JQCM University Of AL-Qadisiyah
Available online at www.qu.edu.iq/journalcm
JOURNAL OF AL-QADISIYAH FOR COMPUTER SCIENCE AND MATHEMATICS
ISSN:2521-3504(online) ISSN:2074-0204(print)

Some connections about sgrs$^{\oplus}$-modules

Narjis Mujtabah Kamil[a], Thaar Younis Ghawi[b]

[a]*Al-Samawah Education Directorate, Iraqi Ministry of Education, Iraq, E-mail: edu-math.post18@qu.edu.iq*

[b]*Department of Mathematics, College of Education, University of Al-Qadisiyah, Iraq, E-mail: thar.younis@qu.edu.iq*

ARTICLE INFO

Article history:
Received: 08/03/2022
Rrevised form: 23/03/2022
Accepted: 11/04/2022
Available online: 22 /04/2022

Keywords:

g-small submdules
⊕-g-supplemented modules
sgrs$^{\oplus}$-modules

ABSTRACT

In this article, we introduced and investigated some relations between the concept of strongly generalized ⊕-radical supplemented module (for short, sgrs$^{\oplus}$-module) and many other types of modules.

MSC. 16D10, 16D70, 16D99

https://doi.org/10.29304/jqcm.2022.14.1.907

1. Introduction

In this work, all modules are unitary left R-modules and R is an associative ring with identity. A submodule $L \leq M$ is said to be essential in M, denoted by $L \trianglelefteq M$, if $N \cap L \neq 0$ for every nonzero submodule N of M [6]. A submodule L of M is called small (g-small), denoted by $L \ll M$ (resp. $L \ll_g M$), if for every (essential) submodule N of M with the property $M = L + N$ implies $N = M$. Recall [18] that the authors renamed a g-small submodule as an e-small submodule. A submodule N of M is known as a generalized maximal submodule of M, if N is an essential and maximal submodule of M. The intersection of all maximal submodules of M, equivalently, the sum of all small submodules of M defined as the radical of a module M, denoted by $Rad(M)$. In [18], Zhou and Zhang defined the generalized radical of a module M (or $Rad_g(M)$) as the intersection of all generalized maximal submodules of M, equivalently, the sum of all g-small submodules of M. A nonzero module M is called uniform if all its nonzero submodules are essential [6]. M is called (generalized) hollow if any proper submodule of M is (g-small) small inside M ([16], resp. [7]), in fact, Hadi and Aidi [7] named a generalized hollow module as an e-hollow module. Assume L and V are two submodules of a module M. Recall [16] that L is a supplement of V in M if it is minimal with respect to property $M = V + L$. Equivalently, L is known as a supplement of V in M if $M = V + L$ and $V \cap L \ll L$. If every submodule of M has a supplement inside M, then M is known as a supplemented module. Moreover, M is named as an ⊕-supplemented module if any submodule of M has a supplement that is a direct summand in M. It is clear that every ⊕-supplemented module is supplemented. Recall ([10] and [16]) the authors defined a submodule V of M as a g-supplement of L in M if, $M = V + L$ and $V \cap L \ll_g L$. A module M is called to be g-supplemented if every submodule of M has a g-supplement in M. Recall [5] that a module M is ⊕-g-supplemented if any submodule of M has a g-supplement that is a direct summand in M. Then M is called a srs-module (srs$^{\oplus}$-module) if any submodule of M contains $Rad(M)$ has a supplement (⊕-supplement) ([2], resp. [15]). Buhphang and Das [4] defined that a module M is strongly generalized radical supplemented (or, sgrs-modules for short) if any submodule of M contains $Rad_g(M)$ has a g-supplement inside M. Obviously, every srs-module is a sgrs-module, in fact $Rad(M) \subseteq Rad_g(M)$.

∗Corresponding author: Narjis Mujtabah Kamil

Email addresses: edu-math.post18@qu.edu.iq

Communicated by : Dr. Alaa Taima Abd Akadhem

However, a module M is called strongly generalized $\oplus$-radical supplemented (or, sgrs$^{\oplus}$-module for short), if for any submodule L of M with $Rad_g(M) \subseteq L$ has a direct summand g-supplement of M, in other words, for any $L \leq M$ with $Rad_g(M) \subseteq L$, there exists a direct summand N of M such that $M = L + N$ and $L \cap N$ is g-small in N [8]. The main goal of the study is to present and investigate a number of outcomes that clarify the relations between the idea of sgrs$^{\oplus}$-modules and a number of other different kinds of modules, such as $\oplus$-g-supplemented modules, g-supplemented modules, sgrs-modules, … etc.

2. Sgrs$^{\oplus}$-modules and related concepts

We will start with the following result.

Proposition 2.1. The following are equivalent for a module M such that $Rad_g(M) = 0$.

(1) M is a $\oplus$-g-supplemented module.
(2) M is a $\oplus$-supplemented module.
(3) M is a g-supplemented module.
(4) M is a supplemented module.
(5) M is a sgrs$^{\oplus}$-module.
(6) M is a srs$^{\oplus}$-module.
(7) M is a sgrs-module.
(8) M is a srs-module.

Proof. Clearly, by definitions (1) ⇒ (3) ⇒ (7) and (1) ⇒ (5) ⇒ (7).
(7) ⇒ (1) Let N be a submodule of M. Since $Rad_g(M) = 0 \subseteq N$, so by assumption, there exists a submodule L of M such that $M = N + B$ and $N \cap B \ll_g B$, also in M. From $N \cap B \subseteq Rad_g(M)$ implies $N \cap B = 0$. Thus, $B \leq^{\oplus} M$ and hence (1) holds.
(1) ⇒ (2) If $N \leq M$, by (1), there is a direct summand K of M such that $M = K + A$ and $K \cap A \ll_g A$. Therefore $K \cap A \subseteq Rad_g(M) = 0$, so that $K \cap A \ll A$. Therefore M is a $\oplus$-supplemented module.
(2) ⇒ (1) Clear.
Now, since $Rad(M) \subseteq Rad_g(M)$, we have that $Rad(M) = 0$. However, by similar technical we can prove (2) ⇔ (4) ⇔ (6) ⇔ (8), as required. ⦀

Recall [17] that a module M is said to be refinable if for all submodules L and V of M with $M = L + V$, there exists a direct summand U of M such that $U \leq L$ and $M = U + V$.

In following, we will give a condition under which sgrs-modules are sgrs$^{\oplus}$-modules.

Proposition 2.2. A refinable module M is a sgrs-module if and only if M is a sgrs$^{\oplus}$-module.
Proof. The sufficiency is clear. Suppose M is a sgrs-module. Let $U \leq M$ with $Rad_g(M) \subseteq U$. Then there exists a submodule H of M such that $M = U + H$ and $U \cap H \ll_g H$. Since M is refinable, $M = U + N$ for a direct summand N of M with $N \leq H$. Clearly, $U \cap N \ll_g M$, so by applying [5, Lemma 2.12(i)], $U \cap N \ll_g N$. Thus U has a g-supplement N that is a direct summand of M. Hence M is a sgrs$^{\oplus}$-module. ⦀

Lemma 2.3. Let M be a non-simple uniform module. If $N \ll_g M$, then G is a proper submodule of M. In particular, $G \ll_g M$ if and only if $G \ll M$.
Proof. Suppose M is a non-simple uniform module and $G \ll_g M$. If $G = M$, then $M = G + N$ for some proper essential submodule N of M, a contradiction with $G \ll_g M$. Thus $G \neq M$. Now, if $G \ll_g M$. Assume that $G + K = M$ for some $K \leq M$. If $K = 0$, then $G = M$, a contradiction. Since M is a uniform module, then $0 \neq K \trianglelefteq M$. Also, $K = M$ and hence $G \ll M$. The converse is clear. ⦀

Proposition 2.4. Let M be a uniform module with $Rad_g(M) \neq M$. The following are equivalent.
(1) M is a $\oplus$-supplemented module.
(2) M is a $\oplus$-g-supplemented module.
(3) M is a srs$^{\oplus}$-module.
(4) M is a sgrs$^{\oplus}$-module.
Proof. It is clear that (1) $\Rightarrow$ (2) $\Rightarrow$ (4) and (3) $\Rightarrow$ (4).
(4) $\Rightarrow$ (3) If M is simple, nothing to prove. Suppose that M is a non-simple module. Let M be a sgrs$^{\oplus}$-module. If $U \leq M$ with $Rad(M) \subseteq U$. We claim that $Rad_g(M) \subseteq Rad(M)$. Assume $m \in Rad_g(M)$, then by [14, Lemma 2.2] $mR \ll_g M$ and hence $mR \ll M$, by Lemma 2.3. Thus, $m \in Rad(M)$ and so $Rad_g(M) \subseteq U$. By hypothesis, there exist submodules V, N of M such that $M = U + V = V \oplus N$ and $U \cap V \ll_g V$, so in M. Again, by Lemma 2.3, we deduce that $U \cap V \ll M$. Since $U \cap V \leq V \leq^{\oplus} M$, we get $U \cap V \ll V$ by [16, 19.3(5)]. Therefore M is a srs$^{\oplus}$-module.
(4) $\Rightarrow$ (2) Assume $G \leq M$. Since M is a sgrs$^{\oplus}$-module and $Rad_g(M) \subseteq Rad_g(M) + G$, then there exist submodules A, K of M such that $M = Rad_g(M) + G + A = A \oplus K$ and $(Rad_g(M) + G) \cap A \ll_g A$. If $G + A = 0$, then $Rad_g(M) = M$, which is a contradiction. Thus, $0 \neq G + A \leq M$. Since M is a uniform module, [11, Lemma 1.11] implies M is indecomposable. By [8, Proposition 2.12], $Rad_g(M) \ll_g M$, and since $G + A \trianglelefteq M$, we deduce that $G + A = M$. Also, $G \cap A \subseteq (Rad_g(M) + G) \cap A$ implies $G \cap A \ll_g A$. Thus A is a g-supplement of G that is direct summand of M. Hence M is a $\oplus$-g-supplemented module.
(2) $\Rightarrow$ (1) If M is simple, nothing to prove. Suppose that M is a non-simple module. Let $V \leq M$, by (2), there exist a direct summand K of M such that $M = V + K$ and $V \cap K \ll_g K$, also in M. By Lemma 2.3 and [16, 19.3(5)] $V \cap K \ll K$, this ends the proof. ▮

If M is an R-module, then the submodule U of M is called fully invariant if $f(U) \subseteq U$ for all nonzero $f \in End(M)$. In state that any direct summand submodule of M is fully invariant, then M called a weak duo module [12].

However, we have the following:

Proposition 2.5. Let M be a weak duo and uniform module such that $Rad_g(M) \neq M$. Then the following are equivalent.
(1) M is a $\oplus$-g-supplemented module.
(2) M is a sgrs$^{\oplus}$-module.
(3) Every direct summand of M is a $\oplus$-g-supplemented module.
(4) Every direct summand of M is a sgrs$^{\oplus}$-module.
Proof. (1) $\Leftrightarrow$ (2) By Proposition 2.4.
(1) $\Leftrightarrow$ (3) It is clear by [11, Lemma 1.11] and [5, Proposition 3.15].
(2) $\Longrightarrow$ (4) By [8, Proposition 3.14].
(4) $\Longrightarrow$ (2) Clear. ▮

The sufficient condition to make the reverse of the note in [8] that states that "any generalized hollow module is a sgrs$^{\oplus}$-module" true, is as follows:

Proposition 2.6. Let M be a uniform module with $Rad_g(M) \neq M$. If M is a sgrs$^{\oplus}$-module, then M is generalized hollow.
Proof. Suppose that $N \subset M$. Since M is a sgrs$^{\oplus}$-module and $Rad_g(M) \subseteq Rad_g(M) + N$, then there exist submodules H and G of M such that $M = Rad_g(M) + N + H = H \oplus G$ and $(Rad_g(M) + N) \cap H \ll_g H$. If $N + H = 0$, then $Rad_g(M) = M$, a contradiction. Therefore, $0 \neq N + H \leq M$. Since M is a uniform module, [11, Lemma 1.11] implies M is indecomposable. By [8, Proposition 2.12], $Rad_g(M) \ll_g M$, and since $N + H \trianglelefteq M$, we include that $N + H = M$. Also, $N \cap H \subseteq (Rad_g(M) + N) \cap H$ implies that $N \cap H \ll_g H$. As M is indecomposable, so either $H = 0$

or $H = M$. If $H = 0$, then $N = M$, a contradiction. Thus, $H = M$. Form $N \cap H \ll_g H$, we have $N \ll_g M$. Hence M is generalized hollow. ▮

Corollary 2.7. Let M be a uniform module such that $Rad_g(M) \neq M$. The next statements are equivalent.
(1) M is a hollow module.
(2) M is a generalized hollow module.
(3) M is a $\oplus$-supplemented module.
(4) M is a $\oplus$-g-supplemented module.
(5) M is a $srs^{\oplus}$-module.
(6) M is a $sgrs^{\oplus}$-module.

Proof. (1) $\Rightarrow$ (2) Clear.
(2) $\Rightarrow$ (1) Since M is a uniform module, then the proper subclasses small and g-small are coincide.
(3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6) By Proposition 2.4.
(2) $\Rightarrow$ (6) Let T be any submodule of M such that $Rad_g(M) \subseteq T$. If $T = M$, then 0 is trivially a (direct summand) g-supplement of M. Assume $T \neq M$, so T is a g-small submodule. Therefore, $M = T + M$ and $T \cap M = T$ is g-small in M, that is M is a (direct summand) g-supplement of T. Therefore, M is a $sgrs^{\oplus}$-module.
(6) $\Rightarrow$ (2) By Proposition 2.6. ▮

Recall [9] that P which is an R-module is named projective if for any two R-module N, L and for any epimorphism $f: N \to L$ and for any homomorphism $g: P \to L$, there is a homomorphism $h: P \to N$ such that $f \circ h = g$.

Proposition 2.8. Let M be a uniform projective module. The next are equivalent.
(1) M is a hollow module.
(2) M is a generalized hollow module.
(3) M is a $\oplus$-supplemented module.
(4) M is a $\oplus$-g-supplemented module.
(5) M is a $srs^{\oplus}$-module.
(6) M is a $sgrs^{\oplus}$-module.

Proof. If M is simple, nothing to prove. Assume that M is a non-simple module. Since $M \neq 0$ is a projective module, then M has a nonzero maximal submodule, say K, see [16, 22.3(1)]. Then we have that $K \trianglelefteq M$, because M is uniform, that means K is a maximal essential in M. Thus, $Rad_g(M) \neq M$. The result is obtained immediately by Corollary 2.7. ▮

Corollary 2.9. Let R be a uniform ring. The following are equivalent.
(1) R is hollow.
(2) R is generalized hollow.
(3) R is $\oplus$-supplemented.
(4) R is $\oplus$-g-supplemented module.
(5) R is a $srs^{\oplus}$-R-module.
(6) R is a $sgrs^{\oplus}$-R-module.

Proof. Since $R = \langle 1 \rangle$, then R is a free R-module and so it is projective. So, the proof is clear by Proposition 2.8. ▮

Proposition 2.10. Let M be a module. If every submodule of M contains $Rad(M)$ has a uniform $\oplus$-g-supplement, then M is a $srs^{\oplus}$-module.

Proof. Assume that N is a submodule of M where $Rad(M) \subseteq N$. If $N = M$, then N trivially has 0 as a direct summand supplement of M. Let $N \neq M$. By hypothesis, there exist a uniform direct summand C of M with $M = N + C$ and $N \cap C$ g-small in C. Assume $(N \cap C) + K = C$ for some submodule K of C. If $K = 0$, $N \cap C = C$ then $C \subseteq N$ and so $N = N + C = M$, a contradiction. So, $K \neq 0$. As K is essential in C and $N \cap C \ll_g C$, then $K = C$. Thus, $N \cap C$ is small in A. Therefore M is a $srs^{\oplus}$-module. ▮

For any $N \leq M$ since M/N is a finitely generated submodule, then N is a cofinite submodule of an R-module M.

Proposition 2.11. Let M be a module such that any cofinite submodule has a (direct summand) g-supplement of M. If $Rad_g(M)$ is cofinite in M, then M is a sgrs$^\oplus$-module.

Proof. Suppose A is a submodule of M such that $Rad_g(M) \subseteq A$. We have that $(M/Rad_g(M))/(A/Rad_g(M)) \cong M/A$. Since $Rad_g(M)$ is a cofinite submodule of M, that implies $M/Rad_g(M)$ is a finitely generated module, and so $(M/Rad_g(M))/(A/Rad_g(M))$ is finitely generated, hence M/A is finitely generated, i.e. A is a cofinite submodule of M. By assumption, A has a g-supplement that is a direct summand of M. So, M is a sgrs$^\oplus$-module. ⅢⅠ

A submodule A of M is known as a distributive submodule if $A \cap (B + C) = (A \cap B) + (A \cap C)$ or $A + (B \cap C) = (A + B) \cap (A + C)$ for all submodules B and C of M. A module M is called distributive if any submodule of M is distributive [3]. Also, by [9], a module M is said to be Artinian if every nonempty set of submodules possesses with respect to inclusion as ordering, a minimal element. However, a module M is said to have a descending chain condition (for short, DCC) for submodules if, every descending chain of submodules of M is determine.

Proposition 2.12. Let M be a finitely generated distributive (or, projective) module satisfies DCC on g-small submodules. If M is a sgrs$^\oplus$-module, then M is Artinian.

Proof. Assume M is a distributive sgrs$^\oplus$-module. By [8, Theorem 3.31], $M/Rad_g(M)$ is semisimple. Since M is a finitely generated module, then $M/Rad_g(M)$ is finitely generated, so that $M/Rad_g(M)$ is Artinian, see [16, 31.3]. Also, M satisfies DCC on g-small submodules implies that $Rad_g(M)$ is Artinian, according to [13, Theorem 4]. Thus, by [9, Theorem 6.1.2(I)] M is Artinian. By a similar way we can prove when M is projective. ⅢⅠ

However, the following corollary is immediately.

Corollary 2.13. Let R be a ring satisfies DCC on g-small ideals. If R is a sgrs$^\oplus$-ring, then R is Artinian.

Proof. As $R = \langle 1 \rangle$, then R is a finitely generated free R-module and so it is finitely generated projective. So, the result is obtained by Proposition 2.12. ⅢⅠ

Recall [1] that the module M is have the SSP (summand sum property) if the sum of any two direct summands of M is also a direct summand of M.

The next result gives case to make sgrs$^\oplus$-module and ⊕-g-supplemented module are identical.

Proposition 2.14. Let M be a module has the SSP, and $Rad_g(M)$ a ⊕-g-supplemented that is a direct summand. If M is a sgrs$^\oplus$-module, then M is ⊕-g-supplemented.

Proof. Let U be a submodule of M. Since $Rad_g(M) \subseteq Rad_g(M) + U$, so by assumption, $Rad_g(M) + U$ has a g-supplement, say X, that is a direct summand in M. Now, as $Rad_g(M) \cap (X + U) \leq Rad_g(M)$ and $Rad_g(M)$ is ⊕-g-supplemented, then $Rad_g(M) \cap (X + U)$ has a g-supplement, say Y, that is a direct summand in $Rad_g(M)$. Since $Rad_g(M) \leq^\oplus M$, then Y is a direct summand in M. As M has SSP, we have $X + Y$ is a direct summand of M. By [13, Lemma 6], $X + Y$ is a g-supplement of U in M. Therefore M is a ⊕-g-supplemented module. ⅢⅠ

A module M is called semisimple if all its submodules are direct summand.

Finally, we came to the following conclusion at the end of this section:

Proposition 2.15. The following are equivalent for a projective R-module M.

(1) M is a sgrs$^\oplus$-module.

(2) For any $X \leq M$ with $Rad_g(M) \subseteq X$, there is a projective module T and an epimorphism $\rho: T \longrightarrow M/X$ such that $Ker\rho$ g-small in T.

Proof. (1) ⟹ (2) Suppose that M is a sgrs$^\oplus$-module. Assume that $X \leq M$ with $Rad_g(M) \subseteq X$. Thus, $M = X + T$ and $X \cap T \ll_g T$ for a direct summand T of M. From [9, Theorem 5.3.4(b)] T is a projective module. Define $\rho : T \rightarrow M/X$

by $\rho(t) = t + X$ for all $t \in T$. Obviously, ρ is an epimorphism. Also, $Ker\rho = \{t \in T | \rho(t) = X\} = \{t \in T | t + X = X\} = \{t \in T | t \in X\} = X \cap T$. Therefore $Ker\rho$ is g-small in T.
(2) $\Longrightarrow$ (1) Let $X \leq M$ with $Rad_g(M) \subseteq X$. By (2), there is a projective module T and an epimorphism $\rho: T \longrightarrow M/X$ such that $Ker\rho$ g-small in T. Consider a canonical epimorphism map $\pi : M \to \frac{M}{X}$. As M is projective, there exists a homomorphism $h : M \to T$ such that $\rho h = \pi$. Thus, we have that $\frac{M}{X} = \pi(M) = \rho h(M) = \rho(h(M))$, then $\rho^{-1}(M/X) = \rho^{-1}(\rho(h(M))$, that implies $T = h(M) + ker\rho y$ [9, Lemma 3.1.8]. Since $ker\rho$ is g-small in T, by [18, Proposition 2.3] there is a semisimple submodule Y of T with $T = h(M) \oplus Y$. Hence $h(M)$ is projective, by [9, Theorem 5.3.4(b)]. Thus, $kerh$ is a direct summand of M, i.e. $M = kerh \oplus H$ for some $H \leq M$. Since $kerh \leq ker\pi = X$, then $M = X + H$. Clearly, $ker\rho \cap h(H) = h(X \cap H)$. $M = kerh \oplus H$ implies that $h(M) = h(H)$ is a direct summand of T. Since $ker\rho \ll_g T$, then $ker\rho \cap h(H) \ll_g T$ and so $h(X \cap H)$ is g-small in T. By [5, Lemma 2.12(i)] we get $h(X \cap H)$ is g-small in $h(H)$. As h between H and $h(H)$ is an isomorphism, $h^{-1}(ker\rho \cap h(H)) \ll_g H$, but $X \cap H \leq h^{-1}(ker\rho \cap h(H))$, we get $X \cap H$ is g-small in H. Therefore H is a g-supplement of X in M. Hence, the proof is ends. ▮

Corollary 2.16. Let R be a ring. Then R is a sgrs$^{\oplus}$-ring if and only if, for each ideal J of R with $Rad_g(R) \subseteq J$, there is a ring $\grave{R}$ and an epimorphism $\tau: \grave{R} \longrightarrow R/J$ such that $Ker(\tau)$ g-small in $\grave{R}$.
Proof. It follows directly by Proposition 2.15. ▮

Conclusion

We stated a number of relationships between sgrs$^{\oplus}$-module and other classes of modules. Future desire will achieve deeper outcomes on issues raised in this work.

References

[1] M. Alkan and A. Harmanci, On summand sum and summand intersection property of modules, Turk. J. Math. 26(2002), 131-147.
[2] E. Buyükasik and E. Türkmen, Strongly radical supplemented modules, Ukrainian Mathematical Journal,63(8)(2012), 1306-1313.
[3] V. Camillo, Distributive modules, J. of Algebra, 36(1975), 16-25.
[4] S. Das and A. M. Buhphang, Strongly generalized radical supplemented module, General Algebra and Applications,40(2020),63-74.
[5] T. Y. Ghawi, Some Generalizations of g-lifting Modules, Quasigroups and Related Systems, 2022, unpublished.
[6] K. R. Goodearl, Ring theory, *Nonsingular Rings and Modules*, Dekker, Newyork, (1976).
[7] Hadi I. M-A and Aidi S.H., *e*-Hollow modules, Int. J. of Advanced Sci. and Technical Research, issue 5, 3(2015), 453-461.
[8] N. M. Kamil and T. Y. Ghawi, Strongly generalized ⊕-radical supplemented modules, Journal of Discrete Mathematical Sciences & Cryptography, 2022, unpublished.
[9] F. Kasch, *Modules and rings*, Academic press, New York, (1982).
[10] B. Koşar, C. Nebiyev and N. Sökmez, G-supplemented modules, Ukrainian Mathematical Journal, 67 (2015), 861-864.
[11] M. M. Obaid and T. Y. Ghawi, Principally g-supplemented modules, 9th International Scientific Conference of Iraqi Al-Khwarizmi Society, 2022, unpublished.
[12] A. C. Ozcan, A. Harmanci, Duo modules, Glasgow Math. J., 48 (2006), 533-545.
[13] T. C. Quynh and P. H. Tin, Some properties of *e*-supplemented and *e*-lifting modules, Vietnam J. Math., 41 (3) (2013), 303-312.
[14] L. V. Thuyet and P. H. Tin, Some characterization of modules via Essential small submodules, Kyungpook Math. J., 56(2016), 1069-1083.
[15] B. N. Türkmen and A. Pancar, Generalizations of ⊕-supplemented modules, Ukrainian Mathematical Journal, 65(4)(2013), 612-622.
[16] R. Wisbauer, *Foundations of module and ring theory*, University of Dusseldorf, (1991).
[17] R. Wisbauer, *Modules and algebras: bimodule structure and group actions on algebras*, Pitman Monographs and Surveys in Pure and Applied Mathematics, 81, Longman, Harlow, MR1396313 (97i:16002), (1996).
[18] D.X. Zhou and X.R. Zhang, Small-Essential submodules and Morita Duality, Southeast Asian Bulletin of Mathematics, 35(2011), 1051-1062.