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On ArtinCokernel of The Group $D_n \times C_5$ When n is an Odd Number

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Abstract

The group of all Z-valuedgeneralized characters of G over the group of induced unit characters from all cyclic subgroups of G, AC(G)= $\overline{R}$ (G)/T(G) forms a finite abelian group, called ArtinCokernel of G .The problem of finding the cyclic decomposition of Artincokernel AC($D_n \times C_5$) has been considered in this paper when n is an odd number , we find that if n = $p_1^{\alpha_1}.p_2^{\alpha_2}..p_m^{\alpha_m}$, where $p_1,p_2,\ldots,p_m$ are distinct primes and not equal to 2 , then :

$$AC(D_n \times C_5) = \bigoplus_{i=1}^{2((\alpha_1+1)\cdot(\alpha_2+1)\cdots(\alpha_m+1))-1} C_2$$

$$= \bigoplus_{i=1}^{2} AC(D_n) \oplus C_2$$

And we give the general form of Artin's characters table Ar($D_n \times C_5$)when n is an odd number .

Mathematics Subject Classification: 20CXX .

Introduction

For a finite group G the finite abelain factor group $\overline{R}(G)$/T(G) is called Artincokernel of G and denoted by AC(G) where $\overline{R}(G)$ denotes the abelian group generated by Z-valued characters of G under the operation of pointwise addition and T(G) is anormal subgroup of $\overline{R}(G)$ which is generated by Artin'scharacters.Permutationcharcters induce from the principle charcters of cyclic Subgroups . A well-known theorem which is due to Artinasserated that T(G)has a finite index is , i.e [:T(G)] is finite .

The exponent of AC(G) is called Artin exponent of G and denoted by A (G) .

In 1968 , Lam . T .Y [5] gave the definition of the group AC(G) and the studied AC(C_n) .In 1976 , David .G [12] studied A(G) of arbitrary characters of cyclic subgroups. In 1996 , Knwabuez .K [11] studied A(G) of p-groups .

In 2000,H.R.Yassein [4] found AC(G) for the group $\bigoplus_{i=1}^{n} C_p$. In 2002, k.Sekieguchi [12] studied the irreducible Artin characters of p-group and in the Same year H.H.Abbass [10]found ≡*(Dn) .

In 2006, Abid . A .S [6] foundAr(C_n) when C_n is the cyclic group of order n . In 2007 , Mirza .R .N [9] found in herthesis Artincokernel of the dihedral group

In this paper we find the general form of Ar($D_n \times C_5$) and we study AC($D_n \times C_5$) of the non abelian group$D_n \times C_5$ when n is an odd number .

1. Basic Concepts and Notations:

In this section, we recall some basic concepts,about matrix representation , characters and Artin character which will be used in later section .

Definition (1.1): [1]

A matrix representation of a group G is homomorphism T of G into GL (n ,F) , n is called the degree of matrix representation T .T is called a unit representation(principal) if T(g)=1, for all g∈G.

Definition (1.2):[2]

Let T be a matrix representation of a group G over the field F,*the character*χ of a matrixrepresentation T is the mapping χ: G→F defined by χ(g)=Tr(T(g)) refers to the trace of the matrix T(g)(the sum of the elements diagonal of T(g)). The degree of T is called the degree of χ .

Definition (1.3):[3]
Let H be acyclic subgroup of G and let ϕ be a class function on H . The induced class function on G is given by :

$$\phi'(g) = \frac{1}{|H|}\sum_{x \in G} \phi^{\circ}(xgx^{-1}) \qquad ,\forall g \in G$$

Where$\emptyset^{\circ}$is defined by :

$$\phi^{\circ}(h) = \begin{cases} \phi(h) & if \quad h \in H \\ 0 & if \quad h \notin H \end{cases}$$

Theorem (1.4):[4]
Let H be acyclic subgroup of Gand $h_1, h_2, \ldots, h_m$ are chosen representatives for Γ-conjugate classes , Then :

$$1- \quad \phi'(g) = \frac{|C_G(g)|}{|C_H(g)|}\sum_{i=1}^{m} \phi(h_i) \qquad if \quad h_i \in H \cap CL(g)$$

$$2- \quad \phi'(g) = 0 \qquad\qquad if \quad H \cap CL(g) = \phi$$

Definition (1.5):[5]
Let G be a finite group, allcharacters of G induced from the principal character of cyclic subgroups of G is called Artin characters of G.

Definition (1.6):[4]
Artin characters of the finite group can be displayed in a table called Artin characters table of Gwhich is denoted by Ar(G) .

Proposition (1.7):[6]
The number of all distinct Artin characters on a group G is equal to the number of Γ-classes on G .

Definition (1.8):[1]
A rational valued character θ of G is a character whose values are in Z, which is θ(g) ∈ Z, for all g ∈G .

Definition (1.9):[6]

Let T(G) be the subgroup of $\overline{R}(G)$ generated by Artin characters .

T(G) is a normal subgroup of $\overline{R}(G)$ .Then the factor abelian group $\overline{R}(G)/T(G)$ is called Artincokernel of G**,** denoted by AC(G).

Proposition (1.10):[6]

AC(G) is a finitely generated Z – module

Theorem [Artin] (1.11):[7]

Every rational valued character of G can be written as a linear combination of Artin characters with rational coefficient .

2. The Factor Group AC(G):

In this section, we use some concepts in linear Algebra to study the factor group AC(G). We will give the general form of Ar $(D_n \times C_5)$ when n is an odd number . We shall study Ac(G) dihedral group$D_n$and $\equiv^{*}(D_n)$ when n is an odd number.

Definition (2.1):[5]

Let T(G) be the subgroup of $\overline{R}(G)$ generated by Artin characters .

T(G) is a normal subgroup of $\overline{R}(G)$, then the factor abelian group $\overline{R}(G)/T(G)$ is called Artincokernel of G**,** denoted by AC(G) **.**

Definition (2.2): [8]

*A*k-th determinant divisor of M is the greatest common divisor (g.c.d)of all the k – minors of M.This is denoted by $D_k(M)$.

Lemma (2.3)

Let M , P and W be matrices with entries in the principal idealdomain R and p, W be invertible matrices , then :

$D_k(P \cdot M \cdot W) = D_k(M)$ Modulo the group of units of R**.**

Theorem (2.4):[8]

Let M be an $k \times k$ matrix with entries in a principal ideal domain R , then there exits matrices P and W such that :

1 - P and W are invertible .2 - P M W = D .3 - D is a diagonal matrix .

4 -If we denote D*jj* by d_i then there exists a natural number m ; $0 \le m \le k$

such that $j > m$ implies $d_j = 0$ and $j \le m$ implies $d_j \neq 0$ and $1 \le j \le m$

implies $d_j \mid d_{j-1}$.

Definition (2.5):[8]

Let M be matrix with entries in a principal ideal domain R, equivalent to matrix D = diag $\{d_1, d_2, \ldots, d_m, 0, 0, \ldots, 0\}$ such that $d_j \mid d_{j-1}$ for $1 \le j < m$,we

call D the invariant factor matrix of M and $d_1, d_2, \ldots, d_m$ theinvariant factors of M.

Remark(2.6):

According to the Artin theorem (1.12) there exists an invertible matrix$M^{-1}(G)$with entries in the set of rational numbers such that :

$\overset{*}{\equiv}(G) = M^{-1}(G)\,.Ar(G)$and this implies,

$M(G) = Ar(G)\,.\,(\overset{*}{\equiv}(G))^{-1}$

M(G) is the matrix expressing the T(G) basis in terms of the $\overline{R}(G)$ basis.

By theorem (2.5) there exists two matrices P(G) and W(G) with a determinant∓ 1 such that :

$P(G).\,M(G).W(G) = \text{diag}\{d_1, d_2, \ldots, d_l\} = D(G)$

where $d_i = \overset{+}{-} D_i(G) \mid D_{i-1}(G)$and l is the number of Γ-classes.

Theorem (2.7):[4]

AC(G) = $\bigoplus_{i=1}^{m} z$ where $d_i = \overset{+}{-} D_i(G) \,\big|\, D_{i-1}(G)$ where m is the number of all distinct □-classes.

Theorem(2.8):[9]

If n is an odd number such that $n = p_1^{\alpha 1} \cdot p_2^{\alpha 2} \cdots\cdots p_m^{\alpha_m}$, where $p_1, p_2, \cdots\cdots, p_m$ are distinct primes , then :

$$AC(D_n) = \bigoplus_{i=1}^{(\alpha_1+1)\cdot(\alpha_2+1)\cdots(\alpha_m+1)-1} C_2$$

Proposition (2.9): [8]

The rational valued characters table of the cyclic group C_{p^s} of the rank s+1 where p is a prime number which is denoted by ($\equiv^*(C_{p^s})$), is given as follows:

Γ-classes	[1]	$[r^{p^{s-1}}]$	$[r^{p^{s-2}}]$	$[r^{p^{s-3}}]$	…	$[r^{p^2}]$	$[r^p]$	[r]
θ_1	$p^{s-1}(p-1)$	$-p^{s-1}$	0	0	…	0	0	0
θ_2	$p^{s-2}(p-1)$	$p^{s-2}(p-1)$	$-p^{s-2}$	0	…	0	0	0
θ_3	$p^{s-3}(p-1)$	$p^{s-3}(p-1)$	$p^{s-3}(p-1)$	$-p^{s-3}$	…	0	0	0
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
θ_{s-1}	p(p-1)	p(p-1)	p(p-1)	p(p-1)	…	p(p-1)	-p	0
θ_s	p-1	p-1	p-1	p-1	…	p-1	p-1	-1
θ_{s+1}	1	1	1	1	…	1	1	1

Table (2.1)

where its rank s+1 represents the number of all distinct Γ-classes.

<u>Remark (2.10)</u>:[8]

If $n = p_1^{\alpha 1} . p_2^{\alpha 2} . \ldots . p_m^{\alpha_m}$ where $p_1 , p_2 , \ldots , p_m$,are distinct primes , then :

$\equiv^*(C_n) = \equiv^*(C_{p_1^{\alpha 1}}) \otimes \equiv^*(C_{p_2^{\alpha 2}}) \otimes \ldots \otimes \equiv^*(C_{P_m^{\alpha m}})$.

Definition (2.11):[7]

The dihedral group D_n is a certain non- abelian group of order2n .It is usually thought of as a group of transformations of the Euclidean plane of regular n-polygon consisting of rotations (about the origin) with the angle 2kπ/n ,k=0,1,2,…,n-1 and reflections (across lines through the origin).In generalwe can write it as:$D_n=\{ S^j r^k :0 \leq k \leq n-1 , 0 \leq j \leq 1\}$
which has the following properties :

$r^n =1$, $S^2 =1$, $Sr^k S^{-1} = r^{-k}$

Definition (2.12):

The group$D_n \times C_5$is the direct product group $D_n \times C_5$,where $C_5$is a cyclic group of order 5 consisting of elements $\{1, r', r2', r3', r4'\}$with $(r')^5=1$. It is of order 10n .

Theorem(2.13):[10]

The rational valued characters table of $D_n$when n is an odd number is given as follows:

$\equiv^*(D_n)=$

	Γ-classes of C_n	[S]
θ_1	$\equiv^*(C_n)$	0
⋮		⋮
θ_{S-1}	1 1 1 … 1 1	0
θ_S		1
θ_{S+1}	1 1 1 … 1 1	-1

Table (2.2)

Where S is the number of Γ-classes of C_n.

Theorem(2.14):

The rational valued characters table of the group$D_n \times C_5$ when n is an odd number is given as follows:

$\equiv^*(D_n \times C_5) = \equiv^*(D_n) \otimes \equiv^*(C_5)$

Theorem (2.15):[6]

The general form of Artin characters table of C_{p^s} when p is a prime numberand s is positive integer is given by the lower Triangluer matrix

Ar(C $_{p^s}$)=

Γ-classes	$[1]$	$[r^{p^{s-1}}]$	$[r^{p^{s-2}}]$	$[r^{p^{s-3}}]$	…	$[r]$
$\|CL_\alpha\|$	1	1	1	1	…	1
$\|C_{p^s}(CL_\alpha)\|$	p^s	p^s	p^s	p^s	…	p^s
φ'_1	p^s	0	0	0	…	0
φ'_2	p^{s-1}	p^{s-1}	0	0	…	0
φ'_3	p^{s-2}	p^{s-2}	p^{s-2}	0	…	0
⋮	⋮	⋮	⋮	⋮	⋮	⋮
φ'_s	P	P	p	P	…	0
φ'_{s+1}	1	1	1	1	…	1

Table (2.3)

<u>Corollary (2.16)</u>:[4]

Let n any positive integers and $n = p_1^{\alpha 1} \cdot p_2^{\alpha 2} \cdots\cdots p_m^{\alpha_m}$ where $p_1, p_2, \cdots\cdots, p_m$ are distinct primes , then :

$$Ar(C_n) = Ar\left(C_{P_1^{\alpha 1}}\right) \otimes Ar\left(C_{P_2^{\alpha 2}}\right) \otimes \cdots\cdots \otimes Ar\left(C_{P_m^{\alpha_m}}\right)$$

Where $\otimes$ is the tensor product.

Proposition (2.17):[6]
If p is a prime number and s is a positive integer , then M(C*p*) is an upper triangular matrix with unite entries.

$$M\,(C_{p^s}) = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

which is (s+1)×(s+1) square matrix

Proposition (2.18):[2]
The general form of matrices $P(C_{p^s})$ *and* $W(C_{p^s})$ are :

$$P(C_{p^s}) = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ & & & & & \ddots & & \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}$$

which is (s+1)×(s+1) square matrix and $W\left(C_{p^s}\right) = I_{s+1}$ where I_{s+1} is an identity matrix and D(Cp^s)=diag{1,1,…,1}.

Remarks (2.19):

1- In general if $n = p_1^{\alpha 1} \cdot p_2^{\alpha 2} \cdots\cdots p_m^{\alpha_m}$ such that $p_1, p_2, \cdots\cdots, p_m$ are distinct primes and α_i any positive integers for all i = 1, 2, … , m ; then :

$C_n = C_{p_1^{\alpha 1}} \times C_{p_2^{\alpha 2}} \times \ldots \times C_{p_m^{\alpha m}}$.

$$M(C_n) = M(C_{p_1^{\alpha 1}}) \otimes M(C_{p_2^{\alpha 2}}) \otimes \ldots \otimes M(C_{p_m^{\alpha m}}) .$$

So, we can write M (C_n) as:

$$M(C_n) = \begin{bmatrix} & & & & & 1 \\ & & & & & 1 \\ & & R(C_n) & & & \vdots \\ & & & & & 1 \\ 0 & 0 & 0 & \ldots & 0 & 1 \end{bmatrix}$$

Where $R(C_n)$ is the matrix obtained by omitting the last row {0, 0,...,0, 1} and the last column {1,1,…,1} from the tensor product, $M(C_{p_1^{\alpha 1}}) \otimes M(C_{p_2^{\alpha 2}}) \otimes \ldots\ldots \otimes M(C_{p_m^{\alpha m}})$. $M(C_n)$ is,

$(\alpha_1 +1)(\alpha_2 +1)\ldots(\alpha_m +1) \times (\alpha_1 +1)(\alpha_2 +1)\ldots(\alpha_m +1)$ square matrix.

2) $P(C_n) = P(C_{p_1^{\alpha 1}}) \otimes P(C_{p_2^{\alpha 2}}) \otimes \ldots \otimes P(C_{p_m^{\alpha m}})$.

3) $W(C_n) = W(C_{p_1^{\alpha 1}}) \otimes W(C_{p_2^{\alpha 2}}) \otimes \ldots \otimes W(C_{p_m^{\alpha m}})$.

3. The Main Results

In this section we give the general form of Artin characters table of the group $D_n \times C_5$ and the cyclic decomposition of the factor group AC($D_n \times C_5$) when n is an odd number .

Theorem(3.1):

The Artin characters table of the group $D_n \times C_5$ when n is an odd number is given as follows :
Ar($D_n \times C_5$)=

Γ-Classes	$[1,1']$	$[1,r']$	Γ-Classes of $C_n \times C_5$					$[S,1]$	$[S,r']$
$\|CL_\alpha\|$	1	1	2	2	…..	….	2	n	n
$\|C_{D_n \times c_5}(CL_\alpha)\|$	10n	10n	5n	5n	…..	…..	5n	10	10
$\Phi_{(1,1)}$	$2Ar(C_n) \otimes Ar(C_5)$							0	0
$\Phi_{(1,2)}$								⋮	⋮
:								⋮	⋮
$\Phi_{(l,1)}$								⋮	⋮
$\Phi_{(l,2)}$								0	0
$\Phi_{(l+1,1)}$	5n	0	0	…..	…..	…..	0	5	0
$\Phi_{(l+1,2)}$	5n	0	0	…..	…..	…..	0	0	5

Table(3.1)

where l is the number of Γ-classes of C_n and C_5=< r' >={ $1'$, r' }.

*<u>Proof:-</u>*By theorem(2.15)

Ar (C_5) =

Γ- classes	$[1']$	$[r']$
$\|CL_\alpha\|$	1	1
$\|c_{5(CL_\alpha)}\|$	5	5
φ'_1	5	0
φ'_2	1	1

Table (3.2)

Each cyclic subgroup of the group $D_n \times C_5$ is either a cyclic subgroup of $C_n \times C_5$ or $<(S, r')>$ or $<(S, 1')>$ .If H is a cyclic subgroup of $C_n \times C_5$, then :

$H = H_i \times <1'>$ or $H_i \times <r'> = H_i \times C_5$ for all $1 \le i \le l$ where l is the number of Γ-classesof C_n

If $H = H_i \times <1'>$ and $x \in D_n \times C_5$

If $x \notin H$ then by theorem(1.4)

$$\Phi_{(1,i)}(x) = 0 \text{ for all } 0 \le i \le l \quad [\text{since } H \cap CL(x) = \phi]$$

If $x \in H$ then either $x = (1, 1')$ or $\exists S, 0 < S < n$ such that $x = (r^S, 1')$

If $x = (1, 1')$, then :

$$\Phi_{(I,1)}(x) = \frac{|C_{D_n \times C_5}(X)|}{|C_{H(X)}|} \cdot \varphi'(x) \quad [\text{since } H \cap CL(x) = \{(1, 1')\}] ,$$

where φ is the principle character

$$= \frac{10n}{|H_i| . \ |<1'>|} .1 \quad = \frac{10\,n}{|H_i|} = 2. \quad \frac{n}{|H_i|} .1.5 = 2\frac{|C_{C_n}(1)|}{|C_{H_i}(1)|} \cdot \varphi(1) \cdot \varphi'(1')$$

$$= 2 \cdot \varphi_i(1) \cdot \varphi'(1')$$

If $x = (r^S, 1')$ then

$$\Phi_{(i,1)}(x) = \frac{|C_{D_n \times C_5}(X)|}{|C_{H(X)}|} . \sum_1^2 \varphi'\ (x) \quad [\text{since } H \cap CL(x) = \{(r^S, 1'), (r^{-S}, 1')\}]$$

$$= \frac{5n}{|H_{i \times <1'>}|} \ .(1+1)$$

$$= \frac{5n}{|H_{i \times <1'>}|} .\, 2 = \frac{5n}{|H_i|} .\, 2$$

$$= 2. \frac{n}{|H_{i(r^S)}|} .\, 1.5 = 2\frac{|C_{C_n}(r^S)|}{|C_{H_i}(r^S)|} \cdot \varphi(r^S) \cdot \varphi'(1') = 2 \cdot \varphi_i(r^S) \cdot \varphi_1'(1')$$

If H=$H_i \times \langle r' \rangle = H_i \times C_5$

let x $\in D_n \times C_5$

if x $\notin$ H then

$\Phi_{(i,2)}(x)=0$ for all $1 \le i \le l$ [since H $\cap$ CL(x) = ϕ]

If x $\in$ H then either g=$(1,1')$ or x=$(1,r')$ or $\exists S, 0 < S < n$ such that x=(r^S, r')

If x=$(1,1')$

$$\Phi_{(i,2)} = \frac{|C_{D_n \times c_{5(x)}}|}{|C_{H(X)}|} . \varphi(x) \quad \text{[si}r^s\text{nce H} \cap \text{CL(x)} = \{(1,1')\}]$$

$$= \frac{10n}{|H_{i \times C_5}|} = \frac{10n}{2|H_i|} = \frac{5n}{|H_i|} = 5\frac{|C_{C_n}(1)|}{|C_{H_i}(1)|} \cdot \varphi(1) = 5.\varphi_i(1).\varphi_2{}'(1)$$

If x=$(1,r')$ then

$$\Phi_{(i,2)}(x) = \frac{|C_{D_n \times c_{5(x)}}|}{|C_{H(X)}|} . \varphi(x) \quad \text{[since H} \cap \text{CL(x)} = \{(1,r')\}]$$

$$= \frac{10n}{|H_{i \times C_5}|}$$

$$= \frac{10n}{2|H_i|} = \frac{5n}{|H_i|} = 5\frac{|C_{C_n}(1)|}{|C_{H_i}(1)|} = 5.\varphi_i(1).\varphi'_2(r')$$

If x=(r^S, r') then

$$\Phi_{(i,2)}(x) = \frac{|C_{D_n \times C_5}(X|}{|C_{H(X)}|} . \sum_1^2 \varphi'(x) \quad \text{[since H} \cap \text{CL(x)} = \{(r^S, r'), (r^{-S}, r')\}]$$

$$= \frac{5n}{|H_{i \times C_5}|}(1+1) = \frac{10n}{2|H_i|} = \frac{5n}{|H_i|} = 2\frac{|C_{C_n}(r^s)|}{|C_{H_i}(r^s)|} \cdot \varphi(r^s) \cdot \varphi'_2(r') = 5 \cdot \varphi_i(r^s) \cdot \varphi'_2(r').$$

If H=<$(S,1')$> = { $(1,1'),(S,1')$ }then

$$\Phi_{(l+1,1)}((1,1'))=\frac{\left|C_{D_{n\times C_5}(1,1')}\right|}{\left|C_{H(S,1')}\right|}.\varphi(x)=\frac{10n}{2}=5n$$

$$\Phi_{(l+1,1)}((S,1'))=\frac{\left|C_{D_{n\times C_5}(1,1')}\right|}{\left|C_{H(S,1')}\right|}.\varphi(x)[\text{since } H\cap CL((S,1'))=\{(S,1')\}]=\frac{10}{2}=5$$

Otherwise

$$\Phi_{(l+1,1)}(x)=0 \text{ for all } x\in D_n\times C_5, [\text{since } x\notin H]$$

If H=<(S,r')> = { $(1,1'),(S,r')$ }

$$\Phi_{(l+1,2)}((1,1'))=\frac{\left|C_{D_{n\times C_5}(1,1')}\right|}{\left|C_{H(1,1')}\right|}.\varphi(1,1') \quad [\text{since } H\cap CL((1,1'))=\{(1,1')\}]$$

$$=\frac{10n}{2}.1=5n$$

$$\Phi_{(l+1,2)}((S,r'))=\frac{\left|C_{D_{n\times C_5}(s,r')}\right|}{\left|C_{H(s,r')}\right|}.\varphi(s,r')=\frac{10}{2}.1=5$$

Otherwise $\Phi_{(l+1,2)}(x)=0$ for all $x\in D_n\times C_5$ since $H\cap CL(x)=\phi$ ■

<u>Proposition (3.2)</u>:

If $n=p_1^{\alpha 1}\cdot p_2^{\alpha 2}\cdot\cdots\cdots p_m^{\alpha_m}$ where $p_1,p_2,\cdots\cdots,p_m$ are distinct primes and $p_i\neq 2$ for all $1\le i\le m$ and α_i any positive integers, then:

$$M(D_{n\times C_5})=\begin{bmatrix} 2R(C_n)\times M(C_5) & \begin{matrix}1&1&1&1\\1&0&1&0\\1&1&1&1\\\vdots&\vdots&\vdots&\vdots\end{matrix} \\ \begin{matrix}0&0&\cdots\cdots\\0&0&\cdots\cdots\\1&1&\cdots\cdots\\1&1&\cdots\cdots\end{matrix} & \begin{matrix}0&1&1&1&1\\0&1&0&1&0\\1&1&1&0&0\\1&1&0&0&1\end{matrix}\end{bmatrix}$$

which is $2\left[(\alpha_1+1)\cdot(\alpha_2+1)\cdots(\alpha_m+1)+1\right]\times 2\left[(\alpha_1+1)\cdot(\alpha_2+1)\cdots(\alpha_m+1)+1\right]$ square matrix .

Proof:

By theorem(3.1) we obtain the Artin characters table Ar($D_{n\times c_5}$) and from theorem(1.11) we find the rational valued characters table

$$\overset{*}{\equiv}(D_{n\times c_5})\ .$$

Thus by the definition of M(G) we can find the matrix M($D_{n\times c_5}$) :

$$M\,(D_n\times C_5)=Ar(D_n\times C_5).(\overset{*}{\equiv}(D_n\times C_5))^{-1}=\begin{bmatrix} 2 & 2 & 2 & 2 & \cdots & \cdots & \cdots & \cdots & 1 & 1 & 1 & 1 \\ 0 & 2 & 0 & 2 & \cdots & \cdots & \cdots & \cdots & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 2 & \cdots & \cdots & \cdots & \cdots & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 & \cdots & \cdots & \cdots & \cdots & 1 & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & & & & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & & \ddots & & & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & & & \ddots & & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & & & & \ddots & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & \cdots & \cdots & \cdots & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & \cdots & \cdots & \cdots & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} & & & & 1\ 1\ 1\ 1 \\ & & & & 1\ 0\ 1\ 0 \\ 2R(C_n)\otimes M(C_5) & & & & 1\ 1\ 1\ 1 \\ & & & & \vdots\ \vdots\ \vdots\ \vdots \\ & & & & 1\ 0\ 1\ 0 \\ 0 & 0 & \cdots\cdots & & 0\ 1\ 1\ 1\ 1 \\ 0 & 0 & \cdots\cdots & & 0\ 1\ 0\ 1\ 0 \\ 1 & 1 & \cdots\cdots & & 1\ 1\ 1\ 0\ 0 \\ 1 & 1 & \cdots\cdots & & 1\ 1\ 0\ 0\ 1 \end{bmatrix}$$

Which is $2\left[(\alpha_1+1)\cdot(\alpha_2+1)\cdots+1\right]\times 2\left[(\alpha_1+1)\cdot(\alpha_2+1)\cdots+1\right]$ square matrix .

Proposition (3.3):

If $n = p_1^{\alpha 1} \cdot p_2^{\alpha 2} \cdot \cdots \cdot p_m^{\alpha_m}$ such thatg.c.d(p_i, p_j)=1 and $p_i \neq 2$ are prime numbers and α_i any positive integers, then:

$$P(D_{n\times C_5}) = \begin{bmatrix} & & & 0 & 0 \\ & & & 0 & 0 \\ & & & \vdots & \vdots \\ P(C_n)\otimes P(C_5) & & & 0 & 0 \\ & & & -1 & -1 \\ & & & 0 & 0 \\ 0\,0\,0 & \cdots & 0\;0 & 1 & 0 \\ 0\,0\,0 & \cdots & 0\;0 & 0 & 1 \end{bmatrix}$$

And

$$W(\boldsymbol{D_n} \times \boldsymbol{C_5}) = \begin{bmatrix} & & & & & & 0 & 0 & 0 & 0 \\ & & & & & & 0 & 0 & 0 & 0 \\ & & I_k & & & & \vdots & \vdots & \vdots & \vdots \\ & & & & & & \vdots & \vdots & \vdots & \vdots \\ & & & & & & 0 & 0 & 0 & 0 \\ & & & & & & 0 & 0 & 0 & 0 \\ -1 & -1 & \cdots & \cdots & -1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & \cdots & \cdots & 0 & 0 & -1 & 0 & 1 & 0 \\ 1 & 1 & \cdots & \cdots & 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & \cdots & \cdots & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Where $k = 2[(\alpha_1+1)\cdot(\alpha_2+1)\cdot(\alpha_3+1)\cdots(\alpha_m+1)-1]\times 2[(\alpha_1+1)\cdot(\alpha_2+1)\cdot(\alpha_3+1)\cdots(\alpha_m+1)-1]$

They are $2\left[(\alpha_1+1)\cdot(\alpha_2+1)\cdots(\alpha_m+1)+1\right]\times 2\left[(\alpha_1+1)\cdot(\alpha_2+1)\cdots(\alpha_m+1)+1\right]$ square matrix .

Proof **:**

By using theorem(2.5) and taking the form M($D_n \times C_5$) from proposition(3.2) and the above forms of P($D_n \times C_5$) and W($D_n \times C_5$) then we have

$$P(\boldsymbol{D_n}\times\boldsymbol{C_5}).M(\boldsymbol{D_n}\times\boldsymbol{C_5}).W(\boldsymbol{D_n}\times\boldsymbol{C_5})=\begin{bmatrix}2&0&0&0&\cdots&\cdots&0&0&0&0\\0&2&0&0&\cdots&\cdots&0&0&0&0\\0&0&2&0&\cdots&\cdots&0&0&0&0\\0&0&0&2&\cdots&\cdots&0&0&0&0\\\vdots&\vdots&\vdots&\vdots&\ddots&&\vdots&\vdots&\vdots&\vdots\\\vdots&\vdots&\vdots&\vdots&&\ddots&\vdots&\vdots&\vdots&\vdots\\0&0&0&0&\cdots&\cdots&-2&0&0&0\\0&0&0&0&\cdots&\cdots&0&1&0&0\\0&0&0&0&\cdots&\cdots&0&0&1&0\\0&0&0&0&\cdots&\cdots&0&0&0&1\end{bmatrix}$$

$$\mathbf{D(\boldsymbol{D_n}\times\boldsymbol{C_5})=diag\{2,2,2,\ldots..,-2,1,1,1\}}$$

Which is $2\left[(\alpha_1+1)\cdot(\alpha_2+1)\cdots(\alpha_m+1)+1\right]\times 2\left[(\alpha_1+1)\cdot(\alpha_2+1)\cdots(\alpha_m+1)+1\right]$ squarematrix .

<u>Theorem (3.4)</u>:

If $n=p_1^{\alpha 1}\cdot p_2^{\alpha 2}\cdot\cdots\cdots p_m^{\alpha_m}$ where $p_1, p_2,\cdots\cdots, p_m$ are distinct prime numbers such that $p_i\neq 2$ and α_i any positive integersfor all $i, 1\leq i\leq m$, then the cyclic decomposition AC($D_{n\times C_5}$) is :

$$AC(D_{n\times C_5})=\bigoplus_{i=1}^{2((\alpha_1+1)\cdot(\alpha_2+1)\cdots(\alpha_m+1))-1} C_2$$

$$AC(D_{n\times C_5})=\bigoplus_{i=1}^{2} AC(D_n)\oplus C_2$$

<u>Proof</u> :-

From proposition (3.3) we have

$P(D_{n\times C_5})$ .M$(D_{n\times C_5})$ .W$(D_{n\times C_5})$ =diag{2,2,2,…,-2,1,1,1}= $\{d_1,d_2,\ldots\ldots..,$

$d_{2((\alpha_1+1)\cdot(\alpha_2+1)\cdots(\alpha_m+1)-1)}, d_{2((\alpha_1+1)\cdot(\alpha_2+1)\cdots(\alpha_m+1))-1}, d_{2((\alpha_1+1)\cdot(\alpha_2+1)\cdots(\alpha_m+1))}$,

$d_{2((\alpha_1+1)\cdot(\alpha_2+1)\cdots(\alpha_m+1))+1}\, d_{2((\alpha_1+1)\cdot(\alpha_2+1)\cdots(\alpha_m+1))+2}\}$.

By theorem (2.8) we get

$$AC(D_{n\times C_5}) = \bigoplus_{i=1}^{2((\alpha_1+1)\cdot(\alpha_2+1)\cdots(\alpha_m+1))-1} C_{di}$$

$$= \bigoplus_{i=1}^{2((\alpha_1+1)\cdot(\alpha_2+1)\cdots(\alpha_m+1))-1} C_2$$

From theorem(2.9) we have :

$$AC(D_{n\times C_5}) = \bigoplus_{i=1}^{2} AC(D_n) \oplus C_2$$

Example (3.6):

To find the cyclic decomposition of the groups AC($D_{24389\times C_5}$) ,AC($D_{12901781\times C_5}$) and AC($D_{219330277\times C_5}$) .

We can use above theorem :

1- AC($D_{24389\times C_5}$) =AC($29^3 \times c_5$)= $\bigoplus_{i=1}^{2(3+1)-1} C_2 = \bigoplus_{i=1}^{7} C_2 = \bigoplus_{i=1}^{2} AC(D_{29^3}) \oplus$ C2 .

2- AC($D_{219330277} \times c_5$) =AC($D_{29^3 . 23^2} \times c_5$)= $\bigoplus_{i=1}^{2((3+1)\cdot(2+1))-1} C_2 = \bigoplus_{i=1}^{23} C_2$

$$= \bigoplus_{i=1}^{2} AC(D_{29^3 . 23^2}) \oplus C_2 .$$

3-AC($D_{219330277\times C_5}$) =AC($D_{29^3 . 23^2 . 17} \times c_5$)= $\bigoplus_{i=1}^{2((3+1)\cdot(2+1)\cdot(1+1))-1} C_2$

$$= \bigoplus_{i=1}^{47} C_2 \quad = \bigoplus_{i=1}^{2} AC(D_{29^3 . 23^2 . 17}) \oplus C_2 .$$

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حول النواةالمشارك –آرتن للزمرة $D_n \times C_5$ عندما n عدد فردي

باسم كريم محسن
المديرية العامة للتربية في محافظة كربلاء

المستخلص :

ان زمرة كل الشواخص العمومية ذات القيم الصحيحة للزمرة G على زمرة الشواخص المحتثة من الشواخص الأحادية للزمر الجزئية الدائرية $AC(G) = \overline{R}(G)/T(G)$ تكون زمرة ابيلية منتهية و تسمى النواة المشارك –آرتن للزمرة G .إن مسألة إيجاد التجزئة الدائرية لزمرة القسمة AC(G) تم اعتبارها في هذا البحث للزمرة $D_n \times C_5$ عندما n عدد فردي ، وجدنا إذا كانت $n = p_1^{\alpha_1}.p_2^{\alpha_2}..p_m^{\alpha_m}$ و إن $p_1, p_2, \ldots, p_m$ أعداد أولية مختلفة لا تساوي 2 فان :

$$AC(D_n \times C_5) = \bigoplus_{i=1}^{2((\alpha_1+1)\cdot(\alpha_2+1)\cdots(\alpha_m+1))-1} C_2$$

$$= \bigoplus_{i=1}^{2} AC(D_n) \oplus C_2$$

وكذلك وجدنا الصيغة العامة لجدول شواخص آرتن $Ar(D_n \times C_5)$ عندما يكون n عدد فردي .