University Of AL-Qadisiyah

Available online at www.qu.edu.iq/journalcm

JOURNAL OF AL-QADISIYAH FOR COMPUTER SCIENCE AND MATHEMATICS

ISSN:2521-3504(online) ISSN:2074-0204(print)

On Manifold Jain-Beta Operators

Haneen J. Sadiq

University of Basrah, College of Science, Department of Mathematics, Basrah, Iraq

Han.mah2010@gmail.com, haneen.sadiq@uobasrah.edu.iq

ARTICLE INFO

Article history:
Received: 25 /10/2020
Rrevised form: 15 /11 / 2020
Accepted : 24 /12/2020
Available online: 26 /12/2020

Keywords:

Beta operators, Jain operators, Approximation theorem, Voronovskaja Theorem, Korovkin's Theorem, Modulus of Continuity.

ABSTRACT

The aim of this paper is to construct a new sequence of linear and positive operators which called Jain-Beta operators based on a parameter μ. We investigated some properties of these operators when applied the approximation theorems. Moreover, we estimated the rate of convergence relying on concept the modulus of continuity. Finally, established a Voronoviskaja- type asymptotic formula.

MSC. 41A25; 41A35; 41A36.

DOI : https://doi.org/10.29304/jqcm.2020.12.4.721

1-Introduction

In (1970), Jain [1] introduced a generalization of the well known Szãzs – Mirakjan operators using Poissn – type distributin as

$$G_n^{\mu}(f;x) = \sum_{\kappa=0}^{\infty} \mathcal{W}_{\mu}(\kappa, nx)\, f\left(\frac{k}{n}\right), \tag{1}$$

where $\mathcal{W}_{\mu}(\kappa, nx) = nx(nx+\kappa\mu)^{k-1}\frac{e^{-(nx+\kappa\mu)}}{k!}$, (2)

where $f: \mathbb{R}^+ \to \mathbb{R}$, $\mathbb{R}^+ := [0,\infty)$, $n \in \mathbb{N}$, $0 < nx \le \infty$ and $|\mu| < 1$. In particular case if $\mu = 0$, then from $G_n^0(.;x)$ we get the well-known Szãsz-Mirakjan perators. In [1] G. C. Jain defined a new

⁎Corresponding author: Haneen J. Sadiq

Email addresses: han.mah2010@gmail.com , haneen.mahdi@yahoo.com, haneen.sadiq@uobasrah.edu.iq.

Communicated by : *Alaa Hussein Hamadi*

operators with the help of a Pisson type distribution. He studied them convergence properties and gave them degree of approximation. Nowdays, these operators are known as Jain perators, in honor of G. C. Jain who defined them on 1970. In this paper we study the apprximation of continuous functions belong to the space $C(\mathbb{R}^+)$. In (1995), V. Gupta, G. S. Srivastava and A. Sahai [2] defined a new L. P. O. consisting from Szãsz and Beta operators to approximate a Lebesgue intagrable functions on $\mathbb{R}^+$ as follow

$$\mathcal{B}_n(f;x) = \sum_{\kappa=0}^{\infty} \mathcal{P}_{n,\kappa}(x) \int_0^\infty \mathfrak{b}_{n,\kappa}(t)\; f(t)dt\,, \tag{3}$$

where $\mathcal{P}_{n,\kappa}(x) = e^{-nx}\frac{(nx)^\kappa}{\kappa!}$, and $\mathfrak{b}_{n,\kappa}(t) = \frac{1}{\mathcal{B}_n(\kappa+1,n)} \cdot \frac{t^\kappa}{(1+t)^{n+\kappa+1}}$.

The Beta function can be defined as: $\mathcal{B}_n(\kappa+1,n) \equiv \frac{\Gamma(\kappa+1)\Gamma(n)}{\Gamma(n+\kappa+1)} = \frac{\kappa!(n-1)!}{(\kappa+n)!}$; and $\Gamma(\kappa) = (\kappa-1)!$ be the Gamma function and $\mathcal{P}_{n,\kappa}(x)$ be the weight of Szãsz operators, for more details about Beta operators see ([3] and [4]).

2- Preliminaries

In this part, we have defined a new sequence of Beta operators and gived some results whom will be needed to prove the main theorems. So we applied a Korovkin's Theorem [5] on $M_{n,\ell,\kappa}(f(t);x)$ operators, which define as follow:

$$M_{n,\ell,\kappa}(t^m;x) = \frac{1}{n-\ell}\sum_{\kappa=0}^{\infty} \mathfrak{b}_{n,\ell,\kappa}(x) \int_0^\infty \mathfrak{b}_{n,\ell,\kappa}(t)\; t^m dt \tag{4}$$

Lemma 2.1 For $t \in (0,\infty)$ and $n,\ell \in \mathbb{N} = \{1,2,3,\dots\}$, we have

$$\int_0^\infty \mathfrak{b}_{n-\ell,\kappa+\ell}(t) t^m dt = \frac{(m+\ell+\kappa)!(n-\ell-m-1)!}{(\kappa+\ell)!(n-\ell-1)!}. \tag{5}$$

Encoding: For short we denote $\mathfrak{b}_{n-\ell,\kappa+\ell}(x)$ by $\mathfrak{b}_{n,\ell,\kappa}(x)$

Theorem 2.1 For $M_{n,\ell,\kappa}(t^m;x)$ operators and $n > \ell$ then the below statements are hold for every $x \in \mathbb{R}^+$ and $m = \{0,1,2\}$,

1- $M_{n,\ell,\kappa}(t^0;x) = 1;\ (n-\ell) > 0$ (6)

2- $M_{n,\ell,\kappa}(t^1;x) = \frac{(n-\ell+1)}{(n-\ell-1)}x + \frac{\ell(1+x)}{(n-\ell)(n-\ell-1)}\mathfrak{b}_{n,\ell,0}(x) + \frac{1}{(n-\ell-1)};\ (n-\ell) > 1$ (7)

3- $M_{n,\ell,\kappa}(t^2;x) =$

$$\frac{(n-\ell+1)(n-\ell+2)}{(n-\ell-1)(n-\ell-2)}x^2 + 4x\frac{(n-\ell+1)}{(n-\ell-1)(n-\ell-2)} + \frac{2}{(n-\ell-1)(n-\ell-2)} + \frac{\ell(1+x)\mathfrak{b}_{n,\ell,0}(x)(\ell+x(n-\ell+2)+3)}{(n-\ell)(n-\ell-1)(n-\ell-2)},\ (n-\ell) > 3. \tag{8}$$

Proof . it is easy to prove

1- $M_{n,\ell,\kappa}(1;x) = \frac{1}{n-\ell}\sum_{\kappa=0}^{\infty} \mathfrak{b}_{n,\ell,\kappa}(x) \int_0^\infty \mathfrak{b}_{n,\ell,\kappa}(t)\; dt$

$= \frac{1}{(n-\ell)}(n-\ell) = 1$, for $(n-\ell) > 0$.

2- $M_{n,\ell,\kappa}(t;x) = \frac{1}{n-\ell}\sum_{\kappa=0}^{\infty} \mathfrak{b}_{n,\ell,\kappa}(x) \int_0^\infty \mathfrak{b}_{n,\ell,\kappa}(t)\; tdt$

$$= \frac{1}{n-\ell}\sum_{\kappa=0}^{\infty} \mathfrak{b}_{n,\ell,\kappa}(x)\frac{(\kappa+\ell+1)}{(n-\ell-1)}$$

$$= \frac{1}{(n-\ell)(n-\ell-1)}\sum_{\kappa=0}^{\infty} \mathfrak{b}_{n,\ell,\kappa}(x)(\kappa+\ell) + \frac{1}{(n-\ell)(n-\ell-1)}\sum_{\kappa=0}^{\infty} \mathfrak{b}_{n,\ell,\kappa}(x)$$

$= \frac{(n-\ell+1)}{(n-\ell-1)}x + \frac{\ell(1+x)}{(n-\ell)(n-\ell-1)}\mathfrak{b}_{n,\ell,0}(x) + \frac{1}{(n-\ell-1)}$, for $(n-\ell) > 1$.

3- $\mathrm{M}_{n,\ell,\kappa}(t^2;x) = \frac{1}{n-\ell}\sum_{\kappa=0}^{\infty} \mathfrak{b}_{n,\ell,\kappa}(x)\int_0^{\infty} \mathfrak{b}_{n,\ell,\kappa}(t)\ t^2 dt$

$$= \frac{1}{n-\ell}\sum_{\kappa=0}^{\infty} \mathfrak{b}_{n,\ell,\kappa}(x)\frac{(\kappa+\ell+2)!\,(n-\ell-3)!}{(\kappa+\ell)!\,(n-\ell-1)!}$$

$$= \frac{1}{n-\ell}\sum_{\kappa=0}^{\infty} \mathfrak{b}_{n,\ell,\kappa}(x)\frac{(\kappa+\ell+2)(\kappa+\ell+1)}{(n-\ell-1)(n-\ell-2)}$$

$$= \frac{1}{(n-\ell)(n-\ell-1)(n-\ell-2)}\sum_{\kappa=0}^{\infty} \mathfrak{b}_{n,\ell,\kappa}(x)(\kappa+\ell)^2$$

$$+ \frac{3}{(n-\ell)(n-\ell-1)(n-\ell-2)}\sum_{\kappa=0}^{\infty} \mathfrak{b}_{n,\ell,\kappa}(x)(\kappa+\ell)$$

$$+ \frac{1}{(n-\ell)(n-\ell-1)(n-\ell-2)}\sum_{\kappa=0}^{\infty} \mathfrak{b}_{n,\ell,\kappa}(x)$$

$\mathrm{M}_{n,\ell,\kappa}(t^2;x) =$

$\frac{(n-\ell+1)(n-\ell+2)}{(n-\ell-1)(n-\ell-2)}x^2 + 4x\frac{(n-\ell+1)}{(n-\ell-1)(n-\ell-2)} + \frac{2}{(n-\ell-1)(n-\ell-2)} + \frac{\ell(1+x)\mathfrak{b}_{n,\ell,0}(x)(\ell+x(n-\ell+2)+3)}{(n-\ell)(n-\ell-1)(n-\ell-2)}$,

for $(n-\ell) > 2$.

So, the prrof is complete. ■

Theorem 2.2 [1] Let $f \in C(\mathbb{R}^+)$, $x \in \mathbb{R}^+$ and $0 \le \mu < 1$ then the sequence of L. P. O. $\{G_n^{\mu}(f(t);x)\}$ converges to the functions $f(x)$ as follow

1- $G_n^{\mu}(1;x) = 1,$ (9)

2- $G_n^{\mu}(t;x) = \frac{x}{1-\mu},$ (10)

3- $G_n^{\mu}(t^2;x) = \frac{x^2}{(1-\mu)^2} + \frac{x}{n(1-\mu)^2}$. (11)

This work is focused around the definition of new linear positive operators as following

3- Construction of Operators

Firstly, we define a new class of Beta – type operators

$\mathcal{B}_{n,\kappa,\ell}(f;x) = \sum_{\kappa=0}^{\infty} \mathfrak{b}_{n,\ell,\kappa}(x)$ (12)

$\mathfrak{b}_{n,\ell,\kappa}(f;x) = \sum_{\kappa=0}^{\infty} \frac{1}{\mathcal{B}_n(\kappa+\ell+1,n-\ell)} x^{\kappa+\ell}(1+x)^{-(n-\ell+\kappa+\ell+1)} f(x)$, (13)

$\mathcal{B}_n(\kappa+\ell+1, n-\ell) = \frac{(n-\ell+\kappa+\ell)!}{(\kappa+\ell)!(n-\ell-1)!}$,

for $x \in \mathbb{R}^+$ and the parameter $\ell \in \mathrm{N}$.

In [1] G. C. Jain defined a positive linear operators for $0 \leq \mu < 1$ as follows

$G_n^{\mu}(f;x) = \sum_{\kappa=0}^{\infty} \mathcal{W}_{\mu}(\kappa, nx)\, f\left(\frac{\kappa}{n}\right)$,

where $\mathcal{W}_{\mu}(\kappa, nx) = nx(nx+\kappa\mu)^{k-1} \frac{e^{-(nx+\kappa\mu)}}{k!}$, for more details can see the articles [6],[7], [8] and [9].

Now, to construct the new sequence of operators as shown below, need the equations (2) and (13) respectively. Therefore the required operators consisting from summation $\mathcal{W}_{\mu}$ and integral $\mathfrak{b}_{n,\ell,\kappa}$ as following

$\mathcal{J}_{n,\ell,\kappa}^{\mu}(f;x) = \sum_{\kappa=0}^{\infty} \mathcal{W}_{\mu}(\kappa, nx) \int_0^{\infty} \mathfrak{b}_{n,\ell,\kappa}(t)\, f(t)\, dt$ (14)

Where $0 \leq \mu < 1, n \in \mathbb{N}, \ell \in \mathbb{N}^0 := \mathbb{N} \cup \{0\}$ and $x \in \mathbb{R}^+$. Observe that the parameter μ depends on n, and $\mathcal{W}_{\mu}(\kappa, nx)$ defined in (2).

For the functions f belonging to the weighted space $C_{\rho,}$ [10].

$C_{\rho} = \{f \in CB(\mathbb{R}^+): f \text{ is continuous and bounded, where } f(x) = O(1+x^{\rho}) \text{ for some } \rho \in \mathbb{N}\}$

$\|f\|_{\rho} \equiv \|f(.)\|_{\rho} = sup_{x \in \mathbb{R}^+} \frac{|f(x)|}{|1+x^{\rho}|}$.

4- Auxiliary Results

In this part, we applied a Korovkin's conditions to verify the convergence for the test functions $e_i(t) := t^i$, $i = 0,1,2$.

Theorem (4.1) (Korovkin's Theorem):

Let $f \in C_{\rho}$, and $0 \leq \mu < 1$ and suppose that $e_i(t) := t^i$, $i = 0,1,2$. Then for new Jain-Beta operators the following statements are hold for every $x \in \mathbb{R}^+$;

1- $\mathcal{J}_{n,\ell,\kappa}^{\mu}(e_0;x) = 1$, (15)

2- $\mathcal{J}_{n,\ell,\kappa}^{\mu}(e_1;x) = \frac{nx+(\ell+1)(1-\mu)}{(n-\ell-1)(1-\mu)}$; for $n > \ell+1$, (16) 3- $\mathcal{J}^{\mu}_{n,\ell,\kappa}(e_2;x) = \frac{n^2x^2}{(n-\ell-2)(n-\ell-1)(1-\mu)^2} + \frac{nx}{(n-\ell-1)(n-\ell-2)}\left(\frac{1}{(1-\mu)^3} + \frac{2\ell+3}{(1-\mu)}\right)$

$$+\frac{(\ell+1)(\ell+1)}{(n-\ell-2)(n-\ell-1)};\ \text{for } n > \ell + 2. \tag{17}$$

Proof. Quite simply we can prove

1- $\mathcal{J}^{\mu}_{n,\ell,\kappa}(e_0;x) = \sum_{\kappa=0}^{\infty} \mathcal{W}_{\mu}(\kappa,nx)\ \int_0^{\infty} \mathfrak{b}_{n,\ell,\kappa}(t)\ dt = 1$

2- $\mathcal{J}^{\mu}_{n,\ell,\kappa}(e_1;x) = \sum_{\kappa=0}^{\infty} \mathcal{W}_{\mu}(\kappa,nx)\ \int_0^{\infty} \mathfrak{b}_{n,\ell,\kappa}(t)\, t\, dt$

$$= \sum_{\kappa=0}^{\infty} \mathcal{W}_{\mu}(\kappa,nx) \int_0^{\infty} \frac{t^{\kappa+\ell+1}}{(1+t)^{n+\kappa+1}}\ dt$$

$$= \sum_{\kappa=0}^{\infty} \mathcal{W}_{\mu}(\kappa,nx) \frac{(\kappa+\ell+1)!\,(n-\ell-2)!}{(\kappa+\ell)!\,(n-\ell-1)!}$$

$$= \frac{n}{(n-\ell-1)} \sum_{\kappa=0}^{\infty} \mathcal{W}_{\mu}(\kappa,nx)\frac{\kappa}{n} + \frac{\ell+1}{(n-\ell-1)} \sum_{\kappa=0}^{\infty} \mathcal{W}_{\mu}(\kappa,nx)$$

$$\mathcal{J}^{\mu}_{n,\ell,\kappa}(e_1;x) = \frac{n}{(n-\ell-1)}\frac{x}{1-\mu} + \frac{\ell+1}{(n-\ell-1)} = \frac{nx+(\ell+1)(1-\mu)}{(n-\ell-1)(1-\mu)}$$

3- $\mathcal{J}^{\mu}_{n,\ell,\kappa}(e_2;x) = \sum_{\kappa=0}^{\infty} \mathcal{W}_{\mu}(\kappa,nx)\ \int_0^{\infty} \mathfrak{b}_{n,\ell,\kappa}(t)\ t^2 dt$

$$= \sum_{\kappa=0}^{\infty} \mathcal{W}_{\mu}(\kappa,nx) \int_0^{\infty} \frac{t^{\kappa+\ell+2}}{(1+t)^{n+\kappa+1}}\ dt$$

$$= \sum_{\kappa=0}^{\infty} \mathcal{W}_{\mu}(\kappa,nx) \frac{(\kappa+\ell+1)(\kappa+\ell+2)}{(n-\ell-1)(n-\ell-2)}$$

$$= \frac{n^2}{(n-\ell-1)(n-\ell-2)} \sum_{\kappa=0}^{\infty} \mathcal{W}_{\mu}(\kappa,nx)(\frac{\kappa}{n})^2$$

$$+ \frac{n(2\ell+3)}{(n-\ell-1)(n-\ell-2)} \sum_{\kappa=0}^{\infty} \mathcal{W}_{\mu}(\kappa,nx)\frac{\kappa}{n}$$

$$+ \frac{(\ell+1)(\ell+2)}{(n-\ell-1)(n-\ell-2)} \sum_{\kappa=0}^{\infty} \mathcal{W}_{\mu}(\kappa,nx)$$

$$\mathcal{J}^{\mu}_{n,\ell,\kappa}(e_2;x) = \frac{n^2}{(n-\ell-1)(n-\ell-2)}\frac{x^2}{(1-\mu)^2} + \frac{nx}{(n-\ell-1)(n-\ell-2)}\left(\frac{1}{(1-\mu)^3} + \frac{2\ell+3}{(1-\mu)}\right) + \frac{(\ell+1)(\ell+1)}{(n-\ell-2)(n-\ell-1)}.$$

So, the proof is complete. ■

Denote the central moment by $\bar{\bar{\mathcal{J}}}^{\mu}_{n,\ell,\vartheta}(x) := \mathcal{J}^{\mu}_{n,\ell,\kappa}\left((t-x)^{\vartheta};x\right)$, where ϑ be the order of the moment for the sequence of L. P. O. $\{\mathcal{J}^{\mu}_{n,\ell,\kappa}(.\,;x)\}$ viewed in(13).

For every $x \in \mathbb{R}^+$ and $f \in C_\rho$, we define

$$\bar{\bar{\mathcal{J}}}^{\mu}_{n,\ell,\vartheta}(x) := \mathcal{J}^{\mu}_{n,\ell,\kappa}\left((t-x)^{\vartheta};x\right) = \sum_{\kappa=0}^{\infty} \mathcal{W}_{\mu}(\kappa,nx) \int_0^{\infty} \mathfrak{b}_{n,\ell,\kappa}(t)\,(t-x)^{\vartheta}\,dt. \quad (18)$$

So, we can give the next results

Lemma 4.1 Let $\bar{\bar{\mathcal{J}}}^{\mu}_{n,\ell,\vartheta}(x) := \mathcal{J}^{\mu}_{n,\ell,\kappa}\left((t-x)^{\vartheta};x\right)$, $\vartheta = 0,1,2 \ldots$. Then for $0 \le \mu < 1$we have

1- $\bar{\bar{\mathcal{J}}}^{\mu}_{n,\ell,0}(x) = 1,$ (19)

2- $\bar{\bar{\mathcal{J}}}^{\mu}_{n,\ell,1}(x) = x\frac{(n-((n-\ell-1)(1-\mu)))+(\ell+1)(1-\mu)}{(n-\ell-1)(1-\mu)},$ (20)

3- $$\bar{\bar{\mathcal{J}}}^{\mu}_{n,\ell,2}(x) = x^2\left(\frac{n^2}{(n-\ell-1)(n-\ell-2)(1-\mu)^2} - \frac{2n}{(n-\ell-1)(1-\mu)} + 1\right) + x\left(\frac{n\left(1+(2\ell+3)(1-\mu)^3\right)}{(n-\ell-1)(n-\ell-2)(1-\mu)^3} + \frac{2(\ell+1)}{n-\ell-1}\right) + \frac{(\ell+1)(\ell+1)}{(n-\ell-2)(n-\ell-1)}. \quad (21)$$

Proof. We can prove lemma (4.1) above directly depending on the theorem (4.1). ■

5- Direct Results and Asymptotic Formula

Next, we shall study the asymptotic behavior for given operators by applying the main theorem (Voronoviskaja- type theorem) [11].

Theorem 5.1 (Voronoviskaja Theorem) For any function $f \in C_\rho$such that $f', f'' \in C_\rho$ and $0 \le \mu < 1$, we have $\lim_{n\to\infty} n\left[\mathcal{J}^{\mu}_{n,\ell,\kappa}(f;x) - f(x)\right] = f'(x)(\ell+1)(x+1) + \frac{f''(x)}{2}\{x^2 + 2(\ell+2)\}$, for every $x \in \mathbb{R}^+$.

Proof. Let $f, f', f'' \in C_\rho$ and $x \in (\mathbb{R}^+)$ be fixed. By Taylor expansion, we can write

$f(t) = f(x) + (t-x)f'(x) + \frac{(t-x)^2}{2!}f''(x) + \xi(t,x)(t-x)^2$, where $\xi(t,x)$ is a Peano form of the remainder, $\xi(t,x) \in C_\rho$ and $\xi(t,x) \to 0$ as $t \to x$. Applying $\mathcal{J}^{\mu}_{n,\ell,\kappa}$ we get

$n\left[\mathcal{J}_{n,\ell,\kappa}^{\mu}(f;x)-f(x)\right]=$

$f'(x)n\mathcal{J}_{n,\ell,\kappa}^{\mu}\left((t-x);x\right)+\frac{f''(x)}{2!}n\mathcal{J}_{n,\ell,\kappa}^{\mu}((t-x)^2;x)+n\mathcal{J}_{n,\ell,\kappa}^{\mu}(\xi(t,x)(t-x)^2;x),$

$\lim_{n\to\infty} n\left[\mathcal{J}_{n,\ell,\kappa}^{\mu}(f;x)-f(x)\right]=f'(x)\lim_{n\to\infty} n\mathcal{J}_{n,\ell,\kappa}^{\mu}\left((t-x);x\right)+\frac{f''(x)}{2!}\lim_{n\to\infty} n\mathcal{J}_{n,\ell,\kappa}^{\mu}((t-x)^2;x)+$
$\lim_{n\to\infty} n\mathcal{J}_{n,\ell,\kappa}^{\mu}(\xi(t,x)(t-x)^2;x),$

$$=f'(x)\lim_{n\to\infty} n\frac{x\left(n-((n-\ell-1)(1-\mu))\right)+(\ell+1)(1-\mu)}{(n-\ell-1)(1-\mu)}$$
$$+\frac{f''(x)}{2!}\lim_{n\to\infty} n\left\{x^2\left(\frac{n^2}{(n-\ell-1)(n-\ell-2)(1-\mu)^2}-\frac{2n}{(n-\ell-1)(1-\mu)}+1\right)\right.$$
$$\left.+x\left(\frac{n(1+(2\ell+3)(1-\mu)^3)}{(n-\ell-1)(n-\ell-2)(1-\mu)^3}+\frac{2(\ell+1)}{n-\ell-1}\right)\right\}+\frac{(\ell+1)(\ell+1)}{(n-\ell-2)(n-\ell-1)}$$
$$+\lim_{n\to\infty} n\mathcal{J}_{n,\ell,\kappa}^{\mu}(\xi(t,x)(t-x)^2;x),$$

Then where $\mu\to 0$ and $n\to\infty$, we get

$$\lim_{n\to\infty} n\left[\mathcal{J}_{n,\ell,\kappa}^{\mu}(f;x)-f(x)\right]$$
$$=f'(x)(\ell+1)(x+1)+\frac{f''(x)}{2}\{x^2+2(\ell+2)\}+\lim_{n\to\infty} n\mathcal{J}_{n,\ell,\kappa}^{\mu}(\xi(t,x)(t-x)^2;x),$$

Applying Cauchy – Schwartz inequality on the final term $\mathcal{J}_{n,\ell,\kappa}^{\mu}(\xi(t,x)(t-x)^2;x)$, we get

$$0\le n\mathcal{J}_{n,\ell,\kappa}^{\mu}(\xi(t,x)(t-x)^2;x)\le\sqrt{n^2\mathcal{J}_{n,\ell,\kappa}^{\mu}((t-x)^4;x)}\sqrt{\mathcal{J}_{n,\ell,\kappa}^{\mu}(\xi(t,x)^2;x)}$$

Since $\xi^2(x,x)=0$ also $\xi^2(.,x)$ belong to the space C_ρ then we have

$$\lim_{n\to\infty} n\mathcal{J}_{n,\ell,\kappa}^{\mu}(\xi(t,x)(t-x)^2;x)=\xi^2(x,x)=0$$

Therefore

$$\lim_{n\to\infty} n\left[\mathcal{J}_{n,\ell,\kappa}^{\mu}(f;x)-f(x)\right]=f'(x)(\ell+1)(x+1)+\frac{f''(x)}{2}\{x^2+2(\ell+2)\}$$

So the proof is complete.

■

Definition 5.1 (Modulus of Continuity) [12], [13] Let $f \in C_\rho$ be the functions belong to the space of all bounded and continuous real – valued functions on $\mathbb{R}^+$, with the norm $\|f\|_\rho$. If $\delta > 0$,then the usual modulus of continuity for the functions $f \in C_\rho$ is defined by $\omega(f;\delta) := \sup_{\substack{x,y\in(\mathbb{R}^+)\\ |x-y|\le\delta}} |f(x)-f(y)|$.

In the next theorem, we give some upper bounds for the approximation error from where the modulus of continuity of the first kind

Theorem 5.2 Let $f \in C_\rho$ and $0 \le \mu < 1$, then

$$\left|\mathcal{J}^{\mu}_{n,\ell,\kappa}(f;x) - f(x)\right| \le 2\omega(f;\delta) \tag{22}$$

where

$\delta =$

$$\sqrt{x^2\left(\frac{n^2}{(n-\ell-1)(n-\ell-2)(1-\mu)^2} - \frac{2n}{(n-\ell-1)(1-\mu)} + 1\right) + x\left(\frac{n(1+(2\ell+3)(1-\mu)^3)}{(n-\ell-1)(n-\ell-2)(1-\mu)^3} + \frac{2(\ell+1)}{n-\ell-1}\right) + \frac{(\ell+1)(\ell+1)}{(n-\ell-2)(n-\ell-1)}}$$

Proof. Based on the well-known property of modulus of continuity

$$|f(t) - f(x)| \le \omega(f;\delta)\left(\frac{|t-x|}{\delta} + 1\right) \tag{23}$$

Apply the linear and positive operators $\mathcal{J}^{\mu}_{n,\ell,\kappa}(f;x)$ on the two sides we get

$$\left|\mathcal{J}^{\mu}_{n,\ell,\kappa}(f;x) - f(x)\right| \le \omega(f;\delta)\left(\frac{1}{\delta}\mathcal{J}^{\mu}_{n,\ell,\kappa}(|t-x|;x) + 1\right)$$

By using the Cauchy- Schwartz inequality and (21) we get

$$\left|\mathcal{J}^{\mu}_{n,\ell,\kappa}(f;x) - f(x)\right| \le \omega(f;\delta)\left(\frac{1}{\delta}\sqrt{\left(\mathcal{J}^{\mu}_{n,\ell,\kappa}(t-x)^2;x\right)} + 1\right)$$

$$\le \omega(f;\delta)\left(\frac{1}{\delta}\left(\sqrt{\bar{\bar{\mathcal{J}}}^{\mu}_{n,\ell,2}(x)}\right) + 1\right)$$

$$\le \omega(f;\delta)\left(\frac{1}{\delta}\mathrm{H} + 1\right)$$

Suppose $\mathrm{H} = \sqrt{\bar{\bar{\mathcal{J}}}^{\mu}_{n,\ell,2}(x)}$, be the squared root for the second moment.

Using equation (21) we have

H =

$$\sqrt{x^2\left(\frac{n^2}{(n-\ell-1)(n-\ell-2)(1-\mu)^2}-\frac{2n}{(n-\ell-1)(1-\mu)}+1\right)+x\left(\frac{n(1+(2\ell+3)(1-\mu)^3)}{(n-\ell-1)(n-\ell-2)(1-\mu)^3}+\frac{2(\ell+1)}{n-\ell-1}\right)+\frac{(\ell+1)(\ell+1)}{(n-\ell-2)(n-\ell-1)}}$$

Choose $\delta = \text{H}$

Hence, we get $\left|\mathcal{J}^{\mu}_{n,\ell,\kappa}(f;x)-f(x)\right| \le 2\omega(f;\delta)$.

So the proof is complete. ■

Theorem 5.3 Let $f \in C_\rho$ on $(\mathbb{R}^+)$. Then for a real number $\lambda > 0$ and $0 \le \mu < 1$, the limit relation

$$\lim_{n\to\infty} \mathcal{J}^{\mu}_{n,\ell,\kappa}(f;x) = f(x), \tag{24}$$

holds uniformly on the interval $[0,\lambda]$.

Proof. By using (15), (16) and (17) from theorem (4.1) above we can see that:

$$\left\|\mathcal{J}^{\mu}_{n,\ell,\kappa}(1;x)-1\right\|_{C[0,\lambda]} = 0 \tag{25}$$

$$\left\|\mathcal{J}^{\mu}_{n,\ell,\kappa}(t;x)-x\right\|_{C[0,\lambda]} = \max_{x\in[0,\lambda]} \frac{\{\mu(n-1)+\ell(1-\mu)\}x+(\ell+1)(1-\mu)}{(n-\ell-1)(1-\mu)}$$

$$\le \frac{\{\mu(n-1)+\ell(1-\mu)\}\lambda+(\ell+1)(1-\mu)}{(n-\ell-1)(1-\mu)} \to 0 \text{ where } n\to\infty, \text{ and } \mu \to 0. \tag{26}$$

$$\left\|\mathcal{J}^{\mu}_{n,\ell,\kappa}(t^2;x)-x^2\right\|_{C[0,\lambda]}$$
$$= \max_{x\in[0,\lambda]} \frac{nx^2(2\ell+3)-\ell x^2(\ell+3)-2x^2-\mu(\mu-2)x^2(n-\ell-1)(n-\ell-2)}{(n-\ell-1)(n-\ell-2)(1-\mu)^2} + \frac{nx(1+(2\ell+3)(1-\mu)^2)}{(n-\ell-1)(n-\ell-2)(1-\mu)^3}+\frac{(\ell+1)(\ell+1)}{(n-\ell-2)(n-\ell-1)}$$

$$\le \frac{n\lambda^2(2\ell+3)-\ell\lambda^2(\ell+3)-2\lambda^2-\mu(\mu-2)\lambda^2(n-\ell-1)(n-\ell-2)}{(n-\ell-1)(n-\ell-2)(1-\mu)^2}+\frac{n\lambda(1+(2\ell+3)(1-\mu)^2)}{(n-\ell-1)(n-\ell-2)(1-\mu)^3}+\frac{(\ell+1)(\ell+1)}{(n-\ell-2)(n-\ell-1)} \to 0 \text{ for sufficiently large } n, \text{ and } \mu \to 0. \tag{27}$$

So, our proof can be obtained dependent on P. P. Korovkin [5]. ■

6- Conclusions The motive of the present paper is to give a rate of convergence and estimate the error of convergence by modified Jain – Beta operators. We have defined the new sequence of linear positive operators represented by Jain – Beta operators $\mathcal{J}^{\mu}_{n,\ell,\kappa}(.,x)$ from kind summation integral based on the parameter $0 \le \mu < 1$. We investigated some approximation results like the well – known

Korovkin type theorem. Also, we established a Voronoviskaja –type asymptotic formula. Finally, we have find the rate of convergence for the operators by means of the modulus of continuity.

References

[1] Jain G. C., “Approximation of functions by a new class of linear operators”, *J. Aust. Math. Soc.*, vol. 13 No. 3 (1972), pp 271-276.

[2] Vijay Gupta, G. S. Srivastava and A. Sahai, “On simultaneous approximation by Szãsz-Beta Operators”, *Soochow J. Math*, vol. 21(1995), pp 1-11.

[3] Ali J. Mohammad, Ansam A. Abdul-Rahman, “Rate of converges for new family of summation- Integral Beta operators”, *Iraqi Journal of Statistical Science, The Fourth Scientific Conference of the College of Computer Science & Mathematics*, vol. 20 (2011), pp 83-92.

[4] Ali J. Mohammad, Hanadi A. AbdulSattar, Haneen J. Sadiq, “On Truncated of Classical Beta operators”, *Journal of Basrah researches ((Sciences))*, vol. 41 No. 3, A(2015), pp 15-23.

[5] P. P. Korovkin, “Linear Operators and Approximation Theory”, *Hindustan Publ. Corp. Delhi*, 1960 (Translated from Russian Edition) (1960).

[6] O. Dogru, R. N. Mohapatra, M. Orkcu, “Approximation Properties of Generalized Jain Operators”, *published by University of Nis, Serbia, Faculty of Sciences and Mathematics. Doi 10.2298/FIL1609359D, Filomat*, vol. 30 No. 9 (2016), pp 2359-2366.

[7] Prashantkumar Patel, “Some Approximation of new Families of Positive Linear Operators”, *Hindawi Publishing Corporation Journal of Calculus of Variations,* Article ID 489249, 8 pages, http://dx.doi.org/10.1155/2013/489249, vol. 2013 (2013), pp 1-8.

[8] Vishnu Narayan Mishra, Prashant kumar Patel, “Some Approximation of new Families of Positive Linear Operators”, *Hindawi Publishing Corporation Journal of Calculus of Variations,* Article ID 489249, 8 pages, http://dx.doi.org/10.1155/2013/489249, vol. 2013 (2013), pp 1-8.

[9] Gulen Bascanbaz –Tunca, Murat Bodur, Dilek Soylemez, “On Lupas – Jain Operators.”, *Stud. Univ. Babes-Bolyai Math.*, Doi: 10.24193/subbmath.2018.4.08, vol. 63 No.4 (2018), pp 525-537.

[10] Haneen J. Sadiq, “Approximation of Bounded Functions by Positive Linear Operators in C_p”, *Basrah Journal of Science*, vol. 37(3) (2019), pp 412-429.

[11] Voronovskaja E, “Détermination de la forme asymptotiqued’ ápproximation des functions par les polynômes de S. N. Bernstein.”, *C. R. Acad. Sci. USSR* (1932), pp 79-85.

[12] Serhan Varma, Fatma Taşdelen “Százs Type Operators Involving Charlier Polynomials”, *Mathematical and Computer Moduling, Journal homepage: www.elsevier.com/locate/mcm. Doi*: 10. 1016/j.mcm.2011.12.017, vol. 56 (2012) pp 118-122.

[13] Haneen J. Sadiq, “Some Approximation properties of New Family of Baskakov – Type Operators”, *IOP Conference Series: Materials Science and Engineering, 2nd International Scientific Conference of Al-Ayen University* (ISCAU-2020). vol. 928 (2020), pp 1-13.