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Cubic B-splines Method for Solving Singularly Perturbed Delay Partial Differential Equations

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ABSTRACT

In this paper, we use the cubic B-splines method to solve the singular perturbed delay partial differential equations where the propagation term is multiplied by a small perturbation coefficient. In general, solutions to this type of problem have a boundary layer. The accuracy of the method was tested with two numerical examples and the results were compared with exact solutions and other methods.

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1.1. Introduction

Parabolic convection-diffusion equations are singularly emerging in various sciences and engineering divisions. Fluid flows are common examples and appear in related topics, such as water quality concerns in river networks, simulation of the extraction of oil from underground wells, problems with convective heat transport typically, these forms of problems have boundary layers and their highest spatial derivative is multiplied by an arbitrarily tiny one, to the ε parameter [14].

It is hard to solve these problems numerically by using classical problems. Finite-difference /element / volume on a uniform mesh [1]. Numerical methods for parabolic partial differential equations that are singularly perturbed Partial differential equations (PDEs) were extensively studied by several authors (without the time for the delay) many scholars have thoroughly researched it. The theory and the numerical solution of uniquely disturbed DPDEs, however, are still at the primary level. We can find only a few papers in recent years concerned with the numerical solution of use singularly PDEs [2]. Problems defined by differential equations with

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large or small parameters become more complex, and thus it is normal to use them in their study methods asymptotic. The asymptotic analysis for differential operators, however, has an established theory primarily for the case of frequent disturbances, if with regard to the undisturbed operator, the disruptions bear a subordinate character. In some issues, the disturbances are very operational over very narrow regions in which the dependent variable is subjected to very rapid such narrow regions also follow the limits of the domain. of interest, because of the fact that the small parameter multiplies the highest derivative. Accordingly, these are generally referred to in fluid mechanics as boundary layers, solid mechanics as edge layers, electrical applications as skin layers, fluid, and solid mechanics as shock layers, quantum mechanics as transition points, and mathematics as Stokes lines and surfaces. The theory of fluid mechanics was diverging in two mutually exclusive directions by the end of the 19th century: theoretical hydrodynamics and hydraulics. The former developed and reached a high degree of completeness from Euler equations for in viscid flows. Unfortunately, the findings obtained by using this so-called classical science were in stark contrast to the experimental findings. The famous Paradox of d'Alembert is an illustrative case. This prompted the researchers to establish their own hydraulics analytical science, which was focused primarily on a large number of experimental results. Prandtl showed that a body's flow can be handled by splitting it into two regions: a very thin layer near the body (which he called the boundary layer) where frictional effects are prominent, and the rest of the outside area Prandtl stressed the significance of viscous flows on the basis of this theory, without delving into the statistical complexities involved. The foundation stone for modern fluid dynamics has been this boundary-layer theory [10][15].

Thus, at the third International Congress of Mathematicians in Heidelberg in 1904, singulardisturbances were born. The seven-page study by Prandtl was included in the conference proceedings[3]. The term singular disturbances, however, was first used in the work of Friedrichs and Wasow [4] , a paper that followed a popular seminar on nonlinear vibrations at New York University. The solution to problems of singular disruption usually involves layers while Prandtl introduced the terminology boundary layer at this meeting, Wasow's important work achieved much greater generality [5]. Two main approaches to addressing singular disturbance are problems numerical analysis and asymptotic analysis. Since the targets and the groups of the problem are quite different, there has not been much interaction between these strategies. The numerical analysis seeks to provide quantitative information on a specific problem, while asymptotic analysis seeks to gain insight into a family of problems' qualitative behavior, any specific family member details. Numerical approaches are designed for a wide range of issues and seek to mitigate problem solver demands. Asymptotic approaches treat relatively narrow problems and require a certain in terpretation of the behavior of the solution singular disturbances have been flourishing since the mid-1960s. The topic is now usually part of the training of graduate students in applied mathematics and in many fields of engineering. In this field, numerous good textbooks have appeared, either dealing with asymptotic approaches or numerical ones. Some of the books deal with the two. The academic papers on singularly disturbed partial differential equations are included in this survey. Most of the analysis has begun to appear in the singularly perturbed PDEs since the late 1980s . From 1980 onwards, according to their appearance in the different standard international journals/conference proceedings, we gave the survey (chronologically). Nevertheless, refs are some of the important papers (which appeared before 1980) that are either connected to singularly perturbed PDEs or singularly perturbed ODEs but acted as a basis for PDEs [6,7,8,9],[17].

The following are considered singularly perturbed delay parabolic initial-boundary-value problems [11]:

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} - \varepsilon \dfrac{\partial^2 u(x,t)}{\partial x^2} + a(x,t)u(x,t) + b(x,t)u(x,t-\delta) = f(x,t), (x,t) \in [0,1]\times[0,T] \\ u(x,t) = \psi(x,t), (x,t) \in [0,1]\times[-\delta,0] \\ u(0,t) = \varphi_T(t), t \in [0,T] \\ u(1,t) = \varphi_R(t), t \in [0,T] \end{cases} \tag{1}$$

where $0 < \varepsilon << 1$ is perturbed singular parameter, $f(x,t)$, $\psi(x,t), a(x,t), b(x,t), \varphi_T(t)$ and $\varphi_R(t)$ are assumed to be sufficiently smooth and bounded and satisfy;

$$a(x,t) \geq \alpha > 0, b(x,t) \geq \beta > 0 \ .$$

1.2. Cubic B-spline Method

B-spline functions can be represented by the sequence of their nodes either uniformly or asymmetrically. The B-spline curve is uniform if the node spacing between all nodes is equal on the real line. If the curve is uniform, then it will be the active portion of the entire foundation. Functions form the same shape over each interval. To develop an assembly method based on the cubic B-spline functions of determination of impact strength [8],[9],[13].

We define the uniform cubic B-spline for $i = 0,1,\ldots,N$,

$$B_i(x) = \frac{1}{6h^3}\begin{cases} (x - x_{i-2})^3 & if \quad x_{i-2} \le x < x_{i-1} \\ -3(x - x_{i-1})^3 + 3h(x - x_{i-1})^2 + 3h^2(x - x_{i-1}) + h^3 & if \quad x_{i-1} \le x < x_i \\ -3(x_{i+1} - x)^3 + 3h(x_{i+1} - x)^2 + 3h^2(x_{i+1} - x) + h^3 & if \quad x_i \le x < x_{i+1} \\ (x_{i+2} - x)^3 & if \quad x_{i+1} \le x < x_{i+2} \\ 0 & otherwise. \end{cases} \tag{2}$$

We consider a mesh $\Delta = \{a = x_0 < x_1 < \ldots < x_{N-1} < x_N = b\}$ as a uniform partition of the solution domain by the knots x_i with $h = x_{i+1} - x_i = \dfrac{b-a}{N}$, $i = 0,1,\ldots,N$.

Table 1: Coefficient of extended cubic B-splines and its derivatives at knots x_i

x	x_{i-1}	x_i	x_{i+1}	x_{i+2}
$B_i(x)$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{1}{6}$	0
$B_i'(x)$	$-\frac{1}{2h}$	0	$\frac{1}{2h}$	0
$B_i''(x)$	$\frac{1}{h^2}$	$-\frac{2}{h^2}$	$\frac{1}{h^2}$	0

Let $\Omega = \{B_{-1}, B_0, B_1, .., B_{N+1}\}$ and $\Phi_3(\pi) = span\Omega$. The functions Ω are linearly independent on [0, 1], thus $\Phi_3(\pi)$ is dimensional. Where $B_{-1}, B_0, B_1, .., B_{N-1}, B_N, B_{N+1}$ forms a basis over the region $a \le x \le b$. Each cubic B-spline covers four elements, so each element is covered by four cubic B-splines [16].

Now we define

$$U(x,t)=\sum_{i=-1}^{N+1} C_i(t)B_i(x)$$

$$U(x,t)=C_{-1}(t)B_{-1}(x)+C_0(t)B_0(x)+C_1(t)B_1(x)+\ldots+C_N(t)B_N(x)+C_{N+1}(t)B_{N+1}(x), \qquad (3)$$

Where C_i (t) are unknown time-dependent quantities to be determined from the boundary conditions and collocation from the differential equation

1.3. Description of the Numerical Method

Applying the Taylor series to Eq. 1 . We have

$$\frac{\partial u}{\partial t}-\varepsilon\frac{\partial^2 u}{\partial x^2}+a(x,t)u(x,t)+b(x,t)\left[u(x,t)-\delta\frac{\partial u}{\partial t}\right]=f(x,t). \qquad (4)$$

We get

$$\frac{\partial u}{\partial t}\left(1-\delta b(x,\mathrm{t})\right)=\varepsilon\frac{\partial^2 u}{\partial x^2}-u(x,t)\left(a(x,t)+b(x,t)\right)+f(x,t). \qquad (5)$$

Denote the value at the representative mesh point $p(x_j,t_n)$ by

$$U(x,t)=U_j^n$$

The forward difference approximation for

$$\frac{\partial u}{\partial t}\approx\frac{u_j^{n+1}-u_j^n}{\Delta t}. \qquad (6)$$

Substitute $p=u_j$ in Eq. 5 . We have

$$\left(1-\delta b(x_j,t)\right)\frac{p^{n+1}-p^n}{\Delta t}=\varepsilon\frac{\partial^2}{\partial x^2}p^n(x)-p^n(x)\left(a(x_j,t)+b(x_j,t)\right)+f(x_j,t). \qquad (7)$$

Substitute $\Delta t=k$, we have

$$\left(1-\delta b(x_j,t)\right)\left(p^{n+1}-p^n\right)=k\,\varepsilon\frac{\partial^2}{\partial x^2}p^n(x)-kp^n(x)\left(a(x_j,t)+b(x_j,t)\right)+kf(x_j,t), \qquad (8)$$

$$-k\,\varepsilon p_{xx}^n+\left(1-\delta b(x_j,t)\right)p^{n+1}=p^n(1-\delta b(x_j,t)-ka(x_j,t)-kb(x_j,t))+kf(x_j,t). \qquad (9)$$

Suppose

$$p^{n+1}=\sum_{i=-1}^{N+1}C_iB(x_j),$$

$$p_{xx}^n=\sum_{i=-1}^{N+1}C_iB''(x_j),$$

$$p^n=u(x,0)=\psi(x_j). \qquad j=0,\ldots,\mathrm{N}$$

We have

$$\Rightarrow -k\,\varepsilon\left[C_{j-1}B''_{j-1}(x_j)+C_jB''_j(x_j)+C_{j+1}B''_{j+1}(x_j)\right]+\left(1-\delta b(x_j,t)\right)$$
$$\left[C_{j-1}B_{j-1}(x_j)+C_jB_j(x_j)+C_{j+1}B_{j+1}(x_j)\right]$$
$$=(1-\delta b(x_j,t)-ka(x_j,t)-kb(x_j,t))\psi(x_j)+kf\,(x_j,t),\qquad \forall j=0,1,\dots,N$$

$$\Rightarrow C_{j-1}\left[-k\,\varepsilon B''_{j-1}(x_j)+\left(1-\delta b(x_j,t)\right)B_{j-1}(x_j)\right]+C_j\left[-k\,\varepsilon B''_j(x_j)+\left(1-\delta b(x_j,t)\right)B_j(x_j)\right]+$$
$$C_{j+1}\left[-k\,\varepsilon B''_{j+1}(x_j)+\left(1-\delta b(x_j,t)\right)B_{j+1}(x_j)\right]=(1-\delta b(x_j,t)-ka(x_j,t)-kb(x_j,t))\psi(x_j)$$
$$+kf\,(x_j,t).\qquad \forall j=0,1,\dots,N \tag{10}$$

By using the value in Tabel 1. We have

$$\Rightarrow C_{j-1}\left[-6k\,\varepsilon+h^2\left(1-\delta b(x_j,t)\right)\right]+C_j\left[12k\,\varepsilon+4h^2\left(1-\delta b(x_j,t)\right)\right]+$$
$$C_{j+1}\left[-6k\,\varepsilon+h^2\left(1-\delta b(x_j,t)\right)\right]=6h^2\left(1-\delta b(x_j,t)-ka(x_j,t)-kb(x_j,t)\right)\psi(x_j)+6h^2kf\,(x_j,t).$$
$$\forall j=0,1,\dots,N$$

For j=0

$$\Rightarrow C_{-1}\left(-6k\,\varepsilon+h^2\left(1-\delta b(x_0,t)\right)\right)+C_0\left(12k\,\varepsilon+4h^2\left(1-\delta b(x_0,t)\right)\right)+C_1\left(-6k\,\varepsilon+h^2\left(1-\delta b(x_0,t)\right)\right)$$
$$=6h^2\left(1-\delta b(x_0,t)-\mathrm{ka}(x_0,\mathrm{t})-\mathrm{kb}(x_0,\mathrm{t})\right)\psi(x_0)+6h^2kf\,(x_0,t).$$

The boundary condition

$$\mathrm{C}_{-1}+4\,\mathrm{C}_0+\mathrm{C}_1=6\alpha_0. \tag{11}$$

We have

$$36k\,\varepsilon C_0=36k\,\varepsilon\alpha_0-6\alpha_0h^2(1-\delta b(x_0,t))+6h^2(1-\delta b(x_0,t)-ka(x_0,t)-kb\,(x_0,t))\psi(x_0)$$
$$+6h^2k\;f\,(x_0,t),$$

For j=1

$$\Rightarrow C_0\left[-6k\,\varepsilon+h^2\left(1-\delta b(x_1,t)\right)\right]+C_1\left[12k\,\varepsilon+4h^2\left(1-\delta b(x_1,t)\right)\right]+C_2\left[-6k\,\varepsilon+h^2\left(1-\delta b(x_1,t)\right)\right]$$
$$=6h^2\left(1-\delta b(x_1,t)-ka(x_1,t)-kb(x_1,t)\right)\psi(x_1)+6h^2kf\,(x_1,t),$$

for j=2

$$\Rightarrow C_1\left[-6k\,\varepsilon+h^2\left(1-\delta b(x_2,t)\right)\right]+C_2\left[12k\,\varepsilon+4h^2\left(1-\delta b(x_2,t)\right)\right]+C_3\left[-6k\,\varepsilon+h^2\left(1-\delta b(x_2,t)\right)\right]$$
$$=6h^2\left(1-\delta b(x_2,t)-ka(x_2,t)-kb(x_2,t)\right)\psi(x_2)+6h^2kf\,(x_2,t),$$

for j=i

$$\Rightarrow C_{i-1}\left[-6k\,\varepsilon+h^2\left(1-\delta b(x_i,t)\right)\right]+C_i\left[12k\,\varepsilon+4h^2\left(1-\delta b(x_i,t)\right)\right]+C_{i+1}\left[-6k\,\varepsilon+h^2\left(1-\delta b(x_i,t)\right)\right]$$
$$=6h^2\left(1-\delta b(x_i,t)-ka(x_i,t)-kb(x_i,t)\right)\psi(x_i)+6h^2kf\,(x_i,t),$$

for j=N-1

$$\Rightarrow C_{N-2}\left[-6k\varepsilon+h^2\left(1-\delta b(x_{N-1},t)\right)\right]+C_{N-1}\left[12k\varepsilon+4h^2\left(1-\delta b(x_{N-1},t)\right)\right]+C_N\left[-6k\varepsilon+h^2\left(1-\delta b(x_{N-1},t)\right)\right]$$
$$=6h^2\left(1-\delta b(x_{N-1},t)-ka(x_{N-1},t)-kb(x_{N-1},t)\right)\psi(x_{N-1})+6h^2kf(x_{N-1},t)$$

for j=N

$$\Rightarrow C_{N-1}\left[-6k\varepsilon+h^2\left(1-\delta b(x_N,t)\right)\right]+C_N\left[12k\varepsilon+4h^2\left(1-\delta b(x_N,t)\right)\right]+C_{N+1}\left[-6k\varepsilon+h^2\left(1-\delta b(x_N,t)\right)\right]$$
$$=6h^2\left(1-\delta b(x_N,t)-ka(x_N,t)-kb(x_N,t)\right)\psi(x_N)+6h^2kf(x_N,t).$$

The boundary condition

$$\mathrm{C}_{N+1}=-4\mathrm{C}_N-\mathrm{C}_{N-1}+6\alpha_1,$$

then,

$$36k\varepsilon C_N=36k\varepsilon\alpha_1-6\alpha_1h^2(1-\delta b(x_N,t))+6h^2(1-\delta b(x_N,t)-ka(x_N,t)-kb(x_N,t))\psi(x_N)$$
$$+6h^2k\ f(x_N,t),$$

we have matrix, $AC=B$,

$$A=\begin{bmatrix} 36k\varepsilon & 0 & & \cdots & & 0 \\ \gamma & \beta & \gamma & & \cdots & 0 \\ 0 & \gamma & \beta & \gamma & \cdots & 0 \\ 0 & & & \cdots & & \\ 0 & \cdots & & \gamma & \beta & \gamma \\ 0 & & \cdots & & 0 & 36k\varepsilon \end{bmatrix}_{(N+1)\times(N+1)}$$

for j =1 (10

$$C=\begin{bmatrix} C_0 \\ C_1 \\ \vdots \\ C_{N-1} \\ C_N \end{bmatrix}_{(N+1)\times 1},\quad B=6h^2.\begin{bmatrix} Z_0 \\ g_1 \\ \vdots \\ g_{N-1} \\ Z_N \end{bmatrix}_{(N+1)\times 1},$$

where

$$\gamma=-6k\varepsilon+h^2(1-\delta b(x,t)),$$
$$\beta=12k\varepsilon+4h^2(1-\delta b(x,t)),$$
$$g_i=(1-\delta b(x_i,t)-ka(x_i,t)-kb(x_i,t))\psi_i+kf(x_i,t),\ i=0,1,2,\ldots,N-1,N$$
$$Z_0=6h^2g_0+36k\varepsilon\alpha_0-6\alpha_0h^2(1-\delta b(x_0,t)),$$
$$Z_N=6h^2g_N+36k\varepsilon\alpha_1-6\alpha_1h^2(1-\delta b(x_N,t)).$$

We can see that the system is diagonally and hence nonsingular. So we can solve the system for $C_0, C_1, \ldots, C_N$ and substitute into the boundary conditions (11) and (12) to obtain C_{-1} and C_{N+1} .

1.4. Numerical Result

We now consider tow numerical examples to illustrate the comparative performance of our method. All calculations are implemented by Maple (2018) .

Example 1: Consider the following equation [11]:

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} - \varepsilon \dfrac{\partial^2 u(x,t)}{\partial x^2} = -2e^{-1}u(x,t-1), (x,t) \in (0,1)\times(0,2) \\ \\ u(x,t) = e^{-\left(t+\frac{x}{\sqrt{\varepsilon}}\right)}, (x,t) \in [0,1]\times[-1,0] \\ u(0,t) = \mathrm{e}^{-t}, \mathrm{t} \in [0,2] \\ u(1,t) = e^{-t-\frac{1}{\sqrt{\varepsilon}}}, t \in [0,2] \end{cases}$$

The exact solution is:

$$u_T(x,t) = e^{-\left(t+\frac{x}{\sqrt{\varepsilon}}\right)}.$$

In Example 1, the accuracy of the method is measured by $L^\infty -$ norm error defined as:

$$L^\infty = \underset{i}{Max} \left| U(x_i,t) - u(x_i,t) \right|,$$

where, $U(x_i,t)$ the exact solution and $u(x_i,t)$ the numerical solution this problem. Table 2 shows the error when N= 64 and we compared it with other methods [2], [11] for different values of ε . We apply the scheme (10) to solve this problem for different values of N = 32, 64, and compare with the exact solution as shown in Fig. 1. Figs. 2 and 3 show that the numerical approximation by cubic B-spline method with exact solution.

Table 2: Comparison of the maximum absolute errors of B- spline method with the maximum absolute errors of others methods for Example 1.

ε ↓ N→	Cubic B-spline 64	Modified BICM[12] 64	CUM [1]		FPUM [1]	
			64	256	64	256
2^{-10}	1.942e-7	$3.287e-4$	4.505e-3	6.696e-4	4.505e-3	6.696e-4
2^{-12}	1.01e-11	$3.287e-4$	1.144e-2	1.161e-3	4.718e-3	8.212e-4
2^{-14}	7.245e-15	$3.287e-4$	2.642e-2	3.1e-3	4.718e-3	8.212e-4
2^{-16}	8.204e-14	$3.287e-4$	2.611e-2	1.027e-2	4.505e-3	8.212e-4
2^{-18}	1.310e-12	$3.287e-4$	1.021e-2	2.607e-2	4.505e-3	8.212e-4
2^{-20}	1.647e-14	$3.287e-4$	2.664e-3	2.64e-2	4.505e-3	8.212e-4

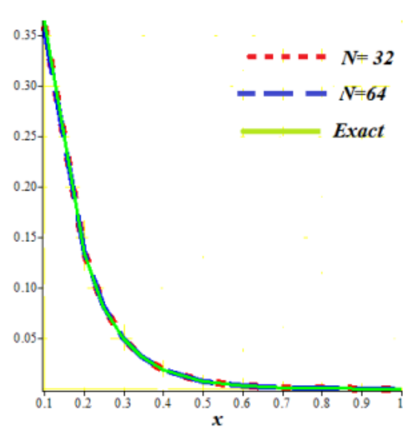

Fig. 1: Comparison between exact solution and numerical approximations at $\varepsilon = 10^{-2}$ and $t = 0.01$, $N = 32, 64$ for Example 1.

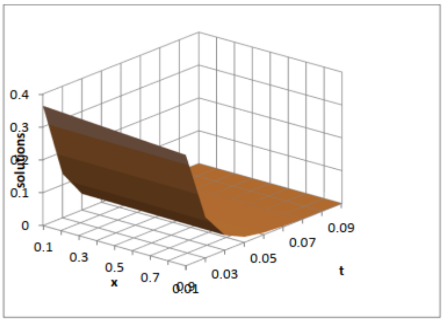

Fig.2 : Exact solution for Example 1at $\varepsilon = 10^{-2}$.

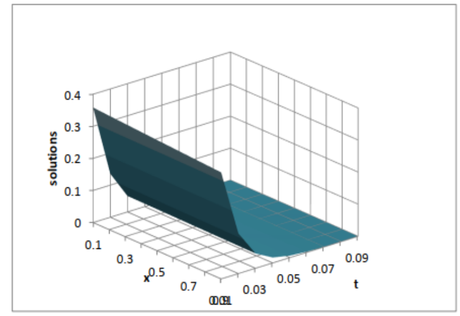

Fig.3 : Numerical solution forExample 1 at

$$\varepsilon = 10^{-2}, \Delta t = 0.01.$$

Example 2: Consider the following equation [11]

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} - \varepsilon \dfrac{\partial^2 u(x,t)}{\partial x^2} + \dfrac{(1+x^2)}{2} u(x,t) + u(x,t-\delta) = t^3, (x,t) \in D : \{0 < x < 1, 0 < t \le 2\} \\ u(x,t) = 0, (x,t) \in D_1 : \{0 \le x \le 1, -\delta < t \le 0\} \\ u(0,t) = 0 \\ u(1,t) = 0 \end{cases}$$

Since the exact solution of Example 2 is not known, to get the numerical precision of a solution and also to show the “uniform convergence of the proposed scheme,” we use a variant of the dual network principle to estimate the numerical errors and convergence rates.

We then estimate the errors for different values of ε, N and Δt by

$$E_\varepsilon^{N,\Delta t} = \left\| \tilde{U}^{N,\Delta t} - \tilde{U}^{2N,\Delta t/2} \right\|_{\bar{D}^{N,N_t}},$$

where $\tilde{U}^{N,\Delta t} = U(x,t)$ (at Eq.3), Parameter-uniformerrors are computed in the following way,

$$E^{N,\Delta t} = \max_\varepsilon E_\varepsilon^{N,\Delta t}.$$

The error value of the different values of is shown in Table 3. Table 4 shows a comparison of the maximum point errors of the B-spline method with another method [11] for different values for Example 2. The maximum point errors of the B-spline method for different values of N are shown in Fig. 4.

Table 3: The maximum point wise errors of numerical solutions for various values of and N for Example 2.

N→ δ ↓	64	128	256	512
0.02	1.2538e-13	6.6876e-13	4.10156e-12	3.317991e-10
0.04	1.2800e-13	6.8267e-13	4.18702e-12	4.428035e-10
0.06	1.3072e-13	6.9717e-13	4.27613e-12	5.878324e-10
0.08	1.3356e-13	7.1234e-13	4.36905e-12	7.777987e-10

Table 4 : Comparing the maximum point-wise errors of the B-spline method with other method for different values for Example 2.

	Cubic B-spline	A Fitted Numerov Method [11]				
ε ↓	N=64	N=64	N=128	N=256	N=512	N=1024
2^{-1}	4.96540896e-5	4.5965e-2	2.3365e-2	1.1780e-2	5.9146e-2	2.9635e-3
2^{-2}	4.43491643e-6	4.8072e-2	2.4416e-2	1.2304e-2	6.1765e-3	3.0944e-3
2^{-4}	5.37084066e-9	4.9092e-2	2.4914e-2	1.2550e-2	6.2989e-2	3.1553e-3
2^{-6}	5.03931589e-9	4.9913e-2	2.5333e-2	1.2763e-2	6.4055e-3	3.2088e-3
2^{-8}	5.00875691e-9	5.2462e-2	2.6615e-2	1.3405e2	6.7272e-3	3.3697e-3
2^{-10}	4.86267825e-9	5.3429e-2	2.7108e-2	1.3653e-2	6.8512e-3	3.4318e-3
2^{-12}	4.53412045e-9	5.3723e-2	2.7268e-2	1.3735e-2	6.8929e-2	3.4529e-3
2^{-14}	4.37377829e-9	1.5475e-2	1.6134e-1	7.8680e-2	2.5410e-2	6.6458e-3
2^{-16}	4.31989273e-9	1.7530e-1	2.6093e-1	2.6474e-1	1.6807e-1	7.8391e-1
2^{-18}	4.29992135e-9	1.7554e-2	2.6692e-2	3.0979e-1	3.0140e-1	2.6136e-1
2^{-20}	4.19992849e-9	1.7554e-1	2.6693e-1	3.1039e-1	3.1017e-1	3.0689e-1

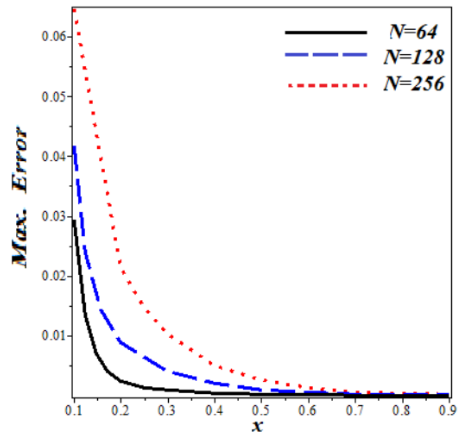

Fig. 4: The Maximum errors for the B-spline method for different values of N.

1.5. Conclusions

The cubic B-spline method is developed for the approximate solution of the singularly perturbed delay partial differential equations in this paper. Two examples are considered for numerical illustration of the method. This method is shown to be convergent methods which are better than other methods. The numerical results are presented in Tables (2-4) and compared with the exact solutions and other methods

The obtained numerical results show that the proposed method maintain a high accuracy which makes them are very encouraging for dealing with the solution of this type of singularly perturbed delay partial differential equations.

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