes \mathcal{F}_2$ can be approximated by a finite disjoint elements of $\mathcal{S}$, and therefore

$\lim_{k\to\infty} \mathbb{P}(E \cap \theta(g_k)^{-1} F) = \mathbb{P}(E)\mathbb{P}(F),$ for all $E, F \in \mathcal{F}_1 \otimes \mathcal{F}_2$.

This complete the proof.

References

[1] J.R. Brown " Ergodic Theory and Topological Dynamics" Academic Press, Inc.(1976).

[2] H. Furstenberg, Y. Katznelson and D. Ornstein," The ergodic Theoretical Proof of Szemeredi's Theorem", American Mathematical Society" 7(3) 527-552,November (1982).

[3] J. Earman and M. Redei " Why Ergodic Theory does not Explain the Succes of Equilibrium Statistical Mechanics", The British J. for Philosophy of Science 47, 63-78 (1996).

[4] H. R. Biswas and M. S. Islam, "Ergodic Theory of One Dimensional Map", Bangladesh J. Sci. Ind. Res. 47(3), 321-326, (2012).

[5] P. J. Mitkowski, W. Mitkowski,"Ergodic Theory Approach to Chaos: Remark and Computational Aspects", Int. J. Appl. Math. Comput. Sci., 22(2), 259–267,(2012).

[6] H.Taqdir " Introduction to Topological groups", W.B Saunders company (1966)

[7] M.Stroppel, "Locally Compact Groups", European Mathematical Society (2006).

[8] J.L. Doob "Measure Theory" Springer-Verlag New York Inc.(1994).

[9] P.R. Halmos " Measure Theory"Springer-Verlag New York Inc.(1974).

[10] L. Arnold "Random Dynamical Systems", Springer-Verlag Berlin Heidelberg Germany (1998).

[11] I.Chueshov " Monotone Random Systems Theory and Applications" Springer-Verlag Berlin Heidelberg Germany (2002).

[12] W. H.Gottschalk , G. A. Hedlund, " Topological Dynamics", American Mathematical Society (1955).

[13] S.H. Abd and I.H.Kadhim,"Generating a Random Dynamical System", Asian Journal of Applied Science and Engineering, 3(3), 7-15,(2014).

Journal of AL-Qadisiyah for computer science and mathematics
Vol.7 No.2 Year 2015

حول السلوك المحاذي للنظم الديناميكية المترية

احسان جبار كاظم
جامعة القادسية
كلية علوم الحاسوب و الرياضيات
قسم الرياضيات

صلاح حمزة عبد
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قسم الرياضيات

المستخلص:

سندرس في هذا البحث السلوك المحاذي للنظم الديناميكية المترية عندما يكون مجال الزمن أي زمرة تبولوجية متراصة محليا. سنتفحص بعض الخواص الجديدة للثبات، الخلط، الخلط الضعيف في النظم الديناميكي المترية.