Available online at www.qu.edu.iq/journalcm
JOURNAL OF AL-QADISIYAH FOR COMPUTER SCIENCE AND MATHEMATICS
ISSN:2521-3504(online) ISSN:2074-0204(print)

Numerical Study of the System of Nonlinear Volterra Integral Equations by Using Spline Method

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ARTICLE INFO

Article history:
Received: 010 /07/2021
Rrevised form: 26 /07/2021
Accepted : 16 /08/2021
Available online: 16 /08/2021

Keywords:

Volterra integral equations,

Non-polynomial spline,

Exact solution.

ABSTRACT

The second order non-polynomial spline function for solving system of two nonlinear Volterra integral equations is proposed in this paper. An algorithm introduced as well to numerical examples to illustrate carry out of this method. Also, we compare the absolute error of quadratic non-polynomial spline method with absolute error of linear non-polynomial spline method and the exact solution.

MSC. 41A25; 41A35; 41A36

DOI : https://doi.org/10.29304/jqcm.2021.13.3.836

1. Introduction

In the several last years, attention has turned to integral equations(IEs), especially the Volterra equation, began in many problem such as particle transport problems in astrophysics, So the population growth models and the mixed problems of continuous media mechanics are summarized into a system of nonlinear (IEs)of the second kind. Many researchers have solved the integral equation (IE) and some they solved especially Volterra integral equation (VIE). Abdou [1] in 2003 studied on the solution of linear and nonlinear (IE). Ayal and Muhammad [4] in 2018 used cubic spline for solving (VIE) with delay. Babolian and Masouri [5] in 2008 submit direct method to solve Volterra integral equation of the first kind (VIE) 1-kind using operational matrix with block-pulse functions. Bani issa et.al [6] in 2019 solving nonlinear (VIEs) by using numerical techniques. Biazar et.al [7] in 2003 Solution of a system of (VIEs) of the first kind by a domian method. Cahlon and Nachman [8] in 1985 introduced numerical solutions of (VIEs) with a

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Communicated by*: Dr. Rana Jumaa Surayh aljanabi.*

solution dependent delay. Costarelli and Spigler [9] in 2013 studied solving Volterra integral equations of the second kind (VIEs) 2-kind by sigmoidal functions approximation. Jumaa and Taqi [10] in 2016 use first order (NPS) method for the solution of the system of two nonlinear (VIEs). Kumar et.al [12] in 2014 used new homotopy analysis transform algorithm to solve (VIE). Lotfi and Mahdiani [13] in 2011 introduced fuzzy galerkin method for solving Fredholm integral equations (FIEs) with error analysis. Mahmoudi [14] in 2005 presented wavelet Galerkin method for numerical solution of nonlinear (IEs). Maleknejad et.al [15] in 2016 studied numerical solution of a non-linear (VIE). Masouri et.al [16] in 2010 gave an expansion–iterative method for numerically solving (VIE) 1-kind. Maturi [17] in 2014 studied numerical solution of system of two nonlinear (VIEs). Muhammad [18] in 2017 numerical solution of (VIE) with delay by using (NPS) function was presented. Mustafa and Harbi [19] in 2014 discussed solution of second kind (VIEs) using non-polynomial spline (NPS) functions. Odibat [20] in 2008 discussed differential transform method for solving (VIE) with separable kernels. Nadjafi and Heidari [21] in 2010 introduced solving nonlinear (IEs) in the urysohn form by Newton–Kantorovich–quadrature method. Saeedi et. al [22] in 2013 presented numerical solution of some nonlinear (VIEs) 1-kind. Yalcinbas and Erdem [24] in 2010 studied approximate solutions of nonlinear Volterra integral equation systems. In this paper, we use second order (NPS) function to solve a system of two integral nonlinear equations for a (VIEs) 2-kind:

$$\eta(x) = g_1(x) + \int_0^x \kappa_{11}(x,t,\eta(t))dt + \int_0^x \kappa_{12}(x,t,\mu(t))dt, \tag{1}$$

$$\mu(x) = g_2(x) + \int_0^x \kappa_{21}(x,t,\eta(t))dt + \int_0^x \kappa_{22}(x,t,\mu(t))dt. \tag{2}$$

Where $\eta(x), \mu(x)$ are unknown functions, while that $g_1(x), g_2(x)$ are given function and it $\kappa_{11}(x,t,\eta(x)), \kappa_{12}(x,t,\eta(x)), \kappa_{21}(x,t,\mu(x))$ and $\kappa_{22}(x,t,\mu(x))$ kernel sintegral nonlinear equations.

2. Non –polynomial spline function (NPSF)

Consider partition $\Delta = \{x_0, x_1, x_2, \ldots, x_n\}$ of $[a,b] \subseteq R$. Let $\psi(\Delta)$ it the set of piecewise polynomials subinterval $I_i = [x_i, x_{i+1}]$ of partition Δ. Let $\xi_i(x)$ be the segment of (NPSF) of order n that has the form:

$$\xi_i(x) = a_i \cos m(x - x_i) + b_i \sin m(x - x_i) + \ldots + y_i (x - x_i)^{n-1} + z_i . \tag{3}$$

Where $a_i, b_i, \ldots, y_i$ and z_i are constants, m is repetition of the trigonometric functions that it makes the method more accurate.

3.Quadratic non-polynomial spline (QNPS) function

(QNPS) function take the form:

$$\phi_i(x) = a_i \cos m(x - x_i) + b_i \sin m(x - x_i) + c_i(x - x_i) + d_i(x - x_i)^2 + e_i, i = 0, \ldots, n. \tag{4}$$

Where $a_i, b_{i,}, c_i, d_i$ and e_i are constants. To find a solution of the equation (1) and (2) we rewrite the (QNPS) $\phi_i(x), i = 0, \dots, n$ as the form

$$\phi_{1i}(x) = a_{1i}\cos m(x - x_i) + b_{1i}\sin m(x - x_i) + c_{1i}(x - x_i) + d_{1i}(x - x_i)^2 + e_{1i} \tag{5}$$

$$\phi_{2i}(x) = a_{2i}\cos m(x - x_i) + b_{2i}\sin m(x - x_i) + c_{2i}(x - x_i) + d_{2i}(x - x_i)^2 + e_{2i}. \tag{6}$$

Differentiate equations (5) and (6) four once with respect to, therefore obtain the following equations:

$$\acute{\phi}_{1\imath}(x) = -ma_{1i}\sin m(x - x_i) + mb_{1i}\cos m(x - x_i) + c_{1i} + 2d_{1i}(x - x_i),$$

$$\phi''_{1i}(x) = -m^2 a_{1i}\cos m(x - x_i) - m^2 b_{1i}\sin m(x - x_i) + 2d_{1i},$$

$$\phi_{1i}{}^{(3)}(x) = m^3 a_{1i}\sin m(x - x_i) - m^3 b_{1i}\cos m(x - x_i),$$

$$\phi_{1i}{}^{(4)}(x) = m^4 a_{1i}\cos m(x - x_i) + m^4 b_{1i}\sin m(x - x_i),$$

$$\acute{\phi}_{2\imath}(x) = -ma_{2i}\sin m(x - x_i) + mb_{2i}\cos m(x - x_i) + c_{2i} + 2d_{2i}(x - x_i),$$

$$\phi''_{2i}(x) = -m^2 a_{2i}\cos m(x - x_i) - m^2 b_{2i}\sin m(x - x_i) + 2d_{2i},$$

$$\phi_{2i}{}^{(3)}(x) = m^3 a_{2i}\sin m(x - x_i) - m^3 b_{2i}\cos m(x - x_i),$$

$$\phi_{2i}{}^{(4)}(x) = m^4 a_{2i}\cos m(x - x_i) + m^4 b_{2i}\sin m(x - x_i).$$

Let $\eta(x)$ be the exact solution of equation (1) and $\mu(x)$ be the exact solution of equation (2), $\psi_{1i}(x)$ be the approximate solution to $\eta_i = \eta(x_i)$ and $\psi_{2i}(x)$ be the approximate solution to $\mu_i = \mu(x_i)$ getten by $\phi_{1i}(x)$ and $\phi_{2i}(x)$. The following relations are achieved:

$$\phi_{1i}(x_i) = a_i + e_{1i} = \eta(x_i) \approx \psi_{1i}(x_i),$$

$$\acute{\phi}_{1\imath}(x_i) = mb_{1i} + c_{1i} = \acute{\eta}(x_i) \approx \acute{\psi}_{1\imath}(x_i),$$

$$\phi''_{1i}(x_i) = -m^2 a_{1i} + 2d_{1i} = \eta''(x_i) \approx \psi''_{1i}(x_i),$$

$$\phi_{1i}{}^{(3)}(x_i) = -m^3 b_{1i} = \eta^{(3)}(x_i) \approx \psi^{(3)}{}_{1i}(x_i),$$

$$\phi_{1i}{}^{(4)}(x_i) = m^4 a_{1i} = \eta^{(4)}(x_i) \approx \psi^{(4)}{}_{1i}(x_i),$$

$$\phi_{2i}(x_i) = a_i + e_{2i} = \mu(x_i) \approx \psi_{2i}(x_i),$$

$$\acute{\phi}_{2\imath}(x_i) = mb_{2i} + c_{2i} = \acute{\mu}(x_i) \approx \acute{\psi}_{2\imath}(x_i),$$

$$\phi''_{2i}(x_i) = -m^2 a_{2i} + 2d_{2i} = \mu''(x_i) \approx \psi''_{2i}(x_i),$$

$$\phi_{2i}{}^{(3)}(x_i) = -m^3 b_{2i} = \mu^{(3)}(x_i) \approx \psi^{(3)}{}_{2i}(x_i),$$

$$\phi_{2i}{}^{(4)}(x_i) = m^4 a_{2i} = \mu^{(4)}(x_i) \approx \psi^{(4)}{}_{2i}(x_i).$$

Now we get the values of $a_{1i}, b_{1i,}, c_{1i}$, d_{1i} and e_{1i} as follows:

$$a_{1i} = \frac{1}{m^4}\phi_{1i}^{(4)}(x_i), \tag{7}$$

$$b_{1i} = -\frac{1}{m^3}\phi_{1i}{}^{(3)}(x_i). \tag{8}$$

$$c_{1i}=\phi_{1i}{}'(x_i)+\frac{1}{m^2}\phi_{1i}{}^{(3)}(x_i), \tag{9}$$

$$d_{1i}=\frac{1}{2}(\phi_{1i}{}''(x_i)+\frac{1}{m^2}\phi_{1i}{}^{(4)}(x_i)), \tag{10}$$

$$e_{1i}=\phi_{1i}(x_i)-\frac{1}{m^4}\phi_{1i}{}^{(4)}(x_i). \tag{11}$$

And we get the values of $a_{2i}, b_{2i,}, c_{2i}$, d_{2i} and e_{2i} as follows:

$$a_{2i}=\frac{1}{m^4}\phi_{2i}^{(4)}(x_i), \tag{12}$$

$$b_{2i}=-\frac{1}{m^3}\phi_{2i}{}^{(3)}(x_i), \tag{13}$$

$$c_{2i}=\phi_{2i}{}'(x_i)+\frac{1}{m^2}\phi_{2i}{}^{(3)}(x_i), \tag{14}$$

$$d_{2i}=\frac{1}{2}(\phi_{2i}{}''(x_i)+\frac{1}{m^2}\phi_{2i}{}^{(4)}(x_i)), \tag{15}$$

$$e_{2i}=\phi_{2i}(x_i)-\frac{1}{m^4}\phi_{2i}{}^{(4)}(x_i). \tag{16}$$

4. Solving method of (QNPS) function of the nonlinear equations for (VIE) 2-kind

To find the numerical approximation of equation (1), differentiate it four once with respect to x , then we substitute in $a=x$:

$$\eta_0=g_1(a), \tag{17}$$

$$\eta_0'=g_{10}'(a)+\kappa_{11}(a,a,\eta(a))+\kappa_{12}(a,a,\mu(a)), \tag{18}$$

$$\begin{aligned}\eta_0''=g_{10}''(a)&+[\frac{\partial}{\partial x}\kappa_{11}(x,t,\eta(t))]_{t=x=0}+[\frac{\partial}{\partial x}\kappa_{12}(x,x,\mu(t))]_{t=x=0}\\&+[\frac{\partial}{\partial x}\kappa_{11}(x,x,\eta(t))]_{x=0}+[\frac{\partial}{\partial x}\kappa_{12}(x,x,\mu(t))]_{x=0},\end{aligned} \tag{19}$$

$$\begin{aligned}\eta_0{}^{(3)}=g_{10}'''(a)&+[\frac{\partial^2}{\partial x^2}\kappa_{11}(x,t,\eta(t))]_{t=x=0}+[\frac{\partial^2}{\partial x^2}\kappa_{12}(x,t,\mu(t))]_{t=x=0}\\&+[\frac{d}{dx}(\frac{\partial}{\partial x}\kappa_{11}(x,t,\eta(t)))_{t=x}]_{x=0}+[\frac{d}{dx}(\frac{\partial}{\partial x}\kappa_{12}(x,t,\mu(t)))_{t=x}]_{x=0}+[\frac{\partial^2}{\partial x^2}\kappa_{11}(x,x,\eta(t))]_{x=0}\\&+[\frac{\partial^2}{\partial x^2}\kappa_{12}(x,x,\mu(t))]_{x=0},\end{aligned} \tag{20}$$

$$\eta_0^{(4)} = g_{10}^{(4)}(a) + [\frac{\partial^3}{\partial x^3}\kappa_{11}(x,t,\eta(t))]_{t=x=0} + [\frac{\partial^3}{\partial x^3}\kappa_{12}(x,t,\mu(t))]_{t=x=0}$$
$$+[\frac{d}{dx}(\frac{\partial^2}{\partial x^2}\kappa_{11}(x,t,\eta(t)))_{t=x}]_{x=0} + [\frac{d}{dx}(\frac{\partial^2}{\partial x^2}\kappa_{12}(x,t,\mu(t)))_{t=x}]_{x=0}$$
$$+[\frac{d^2}{dx^2}(\frac{\partial}{\partial x}\kappa_{11}(x,t,\eta(t)))_{t=x}]_{x=0} + [\frac{d^2}{dx^2}(\frac{\partial}{\partial x}\kappa_{12}(x,t,\mu(t)))_{t=x}]_{x=0} + \quad (21)$$
$$[\frac{\partial^3}{\partial x^3}\kappa_{11}(x,t,\eta(t))]_{x=0} + [\frac{\partial^3}{\partial x^3}\kappa_{12}(x,t,\mu(t))]_{x=0}.$$

To find the numerical approximation of equation (2), differentiate it four once with respect to x , Then we substitute in $x = a$

$$\mu_0 = g_{20}(a) \quad (22)$$

$$\acute{\mu_0} = g_{20}'(a) + \kappa_{21}(a,a,\eta(a)) + \kappa_{22}(a,a,\mu(a)), \quad (23)$$

$$\mu_0'' = g_{20}''(a) + [\frac{\partial}{\partial x}\kappa_{21}(x,t,\eta(t))]_{t=x=0} + [\frac{\partial}{\partial x}\kappa_{22}(x,x,\mu(t))]_{t=x=0}$$
$$+[\frac{\partial}{\partial x}\kappa_{21}(x,x,\eta(t))]_{x=0} + [\frac{\partial}{\partial x}\kappa_{22}(x,x,\mu(t))]_{x=0}. \quad (24)$$

$$\mu_0^{(3)} = g_{20}'''(a) + [\frac{\partial^2}{\partial x^2}\kappa_{21}(x,t,\eta(t))]_{t=x=0} + [\frac{\partial^2}{\partial x^2}\kappa_{22}(x,t,\mu(t))]_{t=x=0}$$
$$+[\frac{d}{dx}(\frac{\partial}{\partial x}\kappa_{21}(x,t,\eta(t)))_{t=x}]_{x=0} + [\frac{d}{dx}(\frac{\partial}{\partial x}\kappa_{22}(x,t,\mu(t)))_{t=x}]_{x=0} + [\frac{\partial^2}{\partial x^2}\kappa_{21}(x,x,\eta(t))]_{x=0}$$
$$+[\frac{\partial^2}{\partial x^2}\kappa_{22}(x,x,\mu(t))]_{x=0}, \quad (25)$$

$$\mu_0^{(4)} = g_{20}^{(4)}(a) + [\frac{\partial^3}{\partial x^3}\kappa_{21}(x,t,\eta(t))]_{t=x=0} + [\frac{\partial^3}{\partial x^3}\kappa_{22}(x,t,\mu(t))]_{t=x=0}$$
$$+[\frac{d}{dx}(\frac{\partial^2}{\partial x^2}\kappa_{21}(x,t,\eta(t)))_{t=x}]_{x=0} + [\frac{d}{dx}(\frac{\partial^2}{\partial x^2}\kappa_{22}(x,t,\mu(t)))_{t=x}]_{x=0}$$
$$+[\frac{d^2}{dx^2}(\frac{\partial}{\partial x}\kappa_{21}(x,t,\eta(t)))_{t=x}]_{x=0} + [\frac{d^2}{dx^2}(\frac{\partial}{\partial x}\kappa_{22}(x,t,\mu(t)))_{t=x}]_{x=0} + \quad (26)$$
$$[\frac{\partial^3}{\partial x^3}\kappa_{21}(x,t,\eta(t))]_{x=0} + [\frac{\partial^3}{\partial x^3}\kappa_{22}(x,t,\mu(t))]_{x=0}.$$

5. Algorithm for (QNPS) method

Step1: set

$$h = \frac{b-a}{n}, x_i = x_o + ih, i = 0,\ldots\ldots,n; x_o = a, x_n = b,$$

$$\eta_o = g(a).$$

Step 2:

1- Evaluate a_{10}, b_{10}, c_{10} , d_{10} and e_{10} by substituting equations (17)-(21) in equations (7)-(11).

2- Evaluate a_{20}, b_{20}, c_{20} , d_{20} and e_{20} by substituting equations (22)-(26) in equations (12)-(16).

Step3:

1-Calculate ϕ_{10} by using step 2 a branch (1) and equation (5) at $i = 0$.

2- Calculate ϕ_{20} by using step2 a branch (2) and equation (6) at $i = 0$.

Step 4: Approximate $\eta_1 \approx \phi_{10}(x_1), \mu_1 \approx \phi_{20}(x_1)$.
Step 5: for $i = 1$to $n - 1$do the following steps:
Step 6:

1-Evaluate $a_{1i}, b_{1i}, c_{1i}, d_{1i}$and e_{1i} by using equations (7)-(11) and replacing $\eta'(x_i), \eta''(x_i)$, $\eta^{(3)}(x_i)$ and $\eta^{(4)}(x_i)$ in $\phi_{1i}{}'(x_i), \phi_{1i}{}''(x_i)$, $\phi_{1i}{}^{(3)}(x_i)$ and $\phi_{1i}{}^{(4)}(x_i)$.

2- Evaluate a_{2i}, b_{2i}, c_{2i} d_{2i} and e_{2i} by using equations (12)-(16) and replacing $\mu'(x_i), \mu''(x_i)$, $\mu^{(3)}(x_i)$ and $\mu^{(4)}(x_i)$ in $\phi_{2i}{}'(x_i), \phi_{2i}{}''(x_i), \phi_{2i}{}^{(3)}(x_i)$ and $\phi_{2i}{}^{(4)}(x_i)$.

Step 7: Approximate $\eta_{i+1} \approx \phi_{1i}(x_{i+1})$ and $\mu_{i+1} \approx \phi_{2i}(x_{i+1})$.

6. Numerical example

In this section, we show the some of the examples So that we explain the methods to solve the nonlinear (VIE). The exact solution is specified and used for display that the numerical solution that we get with methods above is correct. To solve the examples, we used Maple.

Example 1: [10]

Consider the following System of two nonlinear (VIEs) 2-kind:

$$\eta(x) = x - \frac{2}{3}x^3 + \int_0^x (\eta^2(t) + \mu(t))dt, \tag{27}$$

$$\mu(x) = x^2 - \frac{1}{4}x^4 + \int_0^x \eta(t)\mu(t)dt. \tag{28}$$

With the exact solution $(\eta(x), \mu(x)) = (x, x^2)$.

In this example, it will be shown how to implement the method through the algorithm:

1- We calculated a_{10}, b_{10}, c_{10} , d_{10} , e_{10} , a_{20}, b_{20}, c_{20} , d_{20} and e_{20} by differentiate equation (27) and (28) four once with respect to x then set $x = a$ and substitute it in equation (7) to (11) and from (12) for (16) this leads to finding $\eta_1 \approx \phi_{10}(x_1), \mu_1 \approx \phi_{20}(x_1)$.

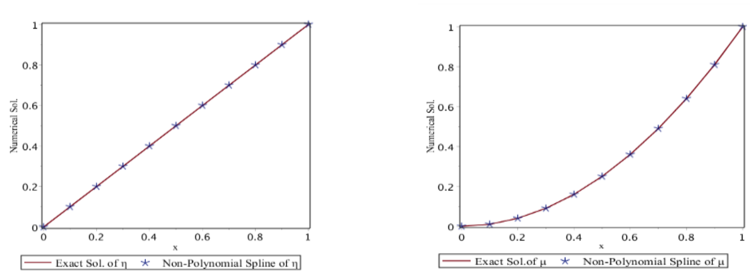
2- Now, we can be find $a_{1i}, b_{1i}, c_{1i}, d_{1i}, e_{1i}$, a_{2i}, b_{2i}, c_{2i} d_{2i} and e_{2i} by using equations (7)-(11) and (12)-(16) and replacing $\eta'(x_i), \eta''(x_i)$, $\eta^{(3)}(x_i), \eta^{(4)}(x_i)$ in $\phi_{1i}{}'(x_i), \phi_{1i}{}''(x_i)$, $\phi_{1i}{}^{(3)}(x_i)$ and $\phi_{1i}{}^{(4)}(x_i)$.

3- Then calculated $\mu'(x_i), \mu''(x_i), \mu^{(3)}(x_i), \mu^{(4)}(x_i)$ in $\phi_{2i}'(x_i), \phi_{2i}''(x_i), \phi_{2i}^{(3)}(x_i), \phi_{2i}^{(4)}(x_i)$ then calculated $\eta_{i+1} \approx \phi_{1i}(x_{i+1})$ and $\mu_{i+1} \approx \phi_{2i}(x_{i+1})$ so we get $\mu(x),\ \eta(x)$.

Table (1) provides a comparison between the Absolute Error of linear non-polynomial spline(LNPS)[10] and absolute error of (QNPS) for Example (1) by applying (QNPS) algorithm .where $h = 0.1$ and $x_i = x_i + ih, n = 10, i = 0,1,\dots,10$. Figure (1) comparison between the exact solution and numerical solution of $\eta(x)$ and $\mu(x)$ by applying (QNPS) method for Example (1).

Table 1: Comparison between the errors of our method with [10] for Example (1) for n=10 at different values of x.

x	**Absolute Error of linear non-polynomial [10]**		**Absolute Error of quadratic non-polynomial**	
	$\eta_i(x)$	$\mu_i(x)$	$\eta_i(x)$	$\mu_i(x)$
0	0	0	0	0
0.1	0	8.33056e-6	0	0
0.2	0	1.33156e-6	0	0
0.3	0	6.72978e-4	0	0
0.4	0	2.12199e-3	0	0
0.5	0	5.16512e-3	0	0
0.6	0	1.06712e-2	0	0
0.7	0	1.9684e-2	0	0
0.8	0	3.34134e-2	0	0
0.9	0	5.32199e-2	0	0
1	0	8.06046e-2	0	0



$(a)\ \eta$ $(b)\ \mu$

Figure 1: The exact solution and numerical solution of $(a)\ \eta$, $(b)\ \mu$ for Example (1).

Example 2: [10]

Consider the following System of two nonlinear (VIEs) 2-kind:

$$\eta(x) = e^x + x - \frac{1}{2}\sinh(2x) + \int_0^x (x-t)(\eta^2(t) - \mu^2(t))dt,$$

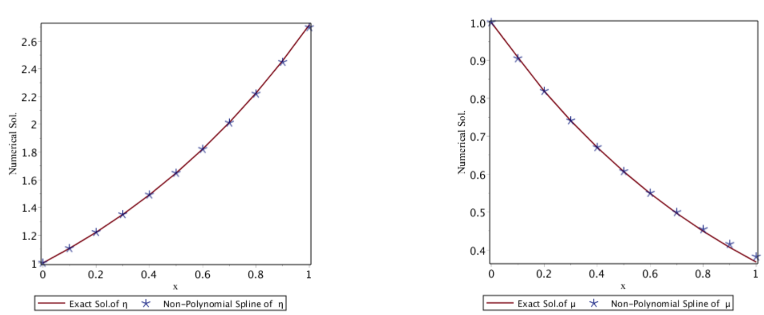
$$\mu(x) = e^{-x} + x - xe^{x} + \int_0^x (x\,\eta^2(t)\,\mu(t))dt.$$

With the exact solution $(\eta(x), \mu(x)) = (e^{x}, e^{-x})$.

Table (2) show that comparison between absolute error of (LNPS) [10] and absolute error of **(QNPS)** for Example (2) where $h = 0.1$ and $x_i = x_i + ih, n = 10, i = 0,1,\ldots,10$. In Figure (2) comparison between exact solution and numerical solution of $\eta(x)$ and $\mu(x)$ for Example (2).

Table 2: Comparison between the errors of our method with [10] for Example (2) for n=10 at different values of x.

x	Absolute Error of linear non-polynomial [10]		Absolute Error of quadratic non-polynomial	
	$\eta_i(x)$	$\mu_i(x)$	$\eta_i(x)$	$\mu_i(x)$
0	0	0	0	0
0.1	8.5e-6	8.16667e-6	1.69444 e-7	1.63889e-7
0.2	1.38667e-4	1.28e-4	5.51111e-6	5.15556e-6
0.3	7.15503e-4	6.34503e-4	4.25251e-5	3.84751e-5
0.4	2.30403e-3	1.9627e-3	1.82046e-4	1.5929e-4
0.5	5.72937e-3	4.68768e-3	5.64247e-4	4.77441e-4
0.6	1.20969e-2	9.50478e-3	1.42566e-3	1.16645e-3
0.7	2.28126e-2	1.72098e-2	3.12821e-3	2.47457e-3
0.8	3.96037e-2	2.86796e-2	6.19031e-3	4.73384e-3
0.9	5.64247e-2	4.48527e-2	1.132e-2	8.36722e-3
1	1.00055e-1	6.67108e-2	1.94505e-2	1.38938e-2



$(a)\ \eta$ $(b)\ \mu$

Figure 2: The exact solution and numerical solution of $(a)\ \eta, (b)\ \mu$ for Example (2).

Example 3: [2]

Consider the following System of two nonlinear (VIEs) 2-kind:

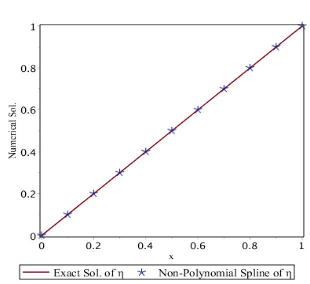
$$\eta(x) = x - \frac{x^5}{3} - \frac{x^4}{4} + \frac{x^3}{3} + \int_0^x (x^2 - t)\,\eta(t)dt - \int_0^x (x^2 - t)\,\mu(t)dt,$$

$$\mu(x) = x^2 - \frac{x^3}{2} - \frac{x^4}{3} + \int_0^x x\,\eta(t)dt + \int_0^x x\,\mu(t)dt, \qquad x \in [0,1].$$

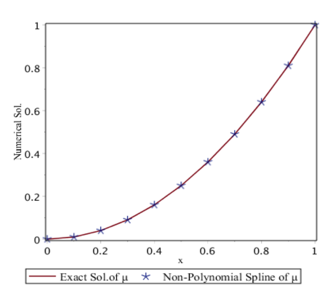
With the exact solution $(\eta(x), \mu(x)) = (x, x^2)$.
Table (3) provides a comparison between the Absolute error of OHAM [2] and absolute error of (QNPS) for Example (3) by applying (QNPS) algorithm .where $h = 0.1$ and $x_i = x_i + ih, n = 10, i = 0,1,\ldots,10$. Figure (3) show that comparison between the exact solution and numerical solution of $\eta(x)$ and $\mu(x)$ for Example (3).

Table 3: Comparison between the errors of our method with [2] for Example (3) for n=10 at different values of x.

x	Absolute Error of OHAM [2]		Absolute Error of quadratic non-polynomial	
	$\eta_i(x)$	$\mu_i(x)$	$\eta_i(x)$	$\mu_i(x)$
0	0	0	0	0
0.1	6.76544e-8	4.69687e-7	0	0
0.2	3.88365e-7	3.66185e-6	0	0
0.3	6.09156e-7	1.08757e-5	0	0
0.4	2.25143e-7	1.93236e-5	0	0
0.5	2.49502e-6	2.10611e-5	0	0
0.6	4.1888e-6	8.03933e-6	0	0
0.7	1.61514e-6	1.39203e-5	0	0
0.8	3.3662e-6	1.22626e-5	0	0
0.9	2.1935e-6	3.63494e-5	0	0
1	1.3456e-5	7.10952e-5	0	0



(a) η



(b) μ

Figure 3: The exact solution and the numerical solution of $(a)\ \eta, (b)\ \mu$ for Example (3).

7. Conclusions

The (QNPS) method was used to approximate the non-linear system (VIEs) 2-kind in this paper. To prove the accuracy of the method, we applied three numerical examples. This method turned out to be preferable to the (LNPS). The numerical results are presented in Tables (1-3) compared to the exact solutions and other methods. The proposed method maintains high accuracy through the numerical results obtained which makes it very encouraging for dealing with this type of system. The method illustrated by three test examples which verify that of the presented method is applicable and considerable accurate.

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