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ON Locally Lindelöf Spaces,Co-Lindelöf Topologies and Locally LC-Spaces

Reyadh Delfi Ali*

Adam Abdulla Abker**

*** Department of Mathematics, Faculty of Science Omdurman Islamic University, Sudan**

Reyadh_delphi@yahoo.co.uk

** Department of Mathematics, Faculty of Education Wadi Al Nile University, Sudan

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Abstract:

The aim of this paper is to continue the study of locally Lindelöf spaces,Co-Lindelöf topologies and locally $LC-spaces$. We also study their relationships to $L_i-spaces \quad i=1,2,3,4$.

المستخلص

تهدف هذه الورقة دراسة فضاءات locally Lindelöf **وتبولوجيات** Co- Lindelöf **و فضاءات** $LC-$ locally **وايضا درسنا علاقة علاقة هذه الفضاءات مع الفضاءات** $L_i -$ $i=1.2,3,4$.

KEYWORDS: $LC-space$, $P-space$, Lindelöf, locally Lindelöf spaces, Co- Lindelöf topologies, locally $LC-spaces$

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Mathematics Subject Classification :54xx

1. Introduction:

Dontchev, Ganster and Kanibir $[2]$introduced the class of locally Lindelöf and weakly locally Lindelöf by definitions, a topological space (X,T) is called locally Lindelöf (resp. weakly locally Lindelöf) if each point of X has a closed Lindelöf (resp. Lindelöf) neighborhood.

In 1984, Gauld, Mrsevic, Reilly and Vamanamurthy$[8]$ introduced the Co- Lindelöf topology of a given space(X,T).They showed that $l(T)=\{\phi\}\cup\{G\in T: X-G \quad is \quad Lindelof \quad in \quad (X,T)\}$ is a topology on X with $l(T)\subseteq T$, called the Co- Lindelöf topology of (X,T).

Ganster, Kanibir and Reilly$[6]$ introduced the class of locally $LC-spaces$ .By definition, a topological space (X,T) is called a Locally $LC-space$ if each point of X has a neighborhood which is an $LC-subspace$. In$[6]$, the authors proved that a space (X,T) is an $LC-space$ if each point of X has a closed neighborhood that is an $LC-subspace$.Thus every regular locally $LC-space$ is an $LC-space$,a result first proved by Hdeib and Pareek in$[10]$.

A set F in a topological space is called $F_\sigma-closed$ if it is the union of at most countably many closed sets. A set G is called a $G_\sigma-open$ if it is the intersection of at most countably many open sets$[4]$.

In this paper, we consider and study of locally Lindelöf spaces, Co- Lindelöf topologies and locally $LC-spaces$ .Furthermore, basic properties, preservation theorems and relationships of locally Lindelöf spaces, Co- Lindelöf topologies and locally $LC-spaces$, are investigated. Moreover, to obtain several characterization and properties of locally Lindelöf spaces, Co- Lindelöf topologies and locally $LC-spaces$.

Our terminology is standard. The closure of a subset A of a space (X,T) is denoted by clA .The set of all positive integer is denoted by ω .

2. locally Lindelöf and weakly locally Lindelöf :

Definition2.1 [2]: A topological space (X,T) is called locally Lindelöf (resp. weakly locally Lindelöf) if each point of X has a closed Lindelöf (resp. Lindelöf) neighborhood. It follows immediately from the definition that every locally Lindelöf space is a weakly locally Lindelöf.

Note that a weakly locally Lindelöf space need not be a locally Lindelöf space.

Definition2.2: A topological space (X,T) is an $LC-space$ if every Lindelöf subset of X is closed $[7],[13]$.Notice that $LC-space$ is also known under the name $L-closed$ $[9],[11]$ and $[14]$.

Definition2.3[12]: A topological space (X,T) is called $P-space$ if every $G_\sigma-open$ set in X is open .

Definition2.4 [2]: A topological space (X,T) is called

(1) an $L_1-space$ if every Lindelöf $F_\sigma-closed$ is closed,

(2) an $L_2-space$ if clL is Lindelöf whenever $L\subseteq X$ is Lindelöf,

(3) an $L_3-space$s if every Lindelöf subset L is an $F_\sigma-closed$,

(4) an $L_4-space$ if whenever $L\subseteq X$ is Lindelöf, then there is a Lindelöf $F_\sigma-closed$ F with $L\subseteq F\subseteq clL$.

Theorem2.5 [2]:

(i) If (X,T) is an $LC-space$, then (X,T) is a $L_i-space$, i=1,2,3,4.

(ii) If $(X,T)$is an $L_1-space$ and an $L_3-space$, then(X,T) is an $LC-space$.

(iii)) Every space which is $L_1-space$ and $L_4-space$ is an $L_2-space$.

(iv) Every $L_2-space$ is an $L_4-space$ and every $L_3-space$ is an $L_4-space$.

(v) Every $L_3-space$ is T_1.

(vi) Every Lindelöf space is an $L_2-space$,and every $L_2-space$ having a dense Lindelöf

Subset is Lindelöf.

(vii) Every $P-$ space is an $L_1-space$.

Definition2.6 [2]: A topological space (X,T) is called a $Q-set$ space if each subset of X is an $F_\sigma-closed$ sets.

Theorem2.7 [12]: Every Huasdorff $P-space$ is an $LC-space$.

Corollary2.8: Every Tychonoff $P-space$ is an $LC-space$.

Proof. Obvious.

Proposition2.9 [2]: Every weakly locally Lindelöf $L_2-space$ is locally Lindelöf,and so Every weakly locally Lindelöf space which is $L_1 and L_4$ is locally

Lindelöf.

Theorem2.10 [2]:

Every locally Lindelöf space(X,T) is an $L_1-space$ if and only if it is a $P-space$.

Corollary2.11 [2]: Every Huasdorff , locally Lindelöf $L_1-space$ is an $LC-space$.

Corollary2.12 [2]: Every weakly locally Lindelöf $LC-space$ (X,T) is a $P-space$.

Corollary 2.13: For a Lindelöf space X the following are equivalent:

(a) X is locally Lindelöf.

(b) X is a weakly locally Lindelöf.

Proof. This is obvious by definition 2.1.

Corollary 2.14: For an $L_2 - space$ X the following are equivalent:

(a) X is locally Lindelöf.

(b) X is a weakly locally Lindelöf .

Proof. (a) $\Rightarrow$(b): This is obvious by definition 2.1.
(b) $\Rightarrow$(a): This is obvious by proposition 2.9.

Corollary 2.15: For a $LC - space$ X the following are equivalent:

(a) X is locally Lindelöf .

(b) X is a weakly locally Lindelöf .

Proof. (a) $\Rightarrow$(b): This is obvious by definition 2.1.
(b) $\Rightarrow$(a) : This is obvious by theorem 2.5(i)and proposition 2.9.

Theorem 2.16: For a Hausdorff locally Lindelöf space X the following are equivalent:

(a) X is an $LC - space$.

(b) X is an $L_1 - space$.

(c) X is a $P - space$.

Proof. (a) $\Rightarrow$(b): This is obvious by theorem 2.5(i).
(b) $\Rightarrow$(a): Let X be an $L_1 - space$, since X is a Hausdorff locally Lindelof space,
then X is an $LC - space$ by corollary 2.11.
(b) $\Rightarrow$(c) : This is obvious by theorem 2.10.
(c) $\Rightarrow$(b) : This is obvious by theorem 2.10.

Theorem2.17: For Hausdorff weakly locally Lindelöf space X the following are equivalent:

(a) X is an $LC - space$.

(b) X is a $P - space$.

Proof. (a) $\Rightarrow$(b):This is obvious by corollary 2.12.
(b) $\Rightarrow$(a):This is obvious by theorem 2.7.

Theorem2.18: Every P $Q-set$ space X is an $LC-space$.

Proof. If L is a Lindelöf subset in X ,which is a $Q-set$ space, then L is an $F_\sigma-closed$ set, but X is a $P-space$, so L is a closed set , hence X is an $LC-space$.

Theorem2.19: For a weakly locally Lindelöf $Q-set$ space X the following are equivalent:

(a) X is an $LC-space$.

(b) X is a $P-space$.

Proof. (a) $\Rightarrow$(b): This is obvious by corollary 2.12.
(b) $\Rightarrow$(a): This is obvious by theorem 2.18.

Theorem2.20: For a locally Lindelöf $Q-set$ space X the following are equivalent:

(a) X is an $LC-space$.

(b) X is a $P-space$.

(c) X is an $L_1-space$.

Proof. (a) $\Rightarrow$(b): This is obvious by theorem 2.5(i)and theorem2.10.
(b) $\Rightarrow$(a): This is obvious by theorem 2.18.
(b) $\Rightarrow$(c): This is obvious by theorem 2.10.
(c) $\Rightarrow$(b): This is obvious by theorem 2.10.

Corollary2.21: For a weakly locally Lindelöf $L_2-space$ X the following are equivalent:

(a) X is an $L_1-space$.

(b) X is a $P-space$.

Proof. This is obvious by proposition 2.9 and theorem 2.10.

Theorem 2.22: For a locally Lindelöf $L_3-space$ X the following are equivalent:

(a) X is an $LC-space$.

(b) X is an $L_1-space$.

(c) X is a $P-space$.

Proof. (a) $\Rightarrow$(b): This is obvious by theorem 2.5(i).
(b) $\Rightarrow$(a): This is obvious by theorem 2.5(ii).
(b) $\Rightarrow$(c): This is obvious by theorem 2.10.
(c) $\Rightarrow$(b): This is obvious by theorem 2.10.

Corollary2.23:

(i) Every weakly locally Lindelöf $LC-space$ is locally Lindelof.

(ii) Every $LC-space$ having a dense Lindelöf Subset is locally Lindelof.

Proof. Obvious.

Corollary2.24: For a regular locally Lindelöf $L_1-space$ X the following are equivalent:

(a) X is an $LC-space$.

(b) X is T_1.

Proof. (a) $\Rightarrow$(b): Obvious.

(b) $\Rightarrow$(a): Let X be a $T_1-space$,since X is a regular, then X is a Hausdorff .

Since X is a locally Lindelöf $L_1-space$, then X is an $LC-space$ by corollary 2.11.

Definition2.25[5]: A topological space $(X,T)$is a $R_1-space$ if x and y have disjoint neighborhoods whenever $cl\{x\}\neq cl\{y\}$ .Clearly a space is Hausdorff if and only if its T_1 and R_1.

Corollary2.26: For R_1 locally Lindelöf $L_1-space$ X the following are equivalent:

(a) X is an $LC-space$.

(b) X is T_1.

Proof. (a) $\Rightarrow$(b): Obvious.

(b) $\Rightarrow$(a): Let X be a $T_1-space$,since X isa R_1,then X is a Hausdorff by definition2.25.

Since X is a locally Lindelöf $L_1-space$,then X is an $LC-space$ by corollary 2.11.

Theorem2.27:

For a Tychonoff weakly locally Lindelöf space X the following are equivalent:

(a) X is an $LC-space$.

(b) X is a $P-$ space.

Proof. (a) $\Rightarrow$(b):This is obvious by corollary 2.12.
(b) $\Rightarrow$(a):This is obvious by corollary 2.8.

Theorem2.28: For $P-space$ X the following are equivalent:
(a) X is an $LC-space$.

(b) X is an $L_3-space$.

Proof. (a) $\Rightarrow$(b): This is obvious by theorem 2.5(i).
(b) $\Rightarrow$ (a): Let L be a Lindelöf subset of X ,then L is $F_\sigma-closed$ set (since X is an $L_3-space$),so L is closed set(since X is a $P-space$) ,hence X is an $LC-space$.

Theorem2.29: For a weakly locally Lindelöf $L_3-space$ X the following are equivalent:
(a) X is an $LC-space$.

(b) X is a $P-space$.

Proof. (a) $\Rightarrow$(b): This is obvious by corollary 2.12.
(b) $\Rightarrow$(a):This is obvious by theorem 2.28.

Corollary2.30: Every weakly locally Lindelöf P $L_3-space$ is locally Lindelöf.
Proof. Obvious.

Theorem2.31: For a Hausdorff locally Lindelöf space X the following are equivalent:
(a) X is an $LC-space$.

(b) X is a $P-space$ and an $L_2-space$.

Proof. (a) $\Rightarrow$(b): Let X be an $LC-space$, then X is an $L_2-space$ and an $L_1-space$

. Since X is a locally Lindelöf, then X is a $P-space$ by theorem 2.10.
(b) $\Rightarrow$(a):Let L be a Lindelöf subset of (X,T) and let $x\notin L$ .Since (X,T) is Hausdorff, for each $y\in L$ there exist an open set V_y containing y with $x\notin clV_y$.

Clearly $\{V_y : y \in L\}$ is a cover of L and so there exists a countable set $C \subseteq L$ such that $L \subseteq \bigcup_{y \in C} V_y \subseteq \bigcup_{y \in C} clV_y$ .For each $y \in C$, $L \cap clV_y$ is Lindelöf and so $cl(L \cap clV_y)$ is Lindelöf since (X,T) is an $L_2 - space$.

Furthermore, if $W = \bigcup_{y \in C} cl(L \cap clV_y)$ then W is a Lindelöf $F_\sigma - closed$ set and, since (X,T) is a $P - space$, W is a closed Lindelöf set not containing x .Thus $x \notin clL$ .This shows that L is closed in (X,T).

Theorem2.32: Every $Q - set$ $L_1 - space$ is an $L_2 - space$.

Proof. Let L be a Lindelöf subset of X , then L is an $F_\sigma - closed$ set (since X is a $Q - set$ space),so L is closed set(since X is an $L_1 - space$),then $L = clL$ and clL is a Lindelöf. Hence X is an $L_2 - space$.

Theorem2.33: For a locally Lindelöf $Q - set$ space X the following are equivalent:

(a) X is an $L_1 - space$.

(b) X is a $P - space$ and an $L_2 - space$.

Proof. (a) $\Rightarrow$(b): Let X be an $L_1 - space$,since X is a $Q - set$ space, then X is an $L_2 - space$ by theorem 2.32. Since X is a locally Lindelöf $L_1 - space$, then X is a $P - space$ by theorem 2.10.

(b) $\Rightarrow$(a): This is obvious by theorem 2.5(vii).

Definition2.34 [2]: A topological space (X,T) is called aweak $P - space$ if any countable union of regular closed sets is closed. One can show easily that (X,T) is aweak $P - space$ if and only if for every countable family $\{U_n : n \in \omega\}$ of open sets, $cl\left(\bigcup_{n \in \omega} U_n\right) = \bigcup_{n \in \omega} clU_n$.

Corollary2.35: Every $P - space$ (X,T) is a weak $P - space$.

Proof. Let F be a countable union of regular closed sets in $P - space$ (X,T), then F is an is an $F_\sigma - closed$ set, so F is a closed set(since X is a $P - space$),hence X is a weak $P - space$.

Corollary2.36:
(i) Every locally Lindelöf $LC-space\ (X,T)$ is weak $P-space$.
(ii) Every weakly locally Lindelöf $LC-space\ (X,T)$ is weak $P-space$.
(iii) Every Lindelöf $LC-space\ (X,T)$ is weak $P-space$.
Proof. Obvious.

Corollary2.36: For a Hausdorff locally Lindelöf space X the following are equivalent:

(a) X is an $LC-space$.

(b) X is a $P-space$ and an $L_2-space$.

(c) X is an $L_1-space$.

Proof. This is obvious by theorem 2.16 and theorem 2.33.

Theorem2.37: Every locally Lindelöf weak $P-space\ (X,T)$ is $L_4-space$.
Proof. Let L be a Lindelöf subset of (X,T). Each point of L has an open neighborhood U_x such that clU_x is Lindelöf. Pick accountable subset C of L such that $L \subseteq \bigcup_{x\in C} U_x$ .Since (X,T) is aweak $P-space$ we have $clL \subseteq \bigcup_{x\in C} clU_x = W$ .Since W is Lindelöf we conclude that clL is Lindelöf and closed ,so clL is Lindelöf $F_\sigma-closed$ set ,hence (X,T) is an $L_4-space$.

3. Co- Lindelöf Topologies:

Theorem3.1 [2]: For a $space\ (X,T)$ the following are equivalent:

(a) (X,T) is an $L_1-space$.

(b) $(X,l(T))$ is a $P-space$.

Corollary3.2: If (X,T) is Lindelöf space then $l(T)=T$.
Proof: Obvious.

Corollary3.3: If (X,T) is an $LC-space$ then $(X,l(T))$ is a $P-space$.
Proof: This is obvious by theorem 2.5(i) and theorem 3.1.

Theorem3.4: For a $L_3-space$ (X,T) the following are equivalent:

(a) (X,T) is an $LC-space$.

(b) $(X,l(T))$ is a $P-space$.

Proof. (a) $\Rightarrow$(b): This is obvious by corollary 3.3.

(b) $\Rightarrow$(a): Let $(X,l(T))$ be a $P-space$,then $(X,T)$is an $L_1-space$ by theorem 3.1.

Since (X,T) is an $L_3-space$, then (X,T) is an $LC-space$ by theorem 2.5(ii).

Theorem3.5 [1]: Every $Q-set$ space is an $L_3-space$.

Corollary3.6: Every $Q-set$ $L_1-space$ is an $LC-space$.

Proof. Let X be $Q-set$ space, then X is an $L_3-space$ by theorem3.5,since X is an $L_1-space$,then X is an $LC-space$ by theorem2.5(ii).

Theorem3.7: For a $Q-set$ space (X,T) the following are equivalent:

(a) (X,T) is an $LC-space$.

(b) $(X,l(T))$ is a $P-space$.

Proof. (a) $\Rightarrow$(b): This is obvious by corollary 3.3.

(b) $\Rightarrow$(a): Let $(X,l(T))$ be a $P-space$,then $(X,T)$is an $L_1-space$ by theorem 3.1.

Since (X,T) is a $Q-set$ space, then $(X,T)$is an $LC-space$ by corollary 3.6.

Theorem3.8: For a locally Lindelöf space (X,T) the following are equivalent:

(a) (X,T) is a $P-space$.

(b) $(X,l(T))$is a $P-space$.

Proof. (a) $\Rightarrow$(b): Let (X,T) be a $P-space$.Since (X,T) is a locally Lindelof space,then $(X,T)$is an $L_1-space$ by theorem 2.10,hence $(X,l(T))$ is a $P-space$ by theorem 3.1.

(b) $\Rightarrow$(a): Let $(X,l(T))$ be a $P-space$,then $(X,T)$is an $L_1-space$ by theorem 3.1.

Since (X,T) is a locally Lindelöf space, then $(X,T)$is an $P-space$ by theorem 2.10.

Definition3.9 [1]: A topological space (X,T) is said to be anti – Lindelöf if each Lindelof subset of X is countable.

Theorem3.10 [2]: Every T_1 anti- Lindelöf space is an $L_3-space$ .Hence every T_1 , anti- Lindelöf $L_1-space$ is an $LC-space$.

Theorem3.11: For a T_1 anti- Lindelöf space (X,T) the following are equivalent:

(a) (X,T) is an $LC-space$.

(b) $(X,l(T))$ is a $P-space$.

Proof. (a) $\Rightarrow$(b): This is obvious by corollary 3.3.

(b) $\Rightarrow$(a): Let $(X,l(T))$ be a $P-space$,then (X,T) is an $L_1-space$ by theorem 3.1. Since (X,T) is T_1 anti- Lindelöf space, then (X,T) is an $LC-space$ by theorem 3.10.

Theorem3.12 [2]: For a Hausdorff space X the following are equivalent:

(a) X is an $LC-space$.

(b) X is an $L_1-space$ and an $L_2-space$.

Theorem3.13: For a Hausdorff $L_2-space$ (X,T) the following are equivalent:

(a) (X,T) is an $LC-space$.

(b) $(X,l(T))$ is a $P-space$.

Proof. (a) $\Rightarrow$(b): This is obvious by corollary 3.3.

(b) $\Rightarrow$(a): Let $(X,l(T))$ be a $P-space$,then (X,T) is an $L_1-space$ by theorem 3.1.

Since (X,T) is a Hausdorff $L_2-space$, then (X,T) is an $LC-space$ by theorem 3.12.

Theorem3.14 [4]: Acountable union of Lindelöf subset is Lindelöf.

Theorem3.15: Every Lindelöf $L_1-space$ is a $P-space$.

Proof. For each $n \in \omega$, let A_n be closed in Lindelöf $L_1-space$ X and $A=\bigcup_{n\in\omega} A_n$,then A_n is a Lindelöf subset in X and thus A is a Lindelöf subset in X by theorem3.14. Since X is an $L_1-space$, then A is closed in X ,hence X is a $P-space$.

Theorem3.16: For a Hausdorff Lindelöf space X the following are equivalent:

(a) X is an $LC-space$.

(b) X is an $L_1-space$.

Proof. (a) $\Rightarrow$(b): This is obvious by theorem 2.5(i).

(b) $\Rightarrow$(a): Let X be an $L_1-space$, since X is a Lindelöf space ,then X is a $P-space$ by theorem 3.15.Since X is a Hausdorff space, then X is an $LC-space$ by theorem 2.7.

Theorem3.17: For a Hausdorff Lindelöf space $(X,T)$the following are equivalent:

(a) (X,T) is an $LC-space$.

(b) $(X,l(T))$ is a $P-space$.

Proof: (a) $\Rightarrow$(b): This is obvious by corollary 3.3.

(b) $\Rightarrow$(a): Let $(X,l(T))$ be a $P-space$,then $(X,T)$is an $L_1-space$ by theorem 3.1.

Since(X,T) is a Hausdorff Lindelöf space, then$(X,T)$is an $LC-space$ by theorem3.16.

Theorem3.18: For a Hausdorff locally Lindelöf space $(X,T)$the following are equivalent:

(a) (X,T) is an $LC-space$.

(b) $(X,l(T))$is a $P-space$.

Proof: (a) $\Rightarrow$(b): This is obvious by corollary 3.3.

(b) $\Rightarrow$(a): Let $(X,l(T))$ be a $P-space$,then $(X,T)$is an $L_1-space$ theorem 3.1.

Since (X,T) is a Hausdorff locally Lindelöf space, then $(X,T)$is an $LC-space$ by corollary 2.11.

Corollary3.19: If $(X,T)$is an $L_1-space$ then$(X,l(T))$ is an $L_1-space$.

Proof. Let (X,T) be an $L_1-space$, then $(X,l(T))$is a $P-space$ by theorem3.1.Hence $(X,l(T))$ is an $L_1-space$ by theorem 2.5(vii).

Definition3.20[7]: A topological space(X,T) is $cid-space$ if every countable subset of X is closed and discrete.

Remark3.21[7]: Every $LC-space$ is $cid-space$.

Theorem3.22: For anti – Lindelöf space X the following are equivalent:

(a) X is an $LC-space$.

(b) X is $cid-space$.

Proof. (a) $\Rightarrow$(b): This is obvious by remark 3.21.

(b) $\Rightarrow$ (a): Let L be a Lindelöf subset of X, then L is countable (since X is anti – Lindelöf), so L is a closed set(since X is $cid-space$), hence X is an $LC-space$.

Corollary3.23: If (X,T) is a anti- Lindelöf $cid-space$ then $(X,l(T))$ is a $P-space$.

Proof: Let (X,T) be a anti- Lindelöf $cid-space$, then (X,T) is an $LC-space$ by theorem 3.22.Hence $(X,l(T))$ is a $P-space$ by corollary 3.3.

Theorem3.24 [2]: Every $T_1\,L_1$ $-$ space is cid.

Theorem3.25: For a T_1 anti- Lindelöf space (X,T) the following are equivalent:

(a) (X,T) is $cid-space$.

(b) $(X,l(T))$ is a $P-space$.

Proof. (a) $\Rightarrow$(b): Let (X,T) be $cid-space$.Since (X,T) is a anti-Lindelof space, Then (X,T) is an $L_1-space$ by theorem 3.22and theorem 2.5(i), hence $(X,l(T))$ is a $P-space$ by theorem 3.1.

(b) $\Rightarrow$(a): Let $(X,l(T))$ be a $P-space$,then (X,T) is an $L_1-space$ by theorem3.1.

Since (X,T) is T_1 space, then (X,T) is $cid-space$ by theorem 3.24.

Theorem3.26: If $(X,l(T))$ is a Lindelöf $LC-space$ then (X,T) is an $L_1-space$.

Proof. For each $n\in\omega$, let A_n be closed and Lindelöf in (X,T) and let $A=\bigcup_{n\in\omega}A_n$.

Since $(X,l(T))$ is a Lindelöf $LC-space$, then each A_n is closed and Lindelöf in $(X,l(T))$ and so A is also closed and Lindelöf in $(X,l(T))$ by theorem3.14. .Hence A is closed in (X,T) and so (X,T) is an $L_1-space$.

4. Locally LC- spaces:

Definition4.1 [6]: A topological space (X,T) is called a Locally $LC-space$ if each point of X has a neighborhood which is an $LC-subspace$.

Clearly every $LC-space$ is locally $LC-space$ .In general the converse needs not be true[5], however every regular locally $LC-space$ is $LC-space$.

Definition4.2[6]: A topological space (X,T) is called an $LC-space$ if each point of X has a closed neighborhood that is an $LC-subspace$.

Theorem4.3[3]: Every locally $LC-space$ is T_1.

Theorem4.4: For a regular $P-space$ X the following are equivalent:

(a) X is an $LC-space$.

(b) X is a locally $LC-space$.

(c) X is an $T_1-space$.

Proof. (a) $\Rightarrow$(b): This is obvious by definition4.1.

(b) $\Rightarrow$ (a): This is obvious by definition4.1.

(b) $\Rightarrow$ (c):This is obvious by theorem 4.3.

(c) $\Rightarrow$ (b):Let X be a $T_1-space$,since X is a regular ,then X is a Hausdorff. Since X is a $P-space$, then X is an $LC-space$ by theorem 2.7, hence X is a locally $LC-space$ by definition4.1.

Corollary4.5: For a Finite topological $space$ X the following are equivalent:

(a) X is an $LC-space$.

(b) X is a locally $LC-space$.

Proof. Obvious .

Theorem4.6: For a R_1 $P-space$ X the following are equivalent:

(a) X is an $LC-space$.

(b) X is a locally $LC-space$.

(c) X is $T_1-space$.

(d) X is a Hausdorff space .

Proof. (a) $\Rightarrow$(b): This is obvious by definition4.1.

(b) $\Rightarrow$ (a): This is obvious by theorem 4.3, definition2.25and definition4.1.

(b) $\Rightarrow$(c): This is obvious by theorem 4.3.

(c) $\Rightarrow$ (b): Let X be a $T_1-space$,since X is a R_1 ,then X is a Hausdorff by definition 2.25.Since X is a $P-space$, then X is an $LC-space$ by theorem 2.7,

hence X is a locally $LC-space$ by definition4.1.

(c) $\Rightarrow$ (d):This is obvious by definition2.25.

(d) $\Rightarrow$ (c):Obvious .

Theorem4.7: For a regular L_1 $L_2-space$ X the following are equivalent:

(a) X is a locally $LC-space$.

(b) X is $T_1-space$.

Proof. (a) $\Rightarrow$(b): This is obvious by theorem 4.3.

(b) $\Rightarrow$ (a): Let X be a $T_1-space$,since X is a regular L_1 $L_2-space$, then X is an $LC-space$ by theorem 3.12, hence X is a locally $LC-space$ by definition4.1.

Theorem4.8: For a regular locally Lindelöf $L_1-space$ X the following are equivalent:

(a) X is a locally $LC-space$.

(b) X is $T_1-space$.

Proof. (a) $\Rightarrow$(b): This is obvious by theorem 4.3.

(b) $\Rightarrow$ (a) : Let X be a $T_1-space$,since X is a regular locally Lindelöf $L_1-space$, then X is an $LC-space$ by corollary 2.11,hence X is a locally $LC-space$ by definition4.1.

Theorem4.9 [3]: Every locally compact Hausdorff space is T_3 .

Theorem4.10: For a locally compact $R_1-space$ X the following are equivalent:

(a) X is an $LC-space$.

(b) X is a locally $LC-space$.

Proof. (a) $\Rightarrow$(b): This is obvious by definition4.1.

(b) $\Rightarrow$ (a) : Let X be a locally $LC-space$,then X is a $T_1-space$ by theorem 4.3.

Since X is a $R_1-space$, then X is a Hausdorff by definition2.25. Since X is a locally compact, so X is a regular by theorem 4.9 , hence X is an $LC-space$ by definition 4.1.

Proposition4.11 [3]: For a $space$ X the following are equivalent:

(a) X is a locally $LC-space$.

(b) Every point of X has an open neighborhood, which is an $LC-subspace$ of X .

Theorem4.12: Every 2^{nd} countable (C_{11}), a locally $LC-space$ is discrete.

Proof. Let (X,T) be 2^{nd} countable and a locally $LC-space$. We may assume that every point $x \in X$ has an open neighborhood U that is both hereditarily Lindelöf and an $LC-space$ (since X is a 2^{nd} countable and by Proposition 4.11).But this means that U is an open discrete subspace of (X,T).Hence (X,T) is discrete.

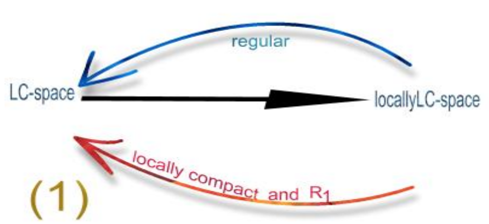
Theorem4.13: If (X,T) a regular space has an open cover by locally $LC-subspaces$,
then X is an $LC-space$.

Proof. Let $X = \bigcup_{i \in I} G_i$ be an open cover of X where each G_i is a locally $LC-space$,and let $x \in X$ .Choose $j \in I$ such that $x \in G_j$ .If U_J is an open and closed neighborhood(since X is a regular)of x in G_j such that U_J is an $LC-space$ of G_j,then U_J is also open and closed in (X,T). By definition4.2, (X,T) is an $LC-space$.

Theorem4.14: If (X,T) a regular space has an open cover by $LC-subspaces$,
then X is an $LC-space$.

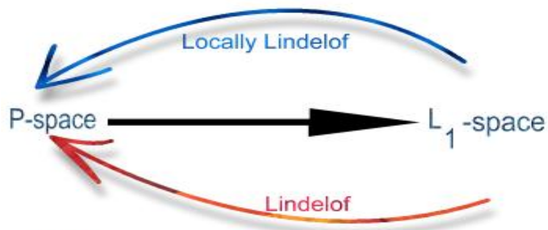
Proof. Let $X = \bigcup_{i \in I} G_i$ be an open cover of X where each G_i is $LC-space$,and let $x \in X$ .Choose $j \in I$ such that $x \in G_j$ .If U_J is a closed neighborhood(since X is a regular)of x in G_j such that U_J is an $LC-space$ of G_j,then U_J is also closed in (X,T). By definition4.2, (X,T) is an $LC-space$.

We have the following diagrams:



(1)

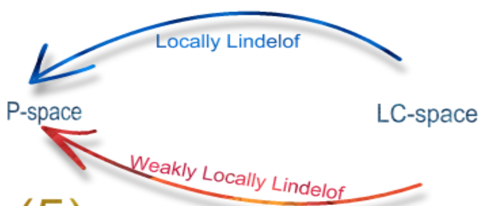
(2)


Locally Lindelof
P-space
L 1 -space
Lindelof

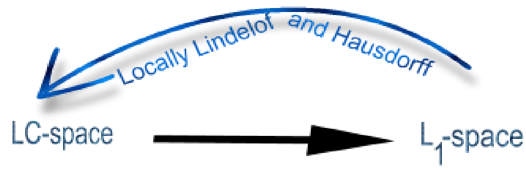
(3)


LC-space
Haudorff-space
L 1 Locally Lindelof

(4)


Locally Lindelof
P-space
LC-space
Weakly Locally Lindelof

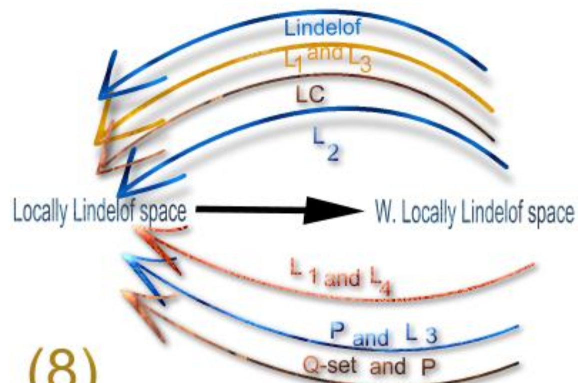
(5)


Locally Lindelof and Hausdorff
LC-space
L1-space

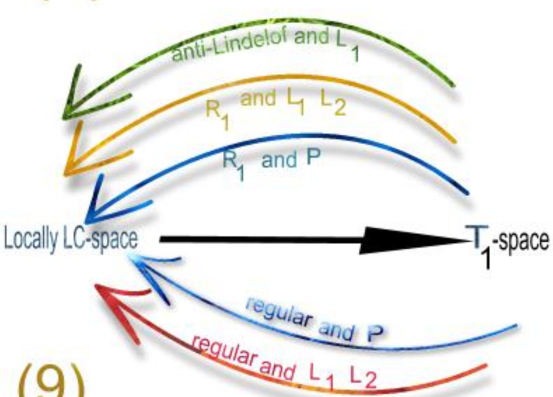
(6)


Locally Lindelof regular L 1
LC-space
T1-space

(7)


Lindelof
L1 and L3
LC
L 2
Locally Lindelof space
W. Locally Lindelof space
L 1 and L4
P and L 3
Q-set and P

(8)


anti-Lindelof and L 1
R1 and L1 L2
R1 and P
Locally LC-space
T1-space
regular and P
regular and L 1 L2

(9)

Refereces

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