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Duo Submodule and C_1-module

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ABSTRACT

In this paper, we will give high priority to some important results about the duality property of submodule. The main reason for choosing this property is that duo is one of the important applications of extending modules. Note that any module $\mathcal{M}$ will be chosen we will deal with it as a submodule on itself.

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1. Introduction

In [1], "let $\mathcal{M}$ be an R-module and let $W: \mathcal{M} \rightarrow \mathcal{M}$. If $W(\mathcal{N}) \subseteq \mathcal{N}$, then $\mathcal{N}$ is called fully invariant (FI) such that $\mathcal{N} \leq \mathcal{M}$". Note that if $\mathcal{M}$ *equal* $\mathcal{N}$, this means $\mathcal{M}$ is also fully invariant. In [2], "the right R-module $\mathcal{M}$ is called a duo module provided every submodule $\mathcal{N} \leq \mathcal{M}$ is fully invariant". Moreover; $\mathcal{M}$ and $\{0\}$ are called Duo submodule. In [3], "the heart submodule of $\mathcal{M}$, denoted by $H(\mathcal{M})$, is defined by the intersection of all nonzero submodules of $\mathcal{M}$". However $H(\mathcal{M})$ is a minimal submodule contained in every submodule is non zero when $H(\mathcal{M})$ is nonzero. In [4], "the right R-module $\mathcal{M}$ is called a multiplication module if for every submodule $\mathcal{N} \leq \mathcal{M}, \exists$ an ideal I of $R \ni \mathcal{N} = I\mathcal{M}$". In [5], "an R-module $\mathcal{M}$ is a projective module if there exists an R-module Q such that $\mathcal{M} \oplus Q$ is a free R-module". In [6], "a module $\mathcal{M}$ is called uniform if $\mathcal{N}_1$ and $\mathcal{N}_2$ are non-zero submodules of $\mathcal{M}$; $\mathcal{N}_1 \cap \mathcal{N}_2 \neq 0$ the intersection of any two non-zero submodules is nonzero, equivalently, $\mathcal{M}$ is uniform if $0 \neq \mathcal{N} \leq_{ess} \mathcal{M}$". In [7], an R-module $\mathcal{M}$ is called extending (C_1-module) if every submodule of $\mathcal{M}$ is essential in a direct summand of $\mathcal{M}$.

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2. The Main Results

In this paper, we study duo property of any submodule $\mathcal{N}$ of C_1-module.

Lemma (2.1):

Consider $\mathcal{M}$ as a submodule of $\mathcal{M}$ over ring R. If $g: \mathcal{M} \to \mathcal{M}$ and $x \in \mathcal{M}$, there exist $r \in R$, then $\mathcal{M}$ is a duo submodule of $\mathcal{M}$.

Proof:

Note that $g(\mathcal{M}) \subseteq \mathcal{M}$ such that $\mathcal{M} \leq \mathcal{M}$. Thus $\mathcal{M}$ is a duo submodule and so is duo-C_1-module.

Examples (2.2): [2]

1- Simple module is Duo module.

2- Multiplication module with projective module is Duo module.

Theorem (2.3):

Let as $\mathcal{M}$ is C_1-module. Consider $\mathcal{M} \leq \mathcal{M}$. If $\mathcal{M}$ has (A.C.C.) property on *cyclic* submodule, then a submodule $\mathcal{M}$ is duo. So a C_1-module $\mathcal{M}$ is duo.

Proof:

Suppose that $\mathcal{M}$ has (A.C.C.) property on *cyclic* submodule. Let $x \in \mathcal{M} \ni x \neq 0$ and let $g: \mathcal{M} \to \mathcal{M}$ be a homomorphism. If $g(x) \notin xR$, then $x \in g(x)R$ and so $x = g(x)r$, $r \in R$. Hence $g^n(x) = g^{n+1}(x)r$; n is positive integer. So

$$xR \subseteq g(x)R \subseteq g^2(x)R \subseteq \dots \dots \dots \dots,$$

$$\exists \, integer \; k^+ \ni g^k(x)R = g^{k+1}(x)R.$$

There exists $r_1 \in R$ such that

$$g^{k+1}(x) = g^k(x)r_1 = g^k(xr_1).$$

$$g^k g(x) = g^k(xr_1).$$

$$g(x) - xr_1 \in ker(g^k).$$

If $ker(g^k) \subseteq xR$, then $g^k(x) = 0$ and so $x = g^k(x)r^k = 0$. ....C!

Therefore $ker(g^k) \subseteq xR$ and hence $g(x) - xr_1 \in xR(g(x) \in xR)$. So $g(x) \in xR$. Then a submodule $\mathcal{M}$ is a duo, so a C_1-module $\mathcal{M}$ is duo.

Remark (2.4):

We can show that some submodules not duo, so a C_1-module $\mathcal{M}$ is not duo, for example:

If R_1 subring of R_2, then any right R_1-module R_2 is not duo module because if $r_2 \in R_2 \ni r_2 \notin R$. So $\varphi: R_2 \to R_2$ defined by: $\varphi(d) = r_2 d \;\forall d \in R_2$ is an R_1-homomorphism. We have $r_2 = \varphi(1)$, then a submodule R_1 is not fully-invariant of *R_1-module R_2*.

Definition (2.5): [3]

"Let $\mathcal{N}$ be a submodule of an R-module $\mathcal{M}$. Then $\mathcal{M}$ is called an essential extension of $\mathcal{N}$ If $\mathcal{N} \cap \mathcal{M} \neq 0$ or $\mathcal{N} \cap \mathcal{M} = 0$, then $\mathcal{M} = 0$".

Definition (2.6): [3]

"A submodule $\mathcal{N}$ of an R-module $\mathcal{M}$ which has no proper essential extension in $\mathcal{M}$ is called a closed submodule of $\mathcal{M}$".

Now we study another property of submodule namely heart submodule $H(\mathcal{M})$, is defined by the intersection of all non-zero submodules of $\mathcal{M}$.

$H(\mathcal{M})$ is a minimal submodule contained in every submodule $\neq 0$ when $H(\mathcal{M}) \neq 0$.

Theorem (2.7):

Let as $\mathcal{M}$ be a C_1-*R-module*. Consider $H(\mathcal{M})$ is heart submodule of $\mathcal{M}$. Then $H(\mathcal{M})$ is fully invariant and so $\mathcal{M}$ is duo ($\mathcal{M}$ is duo-C_1-module).

Proof:

From definition of $H(\mathcal{M})$ we get $H(\mathcal{M}) \leq \mathcal{M}$. Take any homomorphism $g \in End(\mathcal{M})$, $g: \mathcal{M} \to \mathcal{M}$. To prove $g(H(\mathcal{M})) \subseteq H(\mathcal{M})$.

If $H(\mathcal{M}) = 0$, then $H(\mathcal{M})$ already invariant submodule.

Let $H(\mathcal{M}) \neq 0$. Therefore $H(\mathcal{M})$ is simple and hence $H(\mathcal{M}) = soc(H(\mathcal{M}))$. So

$$g(H(\mathcal{M})) = g(soc(H(\mathcal{M}))) \subseteq soc(H(\mathcal{M})) = H(\mathcal{M}).$$

Then $H(\mathcal{M})$ is fully invariant. Thus a C_1-module $\mathcal{M}$ is duo.

Recall that if $\mathcal{M}$ is any *R-module*, then the socle of $\mathcal{M}$ can defined by

$$\mathrm{soc}(\mathcal{M}) = \sum \{\mathcal{N} \leq \mathcal{M}: \mathcal{N} \text{ is simple}\}.$$

Remark (2.8):

- $H(\mathcal{M}) \subseteq soc(\mathcal{M})$ for any right R-module $\mathcal{M}$.
- $H(\mathcal{M}) = soc(\mathcal{M})$ if $\mathcal{M}$ has simple socle.

Theorem (2.9):

Let as $\mathcal{M}$ be a C_1-*module*. If $H(\mathcal{M}) \subseteq \mathcal{M}$ is intersection of all submodules of $\mathcal{M} \ni$ every submodule is fully invariant, then a C_1-module $\mathcal{M}$ is duo.

Proof:

Suppose that $H(\mathcal{M}) = 0$, then $\mathcal{M}$ has a submodule is fully invariant ($\mathcal{M}$ is duo) also is closed. Thus $\mathcal{M}$ is closed duo-C_1-module.

Now suppose that $H(\mathcal{M}) \neq 0$. Then

$$H(\mathcal{M}/H(\mathcal{M})) = \cap\ (B_i/(\cap\ B_i)), i \in I\ \forall\ 0 \neq B_i \leq \mathcal{M}.$$

Then

$$H(\mathcal{M}/H(\mathcal{M})) = 0.$$

So $H(\mathcal{M})$ is h-closed ($\mathcal{M}$ is closed duo-C_1-module).

Remark (2.10): [3]

"Let $\mathcal{M}$ be an R-module and $\mathcal{N} \leq \mathcal{M}$. We called $\mathcal{N}$ is h-closed submodule of $\mathcal{M}$ if $H(\mathcal{M}/\mathcal{N}) = 0$".

Corollary (2.11):

Let as $\mathcal{M}$ be a C_1-R-module and $\mathcal{M}_1, \mathcal{M}_2 \leq \mathcal{M}$. If $H(\mathcal{M}/\mathcal{M}_1) = 0$ and $H(\mathcal{M}/\mathcal{M}_2) = 0$, then $H(\mathcal{M}_R/\mathcal{M}_1) = 0$ (h-closed) and so $\mathcal{M}$ is duo-C_1-module.

Proof:

Assume that $\mathcal{M}_1$ be a h-closed submodule of $\mathcal{M}_2$ (H$(\mathcal{M}_2/\mathcal{M}_1) = 0$) and

$H(\mathcal{M}/\mathcal{M}_2) = 0$. But $(\mathcal{M}_2/\mathcal{M}_1) \subseteq (\mathcal{M}/\mathcal{M}_1)$. So $H(\mathcal{M}/\mathcal{M}) \subseteq H(\mathcal{M}/\mathcal{M})$

(by def. of $H(\mathcal{M})$) $\ni H(\mathcal{M}_2/\mathcal{M}_1) = 0$.

Hence $\mathcal{M}_1$ is a h-closed submodule of $\mathcal{M}$, $(H(\mathcal{M}/\mathcal{M}_1)) = 0$. Therefore,

$H(\mathcal{M}/\mathcal{M}_1)$ is fully invariant. Thus $\mathcal{M}$ is duo-C_1-module.

Example (2.12):

Let F be a field and V_F a vector space over F such that $dim(V_F) = 2$. Consider R subring of $\mathcal{M}_2$(F);

$$R = \begin{bmatrix} F & V \\ 0 & F \end{bmatrix} = \left\{ \begin{bmatrix} f & v \\ 0 & f \end{bmatrix}; f \in F, v \in V \right\}.$$

There are only three submodules of R:

$$\mathcal{N}_1 = \begin{bmatrix} 0 & v_1 f \\ 0 & 0 \end{bmatrix}, \mathcal{N}_2 = \begin{bmatrix} 0 & v_2 f \\ 0 & 0 \end{bmatrix}, \mathcal{N}_3 = \begin{bmatrix} 0 & v_1 f + v_2 f \\ 0 & 0 \end{bmatrix} \ni v_1, v_2 \in V, f \in F.$$

We have $H(R) = 0$ and $H(R/R) = 0$, therefore 0 and R are h-closed submodule and then $H(R/R)$ is fully invariant. Thus $\mathcal{M}$ is duo-C_1-module.

Remark (2.13):

Note that $\mathcal{N}_1, \mathcal{N}_2$ and $\mathcal{N}_3$ not imply $\mathcal{M}$ is duo-C_1-module, because

$$H(R/N_3) \cong H(F) \neq 0,$$

and

$$\begin{aligned} H(R/\mathcal{N}_1) &= (R/\mathcal{N}_1) \cap (\mathcal{N}_3/\mathcal{N}_1) \\ &= \mathcal{N}_3/\mathcal{N}_1 \\ &\neq 0. \end{aligned}$$

Also

$$H(R/\mathcal{N}_2) \neq 0.$$

Example (2.14):

Let $\mathcal{N}_1 = \begin{bmatrix} 0 & v_1 f \\ 0 & 0 \end{bmatrix}$ be R-submodule of R_R. Clear that $\mathcal{N}_1$ not complement submodule of R_R, therefore it is not h-closed and so not fully invariant. Then a C_1-module $\mathcal{M}$ is not duo.

Example (2.15):

Let $\mathcal{M}_z = Z$ and $\mathcal{N} = 6Z$ be a submodules of $\mathcal{M}$. So $H(\mathcal{M}/\mathcal{N}) = H(Z/6Z) = 0$ and then it is h-closed $\{H(Z/6Z)$ fully invariant$\}$. Thus $\mathcal{M}$ is duo-C_1-module.

The next theorem explain that a direct summand of h-closed submodule gives duo-C_1-module.

Theorem (2.16):

Let $\mathcal{M}$ be a C_1-module. Then direct summand of h-closed submodules is fully invariant and so a C_1-module $\mathcal{M}$ is duo.

Proof:

Suppose that $\mathcal{M}$ is a direct summand of h-closed. So

$$\mathcal{M} = D_1 + D_2 \ \forall D_1, D_2 \leq \mathcal{M}.$$

Assume that $K \leq D_1 \ni$ it is h-closed of D_1 $(H(D_1/K) = 0)$. Then

$$\mathcal{M}/(K \oplus D_2) \cong D_1/K. So\ H(\mathcal{M}/K \oplus D_2)) = 0$$

Because $if\ X \cong Y \rightarrow H(X) \cong H(Y)$. Hence $K \oplus D_2$ is h-closed submodule,

$(H(K \oplus D_2))$ is fully invariant and so a C_1-module $\mathcal{M}$ is duo.

Corollary (2.17):

Let $\mathcal{M}$ be a C_1-module. If $\mathcal{M}$ has a socle not equal zero, then C_1-module $\mathcal{M}$ is duo.

Proof:

Assume that $soc(\mathcal{M}) \neq 0$. Since $\mathcal{N} \leq \mathcal{M}$ is a simple, then $soc(\mathcal{M}) = H(\mathcal{M})$. but $H(\mathcal{M})$ is a fully-invariant, then a C_1-module $\mathcal{M}$ is duo.

Corollary (2.18):

Let $\mathcal{M}$ be a multiplication C_1-module. If $\forall\, \mathcal{N} \leq \mathcal{M}$; R-*monomorphism* $g: \mathcal{N} \rightarrow \mathcal{M}$ can be extended to an R-*endomorphism* of $\mathcal{M}$ $(h: \mathcal{M} \rightarrow \mathcal{M})$, then a C_1-module $\mathcal{M}$ is duo.

Proof:

Suppose that $\mathcal{N} \leq \mathcal{M}$ such that R-*monomorphism* $g: \mathcal{N} \rightarrow \mathcal{M}$ can be extended to an R-*endomorphism* of $\mathcal{M}$. Let $\mathcal{M}$ be a multiplication module over R. So $\mathcal{N} = I\mathcal{M}$. Let $f: \mathcal{M} \rightarrow \mathcal{M}$ be a R-*endomorphism*. If $f(\mathcal{N}) = f(I\mathcal{M})$, then $If(\mathcal{M}) \subseteq I\mathcal{M} = \mathcal{N}$. Hence $(\mathcal{N}) = \mathcal{N}$ $(f(\mathcal{N}) \subseteq \mathcal{N})$. Then a submodule $\mathcal{N}$ of $\mathcal{M}$ is fully invariant. Thus a C_1-module $\mathcal{M}$ is duo.

Recall that a submodule $\mathcal{N}$ of a module $\mathcal{M}$ is essential in case $A \cap \mathcal{N} \neq 0$ for every submodule $A \neq 0$.

Corollary (2.19):

Let $\mathcal{M}$ be a C_1-module. If $\mathcal{N} \leq_{ess} \mathcal{M} \ni$ R-*monomorphism* from $\mathcal{N} \rightarrow \mathcal{M}$ can be extended to an R-*endomorphism* of $\mathcal{M}$, then a C_1-module $\mathcal{M}$ is duo.

Proof:

Let $f: \mathcal{M} \rightarrow \mathcal{M}$ be a *monomorphism* and $H = \{x \in \mathcal{N}: f(x) \in \mathcal{N}\}$. Then

$H = f^{-1}(\mathcal{N})$. Since $\mathcal{N} \leq \mathcal{M}$ is pseudo-injective, then $\exists$ an R-*homomorphism* $g: \mathcal{N} \rightarrow \mathcal{N} \ni$ extends f. Also, we have $\mathcal{M}$ is pseudo-injective, then $\exists$ an R-*homomorphism* $h: \mathcal{M} \rightarrow \mathcal{M} \ni$ extends g. Let us claim that $(h - f)(\mathcal{N}) = 0$. Assume that $(h - f)(\mathcal{N}) \cap \mathcal{N} \neq 0$. But $\mathcal{N} \leq_{ess} \mathcal{M}$. So

$$(h - f)(\mathcal{N}) \cap \mathcal{N} \neq 0.$$

Hence

$$(h - f)(n) = l,\ n, l \in \mathcal{N}.$$

So

$$(h-f)(n)=l.$$

Therefore

$$(g-f)(n)=f, so\ f(n)=g(n)-l.$$

Hence $n\in H$. Then $l=(h-f)(n)=0$; which is contradicts with assumption, then $(h-f)=0$. Hence

$$h(\mathcal{N})=f(\mathcal{N}).$$

But

$$f(\mathcal{N})=h(\mathcal{N}) = g(\mathcal{N}) \subseteq \mathcal{N}.$$

So

$$f(\mathcal{N}) \subseteq \mathcal{N}.$$

Hence $\mathcal{N}$ is a fully-invariant. Thus a C_1-module $\mathcal{M}$ is duo.

Definition (2.20): [4]

"any *module* $\mathcal{N}\in\sigma[\mathcal{M}]$ is called $\mathcal{M}$-*multiplication module* if for every submodule of $\mathcal{N}$, $\exists\, I\leq\mathcal{M}\ni L=I_{\mathcal{M}}\mathcal{N}$, where $\sigma[\mathcal{M}]$ is all *multiplication modules*".

Theorem (2.21):

Let as $\mathcal{M}$ be D_1-*module* and $\mathcal{N}\in\sigma[\mathcal{M}]$. If $\mathcal{N}$ is an $\mathcal{M}$-*multiplication module*, then a C_1-*module* $\mathcal{M}$ is duo.

Proof:

Since $\mathcal{M}$ is D_1-*module*, then $\mathcal{M}$ has a decomposition $\mathcal{M}=\mathcal{M}_1\oplus\mathcal{M}_2 \ni \mathcal{M}_1\leq\mathcal{N}$ and $\mathcal{M}_2\cap\mathcal{N}\ll\mathcal{M}_2$ where $\mathcal{N}\leq\mathcal{M}$ ($\mathcal{M}$ is a lifting). So $\mathcal{M}$ is C_1-*module*. Assume that $L\leq\mathcal{N}$. Since $\mathcal{N}$ is an $\mathcal{M}$-*multiplication module*, then there exists $K\leq\mathcal{M}\ni f(k)\subseteq K$ and $\mathrm{K}_{\mathcal{M}}\mathcal{N}=\mathrm{L}$ ($\mathrm{f}:\mathcal{M}\to\mathcal{M}$). $\mathrm{KN}=\sum\mathrm{f(K)}\Rightarrow\mathrm{L}=\sum\mathrm{f(K)}$. Then if $\beta:\mathcal{N}\to\mathcal{N}\Rightarrow\beta(L)=\beta\sum f(K)=\sum(\beta\circ f)(K)\subseteq L$, then L is fully invariant of $(f(L)\subseteq L)$. Then $\mathcal{M}$ is duo-C_1-module.

Example (2.22): [4]

"Let $\mathcal{M}=Z_p\infty$. Let $\mathcal{N}\leq_p\mathcal{M}$ (proper), then $\mathcal{N}$ is a duo submodule".

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