Some Topological Indices and (Hosoya and Schultz) Polynomial of Subgroup intersection graph of a group $Z_r$

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\[ \text{Abstract} \]

Let $Z_r$ be any group with identity element (e). A subgroup intersection graph of a group $Z_r$ is the Graph with $V(G) = Z_r - e$ and two separate vertices $a$ and $b$ contiguous for $a$ and $b$ if and only if $|(a) \cap (b)| > 1$, Where $(a)$ is a cyclic subgroup of $Z_r$ generated by $a \in Z_r$. We found some topological indices in this paper and (Hosoya and Schultz) Polynomial of $g_S(Z_r)$, where $r \geq 3, r$ is a prime number.

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1. Introduction

The number associated with the graph is the topological indicator of the finite graph, and under automatic morphology this number is constant [1]. We often referred to such a theoretical graph by a descriptor [2,3,4] or descriptor of the molecular structure of the graph [10], topological indicator. In biology and chemistry, many topological indicators have been used to solve problems.
In this paper we study some topological indices depending on the degree of the examples Eccentric connectivity index[5] , connectivity index[6] ,Sum connectivity index[7 – 10], Zagreb index[11 – 16], forgotten index[17 – 20], The index of geometric-arithmetic [21 – 23], Index of Atom-Bond Connectivity [21], Harmonic index[24 – 27].

One graphical concept was obtained from the group is the concept of a subgroup intersection graph of a group that was introduced by M. Aschbacher [28].Referring to the definition of subgroup intersection graph of a group by T. Tamizh Chelvam and M. Sattanathan [28], let G be a group the subgroup intersection graph rSI(G) of G is a graph with V(rSI(G)) = G − e and two distinct vertices a and b are adjacent in rSI(G) if |⟨a⟩ ∩ ⟨b⟩| > 1, Where ⟨a⟩ is a cyclic subgroup of G generated from a ∈ G.


The search for the topological index was initially related to the graph of biological activity or chemical composition and reactivity, and research in this regard continued. On the other hand, many studies began to examine the topological index of graphs that are not from chemical composition, interaction, or biological. When many researchers introduced new concepts about graphs obtained from an algebraic structure, the search for topological indicators on these graphs began in Emerge.

2. Method and Materials

In the present article, all the graphs are simple, finite, connected, and un directed. For G = (V(G), E(G)) a graph, the order of G is p(G) = V(G) and the scale of G is q(G) = E(G). Let deg(u) denote the degree of vertex u in G. If deg(u) = 0 then u is an isolated vertex. If deg(u) = 1 is an end vertex, then u is an end vertex. Let d(u, v) indicate the distance between vertices u and v in G. The eccentricity e(u) of the vertex u is e(u) = sup{d(u, v): v ∈ V(G)}.

The following definition refers to a graph G = (V(G), E(G)).

Eccentric connectivity index of G is [5] \( \xi^C(G) \) = \( \sum_{u \in V(G)} \text{deg}(u).e(u) \)

The index of connectivity of G is[6] \( X(G) = \sum_{uv \in E(G)} \frac{1}{\text{deg}(u).\text{deg}(v)} \)

The summation of the index connectivity of G is [7] \( S(G) = \sum_{uv \in E(G)} \frac{1}{\text{deg}(u)+\text{deg}(v)} \)

A first Zagreb index of G is [11] \( M_1(G) = \sum_{u \in V(G)} (\text{deg}(u))^2 \)

A second zagreb index of G is [11] \( M_2(G) = \sum_{uv \in E(G)} \text{deg}(u).\text{deg}(v) \)

The forgotten index of G is [20] \( F(G) = \sum_{u \in V(G)} (\text{deg}(u))^3 \)

Atom Bond connectivity index of G is [21] \( ABC(G) = \sum_{uv \in E(G)} \sqrt{\frac{\text{deg}(u)+\text{deg}(v)-2}{\text{deg}(u).\text{deg}(v)}} \)
Geometric-Arithmetic index of $G$ is [21] $GA(G) = \sum_{uv \in E(G)} \frac{2 \sqrt{\text{deg}(u) \cdot \text{deg}(v)}}{\text{deg}(u) + \text{deg}(v)}$

Harmonic index of $G$ is [24] $H(G) = \sum_{uv \in E(G)} \frac{2}{\text{deg}(u) + \text{deg}(v)}$

3. The main result

For a good outlook, $r \geq 3$, a subgroup intersection graph of a group $Z_r$ is $(r_{SL}(Z_r))$ is a graph with $(r_{SL}(Z_r)) = Z_r - e$, any two different vertices $a$ and $b$ are adjacent iff $|\langle a \rangle \cap \langle b \rangle| > 1$, where $\langle a \rangle$ is the subgroup generated by $a \in Z_r$.

**Theorem 3.1**

Let $Z_r$ be a group with $r \geq 3$, $r$ is a prime number then the Eccentric connectivity index of $r_{SL}(Z_r)$ is $\xi^c(r_{SL}(Z_r)) = r^2 - 3r + 2$

**Proof:**

$$\text{deg}(u) = r - 2, \forall u \in V(\ r_{SL}(Z_r)), \ u = 1,2,\ldots,r - 1$$

$$e(u) = 1, \forall u \in V(\ r_{SL}(Z_r)), \ u = 1,2,\ldots,r - 1$$

$$\xi^c(r_{SL}(Z_r))) = \sum_{u \in V(r_{SL}(Z_r))} \text{deg}(u).e(u)$$

$$= \text{deg}(1).e(1) + \cdots + \text{deg}(r - 1).e(r - 1)$$

$$= (r - 1)(r - 2)$$

$$= r^2 - 3r + 2$$

**Theorem 3.2**

Let $Z_r$ be a group with $r \geq 3$, $r$ is a prime number then the connectivity index of $r_{SL}(Z_r)$ is $X(r_{SL}(Z_r)) = 1 + \frac{\sum_{i=3}^{r-1}(r-i)}{(r-2)}$

**Proof:**

$$\text{deg}(u) = r - 2, \forall u \in V(\ r_{SL}(Z_r)), \ u = 1,2,\ldots,r - 1$$

$$X(r_{SL}(Z_r)) = \sum_{uv \in E(r_{SL}(Z_r))} \frac{1}{\sqrt{\text{deg}(u) \cdot \text{deg}(v)}}$$
\[
\frac{1}{\sqrt{\text{deg}(1) \cdot \text{deg}(2)}} + \cdots + \frac{1}{\sqrt{\text{deg}(1) \cdot \text{deg}(r-1)}}
\]

\[(r-2)\text{times}\]

\[
\frac{1}{\sqrt{\text{deg}(2) \cdot \text{deg}(3)}} + \cdots + \frac{1}{\sqrt{\text{deg}(2) \cdot \text{deg}(r-1)}}
\]

\[(r-3)\text{times}\]

\[
\frac{1}{\sqrt{\text{deg}(3) \cdot \text{deg}(4)}} + \cdots + \frac{1}{\sqrt{\text{deg}(3) \cdot \text{deg}(r-1)}}
\]

\[(r-4)\text{times}\]

\[
\vdots
\]

\[
\frac{1}{\sqrt{\text{deg}(r-2) \cdot \text{deg}(r-1)}}
\]

\[
= \frac{(r-2)}{(r-2)} + \frac{(r-3)}{(r-2)} + \frac{(r-4)}{(r-2)} + \cdots + \frac{1}{(r-2)}
\]

\[
= 1 + \frac{\sum_{i=3}^{r-1}(r-i)}{(r-2)}
\]

**Theorem 3.3**

Let \( Z_r \) be a group with \( r \geq 3 \), \( r \) is a prime number then the Summation connectivity index of \( r_{SI}(Z_r) \) is \( S(r_{SI}(Z_r)) = \frac{\sum_{i=2}^{r-1}(r-i)}{\sqrt{2r-4}} \)

**Proof:**

\[
\text{deg}(u) = r-2 , \forall u \in V( r_{SI}(Z_r)), u = 1,2,\ldots,r-1
\]

\[
S(r_{SI}(Z_r)) = \sum_{uv \in E(r_{SI}(Z_r))} \frac{1}{\sqrt{\text{deg}(u) + \text{deg}(v)}}
\]

\[
= \frac{1}{\sqrt{\text{deg}(1) + \text{deg}(2)}} + \cdots + \frac{1}{\sqrt{\text{deg}(1) + \text{deg}(r-1)}}
\]

\[(r-2)\text{times}\]

\[
+ \frac{1}{\sqrt{\text{deg}(2) + \text{deg}(3)}} + \cdots + \frac{1}{\sqrt{\text{deg}(2) + \text{deg}(r-1)}}
\]

\[(r-3)\text{times}\]

\[
+ \frac{1}{\sqrt{\text{deg}(3) + \text{deg}(4)}} + \cdots + \frac{1}{\sqrt{\text{deg}(3) + \text{deg}(r-1)}}
\]

\[(r-4)\text{times}\]
\[
+ \cdots + \frac{1}{\sqrt{\text{deg}(r-2) + \text{deg}(r-1)}}
\]

\[
= \frac{(r-2)}{\sqrt{2r-4}} + \frac{(r-3)}{\sqrt{2r-4}} + \frac{(r-4)}{\sqrt{2r-4}} + \cdots + \frac{1}{\sqrt{2r-4}}
\]

\[
= \sum_{i=2}^{r-1} \frac{(r-i)}{\sqrt{2r-4}}
\]

**Theorem 3.4**

Let \( Z_r \) be a group with \( r \geq 3 \), \( r \) is a prime number then the first zagreb index of \( r_{SI}(Z_r) \)

is \( M_1(r_{SI}(Z_r)) = (r-1)(r-2)^2 \)

**Proof:**

\( \text{deg}(u) = r-2 \), \( \forall u \in V(r_{SI}(Z_r)) \), \( u = 1,2,\ldots,r-1 \)

\[
M_1(r_{SI}(Z_r)) = \sum_{u \in V(r_{SI}(Z_r))} (\text{deg}(u))^2
\]

\[
= (\text{deg}(1))^2 + \cdots + (\text{deg}(r-1))^2
\]

\[\text{times} \]

\[
= (r-1)(r-2)^2
\]

**Theorem 3.5**

Let \( Z_r \) be a group with \( r \geq 3 \), \( r \) is a prime number then the second zagreb index of \( r_{SI}(Z_r) \)

is \( M_2(r_{SI}(Z_r)) = (r-2)^3 + (r-2)^2 \sum_{i=3}^{r-1} (r-i) \)

**Proof:**

\( \text{deg}(u) = r-2 \), \( \forall u \in V(r_{SI}(Z_r)) \), \( u = 1,2,\ldots,r-1 \)

\[
M_2(r_{SI}(Z_r)) = \sum_{uv \in E(r_{SI}(Z_r))} \text{deg}(u), \text{deg}(v)
\]
\[
= \underbrace{\text{deg}(1) \cdot \text{deg}(2) + \cdots + \text{deg}(1) \cdot \text{deg}(r-1)}_{(r-2)\text{times}} + \underbrace{\text{deg}(2) \cdot \text{deg}(3) + \cdots + \text{deg}(2) \cdot \text{deg}(r-1)}_{(r-3)\text{times}} + \underbrace{\text{deg}(3) \cdot \text{deg}(4) + \cdots + \text{deg}(3) \cdot \text{deg}(r-1)}_{(r-4)\text{times}} + \cdots + \underbrace{\text{deg}(r-2) \cdot \text{deg}(r-1)}_{(r-2)\text{times}}
\]

\[
= (r - 2)^3 + (r - 3)(r - 2)^2 + (r - 4)(r - 2)^2 + \cdots + (r - 2)^2
\]

\[
= (r - 2)^3 + (r - 2)^2 \sum_{i=3}^{r-1} (r - i)
\]

**Theorem 3.6**

Let \( Z_r \) be a group with \( r \geq 3 \), \( r \) is a prime number then the forgotten index of \( r_{SI}(Z_r) \) is \( F(r_{SI}(Z_r)) = (r - 1)(r - 2)^3 \)

**Proof:**

\[
\text{deg}(u) = r - 2, \forall u \in V(r_{SI}(Z_r)), \ u = 1, 2, \ldots, r - 1
\]

\[
F(r_{SI}(Z_r)) = \sum_{u \in V(r_{SI}(Z_r))} \text{deg}(u)^3
\]

\[
= (\underbrace{\text{deg}(1)^3 + \cdots + \text{deg}(r-1)^3}_{(r-1)\text{times}})
\]

\[
=(r - 1)(r - 2)^3
\]

**Theorem 3.7**

Let \( Z_r \) be a group with \( r \geq 3 \), \( r \) is a prime number then the Atom Bond connectivity index of \( r_{SI}(Z_r) \) is \( ABC(r_{SI}(Z_r)) = \sqrt{2r - 6} + \frac{\sqrt{2r - 6} - 6 \sum_{i=3}^{r-2}(r-i)}{(r-2)} \)

**Proof:**

\[
\text{deg}(u) = r - 2, \forall u \in V(r_{SI}(Z_r)), \ u = 1, 2, \ldots, r - 1
\]
Theorem 3.8

Let $Z_r$ be a group with $r \geq 3$, $r$ is a prime number then the Geometric–Arithmetic index of $r_{SI}(Z_r)$ is $GA(r_{SI}(Z_r)) = (r - 2) + \sum_{i=3}^{r-1} (r - i)$

Proof:

\[
deg(u) = r - 2, \forall u \in V(r_{SI}(Z_r)), u = 1, 2, \ldots, r - 1
\]

\[
GA(r_{SI}(Z_r))) = \sum_{uv \in E(r_{SI}(Z_r))} \frac{2 \sqrt{deg(u) \cdot deg(v)}}{deg(u) + deg(v)}
\]
Proof:

\[
\begin{align*}
\deg(u) &= r - 2, \quad \forall u \in V(\mathcal{r}_{SI}(Z_r)), \ u = 1, 2, \ldots, r - 1 \\
H(\mathcal{r}_{SI}(Z_r))) &= \sum_{u \in V(\mathcal{r}_{SI}(Z_r))} \frac{2}{\deg(u) + \deg(v)} \\
&= \frac{2}{\deg(1) + \deg(2)} + \frac{2}{\deg(1) + \deg(r - 1)} + \frac{2}{\deg(2) + \deg(3)} + \frac{2}{\deg(2) + \deg(r - 1)} + \frac{2}{\deg(3) + \deg(4)} + \frac{2}{\deg(3) + \deg(r - 1)} + \cdots + \frac{2}{\deg(r - 2) + \deg(r - 1)} \\
&= \frac{2(r - 2)}{2r - 4} + \frac{2(r - 3)}{2r - 4} + \frac{2(r - 4)}{2r - 4} + \cdots + \frac{2}{2r - 4} \\
&= (r - 2) + \sum_{i=3}^{r-1} (r - i)
\end{align*}
\]

Theorem 3.9

Let \(Z_r\) be a group with \(r \geq 3\), \(r\) is a prime number then the Harmonic index of \(\mathcal{r}_{SI}(Z_r)\) is

\[
H(\mathcal{r}_{SI}(Z_r))) = 1 + \frac{2}{2r - 4} \sum_{i=3}^{r-1} (r - i)
\]
\[ 1 + \frac{2}{2r - 4} \sum_{i=3}^{r-1} (r - i) \]

4. (Hosoya and Schultz) Polynomial of \( r_{SI}(Z_r) \)

In this section, we found (Hosoya and Schultz) Polynomial of \( r_{SI}(Z_r) \)

**Definition 4.1(33):**

Let \( G \) be a connected graph, then a Hosoya Polynomial of graph \( G \) is defined by \( H(G; x) = \sum_{diam(G)} d(G, k) x^k \), where \( d(G, k) \) is the number of pairs of vertices of a graph \( G \) that are at distance \( k \) apart, for \( k = 0, 1, 2, ..., diam(G) \), where \( diam(G) = \max_{u, v \in V(G)} d(u, v) \).

**Note 4.2(34):**

1. \( d(G, 0) = p(G) \)
2. \( d(G, 1) = q(G) \)

**Definition 4.3(35):**

Let \( G \) be a connected graph, then a Schultz Polynomial of a graph \( G \) is defined by \( Sc(G; x) = \sum_{u, v \in V(G)} (deg(u) + deg(v)) x^{d(u, v)} \), where \( deg(u) \) is the degree of the vertices \( u \) and \( deg(v) \) is the degree of vertices \( v \), \( d(u, v) \) is the distance between \( u \) and \( v \).

**Theorem 4.4:**

\[ H( r_{SI}(Z_r); x) = c_0 + c_1 x, \text{ where } r \geq 3, \text{ } r \text{ is a prime number and} \]

\[ c_0 = r - 1, c_1 = \sum_{i=2}^{r-1} r - i \]

**Proof:**

For every \( r \geq 3, r \) is a prime number, we noticed that every vertex of graph \( r_{SI}(Z_r) \) is adjacent to all the vertices of graph \( r_{SI}(Z_r) \), then \( diam( r_{SI}(Z_r) ) = 1 \), it is mean \( H( r_{SI}(Z_r), x) = c_0 + c_1 x, \) where \( c_i = d( r_{SI}(Z_r), i ), \forall i = 0, 1 \)

It's clear that \( c_0 = d( r_{SI}(Z_r), 0 ) = | r_{SI}(Z_r) | = r - 1 \)

Now to find the size of \( r_{SI}(Z_r) \), we noticed that there exist \( m_{r-1} \) of edges s.t \( m_1 = r - 2, m_2 = r - 3, ..., m_{r-1} = 1 \)

Then \( c_1 = m_1 + m_2 + \cdots + m_{r-1} \)

We can write:

\[ c_1 = \sum_{i=2}^{r-1} r - i \]
Theorem 4.5:

\[ Sc(r_{SI}(Z_r);x) = \sum_{i=2}^{r-1} (r-i)(2r-4)x \], where \( r \geq 3 \), \( r \) is a prime number.

Proof:

\[ \text{deg}(u) = r - 2, \forall u \in V(r_{SI}(Z_r)), \ u = 1,2,\ldots,r - 1 \]
\[ d(u,v) = 1, \forall u, v \in V(r_{SI}(Z_r)) \]

\[ Sc(r_{SI}(Z_r);x) = \sum_{u,v \in V(r_{SI}(Z_r))} (\text{deg}(u) + \text{deg}(v))x^{d(u,v)} \]

\[ = \frac{\text{(deg}(1) + \text{deg}(2))x + \cdots + (\text{deg}(1) + \text{deg}(r-1))x}{(r-2)\text{times}} + \frac{\text{deg}(2) + \text{deg}(3)x + \cdots + (\text{deg}(2) + \text{deg}(r-1))x}{(r-3)\text{times}} + \text{deg}(r-2)x \]

\[ = \sum_{i=2}^{r-1} (r-i)(2r-4)x \]

5. Conclusions

This article has presented the formulae of some degree-based and eccentric-based topological indices of subgroup intersection graph of a group \( Z_r \), where \( r \) is a prime number. For further research, examining on subgroup intersection graph of a group \( Z_{pq} \), where \( p, q \) are a prime number.

REFERENCES PROVISION