Outcomes for the skew-partition $(6, 6, 3) / (1, 1, 0)$

Alaa Abbas Mansour\textsuperscript{a}, Haytham Razooki Hassan\textsuperscript{b}

\textsuperscript{a} The Ministry of Education, Directorate General of Education in Wasit, Wasit, Iraq, sosoalamy@gmail.com

\textsuperscript{b} Department of Mathematics, College of Science, Mustansiriyah University, Baghdad, Iraq, haythamhassaan@uomustansiriyah.edu.iq:

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ABSTRACT

The aim of this work is to find the complex of characteristic zero (c.c.z.) in the case of skew-partition $(6, 6, 3) / (1, 1, 0)$ as a diagram by utilizing the concepts power of place polarization and CaPELLi identities.

1. Introduction

Let $R$ be a commutative ring with identity, $F$ be a free $R$-module and $D_bF$ is the divided power of degree $b$. The (c.c.z.) with in the below cases, $(2,2,2)$, $(3,3,3)$ and $(4,4,3)$ are studied by the authors in [1], [2] and [3], once the author in [4] display the (c.c.z.) in case of partitions $(8,7,3)$. Other articles [5, 6] found the resolution of Weyl module for (c.c.z.) in the case of the partition $(8, 7, 3)$ by using the mapping Cone as in [7], the partition $(7,7,3)$, $(6,5,3)/(t,0,0)$; where $t=1,2$ and the skew shape $(9,7,3)/(1,0)$ studied in [8-10] respectively.

In this work we find the complex of skew-partition $(6, 6, 3)/(1, 1, 0)$ as a diagram after we illustrate the terms of that complex. The map $\partial_{ij}^{(f)}$ means the divided power of the place polarization $\partial_{ij}$ where $j$ must be less than $i$, with its CaPELLi identities [8]. So we need the identities bellow

\[
\partial_{21}^{(u)} \circ \partial_{21}^{(v)} = \sum_{e \geq 0} (-1)^e \partial_{32}^{(v-e)} \circ \partial_{21}^{(u-e)} \circ \partial_{31}^{(e)}
\]

\[
\partial_{32}^{(v)} \circ \partial_{21}^{(u)} = \sum_{e \geq 0} \partial_{21}^{(u-e)} \circ \partial_{32}^{(v-e)} \circ \partial_{31}^{(e)}
\]

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2. The skew-partition (6,6,3) / (1,1,0)

To find the terms of (c.c.z.) we used the following:

\[ \langle (p + |t| + 2)(q - t_1 - 1)(r - t_2 - 1) \rangle \rightarrow \langle (p)(q + t_2 + 1)(r - t_2 - 1) \rangle \]

\[ \rightarrow \langle (p + t_1 + 1)(q - t_2 + 1)(r - |t| - 2) \rangle \rightarrow \langle (p + t_1 + 1)(q + t_1 - 1)(r) \rangle \]

Where \(|t| = t_1 + t_2\).

For our case (5,5,3;1,0) we have

\[ D_8 \otimes D_5 \otimes D_0 \rightarrow D_8 \otimes D_4 \otimes D_1 \]

\[ D_6 \otimes D_5 \otimes D_1 \]

Consider the following diagram:

\[ D_8 F \otimes D_5 F \otimes D_0 F \xrightarrow{R_1} D_8 F \otimes D_4 F \otimes D_1 F \xrightarrow{R_2} D_6 F \otimes D_4 F \otimes D_3 F \]

Where

- \( R_1(v) : D_8 F \otimes D_5 F \otimes D_0 F \rightarrow D_8 F \otimes D_4 F \otimes D_1 F \), such that
  \[ R_1(v) = \partial_{21} (v) \]
- \( w_1(v) : D_8 F \otimes D_5 F \otimes D_0 F \rightarrow D_6 F \otimes D_7 F \otimes D_0 F \), such that
  \[ w_1(v) = \partial_{21} (v) \]
- \( w_2(v) : D_8 F \otimes D_5 F \otimes D_0 F \rightarrow D_6 F \otimes D_7 F \otimes D_0 F \), such that
  \[ w_2(v) = \partial_{21} (v) \]
- \( S_1(v) : D_5 F \otimes D_7 F \otimes D_0 F \rightarrow D_5 F \otimes D_7 F \otimes D_3 F \), such that
  \[ S_1(v) = \frac{1}{3} \partial_{32} \circ \partial_{21} - \partial_{31} \]
- \( S_2(v) : D_5 F \otimes D_7 F \otimes D_1 F \rightarrow D_5 F \otimes D_5 F \otimes D_3 F \), such that
  \[ S_2(v) = \frac{1}{3} \partial_{32} (v) \]
- \( R_2(v) : D_8 F \otimes D_4 F \otimes D_1 F \rightarrow D_6 F \otimes D_4 F \otimes D_3 F \), such that
  \[ R_2(v) = \frac{1}{3} \partial_{21} (v) \circ \partial_{32} (v) + \frac{1}{2} \partial_{21} \circ \partial_{32} \circ \partial_{31} + \partial_{31} \]
- \( w_3(v) : D_6 F \otimes D_4 F \otimes D_3 F \rightarrow D_5 F \otimes D_5 F \otimes D_3 F \), such that
  \[ w_3(v) = \partial_{21} (v) \]

**Proposition (2.1):** The diagram \( P \) is commutative.

**Proof:** we should proof that \((w_2 \circ R_1)(v) = (S_1 \circ w_1)(v)\)

\[ (w_2 \circ R_1)(v) = \partial_{21} (v) \circ \partial_{32} (v), \text{ and} \]
\[ (S_1 \circ w_1)(v) = \left( \frac{1}{3} \partial_{32} \circ \partial_{21} \circ \partial_{31} \right) \circ \partial_{2}^{(2)} \]
\[ = \left( \frac{1}{3} \partial_{32} \circ \partial_{21} \circ \partial_{2}^{(2)} \right) \circ \partial_{31} \circ \partial_{32} \]
\[ = \partial_{32} \circ \partial_{21}^{(3)} - \partial_{21}^{(2)} \circ \partial_{31} \]
\[ = \partial_{21}^{(3)} \circ \partial_{32} \]
\[ = (w_2 \circ R_1)(v) \]

**Proposition (2.2):** The diagram \( Q \) is commutative.

**Proof:** we should proof that \((w_3 \circ R_2)(v) = (S_2 \circ w_2)(v)\)

\[ (w_3 \circ R_2)(v) = \partial_{21} \left( \frac{1}{3} \partial_{21}^{(2)} \circ \partial_{32}^{(2)} + \frac{1}{2} \partial_{21} \circ \partial_{32} \circ \partial_{31} + \partial_{31}^{(2)} \right) \]
\[ = \partial_{32}^{(3)} \circ \partial_{21}^{(2)} + \frac{1}{2} \partial_{21} \circ \partial_{32} \circ \partial_{31} + \partial_{21} \circ \partial_{31}^{(2)} \]

\[ (S_2 \circ w_2)(v) = \partial_{21}^{(3)} \circ \partial_{21}^{(2)} \]
\[ = \partial_{21}^{(3)} \circ \partial_{32}^{(2)} + \frac{1}{2} \partial_{21} \circ \partial_{32} \circ \partial_{31} + \partial_{21} \circ \partial_{31}^{(2)} \]
\[ = (w_3 \circ R_2)(v) \]

Now, consider the following diagram

\[ D_6 F \otimes D_4 F \otimes D_0 F \xrightarrow{R_1} D_6 F \otimes D_4 F \otimes D_3 F \xrightarrow{R_2} D_6 F \otimes D_4 F \otimes D_3 F \]

Define \( N(v) : D_6 F \otimes D_4 F \otimes D_0 F \rightarrow D_6 F \otimes D_4 F \otimes D_3 F \), such that
\[ N(v) = \partial_{32}^{(3)}(v) ; \ v \in D_6 F \otimes D_4 F \otimes D_0 F \]

**Proposition (2.3):** The diagram \( D \) is commutative.

**Proof:** we should proof that \((R_2 \circ R_1)(v) = (N \circ w_1)(v)\)

Now, \((R_2 \circ R_1)(v) = \partial_{21}^{(2)} \circ \partial_{32}^{(2)} + \frac{1}{2} \partial_{21} \circ \partial_{32} \circ \partial_{31} + \partial_{31}^{(2)} \circ \partial_{32} \]
\[ = \partial_{21}^{(2)} \circ \partial_{32}^{(3)} + \partial_{21} \circ \partial_{32} \circ \partial_{31} + \partial_{21} \circ \partial_{31}^{(2)} \circ \partial_{32} \]

And, \((N \circ w_1)(v) = \partial_{32}^{(3)} \circ \partial_{21}^{(2)} \)
\[ = \partial_{21}^{(2)} \circ \partial_{32}^{(3)} + \partial_{21} \circ \partial_{32} \circ \partial_{31} + \partial_{21} \circ \partial_{31}^{(2)} \circ \partial_{32} \]

**Proposition (2.4):** The diagram \( F \) is commutative.

**Proof:** we should proof that \((w_2 \circ N)(v) = (S_2 \circ S_1)(v)\)

Now, \((w_2 \circ N)(v) = \partial_{21} \circ \partial_{32}^{(3)} \)
\[ = \partial_{32}^{(3)} \circ \partial_{21} - \partial_{32}^{(2)} \circ \partial_{31} \]
\[ = \partial_{32}^{(3)} \circ \partial_{21} - \partial_{31} \circ \partial_{32}^{(2)} \]

And \((S_2 \circ S_1)(v) = \left( \frac{1}{3} \partial_{32} \circ \partial_{21} - \frac{1}{2} \partial_{31} \right) \circ \partial_{32}^{(2)} \)
\[ = \partial_{32}^{(3)} \circ \partial_{21} - \partial_{31} \circ \partial_{32}^{(2)} \]
Lastly, we define the maps $\partial_1$, $\partial_2$ and $\partial_3$ where

$$D_8 \otimes D_4 \otimes D_1$$

$\partial_3: D_8 \otimes D_5 \otimes D_0 \to D_6 \otimes D_7 \otimes D_0$

$\partial_3(\chi) = (R_1(\chi), w_1(\chi))$; where $\chi \in D_8 F \otimes D_5 F \otimes D_0 F$

$D_8 \otimes D_4 \otimes D_1 \quad D_7 \otimes D_5 \otimes D_1$

$\partial_2: \quad \oplus \quad \to \quad \oplus$

$D_6 \otimes D_7 \otimes D_0 \quad D_6 \otimes D_4 \otimes D_3$

$\sigma_2((\chi_1, \chi_2)) = (R_2(\chi_1) - N(\chi_2), S_1(\chi_2) - w_2(\chi_1))$; where $\chi \in D_8 F \otimes D_4 F \otimes D_1 F$, $D_6 F \otimes D_7 F \otimes D_0 F$

$D_7 \otimes D_5 \otimes D_1$

$\partial_1: \quad \oplus \quad \to \quad D_8 \otimes D_5 \otimes D_3$

$D_6 \otimes D_4 \otimes D_3$

; where $\chi \in D_5 F \otimes D_7 F \otimes D_1 F$, $D_6 F \otimes D_4 F \otimes D_3 F$

**Proposition (2.5):**

$$0 \quad \to \quad D_8 \otimes D_6 \otimes D_0 \quad \to \quad D_6 \otimes D_4 \otimes D_1 \quad \to \quad D_7 \otimes D_5 \otimes D_2 \quad \to \quad D_8 \otimes D_4 \otimes D_3$$

Is complex.

**Proof:** from definition of place polarization, we have $\partial_{21}$ and $\partial_{32}$ are injective [11-13], and we get $\partial_3$ is injective, now:

$$\sigma_2 \circ \sigma_3(\chi) = \sigma_2(R_1(\chi), w_1(\chi))$$

$$= \sigma_2(\partial_{32}(\chi), \partial_{21}^{(2)}(\chi))$$

$$= (R_2(\chi) - N(x_2), S_1((x_2)) - w_2((x_1))).$$

$$= (R_2(\partial_{32}(\chi)) - N(\partial_{21}^{(2)}(\chi)), S_1(\partial_{21}^{(2)}(\chi)) - w_1(\partial_{32}(\chi))).$$

So,

$$R_2(\partial_{32}(\chi)) - N(\partial_{21}^{(2)}(\chi)) = (\frac{1}{3} \partial_{21}^{(2)} \circ \partial_{32}^{(2)} + \frac{1}{2} \partial_{21} \circ \partial_{32} \circ \partial_{31} + \partial_{31}^{(2)} \circ \partial_{32} \circ \partial_{21})(x) - \partial_{32}^{(3)} \circ \partial_{21}^{(2)}(x)$$

$$= \partial_{32}^{(2)} \circ \partial_{21}^{(3)} + \partial_{21} \circ \partial_{32} \circ \partial_{31} + \partial_{31}^{(2)} \circ \partial_{32} \circ \partial_{21} - \partial_{21} \circ \partial_{32} \circ \partial_{31} + \partial_{31} \circ \partial_{32} \circ \partial_{31}(x) = 0$$

$$S_1(\partial_{21}^{(2)}(\chi)) = w_1(\partial_{32}(\chi)) = (\frac{1}{3} \partial_{32} \circ \partial_{21} \circ \partial_{31} - \partial_{21}^{(2)} \circ \partial_{21})(\chi) - \partial_{21}^{(3)} \circ \partial_{32}(x)$$

$$= \partial_{32} \circ \partial_{21}^{(3)} - \partial_{21} \circ \partial_{32}^{(2)} - \partial_{21}^{(2)} \circ \partial_{32}(x)$$

$$= \partial_{21}^{(3)} \circ \partial_{32} - \partial_{21}^{(2)} \circ \partial_{31} - \partial_{31} \circ \partial_{21}^{(2)} - \partial_{21}^{(3)} \circ \partial_{32}(x)$$

$$= 0$$

So, $\sigma_2 \circ \sigma_3(\chi) = 0$

And

$$\sigma_1 \circ \sigma_2(\chi) = (x_1, x_2) = \sigma_1(R_2(\chi_1) - N(\chi_2), S_1(\chi_2) - w_2(\chi_1))$$
\[
\begin{align*}
\sigma_1 (\frac{1}{3} \partial^{(2)}_{21} \partial^{(2)}_{32} + \frac{1}{2} \partial_{21} \circ \partial_{32} \circ \partial_{31} + \partial^{(3)}_{31}) (x_1) - \partial^{(3)}_{32} (x_2), & \quad (\frac{1}{3} \partial_{32} \circ \partial_{21} \circ \partial_{31}) (x_2) - \partial_{21} (x_1)) \\
= \partial_{21} \circ (\frac{1}{3} \partial^{(2)}_{21} \partial^{(2)}_{32} (x_1) + \frac{1}{2} \partial_{21} \circ \partial_{32} \circ \partial_{31} + \partial^{(2)}_{31}) (x_2) - \partial^{(3)}_{32} (x_2) + \partial^{(2)}_{32} \circ (\frac{1}{3} \partial_{32} \circ \partial_{21} - \\
\partial^{(1)}_{31} (x_2) + \partial^{(3)}_{21} (x_1)) \\
= (\partial^{(3)}_{21} \circ \partial^{(2)}_{32} + \partial^{(2)}_{21} \circ \partial_{32} \circ \partial_{31} + \partial^{(2)}_{31} - \partial^{(3)}_{32} \circ \partial^{(3)}_{21} (x_1) + \partial^{(3)}_{32} \circ \partial_{21} - \partial^{(2)}_{32} \circ \partial_{31} - \\
\partial^{(3)}_{21} \circ \partial^{(3)}_{32} (x_2) \\
= (\partial^{(3)}_{21} \circ \partial^{(2)}_{32} + \partial^{(2)}_{21} \circ \partial_{32} \circ \partial_{31} + \partial^{(2)}_{31} - \partial^{(3)}_{32} \circ \partial^{(2)}_{21} - \partial^{(2)}_{32} \circ \partial_{31} - \partial^{(3)}_{21} \circ \partial^{(2)}_{31} ) (x_1) + \\
\partial^{(3)}_{21} \circ \partial^{(3)}_{32} + \partial^{(2)}_{32} \circ \partial_{31} - \partial^{(2)}_{21} \circ \partial^{(3)}_{32} (x_2) = 0
\end{align*}
\]

Conclusions

By using a diagram, Capelli identities we find the terms of the complex of characteristic zero for the case (5, 5, 3;1, 0) and we conclusion that the sequence of these terms is complex.

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