Available online at www.qu.edu.iq/journalcm
JOURNAL OF AL-QADISIYAH FOR COMPUTER SCIENCE AND MATHEMATICS
ISSN:2521-3504(online) ISSN:2074-0204(print)

Noetherian, Artinian Regular Modules and Injective Property

Fawzi N. Hamad [a,*], *Majid Mohammed Abed* [b]

[a]*Department of Mathematics, University Of Anbar, Anbar, Iraq, Email: faw19u2014@uoanbar.edu.iq,*
[b]*Department of Mathematics, University Of Anbar, Anbar, Iraq, Email: majid_math@uoanbar.edu.iq*

ARTICLE INFO

Article history:
Received: 02 /02/2021
Rrevised form: 18 /02/2021
Accepted : 01 /03/2021
Available online: 04 /03/2021

Keywords:

Injective module, Regular module, Noetherian module, Artinian module, Cyclic module.

ABSTRACT

In this article we provide that several relationships between some concepts and injective module. We investigate, if M is a cyclic and regular module is injective. Also, if M is regular with N≤ M is finitely generated submodule, so M is injective. Finally, some relationships have been studied about injective module in details.

MSC. 41A25; 41A35; 41A36.

DOI : https://doi.org/10.29304/jqcm.2021.13.1.764

1- Introduction*:*

All ring in this article are commutative and all modules with unit. In [1], "any module M is called regular if (mg)m=m, g:M→R and $m \in M$". From [2], "if R has no infinite direct sum of ideals, so R is called finite dimensional". Also, by [3], "a module M is called finitely generated if it has finite set of generators". In [4], "we found that every projective module is an injective". Also, from [5], "every direct limits of projective module is a projective". From [6], "any module M over prufer domain is linearly compact". In [7], "prufer domain means R integral domain and f-generated ideal of R is projective". "Any ring R is called QF(quasi-Frobenius) if every projective module is injective; or every injective module is discrete". In this paper, we present a new result about injective module depending on other concepts, namely Noetherian, Artinian and regular modules.

*Corresponding author: Fawzi N. Hamad

Email addresses: *faw19u2014@uoanbar.edu.iq*

Communicated by *:* Dr. Rana Jumaa Surayh aljanabi.

2- The Main Result

In this section, we present some new results about the relationship between Noetherian, Artinian and regular rings and injective module. We should start with the following definitions.

Definition (2.1). [1]. "Any module M is called regular if $\forall m \in M$, $\exists$ g$\in Hom(M$,R), then(mg)m=m".

More details of regular property can find it in [2].

In the beginning, we need a basic preliminary in order to proceed from it towards then main objective of the current paper. See the following lemma:

Lemma (2.2). Every Noetherian or Artinian Regular module over abelian ring R with unity is injective module.

Proof.

Any module M fulfills all the conditions in theory, this means M is a finite direct sum of projective module have only two submodule are $\{0\}$ and M. Since R is a commutative ring with identity element, so M is a flat module iff it is injective and a finite direct sum of injective is also injective.(see[8]).

Definition (2.3). An R-module M is called projective if and only if for any *epimorphism* f:C→V such that C,V are any R-modules and for any homomorphism g:M→V ∃ a homomorphism h:M→C such that f∘h=g [9].

Remark (2.4). A ring R is called finite–dimensional if R have no infinite direct sums of ideals. [1]

Theorem (2.5). Let R be a (QF) and perfect ring or finite–dimensional ring. If M is a regular R-module, then M is injective.

Proof.

Let R is a perfect ring. Let T=direct limits of projective-module. Then T projective (see [5]). But M is a direct limit of $N_i \ni N_i$ f. submodules. So M is a projective. But from [4], "for a QF-ring. every projective module is an injective module". Now if R have no infinite direct sums of ideals. Let M be a regular and $\Gamma=\{\sum \oplus Rm_\alpha : m_\alpha \in M\}$ is a partially ordered and $\sum \oplus Rm_\alpha \leq \sum \oplus Rm_\beta$ *iff* $\{m_\alpha\} \subseteq \{m_\beta\}$.So $\exists$ $N=\sum \oplus Rm_\alpha$ is a maximal in Γ (By Zorn's Lemma). Then $N \cap Rm \neq 0$, $0 \neq m \in M$. If $N = M$, suppose that $m \in M$. $Rm \approx I$; I is an ideal of R. So have no infinite direct sums of N_i. But

$$Rm=Rn_1 \oplus \ldots\ldots \oplus Rn_t \ni Rni \text{ simple }, Rni \cap N \neq 0 \ \forall i. \text{ so } Rni \cap N = Rni$$

$$Rm \cap N=(Rn_1 \oplus \ldots\ldots \oplus Rn_t) \cap N \supseteq \sum \oplus (Rn_i \cap N) = \sum \oplus Rn_i = Rm$$

Hence $Rm \subseteq N$ and then M=N. Therefor M is a projective. Thus it is injective module [4].

Lemma (2.6). "Every f-generated regular R-module is a projective". [1]

Recall that for all $m_i \in M$, $i \in J \ni J$ is a some of several generators of M. So we can present a definition of finitely generated module M by the following way:

Definition (2.7). "If M has a f-set of generators, so M is called f-generated

R-module".[٣]

Example (2.8). Any f-dimensional vector space is a f-generated over a field K.

Proposition (2.9). Let E be a regular R-module. If $E\cong\frac{R^n}{N}$, then E is injective module.

Proof.

Since $E\cong\frac{R^n}{N}$, $n \in Z^+$, $N \leq R^n$,so there is a homo.

γ: $R^n \rightarrow \frac{R^n}{N} \cong E \ni \gamma$ (r_1,.....,r_n)$\rightarrow$ (r_1,....., r_n)+N.

Take e_i =(0,….,0,1,…,0) ∍ (1 being at the $i-th$ place).Hence e_i generate R^n, 1≤ i ≤n. so γ (e_i) generate E over R, (1≤ i ≤n).Therefor E is f. g. module. But E is regular module. So E is a projective (Lemma 2.6). Thus E is injective module.

Corollary (2.10). Every cyclic regular R-module over QF-ring is injective.

Proof.

Since M is cyclic regular R-module then it is f-generated and by (Lemma 2.6), M is injective.

Corollary (2.11). Let M_1 and M_2 be an R-modules over (QF)-ring and let

δ: $M_1 \rightarrow M_2$ be an onto homomorphism such that M_2 is a regular and M_1 is

f-generated, so M_2 is injective module.

Proof.

Suppose that M_1 and M_2 are two modules over the ring R. Also suppose $M_1$is f-generated module. To prove that M_2 is injective.

Since M_1 is a f-generated. R-module, so M has a generating set {m_1,…,m_k}. Therefore, we need to show that M_1 generated by the set {δ(m_1),…, δ(m_k)}. ∀ m_2∈M_2 , we have δ is onto, ∃ m_1∈M_1 ∍ δ(m_1)=m_2. But M_1 is a f-generated module, m_1=r_1a_1+………+$r_k\, a_k$, r_1,….., r_k ∈R. So m_2= δ(r_1a_1)+………+ δ($r_k\, a_k$) = r_1 δ(a_1)+…… r_k δ(a_k). Hence M_2 =< δ(a_1) , ……, δ(a_k)>. Then M_2 is a f-generated with regular property imply M_2 is “projective and hence is injective” [6].

Theorem (2.12). Let M be regular module over P.I.D. If M is acyclic module, then it is a Noetherian and is injective module.

Proof.

Since M is a cyclic, then it is a f-generated. So M have a generators m_1,…...m_k. Hence ∃ $\varphi: R^k \rightarrow M$ defined by φ(b_1,……,b_k)=b_1m_1+……,+b_km_k, then $M \cong \frac{R^k}{N}$, But R is a P.I.D, so R is a Noetherian R-module. We have M is a regular module. Thus M is injective (Lemma 2.2).

Corollary (2.13). Let M be a regular R-module. If N and $\frac{M}{N}$ are Noetherian ∍ N is a submodule of M, so M is injective module.

Proof.

Assume that K≤ M. So $Img(K)$ in $\frac{M}{N}$ is f-generated. Hence $K \cap N$ is also f. generated. Let k_1,……k_n ∈K generate $Img(K)$ in $\frac{M}{N}$ and let b_1,……,b_m generate $K \cap N$. ∀ $k \in K$, k ≡r_1k_1+……+r_nk_n , r_i∈R. So K-$\sum r_ik_i$ ∈$K \cap N$. Then K-$\sum r_ik_i$=$\sum t_ib_j$, t_i∈R. Hence K=$\sum r_ik_i$+$\sum t_ib_j$.(K f. g in M) therefore M is Noetherian module with regular property imply M is injective module (Lemma 2.2).

Recall that any R-module M is satisfy the maximal condition for submodules if $\varphi \neq \Gamma$ of submodules have a maximal ($\Gamma \supset H_0 \ni \nexists$ number containing H_0). Therefore it easy present a definition of Noetherian R-module, any module M satisfy the maximal condition $(a.c.c)$ is Noetherian.

The next theorem shows the relationship between Noetherian module and injective module. But before that we need to present the following lemma:

Lemma (2.14). If $N \leq M$ is f-generated, then M is a Noetherian.

Proof.

Suppose that $N \leq M$ is a f-generated. Assume $H_1 \subseteq H_2 \subseteq H_3 \subseteq \ldots$ is a submodules of M

Take $H = \cup H_i$, $i=1,\ldots\ldots,\infty$. So $H \leq M$ and hence H is a f-generated.

Let $H = Rh_1 + \ldots\ldots + Rh_n$. all h_i in one of H_i, $\exists$ m $\ni h_1, \ldots\ldots h_n \in H_m$. But $H = H_m$, $n \geq m$. So M is a Noetherian module.

Theorem (2.15). Let M be an R-module. If M is a regular module and have N is a f-generated submodule of M, then M is injective.

Proof.

By Lemma 2.14, M is Noetherian-module ($N \leq M$ is a f-generated). But M is a regular module. Then from(Lemma 2.2), M is injective.

Theorem (2.16). Let M be a regular module and let

$0 \to M_1 \to M \to M_2 \to 0$ be an exact seq. If M_1 and M_2 are Noetherian, then M is injective module.

Proof.

Suppose $M_1 \leq M$, $M_2 = \frac{M}{M_1}$ and assume that M_1 and $\frac{M}{M_1}$ are Noetherian.

Let $H_1 \subseteq H_2 \subseteq \ldots\ldots, \frac{(H_1+M_1)}{M_1} \subseteq \frac{(H_2+M_1)}{M_1} \subseteq \frac{(H_3+M_1)}{M_1} \subseteq \ldots\ldots$

of M_1 and $\frac{M}{M_1}$, $\exists$ m $\ni H_n \cap M = H_m \cap M_1$ and $H_n + M_1 = H_m + M_1$, $n \geq m$.

So $H_n = H_m \cap (H_n + M_1) = H_n \cap (H_m + M_1) = H_m + (H_n \cap M_1)$ by Modular law,

(Let H, Y, $L \leq M$ and $Y \subseteq H$. So $H \cap (Y+L) = Y + (H \cap L)$).

$= H_m + (H_n \cap M_1) = H_m$.

So M is Noetherian module with regular property, we get M is injective.

Example (2.17). Any module over division ring is injective, because division ring R has only 2 ideals 0 and R itself.

Proposition (2.18). Let M be a regular module. If:

1. S_1 is the set of f-generated submodules of M is Noetherian,

2. $\varphi \neq N_1$ is f-generated and $N_1 \leq M \ni N_1$ has maximal element,

3. $N \leq M$ is a f-generated, then M is injective,

Proof.

Let $S_1 \neq \varphi$ be a set of f -generated ($S_1 \leq M$) If S_1 has no maximal element, so any $s \in S_1$ $\{S_2 \in S_1 : S_2 \supset S_1 , S_2 \neq S_1\} \neq \varphi$,thus we get $a.c.c$ of submodules which is infinite.

Now let $N \leq M$, there is a maximal element N_1. then $N_1 = N$. Now let

$H_1 \subset H_2 \subset \ldots\ldots$ Be $a.c.c$ of submodules of M. So $\cup H_i \subset M$ is a f-generated and all generating elements in $H_i, i \in H$.

Thus $H_i = H_{i+r}$ $\forall\, r \in N$. So M is Noetherian module But every Noetherian module is Artinian with regular property M is injective module.(Lemma2.2).

Recall that if proper f-generated submodule of M is a small in M (P. f-generated N<<M), so M is called semi hollow module such that M is hollow if P-submodule N is small in M. Therefore we present the following theorem.

Theorem (2.19). Let M be a regular R-module. If M is semi hollow and Rad(M) is a Noetherian module, so M is injective.

Proof.

Assume M is semi hollow-module. Let Radical of M not equal M. there are Max(N) $\ni$ N$\leq M$. This means M is also module. Hence Rad(M) is a maximal and Rad<<M.

Hence $\frac{M}{Rad(M)}$ is a simple module and hence is Noetherian.

since

$$0 \rightarrow \text{Rad}(M) \rightarrow M \rightarrow \frac{M}{Rad(M)} \rightarrow 0$$

is a short exact seq. Then M is a Noetherian (M is Artinian-module) with regular property imply M is injective module.

It is possible to rely on the previous example to discuss its content in another way, follows:

Proposition (2.20). Let R be a division ring if:

1- M is a regular module over R.

2- M is a divisible-module; then M is injective.

Proof.

Assume R is a division ring and M is a divisible-module. Let N$\leq M$ such that K is a basis for N. So $\exists$ a basis k_1 of $M \ni k_1 \supset$ K. Assume that $k_2 = K_1 \cap k$ and N_1 is a span of k_2.

Then $N_1 \oplus N = M$. Hence M is a semi simple (M is Artinian). But M is a regular. Thus M is injective.

Example (2.21). Any regular module M of the ring R which has only two ideals {0} and R is Artinian, because R have only two ideals {0} and R imply R is division and hence M is Artinian with regular property we get M is injective.

Proposition (2.22). Let M_1, M_2 and M_3 are three modules if

1- $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$ is short exact of R-modules.

2- M_1 and M_3 are Artinian modules.

3- M_2 is a regular module. Then M_2 is injective.

Proof.

Take a chain M_{2_m} such that M_{2_m} are submodules of M_2. From the projecting to M_3, $Im(M_{2_m})$ is a stabilizes, so if

f:$M_{2_m} \rightarrow M_3$, the $ker(f)$ from chain submodules of M_1 . Hence it is stabilizes. Then M_2 is Artinian module with condition (3), we get M_2 is injective(Lemma2.2).

Remark (2.23). Every homomorphic image of Artinian ring is Artinian.

Theorem (2.24). Let R be Artinian ring. If M is a f-generated. R-module and regular, so M is injective module.

Proof.

We know that $M \cong \frac{R^n}{N}$ such that N≤ R^n and n^+.

But R^n is Artinian ring, so a direct sum of Artinian modules. Hence M is Artinian module (Remark2.23). But M is a regular. Thus it is injective module.

Definition (2.25). The ring R is called $prufer$ domain if R is integral domain and every f-generated. ideal of R is projective (invertible). [7] In the theorem(2.27), we study some conditions over $prufer$ domain in order to get injective module.

Remark (2.26). Any ideal I is injective if $I \cdot I^{-1} = R \ni I^{-1} = \{rI \subseteq R: r \in q(R)$ and q(R) is the field of fractions is the smallest field can be embedded.

(⋆) Let R_2 be a unitary extension ring of R_1. We say R_2 is a p-extension of R_1 if ∀ $r_1 \in R_2$ satisfies $R_1[X]$ one whose coefficients is a unit of R_1(whose coefficients generate a unit ideal of R_1)

Theorem (2.27). Let R is a ring. If

1. R is a prufer domain Krull domain 1,
2. M is a divisible R-module,
3. M is Artinian R-module; M is injective.

Proof.

"Over prufer domain any module is linearly compact and divisible is injective or, Artinian module is linearly compact with divisible property M is injective". From [6].

Proposition (2.28). Let M be an R-module. If:

1. M satisfies $(d.c.c)$,
2. Every element $m \in M$ is a divisible,
3. R is integrally closed domain has quotient field K,
4. K is a p-extension of R, M is injective.

Proof.

Assume that K is P-extension of R. and Let $I_{\varkappa}$ be a maximal ideals in the ring R_1.

Let H=all elements f in $R_1[X] \ni A_f=R_1$. So H is a regular multiplicative in $R_1[X]$.

Hence H= $R_1[X] - \cup I_{\varkappa}[X]$. So if I_1 is an ideals of $R_1[X] \subseteq \cup I_{\varkappa}[X]$. then I_1 contained in one of $I_{\varkappa}[X]$. So $\{I_{\varkappa}[X]\}$ is the set of prime ideals of $R_1[X]$. Hence R is a $prufer$ domain. Thus M is injective module because from condition (1), M is Artinian and by condition (2), M is a divisible module.

Conclusion:

Throughout this paper, we used several concepts of module theory in order to obtain injective module. We proved that any regular module M over perfect ring R is injective module. Also, we investigated that a cyclic regular module M over principal ideal domain satisfies $(a.c.c)$ property and finally this mean M is injective. Another result say when M is a regular and semi-hollow module with radical of M is a Noetherian, this mean M is injective module. Finally, if R is Artinian and M is a finitely generated with regular property give M is injective.

References

[1] Zelmanowitz J., (1972). Regular module, Transaction of the American mathematical society, Vol.163,.
[2] Funayama, N. (1966). Imbedding a regular ring in a regular ring with identity. Nagoya Mathematical Journal, 27(1), 61-64.
[3] Subhash Atal, (2014). Finitely generated Modules, M A498 project I, Department of Mathematics, Indian Institute of Technology Guwahati, Guwahati-781039.
[4] Oshiro, K. (1984). Lifting modules, extending modules and their applications to QF-rings. Hokkaido Math. J, 13(3), 310-338.
[5] Lambek, J. (1966). Lectures on rings and modules, Blaisdell Publ. Com., Waltham, Toronto, London.
[6] Fuchs, L., & Salce, L. (2001). Modules over non-Noetherian domains (No. 84). American Mathematical Soc.
[7] Gilmer, R. (1992). Multiplicative ideal theory. Queen's Papers in Pure and Appl. Math., 90.
[8] Ware R., (1971). Endomorphism rings of projective Module, Trans. Amer. Math. Soc. 155,233-259.
[9] Inaam M.A., Tha'ar Y. G., Small Quasi-Dedekind Modules, Journal of Al-Qadisiyah for Computer Science and Mathematics 3nd. Sinentific Conference 19-20, APRIL-2011 Vol 3 No.2 Year 2011.