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On NM-Hollow Modules

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ABSTRACT

This notion investigates the generalization of hollow module namely near maximal hollow module (NM-hollow module). Any module M is NM-hollow if for every H ≪ M satisfy two properties namely nearly and maximal. Some new results have been studied and more relationships are presented. The main result in this section is; if M is a faithful and multiplication over the ring R, then M is NM-hollow module. Also, a direct-sum of NM-hollow module is also NM-hollow.

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1- Introduction*:*

All the rings in this article are commutative and have 1 and any module is unitary. In [1], "a proper submodule N of M is called maximal, if whenever W is a submodule of M with N ⊆ of W ⊆ M implies that W = M". In [2] "a proper submodule N of an R -module M is called a prime submodule if for each $r \in R$, $x \in$ M, such that $rx \in N$, then either $x \in N$ or $r \in [N\colon \mathrm{M}]$, where $N\colon \mathrm{M} = \{r\colon r \in R, r\mathrm{M} \subseteq N\}$". In [3], "we study nearly prime as a generalization of prime submodules and they define a nearly prime submodule as follows:

a proper submodule N of an R -module M is called nearly prime, if whenever $rx \in N$, $r \in R$, $x \in$ M implies either $x \in N + J(\mathrm{M})$ or $\in N + J$ M: M".

the author in [4] showed "a proper submodule L of a module M is called small (denoted by L ≪ M), if for every proper submodule K of M, L + K ≠ M". In [4] "a non-zero module M is called a hollow if every proper submodule N of M is a small submodule of M (N ≪ M) that is N + W ≠ M for every W < M".

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In [5],"Inaam and Riadh studied a generalization of maximal submodule in details". In [6], " a submodule N of M is called essential, if whenever N∩L= (0), then L = (0) for each submodule L of M". In [1], "Muna did a generalization of maximal submodule namely nearly maximal submodules, but in this paper, we try to do same this generalization for hollow module". In other words; nearly maximal of small submodule in hollow module M (NM-hollow module).

2- The Main Result

In this section, we present several conditions about any submodule N of M to obtain a generalization of hollow module. Before we start, we showed introduce the following definitions and some examples and Remarks.

Definition (2.1). [7] "The submodule N of M is called prime if N≠ M and, given r ∈ R and m ∈ M such that rm∈ N, either m∈ N or r ∈ (N : M)".

Definition (2.2). "An R-module M is called projective if and only if for any *epimorphism* f:C→V such that C,V are any R-modules and for any homomorphism g:M→V ∃ a homomorphism h:M→C such that f∘h=g" [8].

Recall that an indecomposable module with projective module gives local module, also every local module is hollow module.

Note that the maximal submodule means any submodule H of M (H≤ M) is named maximal if W≤ M; H⊂W⊆M, so W=M.

From definition of maximal submodule and nearly maximal submodule, we can introduce the following remarks and some examples:

Remark (2.3). Every maximal submodule is a NM-submodule.

Example (2.4). $\mathcal{Z}$ submodule in $\mathcal{Z}$-module Q is NM-submodule but we have Q is not maximal ∋ Q is rational ring.

Remark (2.5). Every M-hollow is NM-hollow.

Example (2.6). If N= (0)≪ M simple, then N is NM-hollow module.

Remark (2.7). We can say the radical of a module M is:

Rad(M)=∩ {N_i: N_i is maximal submodules of M}

Note that since M is any local module has only one maximal submodule, so M has only one NM-local. Also, if Rad(M) = M, this means M has no maximal submodule. On the other hand, if N is NM-hollow with Rad(M)=0, then N is a maximal of hollow module.

♦ If N_1, N_2 are two small proper submodules of M and $N_1 \cap N_2$ is NM-submodule of M, so $N_1$and N_2 are NM-submodule. Hence M is NM-hollow module.

♦ If M is NM-hollow, then $\frac{M}{H}$ is NM-hollow.

Theorem. (2.8). Let M**,** A and B R-modules and ∅ be a mapping from A into B. If A is NM-hollow module, then ∅(A) is NM-submodule of B (B is NM-hollow module).

Proof.

Take M is and R-module**.** Let H≤B ∋ ∅(k)⊂H, k≤A. To prove that ∅(K) ⊂H and H + Rad(B)=B. Since A is NM-hollow, so N≤A is NM-submodule. Also, ker (∅)⊆N, then ∅(N)≤B.

So

$$\emptyset(N)\subset H \text{ and hence } N\subset \emptyset^{-1}(H).$$

Hence

$$\emptyset^{-1}(H)+Rad(A)=A_1$$

So

$$\emptyset(\emptyset^{-1}(H)+Rad(A)) = \emptyset(A) = B.$$

Then

$$\emptyset(\emptyset^{-1}(H)+\emptyset(Rad(A)) = A.$$

But $\emptyset$ onto so H+Rad(B). F(N) is N$\mathcal{M}$-submodule of B. Thus, B is NM- Hollow module.

Corollary. (2.9). If M is NM-hollow, then $(\frac{M}{H})$ is NM-hollow $\ni$ N, $H\ll M$ and $H\subset N$.

Theorem (2.10). The image of epimorphic of near maximal hollow is also near maximal hollow module.

Proof.

By last Theorem.

Definition (2.11). [6]. "Any module M is called finitely generated if $M=\sum xi\ ri \ni x_i \in M$ and $r_i \in R$".

Proposition (2.12). Let M be an R-module. If:

1- M is a f.generated;
2- M is faithful and N=IM
3- $N<< M$. Then M is NM-hollow when R is NM-hollow as a module.

Proof.

Suppose that R is NM-hollow module. Suppose $0\neq N\ll M$. Since M is a multiplication module, then N=IM. I is NM-ideal of R, so N is NM-submodule of M [1]. Thus, M is NM-hollow module.

Theorem (2.13). A direct sum of NM-hollow is an NM-hollow module.

Proof.

Assume that M is NM-hollow module. Assume that $M = A\oplus B \ni A\leq M, B\leq M$. Take $0\neq N_1$ proper small submodule of A ($N_1\ll A$).

Let $H_1\leq A \ni N_1\subset H_1$ with $N_1\subset H_1+Rad(A)\subset A$. So

$$N_1\oplus B\subset H_1+Rad(A)\oplus B.$$

But $N_1\oplus B$ is NM-small submodule of M. Hence

$$(H_1+Rad(A))\oplus B+Rad(M)= M.$$

Also

$$Rad(M)= Rad(A)\oplus Rad(B).$$

Then

$$H_1+Rad(A)+B+(Rad(A)\oplus Rad(B)) \subseteq H_1+Rad(A)) \oplus B_2.$$

So

$$H_1+Rad(A)=A.$$

Therefore, N_1 is a NM-submodule of A. Thus, A is NM-hollow submodule. That is M is NM-hollow module.

Corollary (2.14). Let M be a hollow R-module with

1- M=A+B;

2- A and B are NM-hollow modules;

3- Ann(A)+ann(B)=R; then M is NM-hollow.

Proof.

Assume that $0\neq K\leq M$ and $H\leq M \ni K\subset H$. We have ann(A)+ann(B)=R. Then

$$K=K_1\oplus K_2 \ni K_1\leq A \text{ and } K_2\leq B \text{ with}$$

$$H=H_1\oplus H_2 \ni H_1\leq A \text{ and } H_2\leq B.$$

Now

$$K_1\oplus K_2\subset H_1\oplus H_2+Rad(A\oplus B) \subseteq A\oplus B.$$

But

$$Rad(A\oplus B) =Rad(A)\oplus Rad(B).$$

Then

$$K_1\subset H_1+Rad(A)\subset A.$$

Hence

$$K_2\subset H_2+Rad(B)\subset B.$$

But $K_1\subseteq A$ and A is NM-hollow modules. So $H_1+ Rad(A)=A$. Similarly

$$H_2+Rad(B)=B.$$

So

$$H_1\oplus H_2+Rad(A+B) =A\oplus B.$$

Then

$$H+Rad(M)= M.$$

Hence $K_1\oplus K_2=K$ is small NM-submodule. of M. Thus, M is NM-hollow module.

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