Available online at www.qu.edu.iq/journalcm
JOURNAL OF AL-QADISIYAH FOR COMPUTER SCIENCE AND MATHEMATICS
ISSN:2521-3504(online) ISSN:2074-0204(print)

On Some Properties in Fuzzy Metric Space

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ARTICLE INFO

Article history:
Received: 20 /11/2020
Rrevised form: 20 /12/2020
Accepted : 07 /01/2021
Available online: 23 /01/2021

Keywords:

Fuzzy closed ball, Fuzzy convergent sequence, Fuzzy metric space, Fuzzy open ball.

ABSTRACT

Fuzzy metric spaces theory has an essential significance in mathematics, statistics, computer science, etc. In this paper, several basic properties of the fuzzy metric space (FM-space) $(\tilde{F}, \widetilde{\mathcal{M}})$ are discussed. The concepts of fuzzy convergent sequence, fuzzy Cauchy sequence, fuzzy open ball and fuzzy closed ball are recalled, then main theorems related to these concepts are proved.

MSC. 41A25; 41A35; 41A36

DOI : https://doi.org/10.29304/jqcm.2021.13.1.745

1. Introduction

The principle of metric spaces is introduced initially in 1906 by Frechet [1]. The distance between two points of a set in a metric space is symbolized by one non-negative number in R. A metric space can be defined as a non-empty set combined with a function of two variables such that the distance between points can be measured. In modern

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Communicated by: Alaa Hussein Hamadi

mathematics, more complicated objects like sets, functions and sequences need to find the distance between their elements.

It is well known that L.Zadeh in 1965 [2] created the theory of fuzzy sets. Fuzzy set theory is the field of measuring almost anything in life as a value between 0 and 1 instead of using exactly the values 0 and 1. In other words, a fuzzy set is an extension of classical set theory where elements have a degree of membership in the interval [0, 1]. The notions such as fuzzy sets, fuzzy orderings, fuzzy languages, etc facilitate the study of the degree of uncertainty in a simple mathematical and formal way. Kramosil and Michalek have introduced the concept of fuzzy metric on ordinary set in 1975 [3]. Kaleva and Seikkala in 1984 [4] generalized the idea of a metric space to present the concept of the fuzzy metrics space by specifying the distance between two points to be a non-negative fuzzy number. Several researchers have investigated and concluded various notions of fuzzy metric space using various points [5-10]. For more applications [15-77].

In this paper, some properties and characterizations of the FM-space are established to demonstrate the relationships between this space and different types of fuzzy concepts. The structure of this paper is as follows. In section 2 some properties and basic notions of the fuzzy metric space (FM-space) are given. Section 3 is devoted to proving some new properties of the fuzzy metric space.

2. Preliminaries [11-14]

In this section, we restate basic results and some definitions.

- **Definition 2.1**

Let U be a universal set, then for any $\alpha \in [0,1]$ *and* $u \in$ U, a fuzzy subset u_α of U is called a fuzzy point in U if

$$u_\alpha(w) = \begin{cases} \alpha & if\ u = w \\ 0 & otherwise \end{cases}$$

for each w ∈ U.

- **Definition 2.2**

The definition of a fuzzy metric space (briefly, FM-space) is an ordered pair $(\tilde{F}, \widetilde{\mathcal{M}})$ where $\tilde{F}$ is a fuzzy set and $\widetilde{\mathcal{M}}$ is a mapping from $\tilde{F} \times \tilde{F} \times (0,1]$ into $I = [0,1]$ so the following five properties hold, for each $(a,\alpha), (b,\beta), (c,\delta) \in \tilde{F}$:

$(\widetilde{\mathcal{M}}1)$ $\widetilde{\mathcal{M}}(a,b,\gamma) > 0$ if $a \neq b$ where $\gamma = \max\{\alpha, \beta\}$

$(\widetilde{\mathcal{M}}2)$ $\widetilde{\mathcal{M}}(a,b,\gamma) = 0$ if and only if a = b.

$(\widetilde{\mathcal{M}}3)$ $\widetilde{\mathcal{M}}(a,b,\gamma) = \widetilde{\mathcal{M}}(b,a,\gamma)$.

$(\widetilde{\mathcal{M}}4)$ $\widetilde{\mathcal{M}}(a,b,\gamma) \leq \widetilde{\mathcal{M}}(a,c,\gamma) + \widetilde{\mathcal{M}}(c,b,\gamma)$, where $\gamma = \max\{\alpha, \beta, \delta\}$

$(\widetilde{\mathcal{M}}5)$ If $0 < \lambda \leq \gamma < 1$ then $\widetilde{\mathcal{M}}(a,0,\gamma) \leq \widetilde{\mathcal{M}}(a,0,\lambda)$ and there exists $0 < \gamma_n < \gamma$ such that $\lim_{n\to\infty} \widetilde{\mathcal{M}}(a_n,0,\gamma_n) = \widetilde{\mathcal{M}}(a,0,\gamma)$.

- **Definition 2.3**

Let $(\tilde{F}, \widetilde{\mathcal{M}})$ be an FM- space , and let $(a,\alpha) \in \tilde{F}$,where $\alpha \in (0,1]$. Given real number $\varepsilon > 0$, then:

(1) $\tilde{O}_\varepsilon(a_1,\alpha_1) = \{(a,\alpha) \in \tilde{F} : \widetilde{\mathcal{M}}(a,a_1,\gamma) < \varepsilon\}$ is called the fuzzy open ball of radius ε where $\gamma = \max\{\alpha, \alpha_1 \in (0,1]\}$.

(2) $\tilde{B}_\varepsilon[a_1,\alpha_1] = \{(a,\alpha) \in \tilde{F} : \widetilde{\mathcal{M}}(a,a_1,\gamma) \leq \varepsilon\}$ is called the fuzzy closed ball of radius ε.

- **Definition 2.4**

Let $(\tilde{F}, \widetilde{\mathcal{M}})$ be an FM- space. A fuzzy subset $\tilde{A} \subseteq \tilde{F}$ is fuzzy open if and only if there exists a fuzzy open ball $\tilde{O}_\varepsilon(a,\alpha)$ centered at every fuzzy point (a,α) in $\tilde{A}$ that are contained in $\tilde{A}$. A fuzzy subset $\tilde{B} \subseteq \tilde{F}$ is called fuzzy closed if $\tilde{B}^c = \tilde{F} - \tilde{B}$ is fuzzy open.

- **Definition 2.5**

Let $(\tilde{F}, \widetilde{\mathcal{M}})$ be an FM- space and let $\widetilde{A} \subseteq \tilde{F}$. Then a fuzzy point $(a, \alpha) \in \tilde{F}$ is called an accumulation fuzzy point of $\widetilde{A}$ if and only if every fuzzy open set of (a, α) contains a fuzzy point of $\widetilde{A}$ distinct from (a, α).The union of the fuzzy set $\widetilde{A}$ and its accumulation fuzzy point is said to be a fuzzy closure of $\widetilde{A}$ and it's defined by $\overline{\widetilde{A}}$ or $\mathrm{Fcl}(\widetilde{A})$.

- **Definition 2.6**

In an FM-space $(\tilde{F}, \widetilde{\mathcal{M}})$, a fuzzy sequence $\{(a_n, \alpha_n)\}$ where $\alpha, \alpha_n \in (0,1]$ is:

i. Convergent if there exists $(a, \alpha) \in \tilde{F}$ so $\lim_{n\to\infty} \widetilde{\mathcal{M}}(a_n, a, \gamma)=0$ where $\gamma = \max\{\alpha_n, \alpha\}$ or simply written $\{(a_n, \alpha_n)\} \to (a, \alpha)$.

ii. Cauchy if for all $\varepsilon > 0$ there is an integer number $N \in \mathbb{N}$ such that $\widetilde{\mathcal{M}}(a_n, a_m, \gamma)<\varepsilon$ for every $n, m \geq N$ where $\gamma = \max\{\alpha_n, \alpha_m\}$.

- **Definition 2.7**

$\tilde{A}$ is a fuzzy set in an FM-space $(\tilde{F}, \widetilde{\mathcal{M}})$ is defined fuzzy bounded if there exists $0< r <1$ such that $\widetilde{\mathcal{M}}(a, b, \gamma) < r$ for each $(a, \alpha), (b, \beta) \in \tilde{F}$, $\gamma = \max\{\alpha, \beta\}$.

- **Definition 2.8**

A fuzzy sequence $\{(a_n, \alpha_n)\}$ in an FM-space $(\tilde{F}, \widetilde{\mathcal{M}})$ is said to be fuzzy bounded if the corresponding fuzzy set is fuzzy bounded.

- **Definition 2.9**

An FM-space $(\tilde{F}, \widetilde{\mathcal{M}})$ is defined complete if all Cauchy fuzzy sequence in $\widetilde{F}$ a fuzzy convergent.

- **Definition 2.10**

Let $(\tilde{F}, \widetilde{\mathcal{M}})$ be an FM- space and let $\{(a_n, \alpha_n)\}_{n\geq1}$be a fuzzy sequence of real numbers. Given $r_1 < r_2 < \cdots < r_n \ldots$ be strictly increasing sequence of natural numbers. Then $\{(a_{nk}, \alpha_{nk})\}_{n\geq1}$ is called a fuzzy subsequence of $\{(a_n, \propto_n)\}_{n\geq1}$.

3. Fuzzy Metric Space: Properties

Some new properties of the fuzzy metric space are proved. In addition, some relationships between fuzzy metric space and other fuzzy concepts are given. A new characterization of a fuzzy point that belongs to the fuzzy closer of fuzzy set $\widetilde{A}$ is given in the following theorem.

- **Theorem 3.1**

Let $(\tilde{F}, \widetilde{\mathcal{M}})$ be an FM- space and $\widetilde{A} \subseteq \tilde{F}$. Then $(a, \alpha) \in \mathrm{Fcl}(\widetilde{A})$ if and only if there is a fuzzy sequence $\{(a_n, \alpha_n)\}$ in $\widetilde{A}$ such that $\{(a_n, \alpha_n)\}$ converges to a fuzzy point (a, α) where $\alpha, \in (0,1]$.

Proof

Let $(a, \alpha) \in \mathrm{Fcl}(\widetilde{A})$. If $(a, \alpha) \in \widetilde{A}$ then we take a fuzzy sequence of the type $((a, \alpha), (a, \alpha), \ldots)$. If $(a, \alpha) \notin \widetilde{A}$ then it is the limit fuzzy point of $\widetilde{A}$. Hence for all $n = 1,2, \ldots$, the fuzzy open ball $\widetilde{O}_{\frac{1}{n}}(a, \alpha)$ consist of an $\{(a_n, \alpha_n)\}$ and $\{(a_n, \alpha_n)\} \to (a, \alpha)$ because $\frac{1}{n} \to 0$ as $n \to \infty$. For the converse, if $\{(a_n, \alpha_n)\}$ is in $\widetilde{A}^n$and $\{(a_n, \alpha_n)\} \to (a, \alpha)$ then $(a, \alpha) \in \widetilde{A}$ or each neighborhood of (a, α) contains a fuzzy point $(a_n, \alpha_n) \neq (a, \alpha)$ so that (a, α) is the limit of $\widetilde{A}$. Hence $(a, \alpha) \in \mathrm{Fcl}(\widetilde{A})$.

In an analogous manner to (Theorem (1.4-5), [12]), the following result is proved over the fuzzy metric space rather than the ordinary metric space.

- **Theorem 3.2**

Let $(\tilde{F}, \widetilde{\mathcal{M}})$ be an FM- space. Then, fuzzy Cauchy defined for all fuzzy convergent sequence in $(\tilde{F}, \widetilde{\mathcal{M}})$.

Proof

Suppose $\{(a_n, \alpha_n)\}$ a fuzzy sequence in $\tilde{F}$ that converges to the fuzzy point $(a_n, \alpha_n) \in \tilde{F}$. Assume that $\varepsilon > 0$ then also $\frac{\varepsilon}{2} > 0$. Because $\{(a_n, \alpha_n)\}$ converges to (a, α) then the positive number is $N \in \mathbb{N}$ so $\widetilde{\mathcal{M}}(a_n, a, \gamma) < \frac{\varepsilon}{2}$. Hence for each n, $m > N$ we have

$$\widetilde{\mathcal{M}}(a_n, a_m, \gamma) \leq \widetilde{\mathcal{M}}(a_n, a, \gamma) + \widetilde{\mathcal{M}}(a, a_m, \gamma)$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Thus $\{(a_n, \alpha_n)\}$ is a fuzzy Cauchy sequence.

- **Proposition 3.3**

Let $(\tilde{F}, \widetilde{\mathcal{M}})$ be an FM- space. If a fuzzy Cauchy sequence $\{(a_n, \alpha_n)\}$ in $\tilde{F}$ consist of a fuzzy convergent subsequence $\{(a_{nk}, \alpha_{nk})\}$, so the fuzzy sequence convergence to the same fuzzy limit as the subsequence.

Proof

Consider $\{(a_n, \alpha_n)\}$ fuzzy Cauchy sequence in $\tilde{F}$, so it is an integer number N with $\widetilde{\mathcal{M}}(a_n, a_m, \gamma) < \varepsilon$ where $\varepsilon > 0$, $m, n > N$. Let $\{(a_{nk}, \alpha_{nk})\}$ be a fuzzy converges subsequence of a fuzzy sequence $\{(a_n, \alpha_n)\}$. That is mean for each $\varepsilon > 0$, $\widetilde{\mathcal{M}}(a_{nm}, a_n, \gamma) < \varepsilon$, $m, n > N$. Now we have $\widetilde{\mathcal{M}}(a, a_n, \gamma) \leq \widetilde{\mathcal{M}}(a, a_{nm}, \gamma) + \widetilde{\mathcal{M}}(a_{nm}, a_n, \gamma) < \widetilde{\mathcal{M}}(a, a_{nm}, \gamma) + \varepsilon$. Letting $m \to \infty$ so we obtain $\lim_{m\to\infty} \widetilde{\mathcal{M}}(a, a_{nm}, \gamma) = 0$ i.e $\widetilde{\mathcal{M}}(a, a_{nm}, \gamma) < \varepsilon$ so the fuzzy sequence $\{(a_n, \alpha_n)\}$ converges to (a, α).

The following proposition describes the behavior of the fuzzy convergence sequence in a fuzzy metric space.

- **Proposition 3.4**

A fuzzy convergence sequence in an FM-space $(\tilde{F}, \widetilde{\mathcal{M}})$ is a fuzzy bounded and it's unique for a fuzzy limit.

Proof

Consider $\{(a_n, \alpha_n)\}$ be a fuzzy sequence that converges to a fuzzy point (a, α). Put $\varepsilon > 0$ we can find a positive integer number N with $\widetilde{\mathcal{M}}(a_n, a, \gamma) < \varepsilon$ where $n > N$. Assume that $s = \min\{\widetilde{\mathcal{M}}(a_1, a, \gamma), \widetilde{\mathcal{M}}(a_2, a, \gamma), \ldots, \widetilde{\mathcal{M}}(a_N, a, \gamma)\}$, then we have $\widetilde{\mathcal{M}}(a_n, a, \gamma) \leq \widetilde{\mathcal{M}}(a_n, a_N, \gamma) + \widetilde{\mathcal{M}}(a_N, a, \gamma) < s + \varepsilon$. Put $r = s + \varepsilon$, hence $\widetilde{\mathcal{M}}(a_n, a, \gamma) < r$ and this shows that $\{(a_n, \alpha_n)\}$ is fuzzy bounded. Now we claim that $\{(a_n, \alpha_n)\}$ has a unique fuzzy limit point. Let $\{(a_n, \alpha_n)\} \to (a, \alpha)$ and $\{(a_n, \alpha_n)\} \to (b, \beta)$ so $\lim_{n\to\infty} \widetilde{\mathcal{M}}(a_n, a, \gamma) = 0$ and $\lim_{n\to\infty} \widetilde{\mathcal{M}}(a_n, b, \gamma) = 0$. Now we obtain $\widetilde{\mathcal{M}}(a, b, \gamma) \leq \widetilde{\mathcal{M}}(a, a_n, \gamma) + \widetilde{\mathcal{M}}(a_n, b, \gamma)$. By taking limit for both sides as $n \to \infty$ we conclude that $\widetilde{\mathcal{M}}(a, b, \gamma) \leq 0 + 0 = 0$. Hence $(a, \alpha) = (b, \beta)$.

- **Theorem 3.5**

An FM-space $(\tilde{F}, \widetilde{\mathcal{M}})$ is a fuzzy topological space.

Proof

Suppose that $(\tilde{F}, \widetilde{\mathcal{M}})$ be an FM-space. Put $\mathcal{T}_{\widetilde{\mathcal{M}}} = \{\tilde{A} \subseteq \tilde{F}: (a, \alpha) \in \tilde{A}$ if and only if there is $\varepsilon > 0$ such that $\tilde{O}_\varepsilon(a, \alpha) \subseteq \tilde{A}\}$. We shall prove that $\mathcal{T}_{\widetilde{\mathcal{M}}}$ is a fuzzy topology on $\tilde{F}$. Clearly $\emptyset$ and $\tilde{F}$ belong $\mathcal{T}_{\widetilde{\mathcal{M}}}$. Let $\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n \in \mathcal{T}_{\widetilde{\mathcal{M}}}$ and put $\tilde{U} = \bigcap_{i=1}^{n} \tilde{A}_i$. We must show that $\tilde{U} \in \mathcal{T}_{\widetilde{\mathcal{M}}}$. Assume that $(a, \alpha) \in \tilde{G}$ so for each $1 \leq i \leq n$, $(a, \alpha) \in \tilde{A}_i$ therefore there is $\varepsilon_1 > 0$ with $\tilde{O}_{\varepsilon_1}(a, \alpha) \subseteq \tilde{A}_i$. Let $t = \min\{\varepsilon_i: 1 \leq i \leq n\}$ hence $t \leq \varepsilon_i$ for each $1 \leq i \leq n$ so $\tilde{O}_{\varepsilon_n}(a, \alpha) \subseteq \tilde{A}_i$. Therefore $\tilde{O}_\varepsilon(a, \alpha) \subset \bigcap_{i=1}^{n} \tilde{A}_i = \tilde{U}$. That is mean $\tilde{U} \in \mathcal{T}_{\widetilde{\mathcal{M}}}$. Now let $\{\tilde{A}_i: i \in I\} \in \mathcal{T}_{\widetilde{\mathcal{M}}}$ and put $\tilde{V} = \bigcup_{i=1}^{n} \tilde{A}_i$ we prove that $\tilde{V} \in \mathcal{T}_{\widetilde{\mathcal{M}}}$. Let $(a, \alpha) \in \tilde{V}$ then $(a, \alpha) \in \bigcup_{i=1}^{n} \tilde{A}_i$ that follows $(a, \alpha) \in \tilde{A}_i$ for some $\in I$. Since $\tilde{A}_i \in \mathcal{T}_{\widetilde{\mathcal{M}}}$ then there is a $\varepsilon > 0$ with $\tilde{O}_\varepsilon(a, \alpha) \subseteq \tilde{A}_i$. We have $\tilde{O}_\varepsilon(a, \alpha) \subseteq \tilde{A}_i \subseteq \bigcup_{i=1}^{n} \tilde{A}_i = \tilde{V}$ and this shows that $\tilde{V} \in \mathcal{T}_{\widetilde{\mathcal{M}}}$. Hence $(\tilde{F}, \widetilde{\mathcal{M}})$ is a fuzzy topological space.

- **Proposition 3.6**

In an FM-space $(\tilde{F}, \widetilde{\mathcal{M}})$ every fuzzy closed ball is a fuzzy closed set.

Proof

Suppose $\widetilde{B}_\varepsilon[a,\alpha]$ fuzzy closed ball in an FM-space $(\tilde{F},\widetilde{\mathcal{M}})$ and let $(b,\beta) \in Fcl(\widetilde{B}_\varepsilon[a,\alpha])$. Then by Theorem (3.1), there is a fuzzy sequence $\{(b_n,\beta_n)\}$ in a fuzzy closed ball $\widetilde{B}_\varepsilon[a,\alpha]$ such that $\{(b_n,\beta_n)\} \to (b,\beta)$. That is mean $\lim_{n\to\infty} \widetilde{\mathcal{M}}(b_n,b,\gamma) = 0$. Now we have $\widetilde{\mathcal{M}}(a,b,\gamma) \le \widetilde{\mathcal{M}}(a,b_n,\gamma) + \widetilde{\mathcal{M}}(b_n,b,\gamma)$ and taking the limit as $n \to \infty$ we get $\widetilde{\mathcal{M}}(a,b,\gamma) \le \lim_{n\to\infty} \widetilde{\mathcal{M}}(a,b_n,\gamma) + \lim_{n\to\infty} \widetilde{\mathcal{M}}(b_n,b,\gamma) < \varepsilon + 0 = \varepsilon$. Hence $(b,\beta) \in \widetilde{B}_\varepsilon[a,\alpha]$ and $\widetilde{B}_\varepsilon[a,\alpha]$ is a fuzzy closed set.

The remainder of this section devoted to proving some theorems and properties of FM-space based on the fuzzy open ball and the fuzzy open ball notions.

- **Theorem 3.7**

In an FM-space $(\tilde{F},\widetilde{\mathcal{M}})$ every distinct fuzzy point of $\tilde{F}$ can be separated by disjoint fuzzy open balls.

Proof

Suppose that $(\tilde{F},\widetilde{\mathcal{M}})$ be an FM-space and let $(a,\alpha),(b,\beta) \in \tilde{F}$ with $(a,\alpha) \ne (b,\beta)$. Letting $\widetilde{\mathcal{M}}(a,b,\gamma) = 2\varepsilon$ where $\gamma = \max\{\alpha,\beta \in (0,1]\}$. Consider two fuzzy open balls say, $\widetilde{O}_\varepsilon(a,\alpha)$ and $\widetilde{O}_\varepsilon(b,\beta)$. If $(c,\delta) \in \widetilde{O}_\varepsilon(a,\alpha) \cap \widetilde{O}_\varepsilon(b,\beta)$, i.e $(c,\delta) \in \widetilde{O}_\varepsilon(a,\alpha)$ implies $\widetilde{\mathcal{M}}(a,c,\gamma) < \varepsilon$ where $\gamma = \max\{\alpha,\delta \in (0,1]\}$ and $(c,\delta) \in \widetilde{O}_\varepsilon(b,\beta)$ implies $\widetilde{\mathcal{M}}(b,c,\gamma) < \varepsilon$, $\gamma = \max\{\beta,\delta \in (0,1]\}$. Thus $\widetilde{\mathcal{M}}(a,c,\gamma) + \widetilde{\mathcal{M}}(c,b,\gamma) < \varepsilon + \varepsilon = 2\varepsilon = \widetilde{\mathcal{M}}(a,b,\gamma)$ and this contradicts the definite of a fuzzy metric so $\widetilde{O}_\varepsilon(a,\alpha) \cap \widetilde{O}_\varepsilon(b,\beta) = \emptyset$. Hence the fuzzy open balls must be disjoint.

- **Proposition 3.8**

Suppose that $(\tilde{F},\widetilde{\mathcal{M}})$ be an FM-space and let $\widetilde{O}_{\varepsilon_1}(a,\alpha)$, $\widetilde{O}_{\varepsilon_2}(a,\alpha)$ be two fuzzy open balls with center $(a,\alpha) \in \tilde{F}$ and radius $\varepsilon_1,\varepsilon_2 > 0$. So either $\widetilde{O}_{\varepsilon_1}(a,\alpha) \subseteq \widetilde{O}_{\varepsilon_2}(a,\alpha)$ or $\widetilde{O}_{\varepsilon_2}(a,\alpha) \subseteq \widetilde{O}_{\varepsilon_1}(a,\alpha)$.

Proof

Suppose $\widetilde{O}_{\varepsilon_1}(a,\alpha)$, $\widetilde{O}_{\varepsilon_2}(a,\alpha)$ be a fuzzy open ball with the center $(a,\alpha) \in \tilde{F}$ and radius $\varepsilon_1,\varepsilon_2 > 0$. If $\varepsilon_1 = \varepsilon_2$ then the proposition holds. Now we assume that $\varepsilon_1 < \varepsilon_2$ and let $(b,\beta) \in \widetilde{O}_{\varepsilon_1}(a,\alpha)$ .That is mean $\widetilde{\mathcal{M}}(a,b,\gamma) < \varepsilon_1$ so $\widetilde{\mathcal{M}}(a,b,\gamma) < \varepsilon_2$. Hence $(b,\beta) \in \widetilde{O}_{\varepsilon_2}(a,\alpha)$ and this shows that $\widetilde{O}_{\varepsilon_1}(a,\alpha) \subseteq \widetilde{O}_{\varepsilon_2}(a,\alpha)$ since by assumption $\varepsilon_1 < \varepsilon_2$. In the same way, we can show that $\widetilde{O}_{\varepsilon_2}(a,\alpha) \subseteq \widetilde{O}_{\varepsilon_1}(a,\alpha)$.

- **Proposition 3.9**

In an FM-space $(\tilde{F},\widetilde{\mathcal{M}})$ any fuzzy open ball is a fuzzy open set.

Proof

Suppose $\widetilde{O}_\varepsilon(a,\alpha)$ be a fuzzy open ball in $\tilde{F}$ and let (b,β) any fuzzy point in a fuzzy open ball $\widetilde{O}_\varepsilon(a,\alpha)$ that is follow $\widetilde{\mathcal{M}}(a,b,\gamma) < \varepsilon$. We must find a fuzzy open ball $\widetilde{O}_{\tilde{\varepsilon}}(b,\beta)$ contained in $\widetilde{O}_\varepsilon(a,\alpha)$.we define $\tilde{\varepsilon} = \varepsilon - \widetilde{\mathcal{M}}(a,b,\gamma)$, then for any $(c,\delta) \in \widetilde{O}_{\tilde{\varepsilon}}(b,\beta)$ implies $\widetilde{\mathcal{M}}(b,c,\gamma) < \tilde{\varepsilon}$. Therefore $\widetilde{\mathcal{M}}(a,c,\gamma) \le \widetilde{\mathcal{M}}(a,b,\gamma) + \widetilde{\mathcal{M}}(b,c,\gamma) < \widetilde{\mathcal{M}}(a,b,\gamma) + \tilde{\varepsilon} < \varepsilon$ and this shows $(c,\delta) \in \widetilde{O}_\varepsilon(a,\alpha)$. Thus $\widetilde{O}_{\tilde{\varepsilon}}(b,\beta) \subseteq \widetilde{O}_\varepsilon(a,\alpha)$.

4. Conclusion

Some properties of the fuzzy metric space have been presented in this paper. Several concepts for example fuzzy convergent sequences, fuzzy Cauchy sequences, fuzzy open ball and fuzzy closed ball and FM-space are introduced and most of the main theorems are proved.

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