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Partial Modular Space

Abdulrahman.A.Mohammed[a] , Laith.K.Shaakir[b]

[a] *Mathematics Department , College of Computer and Mathematics ,Tikrit University, Tikrit ,Iraq , Email :* *Abdulrahman.A.Mohammed35511@st.tu.edu.iq*.

[b] *Mathematics Department , College of Computer and Mathematics ,Tikrit University, Tikrit ,Iraq ,Email: dr.laithkhaleel@tu.edu.iq.*

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ABSTRACT

In this paper we investigate new definitions called Partial Modular (P.M) and Convex Partial Modular (C.P.M) which are generalized of the definitions Modular and Convex Modular respectively . We can satisfy some results and properties of a partial modular (P.M) and we deduced some result in convex partial modular (C.P.M) . Finally we get a new study of relation between (P.M) and modular.

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1. Introduction

Study of modular spaces, was initiated by Nakano [1] in connection with the theory of order spaces which was further generalized by Musielak and Orlicz [2], the fixed point theory for nonlinear mapping functional analysis of modular spaces was initiated by Khamsi [3] and He can widely applied to nonlinear integral equations and differential equations . Modular metric spaces were initiated by Chistyakov [4] , He introduced the notion of modular metric space generated by F-norm and Develop the theory of this spaces , on the same idea he was defined the notion of a modular an arbitrary set and developed the theory of metric spaces generated by modular such that called the modular metric spaces in 2010 [5] , in 2020 Meir- Keeler are introduce the notion of Partial modular metric space and established some fixed point result in this new spaces [6] . Finally Meir-Keeler also introduced the notion modular p-metric (an extended modular b-metric space) and established some fixed point results for a˄ - v –Meir-Keeler contractions in this new space, they using these results and deduced some new fixed point theorems in extended modular metric spaces endowed with a graph and in Partially ordered extended modular metric spaces in 2021 [7].

Modular can be define as[8]: let χ be a real or complex vector space, a mapping $\wp\colon \mathbb{R} \to [0,\infty]$ is called Modular if satisfy the following

1- $\wp(x) = 0$ iff $x = 0$

2- $\wp(\alpha x) = \wp(x), \forall |\alpha| = 1$ or ($\wp(-x) = \wp(x)$ if χ is Real vector space, $\wp(e^{it}x) = \wp(x)$ if χ is Complex vector space[9])

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*Corresponding author: Abdulrahman.A.Mohammed.

Email addresses: Abdulrahman.A.Mohammed35511@st.tu.edu.iq.

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3- $\wp(\alpha x + \beta y) \le \wp(x) + \wp(y)$ where $\alpha, \beta \ge 0$ & $\alpha + \beta = 1$

If replay condition (3) by $\wp(\alpha x + \beta y) \le \alpha\wp(x) + \beta\wp(y)$ where $\alpha, \beta \ge 0$ & $\alpha + \beta = 1$

Then $\wp$ is called convex modular

The vector space $\chi_\wp$ given by $\chi_\wp = \{x \in \chi : \wp(\lambda x) \to 0 as \lambda \to 0\}$ is called modular space. Generally the modular $\wp$ is not sub additive and therefore does not behave as a norm or distance.

Modular space $\chi_\wp$ can be equipped with an F-norm define by $\|x\|_\wp = \inf\{\alpha > 0 : \wp(\frac{x}{\alpha}) \le \alpha\}$ if $\wp$ is convex modular, then

$\|x\|_\wp = \inf\{\alpha > 0 : \wp(\frac{x}{\alpha}) \le 1\}$ defines a norm on the modular space $\chi_\wp$ and called the Luxemburg norm.

define the $\wp$ -ball Centered with radius r , as $B_\wp(x,r) = \{h \in \chi_\wp : \wp(h - x) \le r\}$.

In our study we investigate a new definitions, we discuss some statement by replay first condition of modular space and called it a Partial Modular (P.M) and define a new vector space, finally we can get the relation between modular and partial modular (P.M) .

Through this paper χ denoted to real or complex vector space (F = $\mathbb{R}$ *or* $\mathbb{C}$) .

Preliminaries

Define(2.1) A mapping $\wp: \chi \to [0, \infty]$ is called Partial Modular (P.M) if satisfy the following

1- $\wp(x) = \wp(0)$ iff $x = 0$

2- $\wp(\alpha x) = \wp(x), \forall |\alpha| = 1$

3- $\wp(\alpha x + \beta y) \le \wp(x) + \wp(y)$ where $\alpha, \beta \ge 0$ & $\alpha + \beta = 1$

replace (3) by by $\wp(\alpha x + \beta y) \le \alpha\wp(x) + \beta\wp(y)$ where $\alpha, \beta \ge 0$ & $\alpha + \beta = 1$

then $\wp$ is called convex partial modular (C.P.M)

Lemma(2.2) If $\wp$ is(C.P.M) then $\wp(0) \le \wp(x)$, $\forall x \in \chi$

Proof: let $x \in \chi$

$$\wp(0) = \wp\left(\frac{1}{2}x - \frac{1}{2}x\right) = \wp\left(\frac{1}{2}x + (-\frac{1}{2}x)\right) \le \frac{1}{2}\wp(x) + \frac{1}{2}\wp(-x) = \frac{1}{2}\wp(x) + \frac{1}{2}\wp(x) = \wp(x)$$

$\therefore \wp(0) \le \wp(x)$, $\forall x \in \chi$

Theorem (2.3)

if $\wp$ is (C.P.M) then $\chi_\wp = \{x \in \chi: \wp(\lambda x) \to \wp(0), as\ \lambda \to 0\}$ is subspace of χ

Proof:

suppose x_1 , $x_2 \in \chi_\wp$ to show $x_1 + x_2 \in \chi_\wp$

let $\mathcal{E} > 0$

$\because x_1 \in \chi_\wp \Longrightarrow \exists \delta_1 > 0$ such that if then $|\lambda| < \delta_1$ $|\wp(\lambda x_1) - \wp(0)| < \mathcal{E}$
also $\because x_2 \in \chi_\wp \Longrightarrow \exists \delta_2 > 0$ such that if then $|\lambda| < \delta_2$ $|\wp(\lambda x_2) - \wp(0)| < \mathcal{E}$

Take $\delta = \frac{\min\{\delta_1,\delta_2\}}{2}$ then $2\delta = \min\{\delta_1,\delta_2\}$

Now if $|\lambda| < \delta \Rightarrow |2\lambda| < 2\delta = \min\{\delta_1,\delta_2\}$

$$\wp(\lambda(x_1+x_2)) = \wp(\lambda x_1 + \lambda x_2) = \wp(\frac{1}{2}2\lambda x_1 + \frac{1}{2}2\lambda x_2) \le \frac{1}{2}\wp(2\lambda x_1) + \frac{1}{2}\wp(2\lambda x_2)$$
$$= \frac{1}{2}[\wp(2\lambda x_1) - \wp(0)] + \frac{1}{2}[\wp(2\lambda x_2) - \wp(0)] + \wp(0)$$

$$< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon + \wp(0) = \varepsilon + \wp(0) \Rightarrow \wp(\lambda x_1 + \lambda x_2) < \varepsilon + \wp(0)$$
$$\Rightarrow \wp(\lambda(x_1+x_2)) - \wp(0) < \varepsilon \text{....(1)}$$

$\because \wp(0) \le \wp(\lambda(x_1+x_2))$ for all $x \in \chi \Rightarrow -\varepsilon < 0 \le \wp(\lambda(x_1+x_2)) - \wp(0)$....(2)

From (1) & (2) we have $-\varepsilon < \wp(\lambda(x_1+x_2)) - \wp(0) < \varepsilon \Rightarrow |\wp(\lambda(x_1+x_2)) - \wp(0)| < \varepsilon, where |\lambda| < \delta$

$\therefore x_1 + x_2 \in \chi_\wp$

Let $x \in \chi_\wp$ & $\alpha \in F$ To show $\alpha x \in \chi_\wp$ *i.e* $\lim_{\lambda\to 0}\wp(\lambda\alpha x) = \wp(0)$

Let $\varepsilon > 0$ $\therefore \exists \delta' > 0$ such that if $|\lambda| < \delta'$ to prove that $|\wp(\lambda\alpha x) - \wp(0)| < \varepsilon$

a) if $\alpha = 0$ take $\delta' = \varepsilon > 0 \Rightarrow |\wp(\lambda\alpha x) - \wp(0)| = |\wp(0) - \wp(0)| = 0 < \varepsilon$

b) if $\alpha \ne 0$ take $\delta' = \frac{\delta}{|\alpha|} > 0 \to if |\lambda| < \delta' = \frac{\delta}{|\alpha|} \Rightarrow |\lambda\alpha| < \delta$

$|\wp(\lambda\alpha x) - \wp(0)| < \varepsilon$

$\Rightarrow \alpha x \in \chi_\wp$

$\therefore \chi_\wp$ is subspace of χ .

If $\wp$ is (C.P.M) on χ then $\chi_\wp$ is called convex partial modular space (C.P.M.S)

Proposition(2.4)

if $\wp$ is (C.P.M) and $0 \le \lambda_1 \le \lambda_2$ then $\wp(\lambda_1 x) \le \wp(\lambda_2 x), \forall x \in \chi$

Proof:

if $\lambda_1 = \lambda_2$ it is clear the inequality is satisfy

If $\lambda_1 < \lambda_2$ then $\frac{\lambda_1}{\lambda_2} < 1$

$$\Rightarrow \wp(\lambda_1 x) = \wp(\frac{\lambda_1}{\lambda_2}\lambda_2 x) = \wp(\frac{\lambda_1}{\lambda_2}\lambda_2 x + (1-\frac{\lambda_1}{\lambda_2})0) \le \frac{\lambda_1}{\lambda_2}\wp(\lambda_2 x) + (1-\frac{\lambda_1}{\lambda_2})\wp(0)$$

$$= \frac{\lambda_1}{\lambda_2}(\wp(\lambda_2 x) - \wp(0)) + \wp(0) \le \wp(\lambda_2 x) - \wp(0) + \wp(0) = \wp(\lambda_2 x)$$

$$\therefore \wp(\lambda_1 x) \le \wp(\lambda_2 x)$$

Lemma (2.5) let $\wp$ be a (C.P.M) on χ , then

a) $\wp(\alpha x) = \wp(|\alpha| x), \forall \alpha \in F, \forall x \in \chi$

b) $\wp(\alpha x) \le \wp(x), \forall |\alpha| \le 1, \forall x \in \chi$

c) $\wp(\sum_{i=1}^{n} \alpha_i x_i) \le \sum_{i=1}^{n} \alpha_i \wp(x_i), n \ge 2, \alpha_i \ge 0 \forall i \,\&\, \sum_{i=1}^{n} \alpha_i = 1$

Proof:- a) if $\alpha = 0$ then $\wp(\alpha x) = \wp(0) = \wp(|\alpha| x)$

If $\alpha \ne 0 \Rightarrow \wp(\alpha x) = \wp(\frac{\alpha}{|\alpha|}|\alpha| x) = \wp(|\alpha| x)$

b) $\wp(\alpha x) = \wp(|\alpha| x) = \wp(|\alpha| x + (1-|\alpha|)0)$

$$\le |\alpha| \wp(x) + (1-|\alpha|)\wp(0) = |\alpha|(\wp(x) - \wp(0)) + \wp(0) \le \wp(x) - \wp(0) + \wp(0) = \wp(x)$$

$$\therefore \wp(\alpha x) \le \wp(x) \forall |\alpha| \le 1$$

c) we prove by mathematical induction

when n=2 then $\wp(\sum_{i=1}^{2} \alpha_i x_i) = \wp(\alpha_1 x_1 + \alpha_2 x_2) \le \alpha_1 \wp(x_1) + \alpha_2 \wp(x_2)$

$= \sum_{i=1}^{2} \alpha_i \wp(x_i)$ for every $\alpha_1, \alpha_2 \ge 0$ as $\alpha_1 + \alpha_2 = 1$

the statement is true when n=2

we assume the statement is true as n=k , to show n=k+1 is true

$$\wp(\sum_{i=1}^{k+1} \alpha_i x_i) = \wp(\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_k x_k + \alpha_{k+1} x_{k+1})$$

$\wp(\sum_{i=1}^{k+1} \alpha_i x_i) = \wp\left(\sum_{i=1}^{k} \alpha_i \left(\frac{\alpha_1}{\sum_{i=1}^{k} \alpha_i} x_1 + \frac{\alpha_2}{\sum_{i=1}^{k} \alpha_i} x_2 \ldots + \frac{\alpha_k}{\sum_{i=1}^{k} \alpha_i} x_k\right) + \alpha_{k+1} x_{k+1}\right)$, Since $\sum_{i=1}^{k} \alpha_i + \alpha_{k+1} = 1$ then

$$\wp\left(\sum_{i=1}^{k+1} \alpha_i x_i\right) \le \sum_{i=1}^{k} \alpha_i \, \wp\left(\frac{\alpha_1}{\sum_{i=1}^{k} \alpha_i} x_1 + \frac{\alpha_2}{\sum_{i=1}^{k} \alpha_i} x_2 \ldots + \frac{\alpha_k}{\sum_{i=1}^{k} \alpha_i} x_k\right) + \alpha_{k+1} \wp(x_{k+1})$$

$= \sum_{i=1}^{k} \alpha_i \wp\,(\propto_1 x_1 + \propto_2 x_2 + \cdots + \propto_k x_k) + \alpha_{k+1}\wp(x_{k+1})$ Where $\propto_i = \frac{\alpha_i}{\sum_{i=1}^{k} \alpha_i}$ .Since $\sum_{i=1}^{k} \propto_i = 1$

Then $\wp\left(\sum_{i=1}^{k+1} \alpha_i x_i\right) \le \sum_{i=1}^{k} \alpha_i \,(\propto_1 \wp(x_1) + \propto_2 \wp(x_2) + \cdots + \propto_k \wp(x_k)) + \alpha_{k+1}\wp(x_{k+1})$

Because $\propto_i = \frac{\alpha_i}{\sum_{i=1}^{k} \alpha_i}$.

Then $\wp\left(\sum_{i=1}^{k+1} \alpha_i x_i\right) = \sum_{i=1}^{k} \alpha_i \left(\frac{\alpha_1}{\sum_{i=1}^{k} \alpha_i}\wp(x_1) + \frac{\alpha_2}{\sum_{i=1}^{k} \alpha_i}\wp(x_2) \ldots + \frac{\alpha_k}{\sum_{i=1}^{k} \alpha_i}\wp(x_k)\right) + \alpha_{k+1}\wp(x_{k+1})$

$$= \sum_{i=1}^{k} \alpha_i \left(\frac{1}{\sum_{i=1}^{k} \alpha_i}(\alpha_1\wp(x_1) + \alpha_2\wp(x_2) + \cdots + \alpha_k\wp(x_k))\right) + \alpha_{k+1}\wp(x_{k+1})$$

$$= \alpha_1\wp(x_1) + \alpha_2\wp(x_2) + \cdots + \alpha_k\wp(x_k) + \alpha_{k+1}\wp(x_{k+1}) = \sum_{i=1}^{k+1} \alpha_i\wp(x_i)$$

then $\wp(\sum_{i=1}^{k+1} \alpha_i x_i) \le \sum_{i=1}^{k+1} \alpha_i\wp(x_i)$ is true for $n = k + 1$ & $\sum_{i=1}^{k+1} \alpha_i = 1$

$\therefore \wp(\sum_{i=1}^{n} \alpha_i x_i) \le \sum_{i=1}^{n} \alpha_i\wp(x_i), \forall i$ where $n \ge 2$ & $\sum_{i=1}^{n} \alpha_i = 1$

3-Some properties of $\|x\|_{\wp}$

Theorem (3.1)

if $\wp$ is (C.P.M) on χ and $\|\ \|_{\wp} : \chi_{\wp} \to [0,\infty]$ is define by $\|x\|_{\wp} = \inf\{\alpha > 0 : \wp(\frac{x}{\alpha}) \le \alpha\}$ then

a) $\wp(0) \le \|x\|_{\wp}\ \forall x \in \chi_{\wp}$ b) $\|x\|_{\wp}$ exist $\forall x \in \chi$ c) $\|x\|_{\wp} = \wp(0) \Leftrightarrow x = 0$

d) if $|\alpha| = 1$ then $\|\alpha x\|_{\wp} = \|x\|_{\wp}\quad \forall x \in \chi_{\wp}$

e) $\|x+y\|_{\wp} \le \|x\|_{\wp} + \|y\|_{\wp}, \forall x, y \in \chi_{\wp}$ f) if ℕ is positive integer then $\|Nx\|_{\wp} \le N\|x\|_{\wp}\ \forall x \in \chi_{\wp}$

proof:

a) $\because \wp(0) \le \wp(x), \forall x \in \chi \Rightarrow \wp(0) \le \wp(\frac{x}{\alpha}) \le \alpha, \forall \alpha \in \{\alpha > 0 : \wp(\frac{x}{\alpha}) \le \alpha\}$

$\therefore \wp(0) \le \inf\{\alpha > 0 : \wp(\frac{x}{\alpha}) \le \alpha\} \Rightarrow \wp(0) \le \|x\|_{\wp}, \forall x \in \chi_{\wp}$

b) let $\varepsilon = 1$ $\because x \in \chi_{\wp} \Rightarrow \exists \delta > 0$ such that if $|\lambda| < \delta$ then $\wp(\lambda x) < 1 + \wp(0)$

$\wp\left(\frac{x}{n+\wp(0)}\right) < 1 + \wp(0) < n + \wp(0) \Rightarrow n + \wp(0) \in \{\alpha > 0 : \wp(\frac{x}{\alpha}) \le \alpha\}$

$\because \delta > 0 \Rightarrow \exists n \in N$ such that $0 < \dfrac{1}{n+\wp(0)} < \dfrac{1}{n} < \delta$

From (a) we have $\wp(0) \le \|x\|_{\wp}\ \forall x \in \chi_{\wp} \Rightarrow -\infty \le \|x\|_{\wp} \le \infty$

$\therefore \|x\|_{\wp}$ exist

c) $\Rightarrow$) if $x=0$ then $\|x\|_{\wp}=\|0\|_{\wp}=\inf\{\alpha>0:\wp(\frac{0}{\alpha})\le\alpha\}=\wp(0)$

$\therefore \|0\|_{\wp}=\wp(0)$

$\Leftarrow$) let $\|x\|_{\wp}=\wp(0)$ to show $x=0$

$\because \|x\|_{\wp}=\wp(0)$ then from definition of $\|\chi\|_{\wp}, \exists\alpha_n>0 s.t\{\alpha_n\}\to\wp(0)\ \&\ \wp(\frac{x}{\alpha_n})\le\alpha_n$

Since $\{\alpha_n\}$ converge then $\exists k>0$ such $k\alpha_n<1,\forall n$

$\wp(k^{\cdot}x)=\wp(k^{\cdot}\alpha_n\frac{x}{\alpha_n})=\wp(k^{\cdot}\alpha_n\frac{x}{\alpha_n}+(1-k^{\cdot}\alpha_n)0)$

$\le k^{\cdot}\alpha_n\wp(\frac{x}{\alpha_n})+(1-k^{\cdot}\alpha_n)\wp(0)=k^{\cdot}\alpha_n[\wp(\frac{x}{\alpha_n})-\wp(0)]+\wp(0)$

$\le k^{\cdot}\alpha_n[\alpha_n-\wp(0)]+\wp(0)\le[\alpha_n-\wp(0)]+\wp(0)\le\varepsilon+\wp(0)$ as $\varepsilon\to 0$

$\Rightarrow \wp(0)\le\wp(kx)$

But $\wp(0)\le\wp(kx)\Rightarrow\wp(k^{\cdot}x)=\wp(0)\Rightarrow k^{\cdot}x=0$

$\therefore x=0$ since ($k^{\cdot}>0$)

d) $\|\alpha x\|_{\wp}=\inf\{\lambda>0:\wp(\frac{\alpha x}{\lambda})\le\lambda\}=\inf\{\lambda>0:\wp(\frac{|\alpha|x}{\lambda})\le\lambda\}$ (lemma 1.5 (a))

$=\inf\{\lambda>0:\wp(\frac{x}{\lambda})\le\lambda\}=\|x\|_{\wp}$ (since $|\alpha|$ =1)

$\therefore \|\alpha x\|_{\wp}=\|x\|_{\wp},\forall|\alpha|=1$

e) let $x,y\in\chi_{\wp}$ then from definition of $\|x\|_{\wp}$ $\forall\varepsilon>0,\exists\alpha>0$ s.t $\alpha<\|x\|_{\wp}+\varepsilon\ \&\ \wp(\frac{x}{\alpha})\le\alpha$

$\therefore \wp(\frac{x}{(\|x\|+\varepsilon)})=\wp(\frac{\alpha}{(\|x\|+\varepsilon)}\frac{x}{\alpha})=\wp(\frac{\alpha}{(\|x\|+\varepsilon)}\frac{x}{\alpha}+(1-\frac{\alpha}{(\|x\|+\varepsilon)})0)\le\frac{\alpha}{(\|x\|+\varepsilon)}\wp(\frac{x}{\alpha})+(1-\frac{\alpha}{(\|x\|+\varepsilon)})\wp(0)$

$=\frac{\alpha}{\|x\|+\varepsilon}[\wp(\frac{x}{\alpha})-\wp(0)]+\wp(0)\le\frac{\alpha}{\|x\|+\varepsilon}[\ \alpha-\wp(0)]+\wp(0)\le\alpha-\wp(0)+\wp(0)=\alpha<\|x\|+\varepsilon$

Similarly $\wp(\frac{y}{\|y\|+\varepsilon})<\|y\|+\varepsilon$

Let $u=\|x\|_{\wp}+\varepsilon, v=\|y\|_{\wp}+\varepsilon, \forall\varepsilon>0$

$$\wp(\frac{x+y}{u+v})=\wp(\frac{u}{u+v}\frac{x}{u}+\frac{v}{u+v}\frac{y}{v})\leq\frac{u}{u+v}\wp(\frac{x}{u})+\frac{v}{u+v}\wp(\frac{y}{v})\leq\wp(\frac{x}{u})+\wp(\frac{y}{v})\leq u+v$$

$$\therefore u+v\in\{\alpha>0:\wp(\frac{x+y}{\alpha})\leq\alpha\}\Rightarrow\|x+y\|_{\wp}\leq u+v=\|x\|_{\wp}+\|y\|_{\wp}+2\varepsilon$$

$as\varepsilon\to 0$ we have $\|x+y\|_{\wp}\leq\|x\|_{\wp}+\|y\|_{\wp}$

f) $\|Nx\|_{\wp}=\|x+x+\ldots+x\|_{\wp}$ N-times

$\leq\|x\|_{\wp}+\|x\|_{\wp}+\ldots+\|x\|_{\wp}=N\|x\|_{\wp}$ from (e)

$\therefore\|Nx\|_{\wp}\leq N\|x\|_{\wp}$

Proposition (3.2)

let $\wp$ be (C.P.M)

a) if $x_1, x_2\in\chi_{\wp}$ such that $\wp(\lambda x_1)\leq\wp(\lambda x_2)\forall\lambda\in F$ then $\|x_1\|_{\wp}\leq\|x_2\|_{\wp}$

b) if $0\leq\lambda_1\leq\lambda_2$ then $\|\lambda_1 x\|_{\wp}\leq\|\lambda_2 x\|_{\wp}$

Proof:

a) let $A=\{\alpha>0:\wp(\frac{x_2}{\alpha})\leq\alpha\}, B=\{\alpha>0:\wp(\frac{x_1}{\alpha})\leq\alpha\}$ to show $A\subseteq B$

let $\alpha\in A\Rightarrow\alpha>0\,\&\,\wp(\frac{x_2}{\alpha})\leq\alpha$

$$\because\wp(\frac{x_1}{\alpha})\leq\wp(\frac{x_2}{\alpha})\leq\alpha\Rightarrow\wp(\frac{x_1}{\alpha})\leq\alpha$$

$\therefore\alpha\in B\Rightarrow A\subseteq B$ therefore $\|x_1\|_{\wp}=\inf(B)\leq\inf(A)=\|x_2\|_{\wp}$

Thus $\|x_1\|_{\wp}\leq\|x_2\|_{\wp}$

b) To prove $\{\alpha>0:\wp(\frac{\lambda_2 x}{\alpha})\leq\alpha\}\subseteq\{\alpha>0:\wp(\frac{\lambda_1 x}{\alpha})\leq\alpha\}$

Let $\alpha \in \{\alpha > 0 : \wp(\frac{\lambda_2 x}{\alpha}) \le \alpha\} \Rightarrow \alpha > 0 \,\&\, \wp(\frac{\lambda_2 x}{\alpha}) \le 0$

since $0 \le \lambda_1 \le \lambda_2 \Rightarrow \wp(\frac{\lambda_1 x}{\alpha}) \le \wp(\frac{\lambda_2 x}{\alpha})$ (proposition 1.4)

$\wp(\frac{\lambda_1 x}{\alpha}) \le \wp(\frac{\lambda_2 x}{\alpha}) \le \alpha \Rightarrow \wp(\frac{\lambda_1 x}{\alpha}) \le \alpha$

$\alpha \in \{\alpha > 0 : \wp(\frac{\lambda_1 x}{\alpha}) \le \alpha\} \Rightarrow \{\alpha > 0 : \wp(\frac{\lambda_2 x}{\alpha}) \le \alpha\} \subseteq \{\alpha > 0 : \wp(\frac{\lambda_1 x}{\alpha}) \le \alpha\}$

$\Rightarrow \inf\{\alpha > 0 : \wp(\frac{\lambda_1 x}{\alpha}) \le \alpha\} \le \{\alpha > 0 : \wp(\frac{\lambda_2 x}{\alpha}) \le \alpha\}$

$\therefore \|\lambda_1 x\|_{\wp} \le \|\lambda_2 x\|_{\wp}$

Definition (3.3)

let $\wp$ be (C.P.M) on χ and $x_k \in \chi_{\wp}, \forall k$ then $\|x_k\|_{\wp}$ is bounded if $\exists M > 0 s.t \|x_k\|_{\wp} \le M, \forall k$

Theorem (3.4)

let $\wp$ be (C.P.M)

a) if $\{a_k\} \to 0, \|x_k\|_{\wp} \to \wp(0)$ then $\|a_k x_k\|_{\wp} \to \wp(0)$

b) if $\{a_k\} \to 0$ then $\|a_k x\|_{\wp} \to \wp(0)$

c) if $\{a_k\} \to 0 \,\&\, \|x_k\|_{\wp}$ bounded , then $\|a_k x_k\|_{\wp} \to \wp(0)$

Proof:

a) since $\{a_k\} \to 0 \Rightarrow \exists l \in N, s.t |a_k| < 1 \forall k > l$

let $\varepsilon > 0$. Since $\|x_k\|_{\wp} \to \wp(0)$ then $\exists m \in N s.t \|x_k\|_{\wp} \le \wp(0) + \varepsilon, \forall k > m$

hence $\|a_k x_k\|_{\wp} \le \|x_k\|_{\wp} \forall k > l$ (proposition 2.2 (b))

there is $\wp(0) \le \|a_k x_k\|_{\wp} \le \|x_k\|_{\wp} \le \wp(0) + \varepsilon, \forall k > \max\{l, m\}$

Thus $\|a_k x_k\|_{\wp} \to \wp(0)$

b) let $\varepsilon > 0$ since $x \in \chi_{\wp} \,\&\, \frac{a_k}{\varepsilon + \wp(0)} \to 0$ then $\wp(\frac{a_k x}{\varepsilon + \wp(0)}) \to \wp(0)$

there exist $l \in N$ such $\wp(\frac{a_k x}{\varepsilon + \wp(0)}) - \wp(0) < \varepsilon, \forall k > l$

$\therefore \wp(\frac{a_k x}{\varepsilon + \wp(0)}) < \varepsilon + \wp(0), \forall k > l \Rightarrow \|a_k x_k\|_{\wp} < \varepsilon + \wp(0), \forall k > l$

Also $\wp(0) - \varepsilon < \wp(0) \le \|a_k x\|_{\wp} \forall k > l \Rightarrow \left| \|a_k x\|_{\wp} - \wp(0) \right| < \varepsilon, \forall k > l$

$\therefore \|a_k x\|_{\wp} \to \wp(0)$

c) $\because \|x_k\|_{\wp}$ bounded $\Rightarrow \exists M > 0$ s.t $\|x_k\|_{\wp} < M, \forall k$, therefore from definition of $\|x\|_{\wp}$, there exist

$0 < \lambda_k < M$ such that $\wp(\frac{x_k}{\lambda_k}) \le \lambda_k \forall k$

Let $\varepsilon > 0$

$\wp(\frac{a_k x_k}{\wp(0) + \varepsilon}) = \wp(\frac{|a_k| x_k}{\wp(0) + \varepsilon}) = \wp(\frac{|a_k| \lambda_k}{\wp(0) + \varepsilon} \frac{x_k}{\lambda_k}) \le \wp(\frac{|a_k| M}{\wp(0) + \varepsilon} \frac{x_k}{\lambda_k})$ (proposition 1.4)

$= \wp^{\cdot}(\frac{|a_k| M}{\wp^{\cdot}(0) + \varepsilon} \frac{x_k}{\lambda_k} + (1 - \frac{|a_k| M}{\wp^{\cdot}(0) + \varepsilon})0)$

$\le \frac{|a_k| M}{\wp^{\cdot}(0) + \varepsilon} \wp^{\cdot}(\frac{x_k}{\lambda_k}) + (1 - \frac{|a_k| M}{\wp^{\cdot}(0) + \varepsilon})\wp^{\cdot}(0)$ where $(\frac{|a_k| M}{\wp^{\cdot}(0) + \varepsilon} < 1$ for sufficient large $k)$

$= \frac{|a_k| M}{\wp^{\cdot}(0) + \varepsilon}[\wp^{\cdot}(\frac{x_k}{\lambda_k}) - \wp^{\cdot}(0)] + \wp^{\cdot}(0)$

since $\wp^{\cdot}(\frac{x_k}{\lambda_k}) \le \lambda_k < M, \forall k \;\&\; a_k \to 0, \exists l \in N s.t \frac{|a_k| M}{\wp^{\cdot}(0) + \varepsilon}[\wp^{\cdot}(\frac{x_k}{\lambda_k}) - \wp^{\cdot}(0)] < \varepsilon, \forall k > l$

so that $\wp(\frac{|a_k| x_k}{\wp(0) + \varepsilon}) < \varepsilon + \wp(0) \forall k > l$

therefore $\wp^{\cdot}(\frac{|a_k| x_k}{\wp^{\cdot}(0) + \varepsilon}) < \varepsilon + \wp^{\cdot}(0) \forall k > l$ $\|a_k x_k\|_{\wp^{\cdot}} \le \varepsilon + \wp^{\cdot}(0)$ as $k > l$

$\therefore \|a_k x_k\|_{\wp^{\cdot}} \to \wp^{\cdot}(0)$

4- The Relation Between Modular& (P.M)

Theorem (4.1)

Every (C.P.M) on χ is (P.M) on χ.

Proof:

let $\wp$ be a (C.P.M) on χ then

$\wp(\alpha x + \beta y) \leq \alpha\wp(x) + \beta\wp(y)$ where $\alpha, \beta \geq 0$ & $\alpha + \beta = 1$

But $\alpha\wp(x) + \beta\wp(y) \leq \wp(x) + \wp(y)$ since $\alpha, \beta \leq 1$

$\alpha\wp(x) + \beta\wp(y) \leq \wp(x) + \wp(y)$ where $\alpha, \beta \geq 0$ & $\alpha + \beta = 1$

The first & second conditions are satisfy from define of (C.P.M)

$\therefore \wp$ is (P.M) on χ

The converse not true ((P.M) is not (C.P.M)), for example:- a mapping $\wp: \mathbb{R} \to [0, \infty]$ define by $\wp(x) = \sqrt{|x| + 1}$ is (P.M) as

1) $\wp(x) = \wp(0) \Leftrightarrow \sqrt{|x| + 1} = \sqrt{0 + 1} \Leftrightarrow |x| + 1 = 0 + 1 \Leftrightarrow |x| = 0 \Leftrightarrow x = 0$

2) $\wp(\alpha x) = \sqrt{|\alpha x| + 1} = \sqrt{|\alpha||x| + 1} = \sqrt{|x| + 1} = \wp(x), \forall |\alpha| = 1$

3) $\wp(\alpha x + \beta y) = \sqrt{|\alpha x + \beta y| + 1} \leq \sqrt{\alpha|x| + \beta|y| + (\alpha + \beta)} = \sqrt{\alpha|x| + \alpha + \beta|y| + \beta}$

$$= \sqrt{\alpha(|x| + 1) + \beta(|y| + 1)} \leq \sqrt{\alpha(|x| + 1)} + \sqrt{\beta(|y| + 1)}$$

$$= \alpha^{\frac{1}{2}}\sqrt{|x| + 1} + \beta^{\frac{1}{2}}\sqrt{|y| + 1} = \alpha^{\frac{1}{2}}\wp(x) + \beta^{\frac{1}{2}}\wp(y)$$

Where $\alpha, \beta \leq 1 \Longrightarrow \alpha^{\frac{1}{2}}\wp(x) + \beta^{\frac{1}{2}}\wp(y) \leq \wp(x) + \wp(y)$ then $\wp$ is (P.M)

to show $\wp$ not (C.P.M) take $\alpha = \frac{1}{4}, \beta = \frac{3}{4}$ & $\wp(x) = 2, \wp(y) = 1$

now $\alpha + \beta = 1$ thus $\alpha\wp(x) + \beta\wp(y) = \frac{1}{4}(2) + \frac{3}{4}(1) = \frac{5}{4}$, But

$$\alpha^{\frac{1}{2}}\wp(x) + \beta^{\frac{1}{2}}\wp(y) = \sqrt{\frac{1}{4}}(2) + \sqrt{\frac{3}{4}}(1) = \frac{1}{2}(2) + \frac{\sqrt{3}}{2} = 1 + \frac{\sqrt{3}}{2} \Longrightarrow \alpha^{\frac{1}{2}}\wp(x) + \beta^{\frac{1}{2}}\wp(y) > \alpha\wp(x) + \beta\wp(y)$$

hence $\wp$ not (C.P.M).

Theorem (4.2)

Every Convex Modular is (C.P.M)

Proof:

since $\wp$ is convex modular mapping then $\wp(x) = 0$ iff $x = 0$

So that $\wp(0) = 0$ hence $\wp(x) = \wp(0)$ iff $x = 0$

The second & third conditions of (C.P.M) are clearly satisfy from define.

$\therefore \wp$ is partial modular (C.P.M).

The converse is not true ((C.P.M) not convex modular) for Example : let $\wp: \mathbb{R} \to [0, \infty]$ be a mapping define by $\wp(x) = e^{|x|}$

1) $\wp(x) = \wp(0) \Leftrightarrow e^{|x|} = e^{|0|} \Leftrightarrow |x| = |0| \Leftrightarrow x = 0$

2) $\wp(\alpha x) = e^{|\alpha x|} = e^{|\alpha||x|} = e^{|x|} = \wp(x), \forall|\alpha| = 1$

3) $\wp(\alpha x + \beta y) = e^{|\alpha x+\beta y|} \leq e^{\alpha|x|+\beta|y|} \leq \alpha e^{|x|} + \beta e^{|y|} = \alpha\wp(x) + \beta\wp(y)$

$where\ \alpha, \beta \geq 0\ \&\ \alpha + \beta = 1$

$\therefore \wp$ is (C.P.M) .

Now $\wp$ not convex modular since $\wp(0) = e^{|0|} = 1 \neq 0$

Theorem (4.3)

Every modular is (P.M)

Proof:

since $\wp$ is modular then $\wp(x) = 0$ iff $x = 0$

So that $\wp(0) = 0$ hence $\wp(x) = \wp(0)$ iff $x = 0$

The other conditions of (P.M) is clearly satisfy from definition of modular

$\therefore \wp$ is (P.M)

The Converse is not true. as follows from above example, a mapping $\wp: \mathbb{R} \to [0, \infty]$ whose define by $\wp(x) = e^{|x|}$ is (P.M) from (4.1).

Then $\wp$ is not modular, since $\wp(0) = e^{|0|} = 1 \neq 0$.

Theorem (4.4)

Every Convex Modular is Modular

Proof :

suppose $\wp$ is convex modular from definition the first & second conditions are satisfy

Now from convex modular we have $\wp(\alpha x + \beta y) \leq \alpha\wp(x) + \beta\wp(y)$

But $\alpha\wp(x) + \beta\wp(y) \leq \wp(x) + \wp(y)$ since $\alpha, \beta \leq 1$

$\Longrightarrow \wp(\alpha x + \beta y) \leq \wp(x) + \wp(y)$ where $\alpha, \beta \geq 0\ \&\ \alpha + \beta = 1$

$\therefore \wp$ modular.

The converse is not true as follows a mapping $\wp: \mathbb{R} \to [0, \infty]$ define by $\wp(x) = \sqrt{|x|}$ is modular as

1) $\wp(x) = \wp(0) \Leftrightarrow \sqrt{|x|} = \sqrt{0} \Leftrightarrow |x| = 0 \Leftrightarrow x = 0$

2) $\wp(\alpha x) = \sqrt{|\alpha x|} = \sqrt{|\alpha||x|} = \sqrt{|x|} = \wp(x), \forall|\alpha| = 1$

3) $\wp(\alpha x + \beta y) = \sqrt{|\alpha x + \beta y|} \leq \sqrt{\alpha|x| + \beta|y\ |}$

$$\leq \sqrt{\alpha|x|} + \sqrt{\beta|y|} = \alpha^{\frac{1}{2}}\sqrt{|x|} + \beta^{\frac{1}{2}}\sqrt{|y|} = \alpha^{\frac{1}{2}}\wp(x) + \beta^{\frac{1}{2}}\wp(y)$$

$\wp$ is modular but not convex modular since $\alpha^{\frac{1}{2}}\wp(x) + \beta^{\frac{1}{2}}\wp(y) > \alpha\wp(x) + \beta\wp(y)$ (converse of theorem 4.1).

5-Counclusion

This article has presented the formulae of some result whose toke about (P.M) and (C.P.M), by addition some conditions in the article we can found some properties about (C.P.M), some theory described the relation between (P.M) and modular , we can prove the modular is (P.M) but the converse not true , and (C.P.M) is (P.M) but the converse not true also we can given an examples of the converse.

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