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Bornological Transformation Group

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ABSTRACT

In this paper, the researcher recalls the definitions of bornological set & bornological group and gives some examples in detalis. Additionally, the primary goal of this research is to introduce bornological transformation group, which are formulated on bornological group acts on bornological set. We observe that in the bornological transformation group bornological set can be partitioned into orbits. The main important part was that, researcher shows that the bornological transformation group is bornological isomorphism.

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1. Introduction

A bornology on a set X is a family β of subsets of X, such that : β covers X, also β stable under hereditary and finite union. The elements of β are called bounded subsets of X and (X, β) is called bornological set. It is worth to give some background about some construction of bornological set. We start with the bounded map it is a map f: (X, β) ⟶ (Y, β') is called bounded map if the image of every bounded subset of (X, β) is bounded in (Y, β') and a bijective f: (X, β) ⟶ (Y, β') is said to be a bornological isomorphism if f and f^{-1} are bounded maps.

Historically, the idea of a bounded subset of a topological vector space was introduced by von- Neuman, it played an important role in functional analysis that motivated the concept of more general and abstract classes of bounded set it is called bornology. The concept of bornology on set was defined by H. Hogbe – Niend as a collection of subsets β of a set X such that, β covers X, and have hereditary property and finite union. Therefore, (X, β) is called bornological set and the elements of β are called bounded set. The importance of bornology is to solve the problem of boundedness of sets, function and space in general way not just by norm or usual definition of bounded set for all these structures. Also, there are many important applications of bornological space, such as in spying ware program KPJ, when they want to determine person location or the identity of person from his finger print or from an eye print and in satellite broadcast system to determine the limits of the broadcast area [1]. That means, this structure it is general solution to solve the bounded problem. Researchers start to study abstract algebraic

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bornological structures which it is bornological group, bornological ring, bornological module to solve the problem of boundedness of the group or ring or module instead of a set of elements. Which they define as algebraic structure (group, ring) over bornological set for which required that the structure maps are bounded map. In another word, a bornological group is set with two structures group and bornology such that the product map and inverse map are bounded. For more details in bornological group we refer to [2]. In (2014) Bambozzi by [3] studied basic properties of algebraic bornological structure which is the category theory of bornological group, because the new knowledge put every new structures in category. In (2014) D. P. Pombo Jr. by [4] studied bornological group of continuous mappings. Note that every group can be turned into a bornological group by providing it with the discrete bornology [5]. However, the problem of existence of non-discrete bornology on infinite group which would make them into bornological group. This problem was solved by [6] in 2017, by introducing new structure bornological semi group (BSG). Bornological semi group can be easily done by taken semi group instead of the group and require the product map is bounded, so the problem of boundedness for this kind of group, which cannot bornologize because the inverse map is not bounded was solved. But, there is such kinds of groups which cannot bornologize because the product map is not bounded. This problem was solved by introducing a new structure semi bornological group to solve this kind of problems [7]. When a bornological group acts on a bornological set, this process is calls bornological transformation group. The effect of it is to divide a bornological set on to classes of orbitals and bornological transformation group are used to study bounded symmetries which have many applications. For example, in physics. The main important result is that the bornological transformation group is bornological isomorphism.

The outline of this paper is as follows. Section one, is a brief introduction to bornological set, bornological group and bornological transformation group. In section two, researcher presents examples of bornological set and bornological group. Then researcher shows important result related to bornologicl transformation group that was in section three. Another information instead of (X, β) we used X and instead of (G, β) we used G.

2. Bornological set and bornological group

We recall the definition of bornological set, bornological group and we give some examples in details.

Definition (2. 1) [1]:

A bornological on a set X is, $\beta \subseteq P(X)$ s.t.:

(i)$X = \bigcup_{B\in\beta} B$;

(ii) If $A \subset B$ and $B \in \beta$ then $A \in \beta$;

(iii) If $B_1, B_2 \in \beta$, then $B_1 \cup B_2 \in \beta$.

A pair (X,β) is called a bornological set, and the elements are calls bounded sets.

Definition (2. 2)[1]:

Any sub collection of a bornology collection construct a base for bornology if every element of the bornology is contain in an element of the base. And we denote it by β_0.

<u>Example (2. 3):</u>

Let X= {1, 2, 3}.

$\beta = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, X\}$.

To satisfy that β is a bornology of X, we must satisfy three conditions:

(i) Since $X \in \beta$, then β covers X;

(ii) If $A \subset B$, $B \in \beta$, then $A \in \beta$; Thus being an elements of β is a hereditary property.

(iii) $\bigcup_{i=1}^{n} B_i \in \beta$, $\forall B_1,\dots, B_n \in \beta$.

Since

{1} U {2} = {1, 2} {1, 2} U { 2, 3} = X.

{1} U {3} = {1, 3} {1, 3} U { 2, 3} = X.

{2} U {3} = {2, 3} {2, 3} U { 1, 3} = X.

Then β is a bornology on X.

(the power set of the set X is the set of all subset of X including the empty set and X itself and $2^{|x|}$ is the cardinality of the power set of the set X) , Since β is the set of all subsets of X, i.e. β = p(X) =2^{x}

Now to find a base for β.

Let β_0 = { X }

or β_0 = {{1, 2}, {1, 3}, {2, 3}, X}.

Definition (2. 4)[7]:

A bornological group is a set G with two structures, G is a group and β is bornological set such that, the product map $f: G\times G \to G$ and the inverse map $f^{-1}: G \to G$ are bounded.

Example (2. 5):

Let $GL(n,\mathbb{R})$ = ($\begin{bmatrix} a & b \\ c & d \end{bmatrix}_{n\times n}$, +) be the general linear group which is the set of matrix elements and additive operation.

We can define a bornology β on this group, which is the finite bornology, that is the collection of all finite subsets.

To prove GL(n, F) with finite bornology β is a bornological group (GL(n, ℝ), β). We must prove that,

1) $f: (GL(n,\mathbb{R}),\beta)\times(GL(n,\mathbb{R}),\beta)\to(GL(n,\mathbb{R}),\beta)$.

Let S_1, S_2 be two bounded sets (finite sets) in $(GL(n,\mathbb{R}),\beta)$ we must prove that $f\ (S_1\times S_2)$ is bounded.

$f\ (S_1\times S_2) = S_1 + S_2 = \{s_1 + s_2\colon\ s_1 \in GL(2,\mathbb{R}),\ s_2 \in GL(2,\mathbb{R})\}$

$= \{s_1 + s_2 : s_1 = \begin{bmatrix} h & i \\ j & k \end{bmatrix}$ and $s_2 = \begin{bmatrix} l & a \\ p & r \end{bmatrix}\}$

$= \{\begin{bmatrix} h+l & i+a \\ j+p & k+r \end{bmatrix} : a,\ h.\ l, i,\ j, k, p, r\ \in R\} \subset (GL(2,\ \mathbb{R}),\ \beta).$

Thus, the product map is bounded.

2) $f^{-1}: (GL(n,\ \mathbb{R}),\ \beta) \to (GL\ (n,\mathbb{R}),\ \beta)$.

Let $S \in (GL(n,\mathbb{R}),\ \beta)$, $S = \{\, s\colon s = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\}$ which it is finte set.

Then, $S^{-1} = -S = \{-s\colon -s = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}\} \subset (GL(2,\mathbb{R}),\ \beta)$. So, f^{-1} is bounded.

Then $(GL(n,\ \mathbb{R}),\ \beta)$ is a bornological group.

3. bornological transformation group

In this section, we substantiate some fundamental facts which is it only true for bornological transformation group.

Definition (3. 1) [8]:

A bornological transformation group is a (G, X, α) where G is a bornological group, X is a bornological set and $\alpha : G \times X \longrightarrow X$ is a bounded function satisfy two conditions:

1) $\alpha(g_1, \alpha(g_2, x)) = \alpha(g_1 g_2, x)$ for all $g_1, g_2 \in G$, $x \in X$.

2) $\alpha(e, x) = x$ $\forall\ x \in X$, and $e \in G$.

Then α is bornological transformation group and X is called a left G – bornological set.

Further, the notation $g \bullet x$ (even just $g\,x$) will be used for $\alpha(g, x)$, So that (1) and (2) become

$$g_1 \bullet (g_2 \bullet x) = (g_1 g_2) \bullet x \quad \text{and} \quad e \bullet x = x.$$

<u>Example (3. 2):</u>

(1) Let $G = (\mathbb{Z}, +, \beta_1)$ be a bornological group, where β_1 is the collection of all finite subsets of $\mathbb{Z}$ and $\mathbb{R}^2$ with the usual bornology β_2 be bornological set.

Define

$$\alpha : \mathbb{Z} \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2 .$$

As follows:

$$\alpha(u, (a_1, a_2)) = (u + a_1, u + a_2).$$

For all $u \in \mathbb{Z}$, $(a_1, a_2) \in \mathbb{R}^2$. Then α is bornological transformation group.

Indeed

* $\alpha(0, (a_1, a_2)) = (a_1, a_2)$.

* $\alpha(u_1, \alpha(u_2, (a_1, a_2))) = \alpha(u_1, (u_2 + a_1, u_2 + a_2))$

$$= (u_1 + u_2 + a_1, u_1 + u_2 + a_2)$$

$$= \alpha(u_1 + u_2, (a_1, a_2)) \quad \text{for each } u_1, u_2 \in \mathbb{Z}.$$

* Let $B = D_1 \times D_2$ be a bounded subset of $\mathbb{Z} \times \mathbb{R}^2$.

Then,

$\alpha(B) = \alpha(D_1 \times D_2) = \{(u + a_1, u + a_2) : u \in D_1, (a_1, a_2) \in D_2\}$.

$$= \bigcup_{u \in D_1} \{(u + a_1, u + a_2 : (a_1, a_2) \in D_2\}.$$

$$= \bigcup_{u \in D_1} \alpha(u, (a_1, a_2)).$$

$$= \bigcup_{u \in D_1} \alpha \quad (u, \ D_2).$$

$$= \bigcup_{u \in D_1} u \bullet \ D_2.$$

Which is bounded in $\mathbb{R}^2$ as a union of finite bounded sets $\alpha\,(n, D_2)$.

It is worth to mention that if β_1 is the discrete bornology then the action define above cannot be bornological transformation group.

(2) Let G = ($\mathbb{R}^+$, ·, β) where β is the usual bornology.

Consider,

X = $\mathbb{R}$ is a bornological set.

Define

$$\alpha : \ \mathbb{R}^+ \times \mathbb{R} \ \rightarrow \ \mathbb{R}.$$

By

$$\alpha\,(\text{t}, \ \text{r}) \ = \ \text{r} + \ln t, \qquad \text{r} \in \mathbb{R}, \text{t} \in \mathbb{R}^+.$$

Then

* $\alpha\,(1, \text{ r}) = \text{r} + \ln 1 = \text{r}, \qquad \text{r} \in \mathbb{R}.$

* $\alpha\,(t_1, \ \alpha\,(t_2, \text{r})) = \alpha\,(t_1, \ \text{r} + \ln t_2)$

$= \text{r} + \ln t_1 + \ln t_2 = \text{r} + \ln (t_1\, t_2)$

$= \alpha\,(t_1\, t_2, \text{r})$ where $t_1\, t_2 \in \mathbb{R}^+$.

* α is bounded.

Hence $\mathbb{R}$ is a $\mathbb{R}^+$-bornological set.

It is worth to mention that if we have discrete bornology β on infinite group G acts on any bornological set (X, β′).

3.1. Orbits and Orbital Bornological Set.

The observation of the bornological transformation group is to divide the bornological set into orbits.

Definition (3. 3):

The orbit of x under the action of G is the subset $\text{O}(x) = \{\text{u} \in \text{X} : \ \exists\, g \in \text{G}$ such that, $\text{u} = g \bullet x\}$. By generalization of this, D ⊂ X, the union of all orbits of points of D is $\text{G(D)} = \{ g \bullet b : \ g \in \text{G}, \text{b} \in \text{D}\}$. For a bounded set D ⊂ X, and a subgroup H of G, $\text{H(D)} = \{ h \bullet \text{b} : \ \text{h} \in \text{H}, \ \text{b} \in \text{D}\}$. The action α of G on X defines an equivalence relation as follows. For each x, y ∈ X, xRy if and only if there exists $g \in \text{G}$ such that, $\alpha\,(g, x) = g \bullet x = \text{y}$. The equivalence classes with respect to this equivalence relation are the orbits of the elements of X.

Theorem (3. 4):

The bornological transformation group is bornological isomorphism.

Proof :

Let $\alpha : G \times X \longrightarrow X$.

Be a bornological transformation group of a bornological group G on a bornological set X. Every element $g \in G$ determines a bounded translation α_g of X onto itself,

Defined by : $\alpha_g(x) = \alpha(g, x)$ for each $x \in X$.

Then by (1) from definition (3. 1).

$$\alpha_h \, o \, \alpha_g = \alpha_{hg} .$$

And by (2) from definition (3. 1), $\alpha_e = I_x$, the identity mapping of X onto itself, thus

$$\alpha_g \, o \, \alpha_{g-1} = \alpha_{gg-1} = \alpha_e = \alpha_{g-1} \, o \, \alpha g .$$

Hence, $(\alpha)^{-1} = \alpha_{g^{-1}}$ is the bounded inverse mapping of α_g , which shows that each α_g is an isomorphism of bornological set X .

4. Conclusion

In this paper, the observation that left (right) translation is bounded map that motivate us to consider bornological transformation group. When a bornological group acts on a bornological set, this process is called a bornological transformation group such that the work of the bounded action is to divide a bornological set in to classes of orbitals and bornological transformation group , are used to study bounded symmetries which have many applications, for example, in physics. Finally, the main result is that the bornological transformation group is brnological isomorphism.

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