Numerical Study of the System of Nonlinear Volterra Integral Equations by Using Spline Method

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ABSTRACT

The second order non-polynomial spline function for solving system of two nonlinear Volterra integral equations is proposed in this paper. An algorithm introduced as well to numerical examples to illustrate carry out of this method. Also, we compare the absolute error of quadratic non-polynomial spline method with absolute error of linear non-polynomial spline method and the exact solution.

MSC. 41A25; 41A35; 41A36

1. Introduction

In the several last years, attention has turned to integral equations(IEs), especially the Volterra equation, began in many problem such as particle transport problems in astrophysics, So the population growth models and the mixed problems of continuous media mechanics are summarized into a system of nonlinear (IEs) of the second kind. Many researchers have solved the integral equation (IE) and some they solved especially Volterra integral equation (VIE). Abdou [1] in 2003 studied on the solution of linear and nonlinear (IE). Ayal and Muhammad [4] in 2018 used cubic spline for solving (VIE) with delay. Babolian and Masouri [5] in 2008 submit direct method to solve Volterra integral equation of the first kind (VIE) 1-kind using operational matrix with block-pulse functions. Bani issa et.al [6] in 2019 solving nonlinear (VIEs) by using numerical techniques. Biazar et.al [7] in 2003 Solution of a system of (VIEs) of the first kind by a domain method. Cahlon and Nachman [8] in 1985 introduced numerical solutions of (VIEs) with a

$$\eta(x) = g_1(x) + \int_0^x \kappa_{11}(x,t,\eta(t))dt + \int_0^x \kappa_{12}(x,t,\mu(t))dt,$$

$$\mu(x) = g_2(x) + \int_0^x \kappa_{21}(x,t,\eta(t))dt + \int_0^x \kappa_{22}(x,t,\mu(t))dt.$$  

(1)

(2)

Where \(\eta(x), \mu(x)\) are unknown functions, while that \(g_1(x), g_2(x)\) are given function and it \(\kappa_{11}(x,t,\eta(x)), \kappa_{12}(x,t,\eta(x)), \kappa_{21}(x,t,\mu(x))\) and \(\kappa_{22}(x,t,\mu(x))\) kernel integral nonlinear equations.

2. Non–polynomial spline function (NPSF)

Consider partition \(\Delta = \{x_0, x_1, x_2, \ldots, x_n\}\) of \([a,b] \subseteq R\). Let \(\psi(\Delta)\) it the set of piecewise polynomials subinterval \(I_i = [x_i, x_{i+1}]\) of partition \(\Delta\). Let \(\xi_i(x)\) be the segment of (NPSF) of order \(n\) that has the form:

$$\xi_i(x) = a_i \cos m(x-x_i) + b_i \sin m(x-x_i) + \ldots + y_i (x-x_i)^{n-1} + z_i.$$  

(3)

Where \(a_i, b_i, \ldots, y_i, z_i\) and \(m\) are constants, \(m\) is repetition of the trigonometric functions that it makes the method more accurate.

3. Quadratic non–polynomial spline (QNPS) function

(QNPS) function take the form:

$$\phi_i(x) = a_i \cos m(x-x_i) + b_i \sin m(x-x_i) + c_i(x-x_i) + d_i(x-x_i)^2 + e_i, i = 0, \ldots, n.$$  

(4)
Where $a_i, b_i, c_i, d_i$ and $e_i$ are constants. To find a solution of the equation (1) and (2) we rewrite the (QNPS)

$$\phi_i(x), i = 0, ..., n$$

as follows:

$$\phi_1(x) = a_1 \cos m(x - x_i) + b_1 \sin m(x - x_i) + c_1(x - x_i) + d_1(x - x_i)^2 + e_1$$  \hspace{1cm} (5)$$

$$\phi_2(x) = a_2 \cos m(x - x_i) + b_2 \sin m(x - x_i) + c_2(x - x_i) + d_2(x - x_i)^2 + e_2$$  \hspace{1cm} (6)$$

Differentiate equations (5) and (6) four once with respect to, therefore obtain the following equations:

$$\phi_1'(x) = -ma_1 \sin m(x - x_i) + mb_1 \cos m(x - x_i) + c_1 + 2d_1(x - x_i),$$

$$\phi_1''(x) = -m^2a_1 \cos m(x - x_i) - m^2b_1 \sin m(x - x_i) + 2d_1,$$

$$\phi_2'(x) = -ma_2 \sin m(x - x_i) + mb_2 \cos m(x - x_i) + c_2 + 2d_2(x - x_i),$$

$$\phi_2''(x) = -m^2a_2 \cos m(x - x_i) - m^2b_2 \sin m(x - x_i) + 2d_2.$$

Let $\eta(x)$ be the exact solution of equation (1) and $\mu(x)$ be the exact solution of equation (2),

$$\psi_1(x)$$

be the approximate solution to $\eta_i = \eta(x_i)$ and $\psi_2(x)$ be the approximate solution to $\mu_i = \mu(x_i)$ gotten by $\phi_1(x)$ and $\phi_2(x)$. The following relations are achieved:

$$\phi_1(x_i) = a_i + e_i = \eta(x_i) \approx \psi_1(x_i),$$

$$\phi_1'(x_i) = mb_1 + c_1 = \eta'(x_i) \approx \psi_1'(x_i),$$

$$\phi_1''(x_i) = -m^2a_1 + 2d_1 = \eta''(x_i) \approx \psi_1''(x_i),$$

$$\phi_1^{(3)}(x_i) = -m^3b_1 = \eta^{(3)}(x_i) \approx \psi_1^{(3)}(x_i),$$

$$\phi_1^{(4)}(x_i) = m^4a_1 = \eta^{(4)}(x_i) \approx \psi_1^{(4)}(x_i),$$

$$\phi_2'(x_i) = a_i + e_2 = \mu(x_i) \approx \psi_2(x_i),$$

$$\phi_2'(x_i) = mb_2 + c_2 = \mu'(x_i) \approx \psi_2'(x_i),$$

$$\phi_2''(x_i) = -m^2a_2 + 2d_2 = \mu''(x_i) \approx \psi_2''(x_i),$$

$$\phi_2^{(3)}(x_i) = -m^3b_2 = \mu^{(3)}(x_i) \approx \psi_2^{(3)}(x_i),$$

$$\phi_2^{(4)}(x_i) = m^4a_2 = \mu^{(4)}(x_i) \approx \psi_2^{(4)}(x_i).$$

Now we get the values of $a_i, b_i, c_i, d_i$ and $e_i$, as follows:

$$a_i = \frac{1}{m^4} \phi_i^{(4)}(x_i),$$

$$b_i = -\frac{1}{m^3} \phi_i^{(3)}(x_i).$$

(7)

(8)
\[ c_{li} = \phi_{li}'(x_i) + \frac{1}{m^2} \phi_{li(3)}(x_i), \quad \text{(9)} \]

\[ d_{li} = \frac{1}{2} (\phi_{li}''(x_i) + \frac{1}{m} \phi_{li(4)}(x_i)), \quad \text{(10)} \]

\[ e_{li} = \phi_{li}(x_i) - \frac{1}{m^4} \phi_{li(4)}(x_i). \quad \text{(11)} \]

And we get the values of \( a_{2i}, b_{2i}, c_{2i}, d_{2i} \) and \( e_{2i} \) as follows:

\[ a_{2i} = \frac{1}{m^4} \phi_{2i(4)}(x_i), \quad \text{(12)} \]

\[ b_{2i} = -\frac{1}{m^3} \phi_{2i(3)}(x_i), \quad \text{(13)} \]

\[ c_{2i} = \phi_{2i}'(x_i) + \frac{1}{m^2} \phi_{2i(3)}(x_i), \quad \text{(14)} \]

\[ d_{2i} = \frac{1}{2} (\phi_{2i}''(x_i) + \frac{1}{m} \phi_{2i(4)}(x_i)), \quad \text{(15)} \]

\[ e_{2i} = \phi_{2i}(x_i) - \frac{1}{m^4} \phi_{2i(4)}(x_i). \quad \text{(16)} \]

**4. Solving method of (QNPS) function of the nonlinear equations for (VIE) 2-kind**

To find the numerical approximation of equation (1), differentiate it four times with respect to \( x \), then we substitute in \( a = x' \):

\[ \eta_0 = g_1(a), \quad \text{(17)} \]

\[ \eta'_0 = g'_1(a) + \kappa_{11}(a, \alpha, \eta(a)) + \kappa_{12}(a, \alpha, \mu(a)), \quad \text{(18)} \]

\[ \eta''_0 = g''_1(a) + \frac{\partial}{\partial x} \kappa_{11}(x, t, \eta(t))\big|_{x=0} + \frac{\partial}{\partial x} \kappa_{12}(x, t, \mu(t))\big|_{x=0} \]

\[ + \frac{\partial}{\partial x} \kappa_{11}(x, x, \eta(t))\big|_{x=0} + \frac{\partial}{\partial x} \kappa_{12}(x, x, \mu(t))\big|_{x=0}, \quad \text{(19)} \]

\[ \eta^{(3)}_0 = g^{(3)}_1(a) + \frac{\partial^2}{\partial x^2} \kappa_{11}(x, t, \eta(t))\big|_{x=0} + \frac{\partial^2}{\partial x^2} \kappa_{12}(x, t, \mu(t))\big|_{x=0} \]

\[ + \frac{\partial}{\partial x} \frac{\partial}{\partial x} \kappa_{11}(x, x, \eta(t))\big|_{x=0} + \frac{\partial}{\partial x} \frac{\partial}{\partial x} \kappa_{12}(x, x, \mu(t))\big|_{x=0} \]

\[ + \frac{\partial^2}{\partial x^2} \kappa_{11}(x, x, \eta(t))\big|_{x=0}, \quad \text{(20)} \]
\[ \eta_0^{(4)} = g_{20}^{(4)}(a) + \left[ \frac{\partial^3}{\partial x^3} \kappa_{11}(x, t, \eta(t)) \right]_{x=0} + \left[ \frac{\partial^3}{\partial x^3} \kappa_{12}(x, t, \mu(t)) \right]_{x=0} + \left[ \frac{\partial^2}{\partial x^2} \kappa_{11}(x, t, \eta(t)) \right]_{x=0} + \left[ \frac{\partial^2}{\partial x^2} \kappa_{12}(x, t, \mu(t)) \right]_{x=0} \]

\[ + \left[ \frac{\partial}{\partial x} \kappa_{11}(x, t, \eta(t)) \right]_{x=0} + \left[ \frac{\partial}{\partial x} \kappa_{12}(x, t, \mu(t)) \right]_{x=0} + \left[ \frac{\partial^3}{\partial x^3} \kappa_{11}(x, t, \eta(t)) \right]_{x=0} + \left[ \frac{\partial^3}{\partial x^3} \kappa_{12}(x, t, \mu(t)) \right]_{x=0}. \]

To find the numerical approximation of equation (2), differentiate it four times with respect to \( x \), then we substitute in \( x = a \)

\[ \mu_0 = g_{20}(a) \]  

\[ \mu_0 = g_{20}'(a) + \kappa_{21}(a, a, \eta(a)) + \kappa_{22}(a, a, \mu(a)), \]

\[ \mu_0'' = g_{20}''(a) + \left[ \frac{\partial}{\partial x} \kappa_{21}(x, t, \eta(t)) \right]_{x=0} + \left[ \frac{\partial}{\partial x} \kappa_{22}(x, t, \mu(t)) \right]_{x=0} \]

\[ + \left[ \frac{\partial}{\partial x} \kappa_{21}(x, t, \eta(t)) \right]_{x=0} + \left[ \frac{\partial}{\partial x} \kappa_{22}(x, t, \mu(t)) \right]_{x=0}, \]

\[ \mu_0^{(3)} = g_{20}^{(3)}(a) + \left[ \frac{\partial^2}{\partial x^2} \kappa_{21}(x, t, \eta(t)) \right]_{x=0} + \left[ \frac{\partial^2}{\partial x^2} \kappa_{22}(x, t, \mu(t)) \right]_{x=0} \]

\[ + \left[ \frac{\partial^2}{\partial x^2} \kappa_{21}(x, t, \eta(t)) \right]_{x=0} + \left[ \frac{\partial^2}{\partial x^2} \kappa_{22}(x, t, \mu(t)) \right]_{x=0} + \left[ \frac{\partial^3}{\partial x^3} \kappa_{21}(x, t, \eta(t)) \right]_{x=0} + \left[ \frac{\partial^3}{\partial x^3} \kappa_{22}(x, t, \mu(t)) \right]_{x=0}. \]

5. Algorithm for (QNPS) method

Step 1: set

\[ h = \frac{b-a}{n}, x_i = x_o + ih, i = 0, ..., n; x_o = a, x_n = b, \]

\[ \eta_0 = g(a). \]

Step 2:

1- Evaluate \( a_{10}, b_{10}, c_{10}, d_{10} \) and \( e_{10} \) by substituting equations (17)-(21) in equations (7)-(11).
2 - Evaluate \( a_{20}, b_{20}, c_{20}, d_{20} \) and \( e_{20} \) by substituting equations (22)-(26) in equations (12)-(16).

Step 3:

1-Calculate \( \phi_{10}(x) \) by using step 2 a branch (1) and equation (5) at \( i = 0 \).

2 - Calculate \( \phi_{20}(x) \) by using step2 a branch (2) and equation (6) at \( i = 0 \).

Step 4: Approximate \( \eta_i \approx \phi_{10}(x_i), \mu_i \approx \phi_{20}(x_i) \).

Step 5: for \( i = 1 \) to \( n - 1 \) do the following steps:

Step 6:

1-Evaluate \( a_{1i}, b_{1i}, c_{1i}, d_{1i} \) and \( e_{1i} \) by using equations (7)-(11) and replacing \( \eta'(x_i), \eta''(x_i), \eta^{(3)}(x_i) \) and \( \eta^{(4)}(x_i) \) in \( \phi_{1i}'(x_i), \phi_{1i}''(x_i), \phi_{1i}^{(3)}(x_i) \) and \( \phi_{1i}^{(4)}(x_i) \).

2 - Evaluate \( a_{2i}, b_{2i}, c_{2i}, d_{2i} \) and \( e_{2i} \) by using equations (12)-(16) and replacing \( \mu'(x_i), \mu''(x_i), \mu^{(3)}(x_i) \) and \( \mu^{(4)}(x_i) \) in \( \phi_{2i}'(x_i), \phi_{2i}''(x_i), \phi_{2i}^{(3)}(x_i) \) and \( \phi_{2i}^{(4)}(x_i) \).

Step 7: Approximate \( \eta_{i+1} \approx \phi_{1i}(x_{i+1}) \) and \( \mu_{i+1} \approx \phi_{2i}(x_{i+1}) \).

6. Numerical example

In this section, we show the some of the examples So that we explain the methods to solve the nonlinear (VIE). The exact solution is specified and used for display that the numerical solution that we get with methods above is correct. To solve the examples, we used Maple.

Example 1: [10]

Consider the following System of two nonlinear (VIEs) 2-kind:

\[
\eta(x) = x - \frac{2}{3} x^3 + \int_0^x (\eta^2(t) + \mu(t))dt ,
\]

\[
\mu(x) = x^2 - \frac{1}{4} x^4 + \int_0^x \eta(t) \mu(t)dt .
\]

With the exact solution \( (\eta(x), \mu(x)) = (x, x^2) \).

In this example, it will be shown how to implement the method through the algorithm:

1- We calculated \( a_{10}, b_{10}, c_{10}, d_{10}, e_{10}, a_{20}, b_{20}, c_{20}, d_{20} \) and \( e_{20} \) by differentiate equation (27) and (28) four once with respect to \( x \) then set \( x = a \) and substitute it in equation (7) to (11) and from (12) for (16) this leads to finding \( \eta_i \approx \phi_{10}(x_i), \mu_i \approx \phi_{20}(x_i) \).

2- Now, we can be find \( a_{1i}, b_{1i}, c_{1i}, d_{1i}, e_{1i}, a_{2i}, b_{2i}, c_{2i}, d_{2i} \) and \( e_{2i} \) by using equations (7)-(11) and (12)-(16) and replacing \( \eta'(x_i), \eta''(x_i), \eta^{(3)}(x_i), \eta^{(4)}(x_i) \) in \( \phi_{1i}'(x_i), \phi_{1i}''(x_i), \phi_{1i}^{(3)}(x_i) \) and \( \phi_{1i}^{(4)}(x_i) \).
3- Then calculated \( \mu'(x_i), \mu''(x_i), \mu^{(3)}(x_i), \mu^{(4)}(x_i) \) in \( \phi_{2l}'(x_i), \phi_{2l}''(x_i), \phi_{2l}'(x_i), \phi_{2l}'(x_i) \) then calculated \( \eta_{i+1} \approx \phi_{i+1}'(x_{i+1}) \) and \( \mu_{i+1} \approx \phi_{2l}'(x_{i+1}) \) so we get \( \mu(x), \eta(x) \).

Table 1 provides a comparison between the Absolute Error of linear non-polynomial spline (LNPS) [10] and absolute error of (QNPS) for Example (1) by applying (QNPS) algorithm. Where \( h = 0.1 \) and \( x_i = x_i + ih, n = 10, i = 0, 1, ..., 10 \). Figure (1) comparison between the exact solution and numerical solution of \( \eta(x) \) and \( \mu(x) \) by applying (QNPS) method for Example (1).

**Table 1:** Comparison between the errors of our method with [10] for Example (1) for \( n = 10 \) at different values of \( x \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>Absolute Error of linear non-polynomial [10]</th>
<th>Absolute Error of quadratic non-polynomial</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \eta(x) )</td>
<td>( \mu(x) )</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.1</td>
<td>0</td>
<td>8.33056e-6</td>
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<td>0.2</td>
<td>0</td>
<td>1.33156e-6</td>
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<td>0</td>
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<tr>
<td>1</td>
<td>0</td>
<td>8.06046e-2</td>
</tr>
</tbody>
</table>

**Figure 1:** The exact solution and numerical solution of (a) \( \eta \), (b) \( \mu \) for Example (1).

**Example 2:** [10]

Consider the following System of two nonlinear (VIEs) 2-kind:

\[
\eta(x) = e^x + x - \frac{1}{2} \sinh(2x) + \int_0^x (x - t)(\eta^2(t) - \mu^2(t))dt,
\]
\[ \mu(x) = e^{-x} + x - xe^x + \int_0^x (x \eta^2(t) \mu(t)) dt. \]

With the exact solution \((\eta(x), \mu(x)) = (e^x, e^{-x})\).

Table (2) shows the comparison between the absolute error of (LNPS) [10] and absolute error of (QNPS) for Example (2) where \(h = 0.1\) and \(x_i = x_i + ih, n = 10, i = 0, 1, ..., 10\). In Figure (2) comparison between exact solution and numerical solution of \(\eta(x)\) and \(\mu(x)\) for Example (2).

**Table 2:** Comparison between the errors of our method with [10] for Example (2) for \(n = 10\) at different values of \(x\).

<table>
<thead>
<tr>
<th>(x)</th>
<th>(\eta(x)) Error</th>
<th>(\mu(x)) Error</th>
<th>(\eta(x)) Error</th>
<th>(\mu(x)) Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>8.5e-6</td>
<td>8.16667e-6</td>
<td>1.69444 e-7</td>
<td>1.63889e-7</td>
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<td>1.38938e-2</td>
</tr>
</tbody>
</table>

**Figure 2:** The exact solution and numerical solution of (a) \(\eta\), (b) \(\mu\) for Example (2).

**Example 3:** [2]

Consider the following System of two nonlinear (VIEs) 2-kind:
\[ \eta(x) = x - \frac{x^5}{3} - \frac{x^4}{4} + \frac{x^3}{3} + \int_0^x (x^2 - t) \eta(t) dt - \int_0^x (x^2 - t) \mu(t) dt, \]
\[ \mu(x) = x^2 - \frac{x^3}{2} - \frac{x^4}{3} + \int_0^x x \eta(t) dt + \int_0^x x \mu(t) dt, \quad x \in [0,1]. \]

With the exact solution \((\eta(x), \mu(x)) = (x, x^2)\).

Table 3 provides a comparison between the Absolute error of OHAM [2] and absolute error of (QNPS) for Example (3) by applying (QNPS) algorithm. where \(h = 0.1\) and \(x_i = x_i + ih, n = 10, i = 0,1,\ldots,10\). Figure (3) show that comparison between the exact solution and numerical solution of \(\eta(x)\) and \(\mu(x)\) for Example (3).

**Table 3**: Comparison between the errors of our method with [2] for Example (3) for \(n=10\) at different values of \(x\).

<table>
<thead>
<tr>
<th>(x)</th>
<th>(\eta(x))</th>
<th>(\mu(x))</th>
<th>(\eta(x))</th>
<th>(\mu(x))</th>
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**Figure 3**: The exact solution and the numerical solution of \((a) \eta, (b) \mu\) for Example (3).
7. Conclusions

The (QNPS) method was used to approximate the non-linear system (VIEs) 2-kind in this paper. To prove the accuracy of the method, we applied three numerical examples. This method turned out to be preferable to the (LNPS). The numerical results are presented in Tables (1-3) compared to the exact solutions and other methods. The proposed method maintains high accuracy through the numerical results obtained which makes it very encouraging for dealing with this type of system. The method illustrated by three test examples which verify that of the presented method is applicable and considerable accurate.

References