Journal of AL-Qadisiyah for computer science and mathematics
Vol.7 No.1 Year 2015

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Countability Axioms in Smooth Fuzzy Topological Spaces

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Recived :18\12\2014 Revised : 27\1\2015 Accepted :19\2\2015

Abstract:

The aim of this paper is to introduce the countability axioms in smooth fuzzy topological spaces (smooth fuzzy first countable, smooth fuzzy second countable and smooth fuzzy separable axiom) and try to find relations between them. Such that we will prove the following: smooth fuzzy first countable axiom $\not\leftrightarrows$smooth fuzzy second countable axiom $\not\rightleftarrows$smooth fuzzy separable axiom. Also, we prove or disprove those concepts are hereditary or topological.

Mathematics Subject Classification :54xx

Introduction:

The theory of smooth fuzzy topology or (smooth topology as it named by Ramadan) introduced by Ramadan and his colleagues [1]. Many authors have introduced the concept of smooth fuzzy topology and gave new definitions to some of the properties depend on the fuzzy topology [٢, 3, 4 and 5]. In this paper we will give new definitions built on definitions in fuzzy topology but by different method.

Keywords: smooth fuzzy topology, relative smooth and smooth continuous function.

1-Preliminaries

Definition 1.1 [6, p.332]: A fuzzy topological space (X, T) is called "satisfy axiom of fuzzy first countable" or "fuzzy first countable space", iff for every fuzzy point p_x^r there exists a countable local base, i.e. a countable family of open sets containing p_x^r such that every neighbouehood of p_x^r contains a member of the family.

Definition 1.2 [6, p.332]: A fuzzy topological space (X, T) is called "satisfy axiom of fuzzy second countable" or "fuzzy second countable space", iff the fuzzy topology has a countable base.

Definition 1.3 [7]: A fuzzy topological space (X, T) is called "satisfy axiom of fuzzy separable countable" or "fuzzy separable space", iff there exists a countable sequence $\{p_{x_i}^{r_i}\}, i = 1,2, \ldots$ such that for every member A of T and $A \neq \emptyset$, there exist a $p_{x_i}^{r_i}$ such that $p_{x_i}^{r_i} \in A$.

Note: in the following we will say smooth topology or smooth topological space instead of smooth fuzzy topology or smooth fuzzy topological space as it use in most the papers.

Definition 1.4 [1 and 2]: A smooth topological space (sts) is a pair (X, τ) where X is a nonempty set and $\tau: I^X \to I$ $(I = [0,1])$ is mapping satisfy the following properties:

1- $\tau(X) = \tau(\emptyset) = 1$.
2- $\forall A, B \in I^X, \tau(A \cap B) \geq \tau(A) \wedge \tau(B)$.
3- For every subfamily $\{A_i : i \in J\} \subseteq I^X, \tau(\bigcup_{i \in J} A_i) \geq \bigwedge_{i \in J} \tau(A_i)$.

Such that the symbol $\wedge$ meaning the minimum.

Definition 1.5 [3]: Let (X, τ) be an sts and $A \subseteq X$. Let $\tau_A: P(A) \to I$ is mapping define as following $\tau_A(U) = \vee\{\tau(V): V \in I^X, V \cap A = U\}$, then this mapping is smooth topology on A is called relative smooth topology on A and (A, τ_A) is called subspace of smooth topological space.

Definition 1.6 [2]: let (X, τ) be an sts and $\alpha \in (0,1]$. Then the family $\tau_\alpha = \{A \in I^X : \tau(A) \geq \alpha\}$ is called α-level on X.

Theorem 1.1 [2]: let (X, τ) be an sts. Then the family τ_α is fuzzy topology on X for each $\alpha \in (0,1]$.

Definition 1.7: let (X, τ) be a sts and p_x^r be any arbitrary fuzzy point in X. A class $\mathcal{B}_{p_x^r}$ of fuzzy sets B which containing p_x^r is called smooth local base for p_x^r if for each $\alpha \in (0,1]$, $\mathcal{B}_{p_x^r}$ is fuzzy local base for τ_α. I.e. for each $G \in I^X$ with $\tau(G) \geq \alpha$, for some $\alpha \in (0,1]$, if G containing p_x^r then there exist $B \in \mathcal{B}_{p_x^r}$ with $\tau(B) \geq \alpha$ and $p_x^r \in B \subseteq G$.

Definition 1.8: Let (X,τ) be a sts and $\mathcal{B} \subseteq I^X$ be a collection then $\mathcal{B}$ is called smooth base for smooth topology τ on X iff $\mathcal{B}$ is fuzzy base for fuzzy topology τ_α, for each $\alpha \in (0,1]$, i.e. $\mathcal{B}$ is smooth base for smooth topology τ on X iff for each $G \in I^X$ with $\tau(G) \geq \alpha$, for some $\alpha \in (0,1]$, then G is union members $\{U_j : j \in J\}$ of $\mathcal{B}$ with $\tau(U_j) \geq \alpha$, $\forall j \in J$.

Theorem 1.2: Let (X,τ) be a sts and $\mathcal{B} \subseteq I^X$ be a collection of fuzzy sets then $\mathcal{B}$ is smooth base for smooth topology τ iff for any $G \in I^X$ with $\tau(G) \geq \alpha$, for some $\alpha \in (0,1]$, if $p_x^r \in G$ then there exist $B \in \mathcal{B}$ such that $p_x^r \in B \subseteq G$ and $\tau(B) \geq \alpha$.

Proof:

Let (X,τ) be a sts and $\mathcal{B} \subseteq I^X$ is smooth base for smooth topology τ. Let $G \in I^X$ with $\tau(G) \geq \alpha$, for some $\alpha \in (0,1]$, and $p_x^r \in G$. Then $\exists \{B_j : j \in J\} \subseteq \mathcal{B}$ s.t. $G = \bigcup_{j \in J} B_j$ with $\tau(B_j) \geq \alpha$ so there exist $k \in J$ such that $p_x^r \in B_k \subseteq G$.

Let $\mathcal{B} \subseteq I^X$ be a collection with property: for any $G \in I^X$ with $\tau(G) \geq \alpha$, for some $\alpha \in (0,1]$, if $p_x^r \in G$ then there exist $B \in \mathcal{B}$ such that $p_x^r \in B \subseteq G$ and $\tau(B) \geq \alpha$. Let $G \in I^X$ with $\tau(G) \geq \alpha$, for some $\alpha \in (0,1]$. Since $G = \bigcup_{x \in X} p_x^r$, $\forall x \in X$ and $r \leq \mu_G(x)$, and by hypothesis for any $p_x^r \in G$ there exist $B_{p_x^r} \in \mathcal{B}$ such that $p_x^r \in B_{p_x^r} \subseteq G$ and $\tau(B_{p_x^r}) \geq \alpha$ then $G = \bigcup_{p_x^r \in G} B_{p_x^r}$, therefore $\mathcal{B}$ is smooth base for smooth topology τ.

2- Smooth first countable Space.

Definition 2.1: An sts (X,τ) is called "satisfy axiom of smooth first countable" or "smooth first countable space" if, for each $\alpha \in (0,1]$, (X,τ_α) is fuzzy first countable space. I.e. for each fuzzy point $p_x^r \in X$ there exist smooth countable local base.

In the following, the smooth first countable space is hereditary property.

Theorem 2.1: Let(X,τ) be a sts. Let $A \subseteq X$, if (X,τ) is smooth first countable space then the relative smooth $(A,\tau_A)$is also smooth first countable space.

Proof:

Let $p_x^r \in A$ be arbitrary fuzzy point. $p_x^r \in X$ and since X is smooth first countable space then there exist smooth countable local base $\mathcal{B}_{p_x^r}$.

Now, we want to prove $\mathcal{B}_{p_x^r} \cap A = \{B \cap A : B \in \mathcal{B}_{p_x^r}\}$ is smooth countable local base at p_x^r in A.

It's clear that $\mathcal{B}_{p_x^r} \cap A$ is countable. Let $G \in P(A)$ containing p_x^r such that $\tau_A(\mathrm{G}) \geq \alpha$ for some $\alpha \in (0,1]$ $\Longrightarrow$ there exist $H \in \mathrm{I}^{\mathrm{X}}$ with $\tau(\mathrm{H}) \geq \alpha$ such that $\mathrm{G} = H \cap A$. $p_x^r \in H$ and since X is smooth first countable space then there exist $B \in \mathcal{B}_{p_x^r}$ with $\tau(\mathrm{B}) \geq \alpha$ and $p_x^r \in B \subseteq H \Longrightarrow B \cap A \subseteq H \cap A$. Let $C = B \cap A \Longrightarrow \tau_A(\mathrm{C}) \geq \alpha$ since $\tau(\mathrm{B}) \geq \alpha$,$C \subseteq G$ and so $C \in \mathcal{B}_{p_x^r} \cap A$.

In the following, the smooth first countable space is topological property.

Theorem 2.2: Let $f:(X,\tau) \to (Y,\sigma)$ be a surjective, open and continuous mapping where τ and σ are smooth topology on X and Y, respectively. If (X,τ) is smooth first countable space then so (Y,σ).

Proof:

Let $p_y^k \in Y$ be arbitrary fuzzy point. Since f is surjective then there exist $p_x^r \in X$ such that $f(p_x^r) = p_y^k$. (X,τ) is smooth first countable space $\Rightarrow$ there exist smooth local base $\mathcal{B}_{p_x^r}$.

Now we want to prove $f(\mathcal{B}_{p_x^r})$ is smooth local base in Y at p_y^k. $f(\mathcal{B}_{p_x^r})$ is countable collection. Let $G \in I^Y$ with $\sigma(G) \geq \alpha$ for some $\alpha \in (0,1]$ containing p_y^k, since f is smooth continuous $\Rightarrow f^{-1}(G) \in \tau_\alpha$ and $p_x^r \in f^{-1}(G) \Rightarrow$ there exist $B \in \mathcal{B}_{p_x^r}$ with $\tau(B) \geq \alpha$ and $B \subseteq f^{-1}(G)$. f is open $\Rightarrow \sigma(f(B)) \geq \alpha, f(B) \subseteq G$ and $f(B) \in f(\mathcal{B}_{p_x^r}) \Rightarrow$ $f(\mathcal{B}_{p_x^r})$ is smooth local base at p_y^k.

3- Smooth Second Countable Space.

Definition 3.1: An sts (X, τ) is called "satisfy axiom of smooth second countable" or "smooth second countable space" if, for each $\alpha \in (0,1]$, (X, τ_α) is fuzzy second countable space. I.e. The smooth topological space (X, τ) has a smooth countable base.

Proposition 3.1: A smooth second countable space is smooth first countable space, but the converse is not true in general as in following example.

Example 3.1: Let $X = \{x\}$ singleton set and $\tau: I^X \to I$ be defined as follows:

$$\tau(A) = \begin{cases} 1 & \text{if } A = \emptyset \\ \mu_A(x) & \text{otherwise} \end{cases}$$

it's clear that τ is smooth topology on X. since the interval [0,1] is not countable $\Rightarrow$ there exist no $\mathcal{B}$ is smooth base for this smooth but (X, τ) is smooth first countable space.

Theorem 3.1: Let (X, τ) be a smooth second countable space, then for any family $\zeta = \{G \in I^X : \tau(G) \geq \alpha$ for some $\alpha \in (0,1]\}$ which pair wise disjoint then ζ is countable.

Proof:

Let (X, τ) be a smooth second countable space and $\zeta = \{G \in I^X : \tau(G) \geq \alpha$ for some $\alpha \in (0,1]\}$. Let ζ be pair wise disjoint.

Let $\mathcal{B}$ be smooth countable base for $\tau \Rightarrow \forall G \in \zeta$ there exist at least one fuzzy set $B \in \mathcal{B}$ such that $B \subseteq G$ (theorem 1.2) $\Rightarrow$ there exist corresponding between ζ and subcollection of $\mathcal{B}$ and since $\mathcal{B}$ is countable $\Rightarrow \zeta$ is countable.

In the following, the smooth second countable space is hereditary property.

Theorem 3.2: Let(X, τ) be a sts and $A \subseteq X$, if (X, τ) is smooth second countable space then the relative smooth $(A, \tau_A)$is also smooth second countable space.

Proof:

Let (X, τ) be a smooth second countable space and $\mathcal{B}$ be a smooth countable base for (X, τ). Let $A \subseteq X$.

Now, we want to prove $\mathcal{B} \cap A = \{B \cap A : B \in \mathcal{B}\}$ is smooth countable base for A.

Its clear that $\mathcal{B} \cap A$ is countable. Let $G \in P(A)$ with $\tau_A(\mathrm{G}) \geq \alpha$ for some $\alpha \in (0,1] \implies$ there exist $H \in \mathrm{I}^{\mathrm{X}}$ with $\tau(\mathrm{H}) \geq \alpha$ then there exist $\{B_j : j \in J\}$ with $\tau(B_j) \geq \alpha, \forall\, j \in J$, and $\mathrm{H} = \bigcup_{\mathrm{j} \in \mathrm{J}} \mathrm{B_j} \Rightarrow G = \bigcup_{\mathrm{j} \in \mathrm{J}} \mathrm{B_j} \cap \mathrm{A} = \bigcup_{\mathrm{j} \in \mathrm{J}} (\mathrm{B_j} \cap \mathrm{A})$.

$\mathrm{B_j} \cap \mathrm{A} \in \mathcal{B} \cap A$ and so $\tau_A(\mathrm{B_j} \cap \mathrm{A}) \geq \alpha$, for each $j \in J$since $\tau(B_j) \geq \alpha \implies G$ is union members of $\mathcal{B} \cap A$.

In the following, the smooth second countable space is topological property.

Theorem 3.3: Let $f : (X, \tau) \to (Y, \sigma)$ be a surjective, open and continuous mapping where τ and σ are smooth topology on X and Y, respectively. If (X, τ) is smooth second countable space then so (Y, σ).

Proof:

Let $\mathcal{B}$ be a smooth countable base for τ, $f(\mathcal{B})$ is countable collection in Y. Now we want to prove $f(\mathcal{B})$ is smooth base in Y.

Let $G \in I^Y$ with $\sigma(G) \geq \alpha$ for some $\alpha \in (0,1]$ and $p_y^k \in G$. Since f is surjective $\Rightarrow$ there exist $p_x^r \in X$ such that $f(p_x^r) = p_y^k$ and so $p_x^r \in f^{-1}(G)$ with $\tau(f^{-1}(G)) \geq \alpha$, since f is smooth continuous, then by (theorem 1.2) there exist $B \in \mathcal{B}$ with $\tau(B) \geq \alpha$ such that $p_x^r \in B \subseteq f^{-1}(G)$ but f is open $\Rightarrow f(p_x^r) \in f(B) \subseteq G \Rightarrow f(\mathcal{B})$ is smooth countable base for $Y \Rightarrow (Y, \tau)$ is smooth second countable space.

4- Smooth Separable Space.

Definition 4.1: An sts (X, τ) is called "satisfy axiom of separable" or "smooth separable space" if, for each $\alpha \in (0,1]$, (X, τ_α) is fuzzy separable space. I.e. There exist a countable sequence of fuzzy points $\{p_{x_i}^{r_i}\}, i = 1,2, \ldots$ such that for every fuzzy set $A \in I^X$ with $\tau(A) \geq \alpha$ for some $\alpha \in (0,1]$ and $A \neq \emptyset$, there exist fuzzy point $p_{x_i}^{r_i}$ of this sequence belong to A.

The following theorem, we will need to prove the theorem 4.2.

Theorem 4.1[6, pp.320]: If a fuzzy topological space (X, T) is fuzzy second countable space then it is fuzzy separable space.

Theorem 4.2: If a sts (X, τ) is smooth second countable then it is smooth separable space.

Proof:

Let (X,τ) is smooth second countable space $\Longrightarrow$ (X,τ_α) is fuzzy second countable space $\forall\alpha\in(0,1]$ by (Definition 3.1) $\Longrightarrow$ (X,τ_α) is fuzzy separable space $\forall\alpha\in(0,1]$ by (Theorem 4.1) $\Longrightarrow$ (X,τ) is smooth separable space by (Definition 4.1).

Remark 4.1: The converse is not true in general.

Example 4.1: Let X be any infinite set and $\tau: I^X \rightarrow I$ be defined as follows:

$$\tau(A) = \begin{cases} 1 & \text{if } A = \emptyset \\ r & \text{if } \mu_A(x) = r \ \forall x \in X \end{cases}$$

It's clear that τ is smooth topology on X. (X,τ) has not smooth countable base but the sequence $\{p_x^{\frac{1}{n}}\}_{n=1}^{\infty}$ for fixed $x\in X$, is countable sequence and $\forall A\in I^X$ with $\tau(A)\geq\alpha$ for some $\alpha\in(0,1]$ then there exist member of this sequence belong to A.

In the following, we will explain the property of smooth separable is not hereditary.

Example 4.2: Let X be any uncountable set and $\tau: I^X \rightarrow I$ be defined as follows:

$$\tau(A) = \begin{cases} 1 & \text{if } A = \emptyset \text{ or } X \\ \frac{1}{2} & \text{if } \mu_A(x) \neq 0 \ for\ fixed\ x \in X \end{cases}$$

It's clear that τ is smooth topology on X. (X,τ) is smooth separable since the sequence $\{p_x^{\frac{1}{n}}\}_{n=1}^{\infty}$ is countable sequence and every $A\in I^X$ with $\tau(A)\geq\alpha$ for some $\alpha\in(0,1]$ there exist member of this sequence belong to A.

Let $B\in I^X$ with $\mu_B(x)=0$ and support of B is uncountable set then for any fuzzy point p_y^r in B when $x\neq y$, $\tau_B(\{p_y^r\})\geq\alpha$, if B relative separable $\Longrightarrow$ support of B is countable, but this contradiction since the support of B is uncountable set.

In the following, we will prove the smooth separable is topological property.

Theorem 4.3: Let $f:(X,\tau) \to (Y,\sigma)$ be a smooth continuous and surjective mapping where τ and σ are smooth topology on X and Y, respectively. If (X,τ) is smooth separable space then so (Y,σ).

Proof:

Let $\{p_{x_i}^{r_i}\}_{i=1}^{\infty}$ be a countable sequence in X then $\{f(p_{x_i}^{r_i})\}_{i=1}^{\infty}$ is countable sequence in Y, let $B \in \mathrm{I}^{\mathrm{Y}}$ with $\sigma(B) \geq \alpha$ for some $\alpha \in (0,1] \Longrightarrow f^{-1}(B) \in \mathrm{I}^{\mathrm{X}}$ and $\tau(f^{-1}(B)) \geq \alpha \Longrightarrow$ there exist $p_{x_i}^{r_i} \in \{p_{x_i}^{r_i}\}_{i=1}^{\infty}$ such that $p_{x_i}^{r_i} \in f^{-1}(B) \Longrightarrow f(p_{x_i}^{r_i}) \in B \Longrightarrow (Y,\sigma)$ is smooth separable.

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Journal of AL-Qadisiyah for computer science and mathematics
Vol.7 No.1 Year 2015

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