University Of AL-Qadisiyah

Available online at www.qu.edu.iq/journalcm

JOURNAL OF AL-QADISIYAH FOR COMPUTER SCIENCE AND MATHEMATICS

ISSN:2521-3504(online) ISSN:2074-0204(print)

New Technique for Solving Lane-Emden Equation with Vieta-Fibonacci Polynomials

Semaa Hassan Aziz

[a] *Computer Science Department, University of Technology, Baghdad, Iraq, e-mail:* Semaahassan83@gmail.com, *110131@uotechnology.edu.iq*.

ARTICLE INFO

Article history:
Received: 05 /03/2020
Rrevised form: 05 /04/2021
Accepted : 03 /05/2021
Available online: 09 /05/2021

Keywords:

Vieta-Fibonacci, convergence analysis, operation matrix of derivative, iterative technique, Lane-Emden equation

ABSTRACT

In the present work, new efficient iterative algorithm is presented for solving Lane-Emden equation with initial conditions. The solution is considered as a linear combination of Vieta-Fibonacci polynomials with unknown coefficients. First, the Vieta-Fibonacci polynomials are presented with some new important properties. A new exact formula for finding operation matrix of derivative for the Vieta-Fibonacci polynomials is constructed. Using such new properties the original Lane-Emden equation is transformed to the solution of algebraic equation with small-unknown coefficients. The obtained solution is in the form of a power series with easily computable coefficients. Comparison with some known exact and approximate solutions shows that the proposed solution is highly accurate.

MSC. 41A25; 41A35; 41A36

DOI : https://doi.org/10.29304/jqcm.2021.13.2.791

1. Introduction

The Lane-Emden equation represents the behavior of a spherical cloud of gas acting under the mutual attraction of its molecules depending on thermodynamic rules. They are considered to be as below

$$y''(t) + \frac{k}{t}y'(t) + g\big(y(t)\big) = 0 \qquad (1)$$

where g is given function of y with the initial conditions

$$y(0) = A, y'(0) = 0 \qquad (2)$$

Our aim in this work is to apply new iteration technique for finding the approximate solution to Lane-Emden equation (1-2) with the values $\alpha = 0, 1, 5$ for $g(y) = y^{\alpha}(t)$ and $k = 2$, which is a basic equation in the theory of stellar structure

$$y''(t) + (2/t)y'(t) + y^{\alpha}(t) = 0 \qquad (3)$$

∗Corresponding author: Semaa Hassan Aziz

Email addresses: *110131@uotechnology.edu.iq*.

Communicated by: ***Dr. Rana Jumaa Surayh aljanabi***

Subject to the conditions $y(0) = 1, \acute{y}(0) = 0$ (4)

The numerical and approximate solutions of the Lane-Emden equation together with initial conditions and boundary conditions are considered in many articles. Since Taylor series is useful mathematical tool to approximate solution for nonlinear equation, it was used in [1] to find an approximate solution for Lane-Emden type singular value problem with extremely simple technique. In [2, 7], the Adomian decomposition method was utilized to solve Lane-Emden equation approximately. The singular Lane-Emden equation was treated in [3] using certain analytic method and obtained an exact solution if the solution of such equation is polynomial type. The Homotopy analysis method was applied to obtain the analytic solution for the classic Lane-Emden equation which describing the thermal behavior of a spherical cloud of gas [4]. The authors in [5] used Bernstein polynomials operational matrix of derivative for solving problems arising in astrophysics. While He's homotopy perturbation method was used for treating numerically the singular initial value for treating numerically the singular initial value Lane-Emden problem.

Both the heuristic technique and Galerkin finite element method were applied in [8] and [9] respectively for solving a class of singular Lane-Emden equation. In addition, Bessel matrix representation in [10], was presented to compute a numerical solution for Lane-Emden singular delayed problem the results is a nonlinear fundamental matrix equation.

Our approach based upon the novel Vieta-Fibonacci operational matrix representation for derivative based upon the iterative method which reduces the original model problem into algebraic equations. To testify the validity and applicability of the proposed method, test examples are given. The computational results are accurate as compared with the exact solutions.

The present article is arranged as follows: the definitions of Vieta-Fibonacci polynomials is described in section 2. Some new properties are also included in section 2. Section 3 studies the convergence analysis while section 4 devotes the derivation of operation matrix of derivative for the Vieta-Fibonacci polynomials. A new iterative method is applied to singular initial value Lane-Emden-type equation is discussed in detail in section 5 based upon the obtained Vieta-Fibonacci operational matrix. Numerical results are listed in section 6. Finally, some concluding remarks are appeared in section 7.

2. Vieta-Fibonacci Polynomials

The explicit representation of the Vieta-Fibonacci polynomials $VF(t)$ of the m^{th} degree are defined on the interval $[-2, 2]$ by the following power formula

$$VF_m(t) = \sum_{i=0}^{\left\lfloor\frac{m-1}{2}\right\rfloor} (-1)^i \frac{\Gamma(m-i)}{\Gamma(i+1)F(m-2i)} (x)^{m-2i-1}, m\epsilon z^+ \quad (1)$$

Furthermore, there polynomials satisfy the orthogonally relation corresponding to the inner product

$$< VF_m(t), VF_n(t) \geq \delta_{ij}, \text{ i, j=0, ..., m} \quad (2)$$

where $\delta_{ij} = \int_{-2}^{2} w(t)VF_m(t), VF_n(t)dt = \begin{cases} 0 & m \neq n \\ 2\pi & m = n \end{cases}$

Note that the weight function $w(t) = (4 - t^2)^{1/2}$

Now, assume that the required approximate solution in terms of Vieta-Fibonacci polynomials as follows

$$y(t) = \sum_{i=0}^{\infty} f_{i+1} V\, F_{i+1}(t) \quad (3)$$

Here the function $y(t)\epsilon L_w^2[-2, 2]$ and f_{i+1} are the unknown coefficients. Truncate the series in Eq. 3 to be

$$y_m(t) = \sum_{i=0}^{m} f_{i+1} VF_{i+1}(t) \quad (4)$$

or $y_m(t) = F^T\sigma(t)$ (5)

where $F^T = [f_1, \quad f_2, \quad \dots, \quad f_{m+1}]$ and $\sigma(t) = [VF_1(t), \quad VF_2(t), \quad \dots, \quad VF_{m+1}(t)]^T$

The vector $\sigma(t)$ can be expressed as the following expression:

$$\sigma(t) = VT(t) \tag{6}$$

where $T(t) = [1 \quad t \quad t^2 \quad \dots \quad t^m]^T$, and V is $(m+1) \times (m+1)$ matrix which has the following expression

$$V = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ V_{10} & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ V_{30} & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & V_{31} & 0 & 0 & 0 & 0 & \dots & 0 \\ V_{50} & 0 & V_{42} & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ V_{m0} & V_{m1} & V_{m2} & V_{m3} & V_{m4} & \dots & \dots & \vdots \end{pmatrix}$$

The matrix component elements of V are given by

$$(V_{ij}) = \begin{cases} \cos\frac{(i-1)\pi}{2} & i = 1, 2, \dots, n+1, j = 0 \\ V_{i-1,j-1} - V_{i-2,\ j} & i - j = \text{even} \\ 1 & i = j \end{cases} \tag{7}$$

In addition, Eq.6 can be written as

$$T(t) = V^{-1}\sigma(t) \tag{8}$$

3. Convergence Analysis

The approximate function of y(t) using Vieta-Fibonacci polynomials is given below

$y_m(t) = \sum_{i=1}^{m+1} F_i vF_i(t)$

where $F_m =< y(t), vF_m(t) >$

Since every continuous function on the closed interval is a bounded function, that is there exists a constant η such that

$|y(t)| \le \eta, \forall t \in [-2, 2]$

Therefore; $|F_m| = \left|\int_{-2}^{2} y(t) vF_m(t) dt\right|$

$$|F_m| \le \int_{-2}^{2} |y(t)||vF_m(t)|dt$$

In addition, one can obtain $|F_m| \le \eta \int_{-2}^{2} |vF_m(t)|$

Using Eq. 1, yields

$$|F_m| \le \eta \int_{-2}^{2} \sum_{i=0}^{\left\lfloor\frac{m-1}{2}\right\rfloor} (-1)^i \frac{\Gamma(m-i)}{\Gamma(i+1)\Gamma(m-2i)} \left|t^{m-2i-1}\right| dt$$

The result is given by

$$|F_m| \le \eta \sum_{i=0}^{\left\lfloor \frac{m-1}{2} \right\rfloor} (-1)^i \frac{\Gamma(m-i)}{\Gamma(i+1)\Gamma(m-2i)}$$

4. Differentiation Operational Matrix $\overline{V}$

A special explicit formula for VF(t) of m^{th} degree operation matrix of derivative is defined in this section.

$$\text{Let } \frac{d}{dt}\sigma(t) = D(VT(t)) = VD[T(t)]^T \tag{9}$$

In other words,

$$\frac{d}{dt}\sigma(t) = \overline{V}T(t), \tag{10}$$

where

$$\overline{V} = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ \overline{V}_{10} & 0 & 0 & 0 & \dots & 0 \\ 0 & \overline{V}_{m21} & 0 & 0 & \dots & 0 \\ \overline{V}_{30} & 0 & \overline{V}_{32} & 0 & \dots & 0 \\ 0 & \overline{V}_{41} & 0 & \overline{V}_{43} & \dots & 0 \\ \overline{V}_{50} & 0 & \overline{V}_{52} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \overline{V}_{m0} & \overline{V}_{m1} & \overline{V}_{m2} & \overline{V}_{m3} & & \overline{V}_{mm} \end{pmatrix}$$

The elements of the Vieta-Fibonacci operational matrix of derivative $\overline{V}$ can be defined as follows

$$(\overline{V}_{ij}) = \begin{cases} \frac{i}{2}\cos\left(\frac{(i-2)\pi}{2}\right), & i = 1, 2, \dots, n, j = 0 \\ \overline{V}_{i-1,\ j-1} - \overline{V}_{i-2,\ j} - (j+1), & i - j = \text{even} \\ 1 & i = j \end{cases}$$

5. Iterative Solution Method for the Model (3-4)

In this section, Vieta-Fibonacci iterative method will be proposed to obtain approximate solutions of the following model

$$\acute{\acute{y}}(t) + \frac{2}{t}\acute{y}(t) + y^{\alpha}(t) = 0 \text{ With } y(0) = 1, \acute{y}(0) = 0$$

Assume that Eq. 3 has an approximate solution in the truncated Vieta-Fibonacci series form Eq. 3. In other word, Eq. 5 can be rewritten into a matrix form as

$$y(t) = \sigma(t)F, \quad F = [f_1 \quad f_2 \quad \dots \quad f_{m+1}]^T$$

The Vieta-Fibonacci polynomials $\sigma(t)$ can be represented in matrix form as follows

$$\sigma^T(t) = MT^T(t) \text{ or } \sigma^T(t) = T(t)\, M^T, \text{therefore; } y(t) = T(t)\, M^T F$$

Now, the matrix representation of $\acute{y}(t)$ will be taken the following form

$$\acute{y}(t) = \acute{T}(t)\, M^T F \text{ where } \acute{T}(t) = [0 \quad 1 \quad 2t \quad \dots \quad nt^{m-1}]$$

or $\acute{T}(t) = T(t)B^T$ where

$$B^T = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & N \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

In general, $y^n(t) = T^{(n)}(t)M^TF$

A summary of the suggested Vieta-Fibonacci iterative algorithm can be given by the following steps;

Step 1: Approximate the unknown function y(t) in Eq. 3

$$y_1(t) = F^T\sigma(t) = F^TVT(t) \tag{11}$$

where $F^T = [f_1 \quad f_2 \quad f_3]^T, \sigma(t) = [VF_1(t) \quad VF_2(t) \quad VF_3(t)]^T, T(t) = [1 \quad t \quad t^2]$

and $V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$

Step 2: Construct the matrix representation of $\acute{y}(t)$ and $\acute{\acute{y}}_1(t)$as below

$$\acute{y}_1(t) = F^TV_1T(t)\,, \quad \acute{\acute{y}}_1(t) = F^TV_2T(t) \tag{12}$$

where $V_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 2 & 0 \end{pmatrix}$

Step 3: Use the initial conditions, Eq. 4, yields

$$f_1 = 1 \text{ and } f_2 = 0 \tag{13}$$

Step 4: Substitute relation (13) into Eq. 11 to obtain

$$y_1 = (1 + f_3) + f_3(t^2 - 1), \acute{y}_1(t) = 2f_3t, \ \acute{\acute{y}}_1(t) = 2f_3 \tag{14}$$

Then Put relation 14 into Eq. 3 and get f_3 and the first approximate solution can be defined.

Step 5: Continue the procedure, the following approximate solution in the $(n+1)^{th}$ step is introduced

$$y_{n+1}(t) = y_n(t) + F^TVT(t) \tag{15}$$

where $F^T = [f_n \quad f_{n+1} \quad f_{n+2}]$

6. Numerical Results

It was shown in physics that important values of α are in the interval $0 \le \alpha \le S$ and the exact solution has been known only when $\alpha = 0, 1, 5$. In this section, the problem in Eqns. 3-4 is solved for $\alpha = 1$ using the proposed iterative method based on Vieta-Fibonacci polynomials. To solve Eqns. 3-4 by means of the new iterative method based on Vieta-Fibonacci polynomials, the following approximations are constructed
The following first initial approximation is considered
$y_1(t) = f_1VF_1(t) + f_2VF_2(t) + f_3VF_3(t)$

The next approximation is obtained from (15) is given by

$y_2 = y_1(t) + f_2 VF_2(t) + f_3 VF_3(t) + f_4 VF_4(t)$

The rest of iterates can be obtained from Eq. 15.

If the index $m = 0$, the exact solution is $y(t) = 1 - \frac{t^2}{6}$

To solve Eqns. 3-4 with index m=0 by means of Vieta-Fibonacci polynomials, the following first approximation is constructed

$y_1 = (1 + f_3) + f_3(t^2 - 1),\ \acute{y}_1(t) = 2f_3 t,\ \acute{\acute{y}}_1(t) = 2f_3$

In this case; there is only one unknown coefficient and its value is $f_3 = \frac{-1}{6}$.

If the index $m = 1$, the exact solution is $y(t) = \frac{\sin t}{t}$

Again, the next two iterates approximation is easily obtained from Eq. 15 and is given by

$$y_1(t) = 1 - \frac{t^2}{3!}$$

$$y_2(t) = 1 - \frac{t^2}{3!} + \frac{t^4}{5!}$$

Comparisons between the numerical results of the proposed method with existing exact results are carried out in order to show that the new approximation algorithm provides accurate solutions.

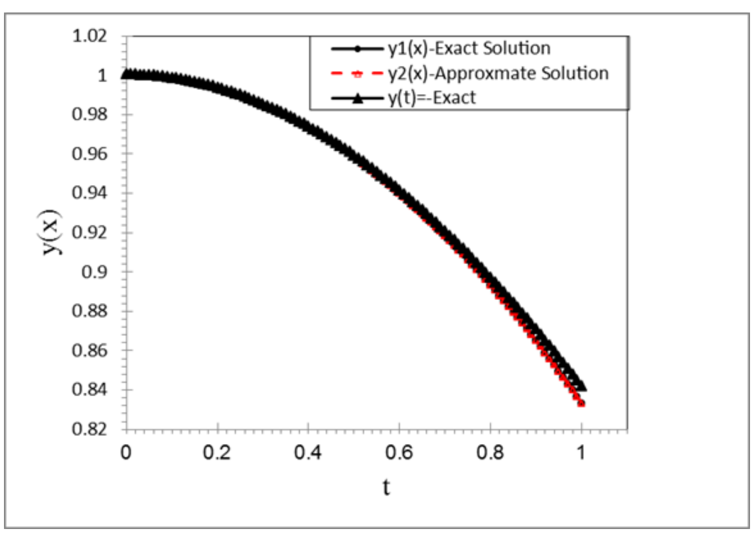

Fig. 1 – Plot of the approximate solution y(x) against the exact with m=0.

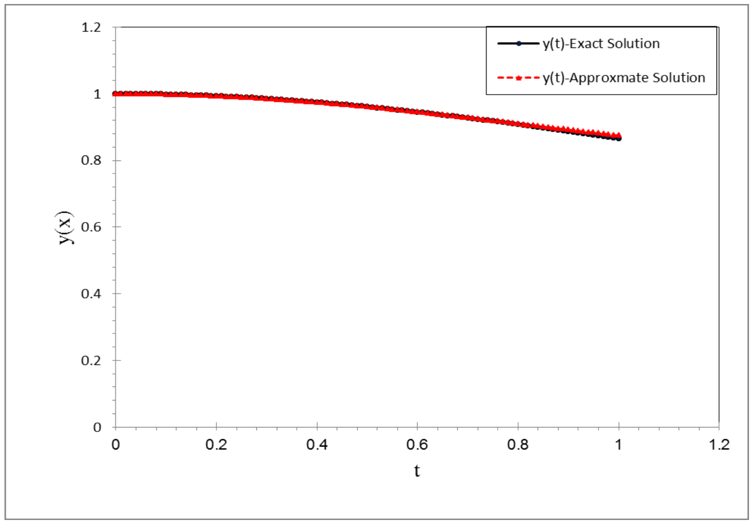

Fig. 2 – Plot of the approximate solution y(x) against the exact with m=1.

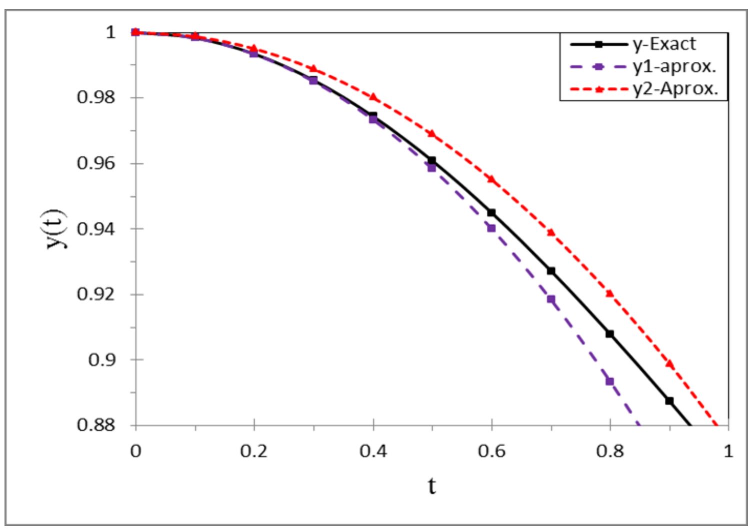

Fig. 3 – Plot of the approximate solution y(x) against the exact with m=5.

The proposed iterative algorithm is shown to be highly accurate as well as only a few terms of Vieta-Fibonacci polynomials are required to obtain accurate computable solutions.

7. Conclusion

In this paper, our aim was to solve the Lane-Emden type equation numerically using Vieta-Fabonacci iterative method, which is a very fast convergent. It was solved for the values of the polytrophic index $m = 0, 1, 5$.

A new iterative method was applied to singular initial-value Lane-Emden-type problems, and the effectiveness and performance of the method is studied. The proposed method depends on Vieta-Fibonacci polynomials, and when the solution is polynomial, our suggested technique produces the exact solution. It can observe that the method is easy to implement, yields excellent results at a minimum computational cost, and requires less time. Computational results of several test problems are presented to demonstrate the viability and practical usefulness of the method. The results reveal that the method is very effective, straightforward and simple.

References

[1] He J H, Ji F Y, "Taylor series solution for Lane–Emden equation", Journal of Mathematical Chemistry, 57(8), (2019), pp.1932-1934.
[2] Shawagfeh N T, "Nonperturbative approximate solution for Lane–Emden equation", Journal of Mathematical Physics, 34(9), (1993), pp. 4364-9.
[3] Abu Arqub O, El-Ajou A, Bataineh A S, Hashim I, "A representation of the exact solution of generalized Lane-Emden equations using a new analytical method", InAbstract and Applied Analysis, 1 (2013).
[4] Van Gorder RA, Vajravelu K, "Analytic and numerical solutions to the Lane–Emden equation", Physics Letters A, 372(39), (2008), pp. 6060-6065.
[5] Pandey R K, Kumar N, "Solution of Lane–Emden type equations using Bernstein operational matrix of differentiation", New Astronomy, 17(3) (2012), pp. 303-308.
[6] Yıldırım A, Öziş T, "Solutions of singular IVPs of Lane–Emden type by homotopy perturbation method", Physics Letters A, 369(1-2) (2007), pp. 70-76.
[7] Wazwaz A M, "A new algorithm for solving differential equations of Lane–Emden type", Applied Mathematics and Computation, 118(2-3), (2001), pp. 287-310.
[8] Sabir Z, Guirao J L, Saeed T, "Solving a novel designed second order nonlinear Lane–Emden delay differential model using the heuristic techniques", Applied Soft Computing, 1(102) (2021), pp. 107105.
[9] Izadi M, "A discontinuous finite element approximation to singular Lane-Emden type equations", Applied Mathematics and Computation. 15(401) (2021), pp. 126115.
[10] Izadi M, Srivastava H M, "An efficient approximation technique applied to a non-linear Lane–Emden pantograph delay differential model", Applied Mathematics and Computation, 15(401) (2021), pp. 126123.