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Trigonometrically Fitted Runge-Kutta Methods for the Numerical Solution of the Oscillatory Problems

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ABSTRACT

In this paper, two trigonometrically methods were established based on classical Runge-Kutta methods of the fourth and fifth-order respectively. The new methods will be applied for the numerical integration of oscillatory problems and have high effectiveness as the results demonstrate. Numerical results show the robustness and competence of the new methods compared to the well-known Runge-Kutta methods in the scientific literature.

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1.1. Introduction

Initial value problems of the system of special first-order ordinary differential equations (ODEs) is of the following form:

$$u'(t) = f(t,u),\ u(t_0) = u_0. \tag{1}$$

Such problems are often observed in a wide of applied sciences, such as quantum chemistry, astronomy, quantum mechanics, electronics, elastics, and chemical physics. Traditional, Runge-Kutta (RK) methods or multi-step methods are used to solve equation (1) (see [1,2,3]). Gautschi [4] and Lyche [5] proposed a first good theoretical basis of the exponentially-fitted technique. Vanden Berghe et al. [6,7] constructed explicit exponentially fitted Runge–Kutta methods for solving first order ordinary differential equations. Paternoster [8] developed Runge-Kutta Nystrom methods by the trigonometric-fitting technique. Trigonometrically-fitted fifth-order Runge-Kutta method for the

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numerical solution of the Schrodinger equation given by Simos et al. [9]. Anastassi et. al [10] derived trigonometrically fitted Runge-Kutta methods for solving the Schrodinger equation. Trigonometrically-fitted two derivative Runge-Kutta method for the numerical solution of the Schrodinger equation presented in Zhang [11]. Fang et al. [12] proposed two derivative Runge-Kutta methods to solve oscillatory problems using the trigonometrically-fitted technique. In this paper, based on the classical fourth and fifth order Runge-Kutta methods respectively, we derive a new explicit Runge-Kutta methods. The paper is associated with the numerical outcomes that show the new methods competence by the oscillatory problems.

1.2. Construct of the proposed method

Consider the m-stage explicit modified Ruge-Kutta (MRK) method given by:

$$u_{n+1} = u_n + h\sum_{i=1}^{m} b_i f(t_n + c_i h, U_i), \tag{2}$$

$$U_i = d_i u_n + h\sum_{j=1}^{m} a_{ij} f(t_n + c_i h, U_i)\,, i = 1, \dots, m. \tag{3}$$

where $b_i, c_i, a_{ij}, i, j = 1, \dots, m$ are real number, h is the step size and the parameter $d_i, i = 1, \dots, m.$ are even function.

Table 1. s-stage modified MRK method

0						
c_2	d_2	a_{21}				
c_3	d_3	a_{31}	a_{32}			
.	.	.	.	.		
.	.	.	.	.		
.	.	.	.	.		
c_s	d_s	a_{s1}	a_{s2}	…	a_{ss-1}	
		b_1	b_2	…	b_{ss-1}	b_s

Theorem: [13]

If MRK method (2)- (3) is satisfying the following condition, then it is said to have exponential order p :

$$cos(v) + i\,sin(v) = 1 + \sum_{k=1}^{s}(iv)^k bA^{k-1}e\,, \tag{4}$$

where $v = w_i h\ for\ i = 0,1,\cdots,p$.

Definition:[13] When applied to the test equation, Runge-Kutta method is told to be trigonometrically-fitted if it completely or equivalently merges the function with the essential frequency of problem.

1.2.1. Five-Stage Fourth-Order Method

A fourth-order MRK method of five-stage is given as follows [14]:

Table 2. Five-stage fourth order MRK method

0						
$\frac{1}{5}$	d_2	$\frac{1}{5}$				
$\frac{2}{5}$	d_3	0	$\frac{2}{5}$			
$\frac{4}{5}$	d_4	$\frac{6}{5}$	$-\frac{12}{5}$	2		
1	1	$-\frac{17}{8}$	5	$-\frac{5}{2}$	$\frac{5}{8}$	
		$\frac{13}{96}$	0	$\frac{25}{48}$	$\frac{25}{96}$	$\frac{1}{12}$

Let $y_n = e^{iwx_n}$ that integrate equation (1), calculating the value of y_{n+1} , y_n' and substitution in (2)-(3) we have:

$$e^v = 1 + iv\sum_{i=1}^{m} b_i d_i - v^2\sum_{i=1}^{m} b_i \sum_{j=1}^{i-1} a_{ij}\left(Y_j e^{-iwx_n}\right) + \ iv^3 \sum_{i=1}^{m} b_i \sum_{j=1}^{i-1} a_{ij}\left(Y_j e^{-iwx_n}\right) \tag{5}$$

where $v = w_i h$. Using the formula:

$$e^v = \cos(v) + i\sin(v) \tag{6}$$

by comparison between the real and imaginary part we have:

$$\cos(v) = 1 - v^2\sum_{i=1}^{m} b_i \sum_{j=1}^{i-1} a_{ij}\left(Y_j e^{-iwx_n}\right), \tag{7}$$

$$\sin(v) = v\sum_{i=1}^{m} b_i d_i + v^3\sum_{i=1}^{m} b_i \sum_{j=1}^{i-1} a_{ij}\left(Y_j e^{-iwx_n}\right) \tag{8}$$

To derive trigonometrically-fitted MRK method, based on five-stage fourth-order MRK method as given in Table 2 and we put d_3 and a_{31} as free parameters, we obtain two equations as follows:

$$cos\ (v) = 1 + \left(\frac{1}{24} + \frac{5}{48}\ a_{31}\right) v^4 + \left(-\frac{3}{16} - \frac{25}{48}\ a_{31} - \frac{5}{16}\ d_3\right) v^2 \tag{9}$$

$$sin\ (v) = \frac{1}{120}\ v^5 - \left(\frac{1}{16} + \frac{5}{16}\ a_{31} + \frac{5}{48}\ d_3\right) v^3 - \left(-\frac{25}{48}\ d_3 - \frac{23}{48}\right) v \tag{10}$$

Now, solving equations (9) and (10), yields

$$d_3 = -\frac{1}{25\,v\,(v^4 - v^2 + 25)}\left(240\, sin(v) v^2 - 2v^7 - 5v^5 - 55v^3 + 720\, v\, cos(v) - 145v - 200\, sin(v)\right) \tag{11}$$

$$a_{31} = \frac{2}{25\,v^2 (v^4 - v^2 + 25)}\left(-360\, v\, sin(v) - 2v^6 + 25v^4 - 60v^2 + 120\, v^2 cos(v) - 600\, cos(v) + 600\right) \tag{12}$$

We obtained the corresponding Taylor series expansion

$$d_3 = 1 + \frac{16}{13125}\ v^6 + \frac{1201}{11812500}\ v^8 - \frac{98501}{2165625000}\ v^{10} - \frac{148953643}{25337812500000}\ v^{12} + \cdots,$$

$$a_{31} = \frac{1}{375}\ v^4 - \frac{43}{175000}\ v^6 - \frac{25897}{236250000}\ v^8 + \frac{4181071}{779625000000}\ v^{10} + \frac{5438986249}{1182431250000000}\ v^{12} + \cdots.$$

The new method is called TMRK4, as $v \to 0$, the original method is transcribed.

1.2.2. Six-Stage Five-Order Method

Consider the following six-stage modified RK method which can be expressed in Butcher tableau [1].

Table 3. Six-stage Five-Order MRK5 method

0							
$\frac{1}{4}$	1	$\frac{1}{4}$					
$\frac{1}{4}$	1	$\frac{1}{8}$	$\frac{1}{8}$				
$\frac{1}{2}$	d_4	0	0	$\frac{1}{2}$			
$\frac{3}{4}$	d_5	$\frac{3}{16}$	$-\frac{3}{8}$	$\frac{3}{8}$	$\frac{9}{16}$		
1	d_6	$-\frac{3}{7}$	$\frac{8}{7}$	$\frac{6}{7}$	$-\frac{12}{7}$	$\frac{8}{7}$	
		$\frac{7}{90}$	0	$\frac{16}{45}$	$\frac{2}{15}$	$\frac{16}{45}$	$\frac{7}{90}$

To construct TFRK5 method, according to the method in Table 3, we set d_3 and a_{54} as free parameters the trigonometrically conditions (7) and (8) become

$$\cos(v) = 1 + \left(-\frac{1}{120} + \frac{2}{45}\, a_{54}\, d_3 + \frac{2}{45}\, a_{54}\right) v^4 - \left(\frac{1}{30} + \frac{4}{15}\, d_3 + \frac{16}{45}\, a_{54}\right) v^2 - \frac{1}{720}\, a_{54}\, v^6, \tag{13}$$

$$\sin(v) = -\left(\frac{1}{960} - \frac{1}{60}\, a_{54}\right) v^5 + \left(-\frac{8}{45}\, a_{54}\, d_3 - \frac{1}{20} + \frac{1}{30}\, d_3 - \frac{4}{45}\, a_{54}\right) v^3 - \left(-\frac{29}{45} - \frac{16}{45}\, d_3\right) v. \tag{14}$$

Solving (13) and (14), we get

$$\begin{aligned} d_3 = \frac{1}{32(3v^2-64)\, v^2} &(-256\, v^2 RootOf(-5760 + 282\, v^4 - 3132\, v^2 + 9\, v^6 + 4320\, v\sin(v) + 540\, v^2\cos(v)\ ) \\ &+5760\cos(v) + (688\, v^2 - 720\, v\sin(v) - 2880\cos(v) + 4928 - 148\, v^4) - Z + (8\, v^4 + 64\, v^2 - 1024) - Z^2) \\ &+4096\, RootOf(-5760 + 282\, v^4 - 3132\, v^2 + 9\, v^2 + 4320\, v\sin(v) + 540\, v^2\cos(v) + 5760\cos(v) + (688\, v^2 \\ &-720\, v\sin(v) - 2880\cos(v) + 4928 - 148\, v^4) - Z + (8\, v^4 + 64\, v^2 - 1024) - Z^2) - 32\, RootOf(-5760 \\ &+282\, v^4 - 3132\, v^2 + 9\, v^2 + 4320\, v\sin(v) + 540\, v^2\cos(v) + 5760\cos(v) + (\ 688\, v^2 - 720\, v\sin(v) \\ &-2880\cos(v) + 4928 - 148\, v^4) - Z + (8\, v^4 + 64\, v^2 - 1024) - Z^2)\, v^4 + 11520\cos(v) - 11520 + 240\, v^4 \\ &+2880\, v\sin(v) + 3\, v^6 - 1472\, v^2\ ), \end{aligned} \tag{15}$$

$$\begin{aligned} a_{54} = \frac{1}{v^2}\, &(RootOf(-5760 + 282\, v^4 - 3132\, v^2 + 9\, v^2 + 4320\, v\sin(v) + 540\, v^2\cos(v) + 5760\cos(v) + (688\, v^2 \\ &-720\, v\sin(v) - 2880\cos(v) + 4928 - 148\, v^4) - Z + (8\, v^4 + 64\, v^2 - 1024) - Z^2)). \end{aligned} \tag{16}$$

Then the Taylor series expansion for d_3, and a_{54} are as follows:

$$d_3 = 1 + \frac{83}{114688} v^6 - \frac{1249}{16515072} v^8 + \frac{1045603}{29066526720} v^{10} - \frac{30950891}{3627502534656} v^{12} + \cdots,$$
$$a_{54} = \frac{9}{16} + \frac{7}{4096} v^4 - \frac{85}{458752} v^6 + \frac{36587}{660602880} v^8 - \frac{3092653}{174399160320} v^{10} + \frac{4503149029}{1015700709703680} v^{12} + \cdots.$$

1.3. Numerical Results

To evaluate the efficiency of the trigonometrically-fitted MRK methods, some oscillatory problems are tested and compared with the several well-known efficient methods. We use the criteria of absolute error to measure the accuracy of the method, which is given by

Absolute error = max $(|y(t_n) - y_n|)$. Where $y(t_n)$ is the true solution and y_n is the numerical solution. Figs. 1-5 demonstrate the competence graphs of $Log10(Max\ Error)$ versus step size h. Integration interval is [0, 1000] for all problems with step sizes $h = 0.1/2^i$, $i = 1,2,3,4$. The following acronyms are used in the comparison:

i.

- TMRK4: The five-stage fourth-order trigonometrically fitted RK method derived in this paper.
- RK4D: the classical fourth-order five-stage RK method presented in [14].
- RK4PF: Fourth order trigonometrically-fitted TRK method given in [15].
- RK5S: fifth-order six-stage RK method proposed in [16].
- RK4MS: the modified four-order RK method given in [17].
- RK4B: fourth-order five-stage RK method of given in [1].

ii.

- TFRK5: Trigonometrically-fitted six-stage fifth-order RK method derived in this paper.
- RK5F: the classical fifth-order six-stage RK method proposed in [1].
- RK5S: Optimized fifth-order RK method given in [2].
- RK5M: Optimized fifth-order RK method presented in [18].
- ORK5SS: Optimized RK method of order five given in [3].
- RK5K: Optimized RK of order five method proposed in [19].

Problem 1: [20]

$$u''(t) = 100\, u(t) + 99 \sin(t), \quad u(0) = 1, \quad u'(0) = 11.$$

Exact solution and frequency are

$$u(t) = \cos(10t) + \sin(10t) + \sin(t), \quad w = 10.$$

Problem 2: [15]

$$u''(t) = -64\, u(t), \quad u(0) = 1, \quad u'(0) = -2.$$

Exact solution and frequency are

$$u(t) = \frac{1}{4}\sin(8t) + \cos(8t), \quad w = 8.$$

Problem 3: [21]

$$u_1''(t) + u_1(t) = 0.001\ \cos(t), \quad u_1(0) = 1, \quad u_1'(0) = 0,$$
$$u_2''(t) + u_2(t) = 0.001\ \sin(t), \quad u_2(0) = 0, \quad u_2'(0) = 0.9995.$$

Exact solution and frequency are

$u_1(t) = \cos(t) + 0.0005\, t \sin(t),$
$u_2(t) = \sin(t) - 0.0005\, t \cos(t), \quad w = 1.$

Problem 4: [22]

$$u''(t) + \begin{pmatrix} \frac{101}{2} & -\frac{99}{2} \\ -\frac{99}{2} & \frac{101}{2} \end{pmatrix} u(t) = \begin{pmatrix} \frac{93}{2}\cos(2t) - \frac{99}{2}\sin(2t) \\ \frac{93}{2}\sin(2t) - \frac{99}{2}\cos(2t) \end{pmatrix},$$

$$u(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad u'(0) = \begin{pmatrix} -10 \\ 12 \end{pmatrix}.$$

Exact solution and frequency are

$$u(t) = \begin{pmatrix} -\cos(10t)(t) - \sin(10t) + \cos(2t) \\ \cos(10t) + \sin(10t) + \sin(2t) \end{pmatrix}, \quad w = 10.$$

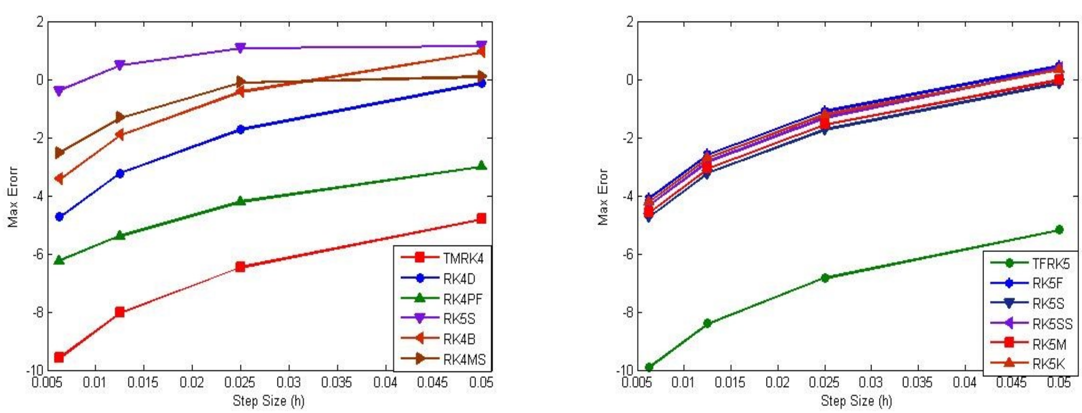
Problem 5: [23] Oscillatory system problem

$$u''(t) + \begin{pmatrix} 13 & -12 \\ -12 & 13 \end{pmatrix} u(t) = \begin{pmatrix} 9\cos(2t) - 12\sin(2t) \\ -12\cos(2t) + 9\sin(2t) \end{pmatrix},$$

$$u(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad u'(0) = \begin{pmatrix} -4 \\ 8 \end{pmatrix}.$$

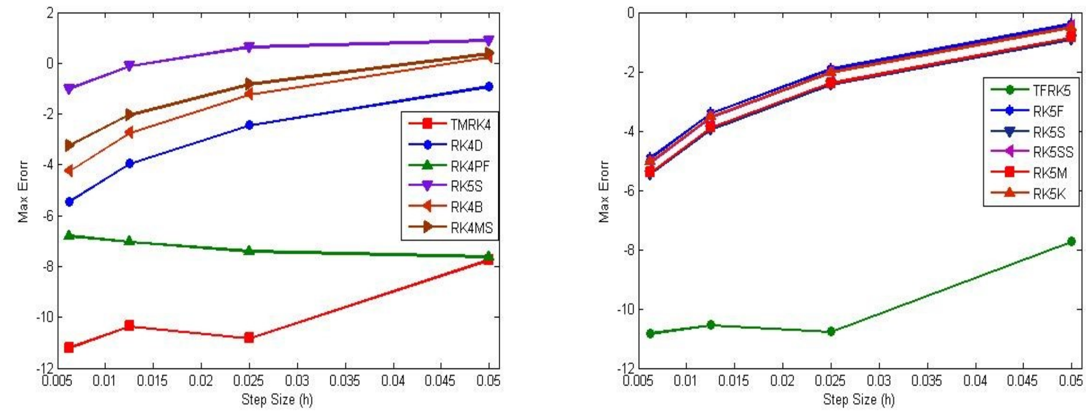
Exact solution and frequency are

$$u(t) = \begin{pmatrix} \sin(t) - \sin(5t) + \cos(2t) \\ \sin(t) + \sin(5t) + \sin(2t) \end{pmatrix}, \quad w = 5.$$



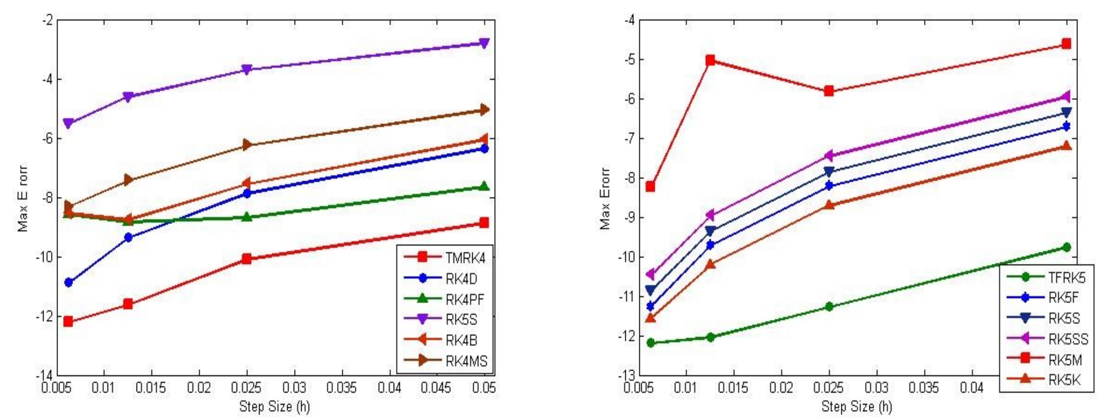
(a) **(b)**

Fig. 1: The competence curves for Problem 1. (a) Comparisons of the methods (i) in Section 3. (b) Comparisons of the methods (ii) in Section 3.



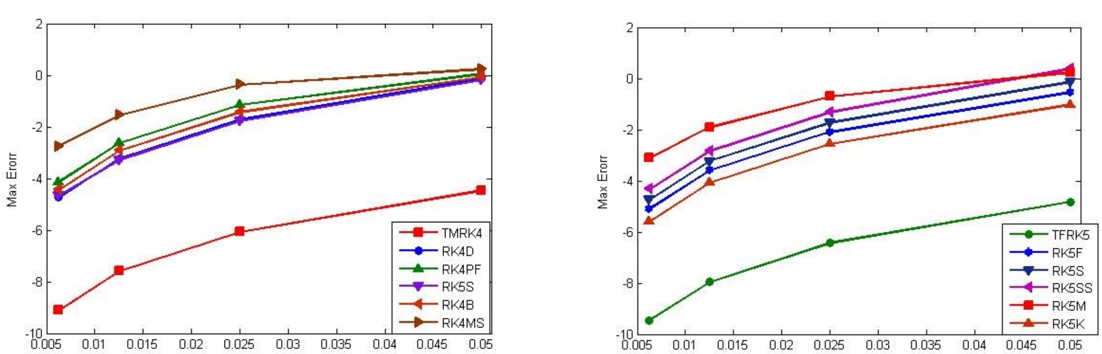
(a) **(b)**

Fig. 2: The competence curves for Problem 2. (a) Comparisons of the methods (i) in Section 3. (b) Comparisons of the methods (ii) in Section 3.



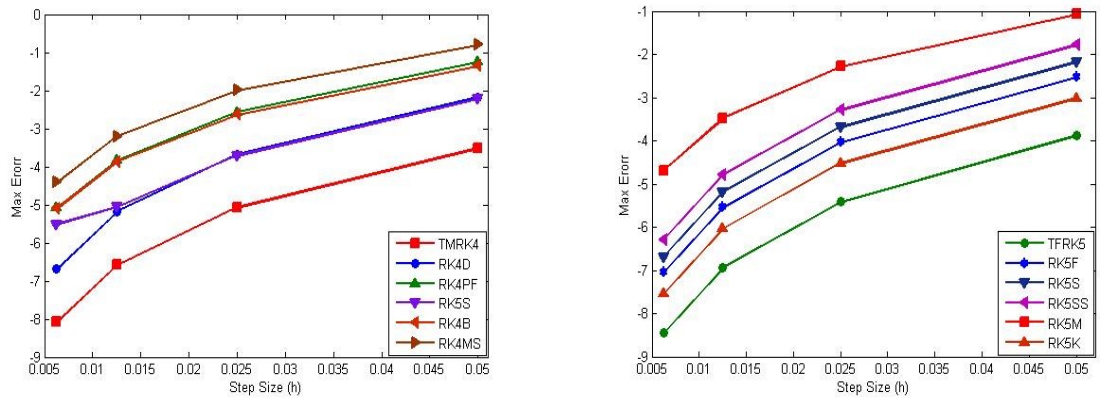
(a) **(b)**

Fig. 3: The competence curves for Problem 3. (a) Comparisons of the methods (i) in Section 3. (b) Comparisons of the methods (ii) in Section 3.



(a) **(b)**

Fig. 4: The competence curves for Problem 4. (a) Comparisons of the methods (i) in Section 3. (b) Comparisons of the methods (ii) in Section 3.



(a) **(b)**

Fig. 5: The competence curves for Problem 5. (a) Comparisons of the methods (i) in Section 3. (b) Comparisons of the methods (ii) in Section 3.

1.4. Conclusions

Trigonometrically-fitted modified Runge-Kutta (RK) methods proposed in this paper. The first one is TMRK4 method based on the fourth order MRK method of five stages and the second is TFRK5 method depend on six stages fifth order MRK5. We conclude from the numerical results that the newly trigonometrically methods are computationally more effective to solve oscillatory problems.

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