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# Bounded Linear Transformations in G-Fuzzy Normed Linear Space

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ABSTRACT

In a host of mathematical applications, the bounded linear transformation arises. The aim of the present work is to report the definition of continuous for linear transformation by using the idea of G-fuzzy normed linear space (GFNLS) with proving the main theorem regarding the continuity. Besides, the notion of a bounded linear transformation depending on GFNLS is presented and some basic properties related to this notion are proved. Furthermore, the extension of a bounded linear transformation is discussed and proved. Finally, a characterization for the notion G-B(X,Y) which is consisting of all bounded linear transformations is presented and proved that this characterization is a complete GFNLS if the space Y is complete.

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## 1. Introduction

A generalization the idea of usual metric and D-metric to translate more results from usual metric to D-metric was attempted by the author B. C. Dhange [1]. But the topological structure of D-metric spaces was proved to be incorrected [2]. The concept of G-metric space is proposed by Mustafa and Sims [3], in which the tetrahedral inequality is substituted by an inequality consisting the repetition of indices. Following such new approach, the concept of Q-fuzzy metric and studied some applications in this space were developed by [4-7].As a result of the strong correlation between fuzzy normed spaces and fuzzy metric spaces theories. The concept of generalized call (G-norm) is presented by K. A. Khan [8]. Depending on the concepts of Q-fuzzy metric and G-norm, the space of G-fuzzy normed is presented in [9]. The fuzzy functional analysis is improved and a wide variety of domains the concept of the fuzzy norm has been used, for examples one can see [10-18].

The main objective of this article is to suggest a new approach for the space of linear transformation defined on a GFNLS. In [19], Bag and Samanta introduced the boundedness of a linear operator as a study in this field and

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different types of linear bounded transformation by distinct authors introduced for more details [20- 24]. This paper is organized as follows: In Section 2, the definition of G-fuzzy norm is recalled and and given as a generalization with development of the idea of G-norm given by Chatterjee, Bag and Samanta. Some basic notions and important properties of G-fuzzy normed linear space are also included in section 2. Section 3 introduces continuous and bounded notions for linear transformation. Moreover, different properties for these notions are proved. In Section 4, a G-fuzzy norm for the notion of bounded linear transformations is defined in order to prove that space $(G\text{-}B(X,Y), G_{NLT}, \Delta)$ is a complete G-fuzzy normed linear. The conclusion is listed in Section 5.

## 2. Preliminaries

Some basic definitions and are given throughout this section.

**Definition** 2.1 [25]
A binary operation $\Delta: [0,1]^2 \to [0,1]$ is a triangular norm if for all $\alpha, \beta, \mu$ and $\sigma \in [0,1]$ the following conditions hold

(1) $\alpha\Delta\beta = \beta\Delta\alpha$

(2) If $\alpha \le \mu$ and $\beta \le \sigma$ then $\alpha\Delta\beta \le \mu\Delta\sigma$

(3) $(\alpha\Delta\beta)\Delta\mu = \alpha\Delta(\beta\Delta\mu)$

(4) $\alpha\Delta 1 = \alpha$

**Remark 2.2** [25]

For any $\alpha > \beta$ there is $\delta, 0 < \delta < 1$ such that $\alpha\Delta\delta \ge \beta$ and for any $\gamma$ there is $\mu$ with $\mu\,\Delta\,\mu \ge \gamma$ where $\alpha, \delta, \beta, \gamma, \mu \in (0,1)$

The GFNLS definition is given as below

**Definition 2.3** [9]

A 3-tuple $(X, G_N, \Delta)$ is GFNLS if X is a linear space, $G_N$ is a fuzzy subset of $X \times X \times X \times \mathbb{R}$ and $\Delta$ represents the general triangular norm then for each $x, y, z \in X$, $c \in \mathbb{F}$, the following properties are satisfied

($G_N$1) For all $t \in \mathbb{R}, t \le 0$, $G_N(x, y, z, t) = 0$

($G_N$2) For all $t \in \mathbb{R}, t > 0$, $G_N(x, y, z, t) = 1$ if and only if $x = y = z = \theta$

($G_N$3) $G_N(x, y, z, t) = G_N(P(x, y, z), t)$, (symmetry) where P is a permutation function of x, y, z

($G_N$4) For all $t \in \mathbb{R}, t > 0$, $G_N(cx, cy, cz, t) = G_N(x, y, z, \frac{t}{|c|})$, if $c \ne 0$

($G_N$5) For all $t, s \in \mathbb{R}$, $x, y, z, \bar{x}, \bar{y}, \bar{z} \in X$

$$G_N(x + \bar{x}, y + \bar{y}, z + \bar{z}, t + s) \ge G_N(x, y, z, t)\Delta G_N(\bar{x}, \bar{y}, \bar{z}, s)$$

($G_N$6) $\lim_{t\to\infty} G_N(x, y, z, t) = 1$

($G_N$7) For all $x, y, z \in X$ $G_N(x + y, \theta, z, t) \ge G_N(x, y, z, t)$

**Example 2.4** [9]

Let $X = C[0,1]$ is the linear space of real-valued continuous function then a function $\|.,.,.\|: X \times X \times X \to \mathbb{R}$ is defined by $\|f, g, l\| = \max_{t\in[0,1]}\{|f(t)| + |g(t)| + |l(t)|\}$ where $f, g, l \in C[0,1]$ .Then $(X, \|.,.,.\|)$ is a GFNLS. Define

$G_N(f,g,l,t) = [\exp(\|f,g,l\|/t)]^{-1}$ for all $f,g,l \in C[0,1]$, $t > 0$, $G_N(f,g,l,t) = 0$ for all $t \leq 0$ and $\alpha\Delta\beta = \alpha\beta$. Hence the linear space $(X, G_N, \Delta)$ is G-fuzzy normed.

**Definition 2.5** [9]

Suppose that the linear space $(X, G_N, \Delta)$ be a GFNLS. For a given x, z belong to X, $\varepsilon \in (0,1)$ and $t > 0$, we define an open ellipse $E_{G_N}(x,z,\varepsilon,t)$ to be a subset of X by

$$E_{G_N}(x,z,\varepsilon,t) = \{y \in X : G_N(x-y, y-z, z-x, t) > 1 - \varepsilon\}$$

**Definition 2.6** [9]

Let the linear space $(X, G_N, \Delta)$ be a GFNLS. For a given x belong to X, $t > 0$ and $\varepsilon \in (0,1)$, we define an open ball $B_{G_N}(x,\varepsilon,t)$ to be a subset of X by

$$B_{G_N}(x,\varepsilon,t) = \{y \in X : G_N(x-y, y-x, \theta, t) > 1 - \varepsilon\}$$

Note that $B_{G_N}(x,\varepsilon,t) = E_{G_N}(x,x,\varepsilon,t)$ for an open ball $B_{G_N}(x,\varepsilon,t)$.

**Definition 2.7** [24]

Let $A \subseteq X$ then set A is said to be closed in a GFNLS $(X, G_N, \Delta)$ if $\lim_{n\to\infty} G_N(x_n - x, x_n - x, x_n - x, t) = 1$ implies that $x \in A$, for all $t > 0$.

**Definition 2.8** [9]

A sequence $\{x_n\}$ in a G-fuzzy normed linear space $(X, G_N, \Delta)$ is converges to the element $x \in X$ if for each $\varepsilon$, $0 < \varepsilon < 1$, $t > 0, \exists N_\circ(\varepsilon,t) \in \mathbb{N}$ such that $G_N(x_n - x, x_m - x, x_\ell - x, t) > 1 - \varepsilon, \forall n, m, \ell \geq N_\circ$

Note that the limit of the sequence $\{x_n\}$ in a GFNLS $(X, G_N, \Delta)$ is unique if $'\Delta'$ is continuous at (1,1).

## 3. A G-Fuzzy Norm for the Notion of Transformations

Some essential definitions and properties are studied in this section. The notion of the linear transformation in terms of a GFNLS is studied and proved. So the definition of a continuous transformation is introduced initially

**Definition 3.1**

The smallest closed set contains A where $A \subset X$ in a GFNLS $(X, G_N, \Delta)$ is called the closure of the set A and denoted by CL(A).

**Lemma 3.2**

If $(X, G_N, \Delta)$ is a G-fuzzy normed linear space such that $\Delta$ is continuous at (1,1) and A be a subset of X, then $a \in CL(A)$ if and only if there is a sequence $\{a_n\}$ in A with $\lim_{n\to\infty} G_N(a_n - a, a_n - a, a_n - a, t) = 1$ for all $t > 0$

**Proof**

Suppose that $a \in CL(A)$, If $a \notin A$, then a is a limit point of A.Since $\Delta$ is continuous at (1,1) , this leads to the limit point a is unique, thus by $G_N(a_n - a, a_n - a, a_n - a, t) > (1 - 1/n)$ for each $(n = 1,2,3, \ldots)$ and $t > 0$ we construct the sequence $\{a_n\}$ in A. $E_{G_N}(a,a,\varepsilon,t)$ is the ball that contains the sequence $\{a_n\}$ in A and this sequence converges to a because $\lim_{n\to\infty} G_N(a_n - a, a_n - a, a_n - a, t) = 1$, for all $t > 0$. If $a \in A$ then the sequence of that type $(a, a, a, \ldots\ldots, a, \ldots)$ is taken

Conversely, let $\{a_n\}$ be a sequence in A and it converges to a then $a \in A$ or every neighborhood of a contains points $a_n \neq a$, hence a is the limit point of A and by definition of CL(A), we obtain $a \in A$

**Definition 3.3**

Let the linear spaces $(X, G_{NX}, \Delta)$ and $(Y, G_{NY}, \Delta)$ are G-fuzzy normed. A transformation $H: (\mathcal{D}(H), G_{NX}, \Delta) \rightarrow (Y, G_{NY}, \Delta)$ where $(\mathcal{D}(H), G_{NX}, \Delta)$ is a subspace of $(X, G_{NX}, \Delta)$ is continuous at the point $x_\circ \in \mathcal{D}(H)$, if for a given $b \in \mathcal{D}(H)$ and $0 < \varepsilon_\circ < 1$ there is $\delta_\circ, 0 < \delta_\circ < 1$ and $d \in \mathcal{D}(H)$ such that $G_{NY}(Hx - Hx_\circ, Hx_\circ - Hb, Hb - Hx), t) > 1 - \varepsilon_\circ \ \ \forall x \in \mathcal{D}(H), t > 0$ for which $G_{NX}(x - x_\circ, x_\circ - d, d - x, t) > 1 - \delta_\circ$. H is said to be is continuous if it is continuous at every point in $\mathcal{D}(H)$

The next theorem gives a characterization of the continuous transformation

**Theorem 3.4**

Let $(X, G_{NX}, \Delta)$ and $(Y, G_{NY}, \Delta)$ are G-fuzzy normed linear spaces. Then $H: (\mathcal{D}(H), G_{NX}, \Delta) \rightarrow (Y, G_{NY}, \Delta)$ where $(\mathcal{D}(H), G_{NX}, \Delta)$ is a subspace of $(X, G_{NX}, \Delta)$ is continuous transformation at the point $x_\circ$ in $\mathcal{D}(H)$ if and only if whenever a sequence $\{x_n\}$ converges to $x_\circ$, then the sequence $\{Hx_n\}$ converges to $Hx_\circ$

**Proof**

Assume that the Transformation $H: (\mathcal{D}(H), G_{NX}, \Delta) \rightarrow (Y, G_{NY}, \Delta)$ is continuous at $x_\circ$ and assume that $\{x_n\}$ be a sequence in $\mathcal{D}(H)$ converges to $x_\circ$. Let $0 < \varepsilon_\circ < 1$ and $b \in \mathcal{D}(H)$ be given. Hence by the continuity of a transformation H at $x_\circ$, there is $0 < \delta_\circ < 1$ and $d \in \mathcal{D}(H)$ Such that $\forall x \in \mathcal{D}(H) \ni t > 0 \ G_{NY}(Hx - Hx_\circ, Hx_\circ - Hb, Hb - Hx, t) > 1 - \varepsilon_\circ$ for which $G_{NX}(x - x_\circ, x_\circ - d, d - x, t) > 1 - \delta_\circ$. Since $\{x_n\}$ converges to $x_\circ$ then $\exists\, N_\circ(\delta_\circ, t) \in \mathbb{N}\ \forall\, 0 < \delta_\circ < 1, t > 0$ such that $G_{NX}(x_n - x_\circ, x_m - x_\circ, x_\ell - x_\circ, t) > 1 - \delta_\circ\ \forall n, m, \ell \geq N_\circ$. Therefore, when $n, m, \ell \geq N_\circ$ implies $G_{NY}(Hx_n - Hx_\circ, Hx_m - Hx_\circ, Hx_\ell - Hx_\circ, t) > 1 - \varepsilon_\circ$. Hence $\{Hx_n\}$ converges to the point $Hx_\circ$. Conversely, suppose that $\forall \{x_n\}$ in $\mathcal{D}(H)$ converging to $x_\circ$ has the property that $\{Hx_n\}$ converges to $Hx_\circ$. Let H is the transformation not continuous at $x_\circ$, it means there is $0 < \varepsilon_\circ < 1, b \in \mathcal{D}(H)$ for which no $\delta_\circ, 0 < \delta_\circ < 1$ and $d \in \mathcal{D}(H)$ can satisfy the requirement that $\forall x \in \mathcal{D}(H), t > 0$ for which $G_{NX}(x - x_\circ, x_\circ - d, d - x, t) > 1 - \delta_\circ$ implies $G_{NY}(Hx - Hx_\circ, Hx_\circ - Hb, Hb - Hx, t) > 1 - \varepsilon_\circ$.This means that for every $\delta_\circ, 0 < \delta_\circ < 1, d \in \mathcal{D}(H)$ such that that $\forall x \in \mathcal{D}(H)$, for which $G_{NX}(x - x_\circ, x_\circ - d, d - x, t) > 1 - \delta_\circ$ but $G_{NY}(Hx - Hx_\circ, Hx_\circ - Hb, Hb - Hx, t) \leq 1 - \varepsilon_\circ$. The sequence $\{x_n\}$ converges to $x_\circ$ but the sequence $\{Hx_n\}$ doesn't converge to $Hx_\circ$. This contradicts the supposition that $\forall \{x_n\}$ in $\mathcal{D}(H)$ converging to $x_\circ$ has the property that $\{Hx_n\}$ converges to $Hx_\circ$, therefore the transformation H must be continuous.

The following definition introduces the concept of a bounded linear transformation.

**Definition 3.5**

Suppose that the linear spaces $(X, G_{NX}, \Delta)$ and $(Y, G_{NY}, \Delta)$ are two G-fuzzy normed. A linear transformation $H: (\mathcal{D}(H), G_{NX}, \Delta) \rightarrow (Y, G_{NY}, \Delta)$ where $(\mathcal{D}(H), G_{NX}, \Delta)$ is a subspace of $(X, G_{NX}, \Delta)$ is said to be bounded if there is a number $\alpha$, $0 < \alpha < 1$ such that $G_{NY}(Hx, Hx, Hz, t) \geq (1 - \alpha)\Delta G_{NX}(x, y, z, t) \ \ \forall x, y, z \in \mathcal{D}(H), t > 0$

The next example explains the concept of a bounded linear transformation

**Example 3.6**

Let $X = C[0,1]$ with $\|f, g, l\| = \max_{t \in [0,1]}\{|f(t)| + |g(t)| + |l(t)|\}$ where $f, g, l \in C[0,1]$. Consider $G_{NX}(f, g, l, t) = [\exp(\|f, g, l\|/t)]^{-1}$ for all $f, g, l \in C[0,1], t > 0$ and $G_{NX}(f, g, l, t) = 0$ for all $t \leq 0$. Then by example 2.4 the linear space $(X, G_{NX}, \Delta)$ is G-fuzzy normed where $\alpha \Delta \beta = \alpha\beta$. Assume that $H: X \rightarrow X$ defined as $h_1 = Hf$, $h_2 = Hg$ and $h_3 = Hl$ with $h_1(t) = \int_0^1 K(t, r)\, f(r)\, dr$, $h_2(t) = \int_0^1 K(t, r)\, g(r)\, dr$ and $h_3(t) = \int_0^1 K(t, r)\, l(r)\, dr$ where $K(t, r)$ is a continuous function on $\hat{G} = [0,1] \times [0,1]$ and $K(t,r)$ is bounded, it means that a positive number $c$ exists such that $|K(t,r)| \leq c$ for all $(t,r) \in \hat{G}$. This transformation H is linear. Now, to prove that H is bounded

Since $\{|f(t)| + |g(t)| + |l(t)|\} \leq \max \{|f(t)| + |g(t)| + |l(t)|\} = \|f,g,l\|$ for all $t \in [0,1]$

Therefore, $\|h_1,h_2,h_3\| = \|Hf,Hg,Hl\| = \max_{t\in[0,1]} \left\{\left|\int_0^1 K(t,r)\, f(r)dr\right| + \left|\int_0^1 K(t,r)\, g(r)dr\right| + \left|\int_0^1 K(t,r)\, l(r)dr\right|\right\}$

$$\leq \max_{t\in[0,1]} \left\{\int_0^1 |K(t,r)||f(r)|dr + \int_0^1 |K(t,r)||g(r)|dr + \int_0^1 |K(t,r)||l(r)|dr\right\}$$

$$\leq \max_{t\in[0,1]} \left\{\int_0^1 c\,|f(r)|dr + \int_0^1 c\,|g(r)| + \int_0^1 c\,|l(r)|dr\right\}$$

$$\leq c\,\|f,g,l\|$$

Hence, $G_{NX}(Hf,Hg,Hl,t) = [\exp(\|Hf,Hg,Hl\|/t)]^{-1} \geq \left(\frac{1}{c}\right)\Delta([\exp(\|f,g,l\|/t)]^{-1}) = \left(\frac{1}{c}\right)\Delta\, G_{NX}(f,g,l,t)$.

Put $1/c = 1-\alpha$, for some $\alpha \in [0,1]$ , it follows that $G_{NX}(Hf,Hg,Hl,t) \geq (1-\alpha)\Delta G_{NX}(f,g,l,t)$

**Definition 3.7**

Suppose that $(X, G_{NX}, \Delta)$ is a GFNLS and $A \subseteq X$. Then A is said to be bounded if there is a number $\alpha$, $0 < \alpha < 1$ such that $G_{NX}(x,y,z,t) \geq (1-\alpha)$ , $\forall\, x,y,z \in A, t > 0$

**Proposition 3.8**

If $(X, G_{NX}, \Delta)$ be a GFNLS. Then:

(a) The sum of any two bounded subsets of X is bounded

(b) The scalar multiple of any bounded subset of X by a real number is bounded

**Proof**

Suppose that the two subsets A, B are bounded of X, we prove that

(a) $A+B$ is bounded. According to our assumption, A and B are bounded, so there are two numbers $\beta$ and $\delta$, where $0<\beta<1$ and $0<\delta<1$ such that $G_{NX}(x,y,z,t) \geq (1-\beta), \forall x,y,z \in A, t>0$ and $G_{NX}(d,h,v,s) \geq (1-\delta), \forall\, d,h,v \in B$. Let $\alpha$ be a number such that $(1-\alpha) \leq (1-\beta)\Delta(1-\delta)$. Let $p,q,r \in A+B$, then there exist $x,y,z \in A$, $d,h,v \in B$ such that $p,q,r = x+d, y+h, z+v$. We have that $G_{NX}(p,q,r,t+s) = G_{NX}(x+d, y+h, z+v, t+s) \geq G_{NX}(x,y,z,t)\Delta G_{NX}(d,h,v,s) \geq (1-\beta)\Delta(1-\delta) \geq (1-\alpha)$. Hence $G_{NX}(p,q,r,t+s) \geq (1-\alpha)$, so $A+B$ is bounded.

(b) $c\,A$ is bounded. Since the subset A is bounded of X, so there is $\beta$, $0<\beta<1$ such that $G_{NX}(x,y,z,t) \geq (1-\beta) \forall x,y,z \in A, t>0$ . Now, if $c \neq 0$, then $G_{NX}(cx,cy,cz,t) = G_{NX}\left(x,y,z,\frac{t}{|c|}\right) \geq (1-\beta)\ \forall\, t \in \mathbb{R}, t>0$ . Thus $G_{NX}(cx,cy,cz,t) \geq (1-\beta)$. Hence $c\,A$ is bounded.

**Remark 3.9**

Suppose that the linear spaces $(X, G_{NX}, \Delta)$ and $(Y, G_{NY}, \Delta)$ are two G-fuzzy normed, then the G-B(X, Y) is the set of all bounded linear transformations in which $\{H: (\mathcal{D}(H), G_{NX}, \Delta) \to (Y, G_{NY}, \Delta) : G_{NY}(Hx,Hy,Hz,t) \geq (1-\alpha)\Delta G_{NX}(x,y,z,t), t>0\}$.

The following lemma demonstrates the addition transformation of two bounded transformations must be bounded transformation.

**Lemma 3.10**

Let $H_1, H_2 \in$ G-B(X, Y) then $H_1 + H_2 \in$ G-B(X, Y), where the linear spaces $(X, G_{NX}, \Delta)$ and $(Y, G_{NY}, \Delta)$ are G-fuzzy normed.

**Proof**

Let $H_1$ and $H_2$ be bounded linear transformations then there are two numbers $p_1, p_2$, with $0 < p_1 < 1$ and $0 < p_2 < 1$ such that $G_{NY}(H_1x,\ H_1y,\ H_1z, t) \geq (1 - p_1)\Delta G_{NX}(x, y, z, t)$ and $G_{NY}(H_2x,\ H_2y,\ H_2z, s) \geq (1 - p_2)\Delta G_{NX}(x, y, z, t)$ for any $x, y, z \in Ɗ(H_1) \cap Ɗ(H_2)$ and $t > 0$. Now

$$\begin{aligned} G_{NY}((H_1 + H_2)(x, y, z), t) &= G_{NY}(\ H_1x + H_2x,\ H_1y + H_2y,\ H_1z + H_2z, t) \\ &\geq G_{NY}\left(H_1x,\ H_1y,\ H_1z, \frac{t}{2}\right)\Delta G_{NY}\left(H_2x,\ H_2y,\ H_2z, \frac{t}{2}\right) \\ &\geq (1 - p_1)\Delta G_{NX}\left(x, y, z, \frac{t}{2}\right)\Delta(1 - p_2)\Delta G_{NX}\left(x, y, z, \frac{t}{2}\right) \\ &= ((1 - p_1)\Delta(1 - p_2))\Delta\left[G_{NX}\left(x, y, z, \frac{t}{2}\right)\right] \end{aligned}$$

We can find $\alpha$ with $0 < \alpha < 1$ such that $(1 - p_1)\Delta(1 - p_2) = (1 - \alpha)$

So $G_{NY}((H_1 + H_2)(x, y, z), t) \geq (1 - \alpha)\Delta G_{NX}(x, y, z, t/2) \forall\ H_1, H_2 \in$ G-B(X, Y).

Therefore, $H_1 + H_2 \in$ G-B(X, Y)

One more characterization for the bounded transformation is assigned in the following theorem

**Theorem 3.11**

Let $(X, G_{NX}, \Delta)$ and $(Y, G_{NY}, \Delta)$ are two G-normed linear spaces. Then H(A) is bounded for every bounded subset A of Ɗ(H) if and only if $H: (Ɗ(H), G_{NX}, \Delta) \rightarrow (Y, G_{NY}, \Delta)$ is a bounded linear transformation

**Proof**

Let that Ɗ(H) be a bounded then H(Ɗ) is bounded, so there is a number $p, 0 < p < 1$ such that $G_{NY}(Hx, Hy, Hz, t) \geq (1 - p)\Delta G_{NX}(x, y, z, t) \forall\ x, y, z \in Ɗ(H), t > 0$. Hence we can find a number $\alpha$ with $(1 - p) \leq (1 - \alpha)\Delta\ G_{NX}(x, y, z, t)$ thus $G_{NY}(Hx, Hy, Hz, t) \geq (1 - \alpha)\ \Delta\ G_{NX}(x, y, z, t)\ \ \forall\ x, y, z \in Ɗ(H)$. Conversely, Let H be bounded, so there is a number $p, 0 < p < 1$ such that $G_{NY}(Hx, Hy, Hz, t) \geq (1 - p)\Delta G_{NX}(x, y, z, t)\ \forall\ x, y, z \in Ɗ(H), t > 0$. Let A be a bounded such that $A \subseteq Ɗ(H)$, then there exists a number $q, 0 < q < 1$ such that $G_{NX}(x, y, z, t) \geq 1 - q, \forall\ x, y, z \in\ A$, we can find a number $\alpha$ with $(1 - \alpha) \geq (1 - p)\Delta(1 - q)$. Therefore $G_{NY}(Hx, Hy, Hz, t) \geq (1 - \alpha)$ .

## 4. The Extension of the Bounded Linear Transformation and the Completeness of all Bounded Linear Transformations Space G-B(X,Y)

This section is devoted to extending the bounded linear transformation according to a GFNLS. The extension of any bounded linear transformation in the case that complete condition is available for a GFNLS $(Y, G_{NY}, \Delta)$ is proved. Moreover, the completeness for the space G-B(X, Y) is proved.

First, a G-fuzzy norm for the space G-B(X, Y) is defined, and that the linear space G-B(X, Y) is proved to be a G-fuzzy normed as follow:

**Theorem 4.1**

Suppose that $(X, G_{NX}, \Delta)$and $(Y, G_{NY}, \Delta)$are two G-fuzzy normed linear spaces, $(\text{G-B}(X, Y), G_{NLY}, \Delta)$ is GFNLS where the function $G_{NLT}(H_1, H_2, H_3, t) = \inf_{x,y,z \in Ɗ(H)} G_{NY}(Hx, Hy, Hz, t)\ \forall\ H \in \text{G-B}(X, Y), t > 0$ where $H_1 = Hx,\ H_2 = Hy$ and $H_3 = Hz$.

**Proof**

$(G_N1)$ $G_{NLT}$ $(H_1, H_2, H_3, t) = 0, \forall\, t \in \mathbb{R}, t \le 0$, since $G_{NY}(Hx, Hy, Hz, t) = 0$ for all $x, y, z \in \mathcal{D}(H)$ and $t \le 0$.

$(G_N2)$ $G_{NLT}$ $(H_1, H_2, H_3, t) = 1$ if and only if $\inf_{x,y,z\in\mathcal{D}(H)} G_{NY}(Hx, Hy, Hz, t) = 1$ if and only if $G_{NY}(Hx, Hy, Hz, t) = 1$ if and only if $Hx = Hy = Hz = \theta$ for all $x, y, z \in \mathcal{D}(H)$ if and only if $H_1 = H_2 = H_3 = \theta$ $\forall\, t \in \mathbb{R}, t > 0$

$(G_N3)$ Since $G_{NY}(Hx, Hy, Hz, t) = G_{NY}(Hy, H\mathfrak{c}, Hx, t) = G_{NY}(Hx, Hz, Hy, t) = \cdots$ for all $x, y, z\, \mathcal{D} \in (H)$ .
Therefore, $G_{NLT}$ $(H_1, H_2, H_3, t) = G_{NLT}$ $(H_2, H_3, H_1, t) = G_{NLT}$ $(H_1, H_3, H_2, t)$.

This means that $G_{NLT}$ $(H_1, H_2, H_3, t) = G_{NLT}$ $(P(H_1, H_2, H_3), t)$, (symmetry) where P is a permutation function

$(G_N4)$ If $c \neq 0$, for all $t \in \mathbb{R}, t > 0$

$$\begin{aligned} G_{NLT}\,(cH_1, cH_2, cH_3, t) &= \inf_{x,y,z\in\mathcal{D}(H)} G_{NY}(\, c(Hx, Hy, Hz), t) \\ &= \inf_{x,y,z\in\mathcal{D}(H)} G_{NY}(\, cHx, cHy, cHz, t) \\ &= \inf_{x,y,z\in\mathcal{D}(H)} G_{NY}(Hx, Hy, Hz, t/|c|\,) \\ &= G_{NLT}\left(Hx, Hy, Hz, \frac{t}{|c|}\right) \\ &= G_{NLT}\,(H_1, H_2, H_3, t/|c|) \end{aligned}$$

$(G_N5)$ For all $t, s \in \mathbb{R}, H_1, H_2, H_3\ , \overline{H}_1, \overline{H}_2, \overline{H}_3 \in$ G-B(X, Y)

$$\begin{aligned} G_{NLT}\,(H_1 + \overline{H}_1, H_2 + \overline{H}_2, H_3 + \overline{H}_3, t + s) &= \inf_{x,y,z\in\mathcal{D}(H)\cap\mathcal{D}(H)} G_{NY}((H + \overline{H})(x), (H + \overline{H})(y), (H + \overline{H})(z), t + s) \\ &= \inf_{x,y,z\in\mathcal{D}(H)\cap\mathcal{D}(H)} G_{NY}(H(x) + \overline{H}(x), H(y) + \overline{H}(y), H(z) + \overline{H}(z), t + s) \\ &\ge \inf_{x,y,z\in\mathcal{D}(H)} G_{NY}(H(x), H(y), H(z), t)\ \Delta\ \inf_{x,y,z\in\mathcal{D}(\overline{H})} G_{NY}(\overline{H}(x), \overline{H}(y), \overline{H}(z), s) \\ &= \inf_{x,y,z\in\mathcal{D}(H)} G_{NY}(Hx, Hy, Hz, t)\ \Delta\ \inf_{x,y,z\in\mathcal{D}(\overline{H})} G_{NY}(\overline{H}x, \overline{H}y, \overline{H}z, s) \\ &\ge G_{NLT}\,(H_1, H_2, H_3, t)\ \Delta\ G_{NLT}(\overline{H}_1, \overline{H}_2, \overline{H}_3, s) \end{aligned}$$

$(G_N6)$ $\lim_{t\to\infty} G_{NLT}\,(H_1, H_2, H_3, t) = \lim_{t\to\infty} \inf G_{NY}(Hx, Hy, Hz, t) = \inf \lim_{t\to\infty} G_{NY}(Hx, Hy, Hz, t) = 1$

$(G_N7)$ For all $x, y, z \in X$,

$$\begin{aligned} G_{NLT}\,(H_1 + H_2, \theta, H_3, t) &= \inf_{x,y,z\in\mathcal{D}(H)} G_{NY}(Hx + Hy, \theta, Hz, t) \\ &\ge \inf_{x,y,z\in\mathcal{D}(H)} G_{NY}(Hx, Hy, Hz, t) \\ &= G_{NLT}\,(H_1, H_2, H_3, t) \end{aligned}$$

Therefore, the linear space (G-B(X, Y), $G_{NLT}$ , Δ) is G-fuzzy normed.

Now we establish that a bounded linear transformation has an extension in the following theorem.

**Theorem 4.2**

Assume that $(X, G_{NX}, \Delta)$ be a GFNLS and $(Y, G_{NY}, \Delta)$ be a complete GFNLS. Let $H: (\mathcal{D}(H), G_{NX}, \Delta\,) \to (Y, G_{NY}, \Delta\,)$ be a bounded linear transformation. Then H has an extension $\widetilde{H}: (CL(\mathcal{D}(H), G_{NX}, \Delta\,) \to (Y, G_{NY}, \Delta\,)$ bounded linear with $G_{NLT}\,(H_1, H_2, H_3, t) = G_{NLT}\left(\widetilde{H}_1, \widetilde{H}_2, \widetilde{H}_3, t\right) \forall\, t > 0$

**Proof**

Assume that $x \in CL(\mathcal{D}(H))$, then by Lemma 3.2 there is a sequence $\{x_n\} \in CL(\mathcal{D}(H))$ with $\{x_n\}$ converges to x. Since H is a bounded linear transformation, then for some $0 < \alpha < 1$ we have $G_{NLT}\,(H_1, H_2, H_3, t) \ge (1 - \alpha)$ and since $\{x_n\}$ converges to x then for each $\varepsilon$, $0 < \varepsilon < 1$, $t > 0$, $\exists N_\circ(\varepsilon, t) \in \mathbb{N}$ such that $G_{NX}(x_n - x, x_m - x, x_\ell - x, t\,) > 1 - \varepsilon, \forall n, m, \ell \ge N_\circ$ . Thus by remark 2.2, there exists $0 < (1 - p) < 1$ with $(1 - \alpha)\Delta\,(1 - \varepsilon) > 1 - p$ .So $G_{NY}(Hx_n - Hx_m, Hx_m - Hx_\ell, Hx_\ell - Hx_n, t\,) = G_{NY}(H(x_n - x_m), H(x_m - x_\ell), H(x_\ell - x_n), t) \ge$

$G_{NLT}(H_1, H_2, H_3, t)\Delta G_{NX}(x_n - x_m, x_m - x_\ell, x_\ell - x_n, t) \geq (1-\alpha)\Delta(1-\varepsilon) > 1 - p\ \forall n, m, \ell \geq N_\circ$, this implies that the sequence $\{Hx_n\}$ in $(Y, G_{NY}, \Delta)$ is Cauchy. By our assumption $(Y, G_{NY}, \Delta)$ is complete for this reason $\{Hx_n\}$ converges to the point y in Y. Define $\widetilde{H}x = y$. Suppose that two sequences $\{x_n\}$ and $\{\bar{x}_n\}$ converges to x then $\{z_n\} = \{x_1, \bar{x}_1, x_2, \bar{x}_2, \ldots\}$ is the sequence that converges to x . Therefore, $\{Hz_n\}$ converges, and the subsequences$\{Hx_n\}, \{H\bar{x}_n\}$ of $\{Hz_n\}$ must have an equal limit by [Remark 1, 24]. This shows that for every x in $CL(Ɗ(H))$, $\widetilde{H}$ is well defined. The transformation $\widetilde{H}$ is linear also for every x in $Ɗ(H)$, $\widetilde{H}x = Hx$ hence $\widetilde{H}$ is an extension of H. Now, we have

$$G_{NY}(H_1x_n, H_2x_n, H_3x_n, t) \geq G_{NLT}(H_1, H_2, H_3, t)\ \Delta\ G_{NX}(x_n, x_n, x_n, t)$$

Let $n \to \infty$ then $Hx_n$ converges to $\widetilde{H}x = y$ this implies that

$$G_{NY}\left(\widetilde{H}_1x, \widetilde{H}_2x, \widetilde{H}_3x, t\right) \geq G_{NLT}(H_1, H_2, H_3, t)\ \Delta\ G_{NS}(x, x, x, t)$$

Thus $\widetilde{H}$ is bounded transformation and $G_{NLT}\left(\widetilde{H}_1, \widetilde{H}_2, \widetilde{H}_3, t\right) \geq G_{NLT}(H_1, H_2, H_3, t)$ but by the definition of G-fuzzy norm which defined by an infimum $G_{NLT}\left(\widetilde{H}_1, \widetilde{H}_2, \widetilde{H}_3, t\right) \leq G_{NLT}(H_1, H_2, H_3, t)$, together we obtain that $G_{NLT}\left(\widetilde{H}_1, \widetilde{H}_2, \widetilde{H}_3, t\right) = G_{NLT}(H_1, H_2, H_3, t)$

The following definition gives the notion of bounded linear functional.

**Definition 4.3**

A linear functional F from the space $(X, G_{NX}, \Delta)$ into the space $(\mathbb{F}, G_{N\mathbb{F}}, \Delta)$ is called bounded if there exist α, $0 < \alpha < 1$such that $G_{N\mathbb{F}}(Fx, Fy, Fz, t) \geq (1-\alpha)\Delta G_{NX}(x, y, z, t)$ for each $x, y, z, \in Ɗ(F)$, $t > 0$ .

Furthermore, a G-fuzzy norm of F is $G_{NLT}(F_1, F_2, F_3, t) = \inf_{x,y,z\, \in Ɗ(F)} G_{N\mathbb{F}}(Fx, Fy, Fz, t)$ and $G_{N\mathbb{F}}(Fx, Fy, Fz, t) \geq G_{N\mathbb{F}}(Fx, Fy, Fz, t)\Delta G_{NX}(x, y, z, t)$.

The set of all bounded linear functional is denoted by $G\text{-}B(X, \mathbb{F})$ with $G\text{-}B(X, \mathbb{F})$ ={F: F is bounded linear functional}.

The Cauchy sequence in a GFNLS, $(X, G_N, \Delta)$ is given in [9] as follows:

A sequence $\{x_n\}$ is Cauchy if for each $\varepsilon, 0 < \varepsilon < 1$, $s > 0$, $\exists N_\circ(\varepsilon, s) \in \mathbb{N}$ such that $G_N(x_n - x_m, x_m - x_\ell, x_\ell - x_n, s) > 1 - \varepsilon, \forall n, m, \ell \geq N_\circ$.In a GFNLS, $(X, G_N, \Delta)$ every convergent sequence is Cauchy.

The completeness of the G-fuzzy normed linear space $(X, G_N, \Delta)$ is satisfied if Cauchyness implies convergence in X. The complete property of the space $G\text{-}B(X, Y)$ is discussed in the following theorem.

**Theorem 4.4**

Let the linear spaces $(X, G_{NX}, \Delta)$ and $(Y, G_{NY}, \Delta)$ be two G-fuzzy normed. If the linear space $(Y, G_{NY}, \Delta)$ is complete G-fuzzy normed where Δ is continuous on (1,1) then $G\text{-}B(X, Y)$ is complete GFNLS.

**Proof**

Assume that the sequence $\{H_n\}$ be Cauchy in $G\text{-}B(X, Y)$, so for each $\alpha, 0 < \alpha < 1, t > 0\ \exists N_\circ(\alpha, t) \in \mathbb{N}$ such that $G_{NY}(H_n - H_m, H_m - H_\ell, H_\ell - H_n, t) > 1 - \alpha$, $\forall n, m, \ell \geq N_\circ$. Now, for the vectors $x, y, z \in X$ and each $n, m, \ell \geq N_\circ$ the following formula say (A) is given:

$$G_{NY}(H_n - H_m, H_m - H_\ell, H_\ell - H_n, t) > (1-\alpha)$$

Now for $x, y, z \in X$ and $n, m, \ell \geq N_\circ$

$G_{NY}(H_n x - H_{m} x, H_{m} y - H_{\ell} y, H_{\ell} z - H_{n} z, t) \geq G_{NY}((H_n - H_{m})(x), (H_{m} - H_{\ell})(y), (H_{\ell} - H_{n})(z), t) > (1 - \alpha)\Delta$ $G_{NX}(x, y, z, t)$.

Hence, for any fixed vectors x, y, z, given $\beta, 0 < \beta < 1$ and from formula (A) the following is obtained:

$G_{NY}(H_n x - H_{m} x, H_{m} y - H_{\ell} y, H_{\ell} z - H_{n} z, t) > (1 - \beta)\Delta\, G_{NX}(x, y, z, t)$.

Therefore, $\{H_{n} x\}$ is Cauchy in Y, since $(Y, G_{NY}, \Delta)$ is complete G-fuzzy normed linear space so $\{H_{n} x\}$ converges to $d \in Y$. Defines an operator $H: (X, G_{NX}, \Delta) \to (Y, G_{NY}, \Delta)$ by $Hx = d$ such that the vector d depends on $x \in X$. We will prove that

1- The transformation H is linear

Since $H(c_1 a + c_2 b) = \lim_{n\to\infty} H_{n}(c_1 a + c_2 b)$

$$= c_1 \lim_{n\to\infty} H_{n} a + c_2 H_{n} \lim_{n\to\infty} b$$
$$= c_1 H(a) + c_2 H(b)$$

2- The transformation H is bounded and the element H is the limit of the sequence $\{H_{n}\}$.

Since the formula (A) holds $\forall m \geq N_\circ$, $H_{m} x$ converges to Hx, $H_{m} y$ converges to Hy and $H_{m} z$ converges to Hz we may allow $m \to \infty$

So for all $x, y, z \in X$, $t > 0$ and $n, \ell \geq N_\circ$ formula (A) gives:

$$G_{NY}((H_n - H)(x), (H_{m} - H)(y), (H_{\ell} - H)(z), t) =$$
$$G_{NY}(H_n x - \lim_{m\to\infty} H_{m} x, H_{m} y - \lim_{m\to\infty} H_{m} y, H_{\ell} z - \lim_{m\to\infty} H_{m} z, t) =$$
$$\lim_{m\to\infty} G_{NY}((H_n - H_{m})(x), (H_{m} - H_{m})(y), (H_{\ell} - H_{m})(z), t) =$$

$\lim_{m\to\infty} G_{NY}((H_n - H_{m})(x), (H_{m} - H_{\ell})(y), (H_{m} - H_{m})(z), t)$ by the conditions $(G_N 3)$ and $(G_N 4)$.

Since $(\Delta)$ is contiuous on (1,1) then the following formula say (B) is obtained

$G_{NY}((H_n - H)(x), (H_{m} - H)(y), (H_{\ell} - H)(z), t) = \lim_{m\to\infty} G_{NY}((H_n - H_{m})(x), (H_{m} - H_{\ell})(y), (H_{m} - H_{m})(z), t) > (1 - \alpha)\Delta\, G_{NX}(x, y, z, t)$

Therefore, $\{H_{n} - H\}$ is bounded but $H_{n}$ is bounded hence $H = H_{n} - (H_{n} - H)$ is bounded transformation, it means that $H \in$ G-B(X, Y)and from formula (B) by taking the infimum ∀ x, y, z gives:

$$G_{NLT}(H_n - H, H_{m} - H, H_{\ell} - H, t) > (1 - \alpha)\Delta G_{NX}(x, y, z, t)$$

Now, we take ε with $\varepsilon \in (0,1)$ with $G_{NX}(x, y, z, t) = 1 - \varepsilon$ for some ε

Hence $(1 - \alpha)\Delta(1 - \varepsilon) = 1 - \varepsilon_\circ$ for some $0 < \varepsilon_\circ < 1$ this implies that

$G_{NLT}(H_n - H, H_{m} - H, H_{\ell} - H, t) \geq 1 - \varepsilon_\circ \quad \forall n, m, \ell \geq N_\circ,\ t > 0$. That is H is the limit of the sequence $\{H_{n}\}$.

According to the previous theorem, proof of the following corollary immediately follows it

**Corollary 4.5**

Suppose that $(X, G_{NX}, \Delta)$ and $(\mathbb{F}, G_{N\mathbb{F}}, \Delta)$ are two G-fuzzy normed linear spaces. Then the space G-B(X, 𝔽)is complete GFNLS when $(\mathbb{F}, G_{N\mathbb{F}}, \Delta)$ is complete

## 5. Conclusion

The continuity property for a linear transformation in terms of GFNLS was introduced; a bounded linear transformation using the concept of GFNLS was defined. An illustrated example of this definition was given and some essential properties are proved. Finally, the class G-B(X, Y) that is consisting of all bounded linear transformations is presented with the proving that this space is complete GFNLS while the space Y is complete. We believe the results established in this work will help authors to generalize further studies and results like the adjoint transformation of bounded transformation.

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