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Principally ⊕-g-supplemented modules

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ABSTRACT

In this paper, we defined and studied the idea of principally ⊕-g-supplemented modules as an advanced concept of ⊕-g-supplemented modules. Many properties, characterizations and examples of these modules are discussed. Also, a number of relations between these modules and other kinds of modules are examined in this work.

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1. Introduction

Throughout this work, R denotes an associative ring with identity, and all modules are unital right R-modules. Let M be a right R-module. The notions $L \subseteq M$, $L \leq M$ and $L \leq^{\oplus} M$ to signify that L is a subset, a submodule, and a direct summand of M. A submodule $K \leq M$ is called to be essential in M, denoted as $K \trianglelefteq M$, if $K \cap L = 0$ implies $L = 0$ for all $L \leq M$ [10]. The socle of an R-module M is denoted by $Soc(M)$ and defined as the sum of all simple submodules of M. If there are no minimal submodules in M we put $Soc(M) = 0$ [24]. Dually for any submodule L of M, if $H + L = M$ implies $L = M$, then the proper submodule $H \leq M$ is called to be small in M and denoted as $H \ll M$. The intersection of all maximal submodules of M is called the Jacobson radical of M, and denoted by $Rad(M)$ or, as in alternative, the sum of all small submodules of M. If M does not contained any maximal submodules, then it is show as $Rad(M) = M$. Zhou [27] introduced δ-small submodules, extended small submodules as follows. A proper submodule N of a module M is called δ-small in M (denoted by $N \ll_{\delta} M$) if whenever $M = N + L$ with M/L singular, we have $M = L$. Recall [28] that a submodule $H \leq M$ is called g-small in M, denoted as $H \ll_g M$ if, whenever $M = H + E$ with $E \trianglelefteq M$, implies $E = M$, in reality, authors Zhou and Zhang put a g-small submodule in place of an *e*-small submodule. When any proper (cyclic) submodule of M is g-small, then M is named as (principally) generalized hollow ([14], resp. [8]). It is clear that any small submodule is g-small. If T is essential and maximal submodule of M then T is said to be a generalized maximal submodule of M. The intersection of all generalized maximal submodules of M is called the generalized radical of M and denoted by $Rad_g(M)$ that also knows as the sum of all g-small submodules in M [28]. Recall [24] that K a supplement of N in M if $M = K + N$ and $K \cap N$ is small in K. A module M is called supplemented if every submodule of M has a supplement in M. A module M is said to be principally (supplemented) δ-supplemented if any cyclic submodule N of M, there exists a submodule X of M such that

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$M = N + X$ and $N \cap X$ is (small) δ-small in X ([1], resp. [11]). Moreover, a module M is said to be principally $\oplus$-supplemented if for all cyclic submodule N of M, there exists a direct summand X of M such that $M = N + X$ and $N \cap X \ll X$ [21]. A module M is called principally $\oplus$-δ-supplemented if for all cyclic submodule N of M, there exists a direct summand H of M such that $M = H + X$ and $H \cap X$ is δ-small in X [22]. Also, A module M called principally semisimple if all it is cyclic submodules are direct summands of M. Also, the author called a principally semisimple module as a regular module [20]. Moreover, the module M called (principally) lifting if, for all (cyclic) submodule N of M, there exists a decomposition $M = A \oplus B$ such that $A \leq N$ and $N \cap B \ll B$ ([6], resp. [24]). A module M is said to be g-lifting if it has a decomposition $M = S \oplus \grave{S}$ such that $S \leq A$ and $A \cap \grave{S} \ll_g M$, if for any submodule $A \leq M$ [18]. Ghawi in [8], recall that M is principally g-lifting module if, for each $m \in M$, M has a decomposition $M = A \oplus B$ such that $A \leq mR$ and $mR \cap B$ is g-small in B. Ghawi in [9], recall that a module M is $\oplus$-g-supplemented if for any submodule of M has a g-supplement that is a direct summand of M, i.e. for any $N \leq M$ there exists a direct summand H of M such that $N + H = M$ and $N \cap H \ll_g H$.

In view of the definitions and concepts by the above it was natural to introduce a new definition of modules named principally $\oplus$-g-supplemented modules as generalization of $\oplus$-g-supplemented modules. A module M is called principally $\oplus$-g-supplemented if every cyclic submodule of M has a principally $\oplus$-g-supplement in M, that is, for each $m \in M$, there exists a submodule L of M such that $M = mR + L = \grave{L} \oplus L$ for some $\grave{L} \leq M$ with $mR \cap L$ is g-small in L. Our work consists of one section in which the notion of principally $\oplus$-g-supplemented modules was presented and studied. Some properties and examples, also the relations between this concept and other different kinds of modules are discussed.

2. Principally $\oplus$-g-supplemented modules

First, we will present the following lemma.

Lemma 2.1. Let M be a module, $m \in M$ and L a direct summand of M. Then the following are equivalent.
(1) $M = mR + L$ and $mR \cap L$ is g-small in L.
(2) $M = mR + L$ and for a proper essential submodule K of L, $M \neq mR + K$.

Proof. (1) $\Longrightarrow$ (2) Let K be an essential submodule of L with $M = mR + K$. Then $L = L \cap (mR + K) = K + (mR \cap L)$. As $mR \cap L$ is g-small in L, we deduce $L = K$.
(2) $\Longrightarrow$ (1) If $L = (mR \cap L) + K$ with $K \trianglelefteq L$, then $M = mR + L = mR + K$. By (2), $K = L$. Hence m$R \cap L$ is g-small in L. ■

Moreover, if M is a right R-module and $m \in M$. we say that a submodule (= direct summand) L of M called a principally $\oplus$-g-supplement of mR in M if, mR and L satisfy Lemma 2.1.

In following, we will present our next main definition.

Definition 2.2. A module M is called principally $\oplus$-g-supplemented if every cyclic submodule of M has a principally $\oplus$-g-supplement in M, that is, for each $m \in M$, there exists a submodule L of M such that $M = mR + L = \grave{L} \oplus L$ for some $\grave{L} \leq M$ and $mR \cap L$ is g-small in L .

Remarks 2.3.
(1) By definitions, it is clear that every $\oplus$-g-supplemented and so every g-lifting module is principally $\oplus$-g-supplemented.
(2) For the same reason in [16, Remark 2.6], we deduce that any cyclic principally $\oplus$-g-supplemented module over a PID is $\oplus$-g-supplemented.

Proposition 2.4. Every principally $\oplus$-g-supplemented module is principally g-supplemented.
Proof. It is clear. ■

Proposition 2.5. Every principally g-lifting module is principally $\oplus$-g-supplemented.

Proof. Suppose M is a principally g-lifting module and $m \in M$. Then there is a decomposition $M = A \oplus B$ such that $A \leq mR$ and $mR \cap A \ll_g M$. Thus $M = mR + B$. As $mR \cap A \subseteq A$ and $A \leq^{\oplus} M$, [9, Lemma 2.12] implies $mR \cap A \ll_g A$. Hence M is principally ⊕-g-supplemented. ■

Examples 2.6. (1) Consider the $\mathbb{Z}$-module $\mathbb{Q}$. Every cyclic submodule of $\mathbb{Q}$ is a small $\mathbb{Z}$-submodule, so is g-small. Therefore $\mathbb{Q}$ is a g-supplement (direct summand) for every cyclic submodule of itself. Thus, $\mathbb{Q}$ is a principally ⊕-g-supplemented $\mathbb{Z}$-module. But, by [16 , Examples 2.4(1)], the $\mathbb{Z}$-module $\mathbb{Q}$ is neither ⊕-g-supplemented nor g-lifting .
(2) Suppose $M = \mathbb{Q} \oplus Z_2$ as $\mathbb{Z}$-module. We prove M is a principally ⊕-g-supplemented module but neither supplemented nor lifting. It is routine to show that $M = (1, \bar{1})Z + (\mathbb{Q} + (\bar{0}))$. Suppose that $(q, \bar{u}) \in M$. If $\bar{u} = \bar{1}$ and $q \neq 1$. In this case we prove $M = (q, \bar{u})Z + (\mathbb{Q} + (\bar{0}))$.
Let $(x, \bar{y}) \in M$. We have two possibilities:
(i) $\bar{y} = \bar{1}$. Then $(x, \bar{y}) = (x, \bar{1}) = (q, \bar{1}) + (x - q, \bar{0}) \in (q, \bar{u})Z + (\mathbb{Q} + (\bar{0}))$.
(ii) $\bar{y} = \bar{0}$. Then $(x, \bar{y}) = (x, \bar{0}) = (q, \bar{1})0 + (x, \bar{0}) \in (q, \bar{u})Z + (\mathbb{Q} + (\bar{0}))$.
Hence $M = (q, \bar{u})Z + (\mathbb{Q} + (\bar{0}))$. As $(q, \bar{u})Z \cap (\mathbb{Q} + (\bar{0}))$ is either zero or isomorphic to $Z \oplus (\bar{0})$ that is small (so is g-small) in $\mathbb{Q} + (\bar{0})$, hence M is a principally ⊕-g-supplemented $\mathbb{Z}$-module. If $M = \mathbb{Q} \oplus Z_2$ were a supplemenetd $\mathbb{Z}$-module, its direct summand $\mathbb{Q}$ would be a supplemented $\mathbb{Z}$-module, that is a contradiction. So M is neither supplemented nor lifting.
(3) Since every principally ⊕-g-supplemented is principally g-supplemented, so we deduce the $\mathbb{Z}$-module $\mathbb{Z}$ is not principally ⊕-g-supplemented, see [16, Example 2.5] .

Proposition 2.7. Consider the following cases for a module M:
(1) $Rad_g(M) = M$;
(2) M is a principally generalized hollow module;
(3) M is a principally g-lifting module;
(4) M is a principally ⊕-g-supplemented module.
Then (1) ⟹ (2) ⟹ (3) ⟹ (4). If M is non-cyclic indecomposable, then (4) ⟹ (1).

Proof. (1) ⟹ (2) Let $mR \subset M$, where $m \in M$. By (1), $m \in Rad_g(M)$ and so $mR \ll_g M$. Hence M is a principally generalized hollow module.
(2) ⟹ (3) By [8, Lemma 3.20]
(3) ⟹ (4) By Proposition 2.5
(4) ⟹ (1) If $m \in M$, by hypothesis, there exists submodules D and H of M such that $M = mR + H = D \oplus H$, $mR \cap H \ll_g H$, so that $mR \cap H \subseteq Rad_g(H)$. As M is an indecomposable module, either $H = 0$ or $H = M$. If $H = 0$, we deduce that $M = mR$, a contradiction. Thus, $D = 0$ and $H = M$. Therefore, $mR \subseteq Rad_g(M)$, $m \in mR \subseteq Rad_g(M)$ and hence $Rad_g(M) = M$. ■

Proposition 2.8. Let R be a non-local commutative domain. Then every injective R-module is principally ⊕-g-supplemented.

Proof. Let M be an injective module over non-local commutative domain R, then M does not contain a maximal submodule, i.e. $Rad(M) = M$ by [2, Lemma 4.4]. Because that $Rad(M) \subseteq Rad_g(M)$, we have $Rad_g(M) = M$. Thus, Proposition 2.7 implies the result. ■

The reverse of Proposition 2.8 may not be true, generally.

Example 2.9. For any prime number $p \in \mathbb{Z}_+$, the $\mathbb{Z}$-module $\mathbb{Z}_p$ is principally ⊕-g-supplemented, because it is simple. While $\mathbb{Z}_p$ as $\mathbb{Z}$-module does not injective.

Corollary 2.10. Let R be a Dedekind domain. Then every injective R-module is principally ⊕-g-supplemented.

Proof. By Proposition 2.8. ■

Theorem 2.11. Let M be an R-module, consider the following cases:
(1) M is principally semisimple.
(2) M is principally g-lifting.
(3) M is principally ⊕-g-supplemented.
(4) M is principally g-supplemented.
Then (1) ⟹ (2) ⟹ (3) ⟹ (4). If $Rad_g(M) = 0$, then (4) ⟹ (1).

Proof. (1) ⟹ (2) Obvious.
(2) ⟹ (3) By Proposition 2.5.
(3) ⟹ (4) By Proposition 2.4.

(4) ⟹ (1) Let $m \in M$. As M is a principally g-supplemented module, then there is a submodule A of M such that $M = mR + A$ and $mR \cap A$ is g-small in A. Since $mR \cap A$ is g-small in M, then $mR \cap A \subseteq Rad_g(M) = 0$, that is $mR \cap A = 0$, so $mR \leq^{\oplus} M$. Therefore, (1) holds. ■

Recall that a module M is called an e-noncosingular module. If $\overline{Z}_e(M) = M$. Where $\overline{Z}_e(M) = \cap\{ker g \mid g \in Hom(M, N), N$ is e-small module$\}$ [19].

However, we have the following consequence.

Corollary 2.12. Let R be an arbitrary ring such that every right R-module is e-noncosingular. Then the following are equivalent for an R-module M.
(1) M is principally semisimple.
(2) M is principally g-lifting.
(3) M is principally ⊕-g-supplemented.
(4) M is principally g-supplemented.

Proof. Since any right R-module is e-noncosingular, we have $Rad_g(M) = 0$ by [19, Proposition 3.7]. This completes the proof by Theorem 2.11. ■

Remarks 2.13. (1) The condition $Rad_g(M_R) = 0$ is necessary in Theorem 2.11. By Examples 2.6(1), $\mathbb{Q}$ is a principally ⊕-g-supplemented $\mathbb{Z}$-module. But, we know that $\mathbb{Q}$ as $\mathbb{Z}$-module is not principally semisimple, in fact $Rad_g(\mathbb{Q}_{\mathbb{Z}}) \neq 0$.
(2) It is well known that the $\mathbb{Z}$-module $\mathbb{Z}$ is not principally semisimple, and it is easy to see that $Rad_g(\mathbb{Z}_{\mathbb{Z}}) = 0$, so by Theorem 2.11 this another reason to make $\mathbb{Z}$-module $\mathbb{Z}$ is neither principally g-supplemented nor principally ⊕-g-supplemented.
(3) Because the example in (2) it can be said that every submodule of a principally ⊕-g-suppl emented module may not be principally ⊕-g-supplemented; that is $\mathbb{Z}$ is not principally ⊕-g-suppl emented in a principally ⊕-g-supplemented $\mathbb{Z}$-module $\mathbb{Q}$.

Proposition 2.14. Let M be a principally ⊕-g-supplemented R-module and L a submodule of M. If any cyclic submodule of M has a ⊕-g-supplement contains L, then M/L is principally ⊕-g-supplemented.

Proof. Let $m \in M$ and consider the submodule $\overline{m}R$ of M/L, then $\overline{m}R = (mR + L)/L$. By hypothesis, there exists a direct summand N of M such that $L \leq N$, $M = mR + N$ and $mR \cap N \ll_g N$. Thus $M = N \oplus K$ for some submodule K of M. Consider a natural map $\pi: M \to M/L$. It is easy to prove that $M/L = N/L \oplus (K + L)/L = N/L + \overline{m}R$. Also, by the modular law and [28, Proposition 2.5], we deduce $(N/L) \cap \overline{m}R = N/L \cap (mR + L)/L = (N \cap (mR + L))/L = (L + (mR \cap N))/L = \pi(mR \cap N)$ is g-small in $\pi(N) = N/L$. This mean that N/L is a g-supplement of $\overline{m}R$ that is a direct summand of M/L, and hence M/L is principally ⊕-g-supplemented. ■

Wisbauer [24] recall that. Let M be an R-module. A submodule N of M is said to be fully invariant if $f(N) \subseteq N$ for all nonzero $f \in End(M)$. If all submodules of M are fully invariant, then M is called a duo module. And also If all direct summand submodules of M are fully invariant, then M is called a weak duo module [17].

Proposition 2.15. Let M be a principally ⊕-g-supplemented R-module. The factor M/L is principally ⊕-g-supplemented for every fully invariant submodule L of M.

Proof. Let L be a fully invariant submodule of M and $\overline{m}R = (mR + L)/L$ be a cyclic submodule of M/L for some $m \in M$. Since M is principally ⊕-g-supplemented, then there exists a direct summand N of M such that $M = mR + N$ and $mR \cap N \ll_g N$. Thus $M = N \oplus K$ for some $K \leq M$. By [22, Lemma 3.3], we have that $M/L = ((N + L)/L) \oplus ((K + L)/L)$. However, we get $M/L = ((N + L)/L) + \overline{m}R$. It is clear that $((N + L)/L) \cap \overline{m}R$ is g-small in $(N + L)/L$. This completes the proof. ■

The next consequence is clear from Proposition 2.15.

Corollary 2.16. Every factor module of a principally ⊕-g-supplemented duo R-module is principally ⊕-g-supplemented.

Corollary 2.17. If M is a principally ⊕-g-supplemented R-module, then so is $M/Rad_g(M)$.

Proof. By [28, Corollary 2.11] $Rad_g(M)$ is fully invariant, so that the result is obtained by Proposition 2.15. ■

Corollary 2.18. Let R be any ring such that every right R-module is e-noncosingular, and let M be a module. Then M is principally $\oplus$-g-supplemented if and only if $M/Rad_g(M)$ principally $\oplus$-g-supplemented.

Proof. By [19, Proposition 3.7], we have $Rad_g(M) = 0$, so that $M/Rad_g(M) \cong M$. This completes the proof. ■

Corollary 2.19. Let M be a weak-duo and principally $\oplus$-g-supplemented module. Then every direct summand of M is principally $\oplus$-g-supplemented.

Proof. Let N be a direct summand of a principally $\oplus$-g-supplemented module M, then $M = N \oplus K$ for some $K \leq M$. Since M is weak-duo, then K is a fully invariant submodule. So, $N \cong M/K$ is principally $\oplus$-g-supplemented by Proposition 2.15. ■

In coming example shows that for a module M and a submodule L, if M/L is a principally $\oplus$-g-supplemented module, then M need not be principally $\oplus$-g-supplemented.

Example 2.20. Consider the $\mathbb{Z}$-module $\mathbb{Z}/p^n\mathbb{Z}$, where p is a prime number and $n \in \mathbb{Z}_+$. By [8] $\mathbb{Z}/p^n\mathbb{Z}$ is principally g-lifting and so principally $\oplus$-g-supplemented, but $\mathbb{Z}$ is not principally $\oplus$-g-supplemented.

In following, we investigate a condition which ensure that a homomorphic image of a principally $\oplus$-g-supplemented module is principally $\oplus$-g-supplemented.

Camillo [5], recall that a module M is called distributive if $X \cap (Y + Z) = (X \cap Y) + (X \cap Z)$ or $X + (Y \cap Z) = (X + Y) \cap (X + Z)$ for all submodules Y, Z of M. A module M is said to be distributive if all submodules of M are distributive.

Theorem 2.21. Let M be a distributive and principally $\oplus$-g-supplemented R-module. Then the homomorphic image of M is principally $\oplus$-g-supplemented.

Proof. Let K be a submodule of M and $(mR + K)/K$ a cyclic submodule of M/K, where $m \in M$. Since M is principally $\oplus$-g-supplemented, then there exists a direct summand A of M such that $M = A \oplus B = mR + A$ for a submodule B of M and $mR \cap A \ll_g A$. So, $M/K = (mR + K)/K + (A + K)/K$ and as M is a distributive module, $(mR + K) \cap (A + K) = (mR \cap A) + K$. Therefore $(mR + K)/K \cap (A + K)/K = ((mR \cap A) + K)/K$ is g-small in $(A + K)/K$ as a homomorphic image of g-small $mR \cap A$ in A under the natural map $\pi: A \to (A + K)/K$ by [28, Proposition 2.5]. Again by distributivity of M and $A \cap B = 0$, we get $M/K = ((A + K)/K) \oplus ((B + K)/K)$. So $(A + K)/K$ is a direct summand of M/K. ■

Kasch [12], recall that an R-module P is called projective if and only if for any two R-module A, B and for any epimorphism $f: A \to B$ and for any homomorphism $g: P \to B$, there is a homomorphism $h: P \to A$ such that $f \circ h = g$.

Proposition 2.22. Let M be a principally $\oplus$-g-supplemented R-module. Then $M/Rad_g(M)$ is a principally semisimple R-module if M has one of the following conditions.

(1) M is a distributive R-module.

(2) M is a projective R-module.

Proof. (1) Suppose that $\overline{m}R$ is a cyclic submodule of $M/Rad_g(M)$ where $m \in M$, then $\overline{m}R = (mR + Rad_g(M))/Rad_g(M)$. By hypothesis, there exists a direct summand A of M such that $M = mR + A$ and $mR \cap A$ is g-small in A. Also $mR \cap A$ is g-small in M, and hence $mR \cap A \subseteq Rad_g(M)$. Hence, $(mR + Rad_g(M))/Rad_g(M) + (A + Rad_g(M))/Rad_g(M) = M/Rad_g(M)$. On the other hand, by distributivity of M, we have that $(mR + Rad_g(M)) \cap (A + Rad_g(M)) = (mR \cap A) + Rad_g(M) = Rad_g(M)$. It follows that $(mR + Rad_g(M))/Rad_g(M) \cap (A + Rad_g(M))/Rad_g(M) = Rad_g(M)$. Hence, $M/Rad_g(M) = (mR + Rad_g(M))/Rad_g(M) \oplus (A + Rad_g(M))/Rad_g(M)$, so that $M/Rad_g(M) = \overline{m}R \oplus (A + Rad_g(M))/Rad_g(M)$.

(2) Let $\overline{m}R$ be any cyclic submodule of $M/Rad_g(M)$, $m \in M$, then $\overline{m}R = (mR + Rad_g(M))/Rad_g(M)$. By hypothesis, there exists submodules X, A of M such that $M = X \oplus A = mR + A$ and $mR \cap A$ is g-small in A. Also $mR \cap A$ is g-small in M, and hence $mR \cap A \subseteq Rad_g(M)$. By projectivity of M and [15, Lemma 4.47], there exists a direct summand N of M such that $M = N \oplus A$ where $N \leq mR$. Therefore $(mR + Rad_g(M))/Rad_g(M) = (N + Rad_g(M))/Rad_g(M)$ and $Rad_g(M) = Rad_g(N) \oplus Rad_g(A)$ implies that $M/Rad_g(M) = \overline{m}R \oplus (A + Rad_g(M))/Rad_g(M)$. So, any principal submodule of $M/Rad_g(M)$ is a direct summand in either case. Therefore $M/Rad_g(M)$ is principally semisimple. ■

Recall that a module M is called refinable if for all submodules U and V of M with $M = U + V$, there is a direct summand $\acute{U}$ of M such that $\acute{U} \subseteq U$ and $M = \acute{U} + V$ [25].

Theorem 2.23. Let M be a projective (or, distributive) R-module. Consider the following cases:
(1) M is principally ⊕-g-supplemented.
(2) $M/Rad_g(M)$ is principally semisimple.
Then (1) ⟹ (2), and (2) ⟹ (1) in case M is a refinable R-module with $Rad_g(M) \ll_g M$.

Proof. (1) ⟹ (2) It follows by Proposition 2.22.
(2) ⟹ (1) Suppose that $m \in M$. Since $\overline{m}R = (mR + Rad_g(M))/Rad_g(M)$ is a cyclic submodule of $M/Rad_g(M)$, so by (2), there exists a submodule U of M such that $M/Rad_g(M) = \overline{m}R \oplus U/Rad_g(M)$, where $Rad_g(M) \subseteq U$. Then $M = mR + U$ and $(mR + Rad_g(M)) \cap U = (mR \cap U) + Rad_g(M) = Rad_g(M)$, by the modular law. Hence $mR \cap U \subseteq Rad_g(M)$, and so $mR \cap U$ is g-small in M. As $M = mR + U$ is refinable, there is a direct summand A of M such that $A \leq U$ and $M = mR + A$. As $mR \cap A \leq mR \cap U$ and $A \leq^{\oplus} M$, so by [9, Lemma 2.12], $mR \cap A$ is g-small in A. Therefore mR has a principally ⊕-g-supplement A in M. This completes the proof. ■

Corollary 2.24. Let M be a projective (or, distributive) R-module. Consider the following cases:
(1) M is principally ⊕-g-supplemented.
(2) $M/Rad_g(M)$ is principally semisimple.
Then (1) ⟹ (2), and (2) ⟹ (1) if M is a refinable finitely generated R-module.

Proof. It follows by [9, Lemma 5.4] and Theorem 2.23. ■

Corollary 2.25. Let R be a commutative ring and M be a projective (or, distributive) R-module. Consider the following cases:
(1) M is principally ⊕-g-supplemented.
(2) $M/Rad_g(M)$ is principally semisimple.
Then (1) ⟹ (2), and (2) ⟹ (1) if M is a refinable and Noetherian R-module.

Proof. Since a Noetherian module implies finitely generated, then the result is obtained by Corollary 2.24. ■

Corollary 2.26. Let R be a ring. Consider the following cases:
(1) R is principally ⊕-g-supplemented.
(2) $R/Rad_g(R)$ is principally semisimple.
Then (1) ⟹ (2), and (2) ⟹ (1) in case R is a refinable R-module.

Proof. Since $R = \langle 1 \rangle$, so the result is followed by Corollary 2.24. ■

Theorem 2.27. Let M be a principally ⊕-g-supplemented module. If K is a submodule of M such that M/K is projective, then K is principally ⊕-g-supplemented.

Proof. Suppose that L is a cyclic submodule of K. By hypothesis, there exists a direct summand N of M such that $M = L + N$ and $L \cap N$ is g-small in N, so in M. Thus, $M = K + N$ and so $K \cap N$ is a direct summand of M, by [13, Lemma 2.3]. So $M = (K \cap N) \oplus H$ for some $H \leq M$. By the modular law, we have $K = K \cap M = K \cap (L + N) = L + (K \cap N)$, also $L \cap (K \cap N) = L \cap N$ is g-small in M. Since $L \cap (K \cap N) \subseteq K \cap N$ and $K \cap N \leq^{\oplus} M$ this implies $L \cap (K \cap N)$ is g-small in $K \cap N$ by [9, Lemma 2.12]. Again by the modular law, we deduce that $K = K \cap ((K \cap N) \oplus H) = (K \cap N) \oplus (K \cap H)$, this mean $K \cap N$ is a direct summand of K, and hence K is principally ⊕-g-supplemented. ■

Recall [15] that a module M is said to have (D_3) property: if for any direct summands A and B of M with $M = A + B$ then $A \cap B$ is also a direct summand of M. If the intersection of any two direct summands of a module M is a direct summand of M, then M is said to have the summand intersection property, and denoted by SIP [23].

Proposition 2.28. Let M be a principally ⊕-g-supplemented module has (D_3), then every direct summand of M is principally ⊕-g-supplemented.

Proof. Assume L is a direct summand of M and $a \in L$. Since M a principally ⊕-g-supplemented module and $a \in M$, $M = aR + B$ and $aR \cap B \ll_g B$ for some direct summand B of M. By the modular law, we have that $L = L \cap M = L \cap (aR + B) = aR + (L \cap B)$. We have L and B are direct summands of M with $M = L + B$, that implies $L \cap B$ is so a direct summand in M, because M has (D_3). Since $aR \cap B \ll_g M$ and $L \cap B \leq^{\oplus} M$, we deduce that $aR \cap (L \cap B) = aR \cap B$ is a g-small submodule in $L \cap B$, by [9, Lemma 2.12]. Hence L is principally ⊕-g-supplemented. ■

Corollary 2.29. Let M be a module has the SIP. Then M is principally ⊕-g-supplemented if and only if every direct summand of M is principally ⊕-g-supplemented.

Proof. ⟹) It is obvious that every module with the summand intersection property has (D_3). So the result is obtained by Proposition 2.28.

⟸) Clear. ■

Recall that a module M is called extending if any closed submodule is a direct summand [7]. A module M said to be polyform if, all it is partial endomorphisms has closed kernel [26].

Corollary 2.30. Let M be an extending polyform R-module. Then M is principally ⊕-g-supplemented if and only if every direct summand of M is principally ⊕-g-supplemented.

Proof. By [3, Lemma 11] , M has the SIP. So by Corollary 2.29., the result is follow. ■

Corollary 2.31. If M is a quasi-projective module, then

(1) M is principally ⊕-g-supplemented if and only if every direct summand of M is principally ⊕-g-supplemented.

(2) M is principally ⊕-δ-supplemented if and only if every direct summand of M is principally ⊕-δ-supplemented.

Proof. By [15, Lemma 4.6] and [15, Proposition 4.38], M has (D_3). Thus (1) and (2) are follows directly by Proposition 2.28, and [22, Proposition 3.6] , respectively. ■

Wisbauer in [24], recall that. If for any two submodules A, B of M with $M = A + B$ there exists an $f \in End_R(M)$ such that $Imf \leq A$ and $Im(1-f) \leq B$. Then M is called π-projective. A submodule A of a module M is weak distributive if $A = (A \cap X) + (A \cap Y)$ for all submodules X, Y of M with $X + Y = M$. A module M is said to be weakly distributive if every submodule of M is a weak distributive submodule of M [4].

Only in certain cases, the classes principally ⊕-g-supplemented modules and principally g-lifting modules are identical as the below theorem shows.

Theorem 2.32. Let M be a principally ⊕-g-supplemented R-module and satisfy any one of the following conditions:

(1) M is duo.

(2) M is weakly distributive.

(3) M is π-projective.

(4) M is refinable and have the SIP.

Then M is a principally g-lifting R-module.

Proof. (1) Let $m \in M$. Since M is a principally ⊕-g-supplemented module, then $M = mR + L$ and $mR \cap L \ll_g L$ for some a direct summand L of M. So $M = L \oplus K$ for some $K \leq M$. Since mR is fully invariant in M, $mR = (mR \cap L) \oplus (mR \cap K)$, and hence $M = (mR \cap K) \oplus L$ where $mR \cap K \leq mR$ and $mR \cap L \ll_g L$. Hence M is a principally g-lifting module. Proof (2) similar to proof (1).

(3) Let $m \in M$. Then $M = mR + L$ and $mR \cap L \ll_g L$ for some a direct summand L of M, as M is principally ⊕-g-supplemented. By π-projectivity for M, there exists $K \leq mR$ such that $M = K \oplus L$, by [24, 41.14(3)]. It follows M is a principally g-lifting module.

(4) As M is a principally ⊕-g-supplemented module and $m \in M$, then $M = mR + L$ and $mR \cap L \ll_g L$ for some a direct summand L of M. Since M is a refinable module, then there exists a direct summand K of M such that $K \leq mR$ and $M = K + L$. Thus $L \cap K$ is a direct summand of M, since M have the SIP. Let $M = (L \cap K) \oplus N$ for some $N \leq M$. By modular law, we deduce $L = (L \cap K) \oplus (L \cap N)$, so $M = K + L = K \oplus (L \cap N)$. It is clear that $mR \cap (L \cap N) \ll_g L \cap N$. Hence completes the proof. ■

Corollary 2.33. If a module M satisfy any one of the following cases:

(1) M is duo.

(2) M is weakly distributive.

(3) M is π-projective.

Then M is a principally ⊕-g-supplemented module if and only if every direct summand of M is principally ⊕-g-supplemented.

Proof. Suppose (1), to prove ⟹) By Theorem 2.32, M is a principally g-lifting module. By [8, Proposition 3.4], any direct summand of M is principally g-lifting, so it is principally ⊕-g-supplemented. ⟸) Clear.

(2) and (3) similar to proof (1). ∎

The proof of following two propositions are exactly analogous to proof [16, Proposition 2.13] and [16, Proposition 2.14], respectively.

Proposition 2.34. Let $M = \bigoplus_{i\in I} M_i$ be an infinite direct sum of principally ⊕-g-supplemented R-modules $\{M_i \mid i \in I\}$. If every cyclic submodule of M is fully invariant, then M is principally ⊕-g-supplemented.

Proposition 2.35. Let $M = M_1 \oplus M_2$ be a direct sum of principally ⊕-g-supplemented modules M_1, M_2. If any cyclic submodule of M is weak distributive, M is principally ⊕-g-supplemented.

Corollary 2.36. Let M be an R-module,

(1) if $M = \bigoplus_{i\in I} M_i$ is a duo infinite direct sum of R-modules $\{M_i \mid i \in I\}$. Then M is principally ⊕-g-supplemented if and only if M_i is principally ⊕-g-supplemented, for $i \in I$.

(2) if $M = M_1 \oplus M_2$ is a weakly distributive direct sum of R-modules M_1, M_2. Then M is principally ⊕-g-supplemented if and only if M_1, M_2 are principally ⊕-g-supplemented.

Proof. (1) It follows directly by Corollary 2.33 and Proposition 2.34.

(2) It follows directly by Corollary 2.33 and Proposition 2.35. ∎

Proposition 2.37. Let M be a principally ⊕-g-supplemented R-module and L a submodule of M. If $L \cap Rad_g(M) = 0$, then L is principally semisimple.

Proof. Let $a \in L$. Since M is a principally ⊕-g-supplemented R-module, then there exists a direct summand A of M such that $M = aR + A$ and $aR \cap A$ is g-small in A. Also $aR \cap A$ is g-small in M, and hence $aR \cap A \subseteq Rad_g(M)$. By the modular law, we have that $L = L \cap (aR + A) = aR + (L \cap A)$. As $aR \cap (L \cap A) \subseteq L \cap Rad_g(M) = 0$, we get $L = aR \oplus (L \cap A)$. Therefore $aR \leq^{\oplus} L$ and L is principally semisimple. ∎

Proposition 2.38. If M is a principally ⊕-g-supplemented module has a cyclic generalized radical. Then $M = M_1 \oplus M_2$ where M_1 is a module with $Rad_g(M_1)$ is g-small in M_1 and M_2 is a module with $Rad_g(M_2) = M_2$.

Proof. Since M is a principally ⊕-g-supplemented module and $Rad_g(M)$ is a cyclic submodule of M, then $Rad_g(M)$ has a g-supplement M_1 in M, i.e. $M = M_1 + Rad_g(M)$ and $M_1 \cap Rad_g(M) \ll_g M_1$, where $M = M_1 \oplus M_2$ for a submodule M_2 of M. As $Rad_g(M_1) \leq M_1 \cap Rad_g(M)$ implies that $Rad_g(M_1) \ll_g M_1$. By [19, Corollary 2.3], $M = M_1 + Rad_g(M) = M_1 + Rad_g(M_1 \oplus M_2) = M_1 + Rad_g(M_1) \oplus Rad_g(M_2)$, so that $M = M_1 \oplus Rad_g(M_2)$. By modular law, $M_2 \cap M = M_2 \cap (M_1 \oplus Rad_g(M_2)) = Rad_g(M_2) \oplus (M_1 \cap M_2)$ that deduce $Rad_g(M_2) = M_2$. ∎

Theorem 2.39. Let M be a principally ⊕-g-supplemented R-module. Then M has a principally semisimple submodule A such that $Soc(A) \trianglelefteq A$ and $Rad_g(M) \oplus A$ is essential in M.

Proof. Since $Rad_g(M) \leq M$, so by [10, Proposition 1.3], there exists a submodule A of M such that $Rad_g(M) \oplus A$ is essential in M. As $A \cap Rad_g(M) = 0$, A is principally semisimple, by Proposition 2.37. Next we show that $Soc(A) \trianglelefteq A$. For this we prove for any $a \in A$, aR has a simple submodule. If aR is simple, the proof is finish. Otherwise, assume $a_1 \in aR$ such that $a_1R \neq aR$. Since M is principally ⊕-g-supplemented, there exists a direct summand C of M such that $M = a_1R + C$ and $a_1R \cap C$ is g-small in C, so in M, and hence $a_1R \cap C \subseteq Rad_g(M)$. Then $a_1R \cap C \subseteq A \cap Rad_g(M) = 0$. Thus $M = a_1R \oplus C$ and then $aR = a_1R \oplus (aR \cap C)$, by the modular law. Obviously, $aR \cap C = \grave{a}_1R$ for some $\grave{a}_1 \in aR$ and $aR = a_1R \oplus \grave{a}_1R$. If a_1R and $\grave{a}_1R$ are simple, then we stop. Otherwise let $a_2 \in a_1R$ such that $a_2R \neq a_1R$. By similar way, there is an $\grave{a}_2 \in a_1R$ such that $a_1R = a_2R \oplus \grave{a}_2R$. Hence $aR = a_2R \oplus \grave{a}_2R \oplus \grave{a}_1R$. If a_2R is simple, then we stop. Otherwise we continue in this way. Since aR is cyclic, this process must terminate at a finite step, say n. At this step all direct summands of aR should be simple. Hence every cyclic submodule of A contains a simple submodule. Therefore the socle of A is essential in A. ∎

Theorem 2.40. Let M be a principally ⊕-g-supplemented module. If M satisfies ascending chain condition on direct summands. Then $M = M_1 \oplus M_2$, where M_1 is a semisimple module and M_2 is a module with $Rad_g(M_2) \trianglelefteq M_2$.

Proof. Since $Rad_g(M) \leq M$, so by [10, Proposition 1.3], there is a submodule M_1 of M with $Rad_g(M) \oplus M_1$ is essential in M. Since $M_1 \cap Rad_g(M) = 0$, Proposition 2.37 implies M_1 is principally semisimple. Let $m_1 \in M_1$. As M is principally ⊕-g-supplemented, there is a direct summand A_1 of M such that $M = m_1R + A_1$ and $m_1R \cap A_1$ is g-small in A_1 and M. Hence $m_1R \cap A_1 \subseteq M_1 \cap Rad_g(M) = 0$ and $M = m_1R \oplus A_1$. By the modular law, $M_1 = M_1 \cap (m_1R \oplus A_1) = m_1R \oplus (M_1 \cap A_1)$. If $M_1 \cap A_1 \neq 0$, let $(0 \neq) m_2 \in M_1 \cap A_1$. There is a direct summand A_2 of M such that $M = m_2R + A_2$ and $m_2R \cap A_2$ is g-small in A_2 and M. Similarly, $m_2R \cap A_2 \subseteq M_1 \cap Rad_g(M) = 0$, and $M = m_2R \oplus A_2$. Since $m_2R \subseteq A_1$, $M = (m_1R \oplus A_1) \cap (m_2R \oplus A_2) = m_1R \oplus (A_1 \cap (m_2R \oplus A_2)) = m_1R \oplus m_2R \oplus (A_1 \cap A_2)$, by the modular law. Also, by the modular law, we have that $M_1 \cap A_1 = (M_1 \cap A_1) \cap M = (M_1 \cap A_1) \cap (m_2R \oplus A_2) = m_2R \oplus (M_1 \cap A_1 \cap A_2)$ and $M_1 = m_1R \oplus (M_1 \cap A_1) = m_1R \oplus m_2R \oplus (M_1 \cap A_1 \cap A_2)$. If $M_1 \cap A_1 \cap A_2 \neq 0$, let $(0 \neq) m_3 \in M_1 \cap A_1 \cap A_2$. There exists a direct summand A_3 of M such that $M = m_3R + A_3$ and $m_3R \cap A_3$ is g-small in A_3 and M.
Similarly, $m_3R \cap A_3 \subseteq M_1 \cap Rad_g(M) = 0$ and $M = m_3R \oplus A_3 = m_1R \oplus m_2R \oplus m_3R \oplus (A_1 \cap A_2 \cap A_3)$. Also, by the modular law, we have that $M_1 \cap A_1 \cap A_2 = (M_1 \cap A_1 \cap A_2) \cap M = (M_1 \cap A_1 \cap A_2) \cap (m_3R \oplus A_3) = m_3R \oplus (M_1 \cap A_1 \cap A_2 \cap A_3)$ and,
$M_1 = m_1R \oplus m_2R \oplus (M_1 \cap A_1 \cap A_2) = m_1R \oplus m_2R \oplus m_3R \oplus (M_1 \cap A_1 \cap A_2 \cap A_3)$. By the hypothesis this procedure stops at a finite number of steps, say r. At this stage we may have
$M = m_rR \oplus A_r = m_1R \oplus m_2R \oplus m_3R \oplus \ldots \oplus m_rR \oplus (A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_r)$and $M_1 = m_1R \oplus m_2R \oplus m_3R \oplus \ldots \oplus m_rR$.
Since M has the ascending chain condition on direct summands, without loss of generality, we may assume that all cyclic submodules $m_1R, m_2R, m_3R, \ldots, m_rR$ to be simple. So by [12, Theorem 8.1.3], M_1 is a semisimple module. Let $M_2 = A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_r$, then $M = M_1 \oplus M_2$. Since M_1 is semisimple, $Rad_g(M_1) = M_1$ and $Rad_g(M) = M_1 \oplus Rad_g(M_2)$. Consider the inclusion map $I: M_2 \longrightarrow M_1 \oplus M_2$. Since $Rad_g(M) \oplus M_1$ is essential in $M = M_1 \oplus M_2$, that means $M_1 \oplus Rad_g(M_2) \trianglelefteq M_1 \oplus M_2$, it follows that $\mathrm{I}^{-1}(\mathrm{M}_1 \oplus \mathrm{Rad_g}(\mathrm{M}_2)) \trianglelefteq \mathrm{M}_2$, hence $\mathrm{Rad_g}(\mathrm{M}_2)$ is essential in M_2.■

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References

[1] U. Acar and A. Harmancı, Principally supplemented modules, Albanian J. Math., 4(3)(2010), 79-88.
[2] R. Alizade, G. Bilhan and Smith PF. Modules whose maximal submodules have supplements, Communications in Algebra, 29(6) (2001), 2389-2405.
[3] M. Alkan, A. Harmanci: On summand sum and summand intersection property of modules, Turk. J. Math., 26(2002), 131-147.
[4] E. Büyükaşik and Y. Demirci, Weakly distributive modules, Applications to supplement submodules, Proc. Indian Acad. Sci. (Math. Sci.), 120(2010), 525-534.
[5] V. Camillo, Distributive modules, J. of Algebra, 36(1975), 16-25.
[6] J. Clark, C. Lomp, N. Vanaja and R. Wisbauer, Lifting modules, supplements and projectivity in module theory, Frontiers in Mathematics, Birkhäuser Verlag, Basel, 2006.
[7] N.V. Dung, D.V. Huyn, P.F. Smith and R. Wisbauer, Extending modules, Pitman Research notes in math. Series, 313, Longman Scientific and Technical: Harlow, 1994.
[8] T. Y. Ghawi, On a class of g-lifting modules, Journal of Discrete Mathematical Sciences & Cryptography, 24(6)(2021), 1857-1872.
[9] T. Y. Ghawi, Some generalizations of g-lifting modules, Quasigroups and Related Systems, 2022, to appear.
[10] K. R. Goodearl, Ring theory, *Nonsingular rings and modules*, Dekker, Newyork, 1976.
[11] I. Hatice, H. Sait and A. Harmanci, A generalization of supplemented modules, Algebra and Discrete Mathematics, 11(1)(2011), 59-74.
[12] F. Kasch, *Modules and rings module*, New York, 1982.
[13] D. Keskin and W. Xue, Generalizations of lifting modules, Acta Math. Hungar., 91(3) (2001), 253-261.
[14] B. Kosar, C. Nebiyev and N. Sökmez, G-supplemented modules, Ukrainian Math. J., 67(6) (2015), 861-864.
[15] S. H. Mohamed and B. J. Müller, Continuous and discrete modules, London Math. Soc. Lecture Note Series, 147, Cambridge University press, 1990.
[16] M. M. Obaid and T. Y. Ghawi, Principally g-supplemented modules, 9th International Scientific Conference of Iraqi Al-Khwarizmi Society, 2022, to appear.
[17] A. C. Ozcan and A. Harmanci, Duo modules, Glasgow Math. J., 48(2006), 533-545.
[18] T. C. Quynh and P. H. Tin, Some properties of e-supplemented and e-lifting modules, Vietnam J. Math., 41(2013), 303-312.
[19] L. V. Thuyet and P. H. Tin, Some characterizations of modules via Essential small submodules, Kyungpook Math. J., 56(2016), 1069-1083.
[20] A. Tuganbaev, Semi distributive modules and rings, Kluwer Academics Publishers, Dordrecht, 1998.
[21] B. Ungor, S. Halicioglu and A. Harmanci, On a class of ⊕-supplemented modules, in Ring Theory and Its Applications, Contemporary Mathematics, Amer. Math. Soc., 2014, 123–136.
[22] B. Ungor, S. Halicioglu and A. Harmanci, On a class of δ-supplemented modules, Bull. Malays. Math. Sci. Soc., 37(3)(2014), 703-717.
[23] G.V. Wilson, Modules with the summand intersection property, comm. In Algebra, 14(1986), 21-38.
[24] R. Wisbauer, *Foundations of module and ring theory*, University of Dusseldorf, 1991.
[25] R. Wisbauer, *Modules and algebras: bimodule structure and group actions on algebras,* Pitman Monographs and Surveys in Pure

and Applied Mathematics, 81, Longman, Harlow. MR1396313 (97i:16002), 1996.
[26] J. M. Zelmanowitz, Representation of rings with faithful polyform modules, Comm. Algebra, 14(1986), 1141-1169.
[27] Y. Zhou, Generalizations of perfect, semiperfect and semiregular rings, Algebra Colloq., 7(3)(2000), 305-318.
[28] D.X. Zhou and X.R. Zhang, Small-essential submodules and Morita Duality, Southeast Asian Bulletin of Mathematics, 35(2011), 1051-1062.