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Unimodality for the free sum and product of Reflexive and Birkhoff polytopes

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ABSTRACT

Scholars have recently become interested in the importance of the reflexive and Birkhoff polytopes in a variety of applications in our daily lives. Unanswered queries and educated guesses abound in reflexive polytopes. We use the free sum and product for reflexives polytopes, as well as the product for two Birkhoff polytopes, and the proven theorem to get specific unimodality results. The computations are acquired via algorithms.

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1. Introduction and Preliminaries

A polytope is the convex hull of finitely many points in R^d, or also can be defined as the solution of system of linear inequality, that is a set $P = \{x \in R^d : Ax \leq b\}$ is said to be a polyhedron and every bounded polyhedron is said to be a polytope [1]. A reflexive polytope is an integral polytope containing the origin in its interior and the dual for it is likewise integral polytope, so they generally show up as a double pair, [2]. Reflexive polytopes were initially characterized given hypothetical material science applications. In the string hypothesis, reflexive polytopes and the related toric assortments assume a vital function in the most quantitatively prescient type of mirror evenness, aside from such physical uses, [3]. More results of reflexive polytopes in [4-5]

The nth Birkhoff polytope is the set of all doubly stochastic $n \times n$ matrices, that is, those matrices with nonnegative real coefficients in which every row and column sums to one. The nth Birkhoff polytopes, also known as

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the set of all n doubly stochastic matrices, are one of the most fascinating objects in combinatorial geometry. B_n is an integer-verticed convex polytope. It has remarkable combinatorial features, and is connected to a wide range of mathematical fields [6-7].

The Ehrhart polynomial of a convex integral polytope is a polynomial of degree d that counts the number of integral points in an integral dilated polytope [8]. Eugene Ehrhart explored the polynomial in the 1960s and named it the Ehrhart polynomial of the polytope P, which is a rich topic in the fields of discrete geometry, number geometry, enumerative combinatorics, and combinatorial commutative algebra [9]. Stanley [10] shown that the Ehrhart series, which is the corresponding generating series, can be expressed as a rational function with a degree d polynomial numerator whose coefficients define the h-vector of P.

One reason combinatorial lists are interesting in unimodality discoveries is that their proofs usually reveal fascinating and striking aspects of combinatorial, mathematics, and algebra objects. In Ehrhart's theory, symmetric h-vectors and their association with reflexive polytopes are particularly important. Many intriguing methods for exploring symmetric unimodal arrangements exist, including techniques from analysis, the Lie hypothesis, and algebraic geometry [11]. Hibi [12] proposed that h(P) is unimodal whenever it is symmetric, and he proved it for d≤5. For each d≥6, however, Payne and Mustata [13] produced counterexamples to the Hibis hypothesis. [14-19] shows a variety of real-world applications of polytopes. In [20] proved the h-vector of the Birkhoff polytope is symmetric as well as unimodal, and its volume is known only for n ≤ 9 and for n ≤ 10, [21].

2. PRELIMINARIES

This section concerning with some basic concepts that are used in our work.

Definition 2.1.[8] A map $L(P,.)$ is defined for the integral polytope, $P \subseteq R^d$ of dimension d, as $L(P,t) = |tP \cap Z^d|$. Also L(P,t) can be represented as $L(P,t) = \sum_{i=0}^{d} c_i t^i$, which is said to be Ehrhart polynomial of an integral d-polytope. c_d is the volume of P, c_{d-1} represents half of the surface area of a polytope while the constant term is one,. The other coefficients of $L(P,t)$are not easily accessible. And the Ehrhart series of P is Ehr P(x) = $1+\sum_{t\in Z\geq 1} L(P,t)x^t$.

Theorem 2.2.[8] Let $P \subseteq R^d$be an integral polytope of dimension d. there exist complex values h_j where $0 \leq j \leq d$, such that the Ehrhart series for P is a rational generating function of the following form: $Ehr_P(x) = \frac{\sum_{j=0}^{d} h_j x^j}{(1-x)^{d+1}}, \sum_j h_j \neq 0$. The vector =($h_0, \ldots, h_d$) of coefficients of the numerator of $Ehr_P(x)$ is called the h-vector for P.

Theorem 2.3.[22] Let P be the Integral polytopes of dimension d, $P \subseteq R^d$ is a and $L(P,t) = \sum_{i=0}^{d} c_i\, t^i = \sum_{j=0}^{d} h_j \binom{t+d-j}{d}$, then :

(i) $h_0 = 1$.

(ii) $h_1 = L(P,1) - d - 1$.

(iii) $h_d = L(P,1) - \#\ (\partial P \cap Z^d)$.

3. Reflexive and Birkhoff polytopes with Properties

This section gives some information about reflexive and Birkhoff polytopes

Definition 3.1.[23] For a polytope P in R^dcontaining $0 \in int(P)$ where $int(P)$ is the set of all points $x \in P$ such that for some $\epsilon > 0$, the $\epsilon - ball\ B_\epsilon(x)$ around x is contained in P, the convex set P^*in R^d is $P^* = \{u \in R^d : \langle u, x\rangle \leq 1\ \forall x \in P\}$ is called the dual of P.

Theorem 3.2.[8] A polytope P is reflexive if and only if P is integral polytope with $0 \in int\ (P\ )$ that satisfies one of the flowing (equivalent) conditions

- $P^*$is an integral polytope.
- $L_{P^0}(t+1) = L_P(t)$ for all t∈ N, that mean all integral points in R^{dP} located on the boundary of some non-negative integral dilate of P.
- $h_i = h_{d-i}$ for all i where h_i is the coefficient in the numerator of the Ehrhart series for reflexive polytopes.

Proposition 3.3.[24] Given polytope P of dimension d with vertices belong to Z^d .define $L(P,t)$, (which is Ehrhart polynomial) as in theorem (2.2)

$$\sum_{t\geq 0} L(P,t)x^i = \frac{h_0 + h_1 x + \cdots + h_s x^s}{(1-x)^{d+1}}$$

where $h_s \neq 0$, that is
$h_0 + h_1 + \cdots + h_i \leq h_s + h_{s-1} + \cdots + h_{s-i}$ $\forall\, 0 \leq i \leq s$.

Stanly in [25] showed the inequalities:

$$h_0 + h_1 + \cdots + h_i \leq h_s + h_{s-1} + \cdots + h_{s-i} \qquad 0 \leq i \leq \tfrac{s}{2} \quad \ldots \qquad (3.1)$$

Make use of the theory of Cohen Macaulay ring. Also, the inequalities

$$h_d + h_{d-1} + \cdots + h_{d-i} \leq h_1 + h_2 + \cdots + h_{i+1} \qquad 0 \leq i \leq \tfrac{d-1}{2} \quad \ldots \qquad (3.2)$$

is hold.

Definition 3.4.[12] The Ehrhart h-vectore of P, $h(P)$ =$(h_0, \ldots, h_d)$ is unimodal if there is an integer number j such that $h_d \leq h_{d-1} \leq \cdots \leq h_j \geq h_{j-1} \geq \cdots \geq h_0 \ldots$ for some j (3.3)

Theorem 3.5.[20] For any positive integer n, we have

$$\sum_{r\geq 0} L_{Bn}(t)x^r = \frac{h_0 + h_1 x + \cdots + h_d x^d}{(1-x)^{(n-1)^{2+1}}}$$

where d=$n^2 - 3n + 2$ and h_i are nonnegative integers satisfying $h_0 = 1$ and $h_{i=}h_{d-i}$ for all i.
For more information see [6], [26].

4. Results

In this section, the free sum and product for two reflexive polytopes in 2-dimension and 3-dimension are given

Definition 4.1[8] For two polytopes $P \subseteq R^{dP}$ and $Q \subseteq R^{dQ}$ of dimension $d_P$and d_Q, the free sum of P and Q is $P \oplus Q = conv\{(0_P \times Q) \cup (P \times 0_Q)\} \subseteq R^{dP+dQ}$.
Definition 4.2.[8] For two polytopes $P \subseteq R^{dP}$ and $Q \subseteq R^{dQ}$ of dimension $d_P$and d_Q, the product of P and Q is $P \times Q = \{(p,q), wher\ p \in P, q \in Q\}\} \subseteq R^{dP+dP}$
Theorem 4.3 [8] If P is a d_P-dimensional reflexive polytope in R^d and Q is d_Q-dimensional Integral polytope in R^{dQ} with $0 \in Q^0$, then $Ehr_{P\oplus Q}$(x) = (1-x)$Ehr_P(x)Ehr_Q(x)$.
Theorem 4.4. [8] If P is a d_P-dimensional integral polytope in R^d and Q is d_Q-dimensional Integral polytope in R^{dQ} the Earhart polynomial of $P \times Q$ =$L_P(t)L_Q(t)$.

4.1. The Free Sum of Two Reflexive Polytopes

Now a theorem for the free sum of two polytopes and the proof of unimodality is introduced.

Theorem 4.5. Let P be a reflexive polytope of dimension d, take $\oplus$ as a basic definition of a free sum, then the free sum of two reflexive polytopes is unimodal.

Proof: Since P is a reflexive polytope, then $h_i = h_{d-i}$,put in equation (3.3) this means that

$$h_d \leq \cdots \leq h_{\frac{d}{2}+2} \leq h_{\frac{d}{2}+1} \leq h_{\frac{d}{2}} \geq h_{\frac{d}{2}-1} \geq \cdots \geq h_0 \qquad (4.1)$$

By $Ehr_{P(x)}Ehr_{P(x)} = \frac{\sum_{i=0}^{d} h_i x^i}{(1-x)^{d+1}} * \frac{\sum_{i=0}^{d} h_i^* x^i}{(1-x)^{d+1}} = \frac{\sum_{i=0}^{d} h_i x^i \sum_{i=0}^{d} h_i^* x^i}{(1-x)^{2d+2}}$

$$= \frac{(h_d x^d + \cdots + h_0)(h_d^* x^d + \cdots + h_0)}{(1-x)^{2d+2}} = \frac{(h_d h_d^* x^{2d} + \cdots + 1)}{(1-x)^{2d+2}}$$

Since h_i and h_i^* satisfying equation (4.1) so is the coefficients of the Ehrhart series for the free sum is unimodal.

Example 4.6. let P be reflexive polytopes in R^3 with the vetices $\{(-1,0,1),(-1,0,-1),(1,1,1),(1,1,-1),(0,-1,1),(0,-1,-1)\}$ and unimodal h-vector (1,8,8,1)

The Ehrhart series of P is $\frac{1+8x+8x^2+x^3}{(1-x)^4}$.

The free sum of $P \oplus P = con\{(0_P \times P) \cup (P \times 0_P)\} \subseteq R^{dP+dP}$

= $conv\{(0,0,0,-1,0,-1),(0,0,0,-1,0,-1),(0,0,,0,1,1,1),(0,0,0,1,1,-1),(0,0,0,0,-1,1),$

$(0,0,0,0,-1,-1) \cup (-1,,0,1,0,0,0),(-1,0,-1,0,0,0)$(1,1,1,0,0,0),(1,1,-1,0,0,0),(0,-1,1,0,0,0)

(0,-1,-1,0,0,0)}

then $Ehr_{P\oplus P}$(x) = (1-x)$Ehr_P(x)Ehr_P(x)$

$$= \frac{1+16x+80x^2+130x^3+80x^4+16x^5+x^6}{(1-x)^7}$$

From the properties of the reflexive polytope $h_i = h_{d-i}$ for all i, $h_0 = h_6, h_1 = h_5, h_2 = h_4, h_3 = h_3$ This means that $h_0t^d + h_1t^{d-1} + h_2t^{d-2} + \cdots h_3t^3 + h_2t^2 + h_1t + h_0$

then $h_6 \leq h_5 \leq h_4 \leq h_3 \geq h_2 \geq h_1 \geq h_0$ is unimodality h-vector.

4.2. The Product of Two Reflexive Polytopes

Now a theorem for the product of two polytopes and the proof of unimodality is given.

Theorem 4.7. Let P be a reflexive polytope of dimension d, take $\times$ as a basic definition of a product of two reflexive polytopes is unimodal.

Proof: Since P is a reflexive polytope, then $h_i = h_{d-i}$, this means that

$$h_d \leq \cdots \leq h_{\frac{d}{2}+2} \leq h_{\frac{d}{2}+1} \leq h_{\frac{d}{2}} \geq h_{\frac{d}{2}-1} \geq \cdots \geq h_0$$

The Ehrhart polynomial for the product of two polytopes is given by

By $L_{P\times Q}(x) = L_p(x)L_Q(x)$ and by using the relation EhrP(x) = 1+$\sum_{t\in Z\geq 1}$ L(P, t)x^t

$$Ehr_{P(x)}Ehr_{P(x)} = \frac{\sum_{i=0}^{d} h_i x^i}{(1-x)^{d+1}} * \frac{\sum_{i=0}^{d} h_i^* x^i}{(1-x)^{d+1}} = \frac{\sum_{i=0}^{d} h_i x^i \sum_{i=0}^{d} h_i^* x^i}{(1-x)^{2d+2}}$$

$$= \frac{(h_dx^d + \cdots + h_0)(h_d^* x^d + \cdots + h_0)}{(1-x)^{2d+2}} = \frac{(h_d h_d^* x^{2d} + \cdots + 1)}{(1-x)^{2d+2}}$$

Since h_i and h_i^* satisfying equation (4.1) so is the coefficients of the Ehrhart series for the product are unimodal

Example 4.8. Let P be the reflexive polytopes in R^2 with the vertices $\{(1,1),(-1,1),(-1,-1),(1,-1)\}$ with unimodal h-vector (1,6,1), and the Ehrhart polynomial $4t^2 + 4t + 1$ and Let P be the reflexive polytopes in R^3 with the vertices $\{(-1,0,1),(-1,0,-1),(1,1,1),(1,1,-1),(0,-1,1),(0,-1,-1)\}$ with unimodal h-vector (1,8,8,1) and the Ehrhart polynomial $\frac{18}{6}t^3 + \frac{27}{6}t^2 + \frac{21}{6}t + 1$

$P \times P = \{ (1,1,-1,01),(1,1,-1,0,1),(1,1,1,1),(1,1,1,1,-1),(1,1,0,-1,1),(1,1,-1,-1),$

(-1,1,-1,0,-1), (-1,1,1,1,1), (-1,-1,1,1.-1), (-1,1,0-1,1), (-1,1,0,-1,-1), (-1,-1,-1,0,1), (-1,-1,-1,0,-1), (-1,-1,1,1,1), (-1,-1,1,1,-1), (-1,-1,0,-1,1),(-1,-1,0,-1,-1),(1.-1,-1,0,1),(1,-1,-1,0.-1),(1,-1,1.1.1),(1,-1,1,1,-1),(1,-1,0,-1,1),(1,-1,0,-1,-1)}$\subseteq R^{dP+dP}$.

$$L_P(t)L_P(t) = (4t^2 + 4t + 1)\ (\frac{18}{6}t^3 + \frac{27}{6}t^2 + \frac{21}{6}t + 1)$$

$$= 12t^5 + 30t^4 + 35t^3 + \frac{45}{2}t^2 + \frac{15}{2}t + 1$$

$f(t) = 12t^5 + 30t^4 + 35t^3 + \frac{45}{2}t^2 + \frac{15}{2}t + 1$, then by using theorem (2.3) to get

$$\sum_{t\in Z\geq 0} f(t) = 12\sum_{t\in Z\geq 0} t^5x^t + 30\sum_{t\in Z\geq 0} t^4x^t + 35\sum_{t\in Z\geq 0} t^3x^t + \frac{45}{2}\sum_{t\in Z\geq 0} t^2x^t + \frac{15}{2}\sum_{t\in Z\geq 0} tx^t + \sum_{t\in Z\geq 0} x^t$$

After some simplifications the result were obtained,

$$= \frac{1+102x+617x^2+617x^3+102x^4+x^5}{(1-x)^6}$$

From the properties of the reflexive polytope $h_i = h_{d-i}$ for all i, $h_0 = h_5, h_1 = h_4, h_2 = h_3$,

This means that $h_0t^d + h_1t^{d-1} + h_2t^{d-2} + \cdots h_3t^3 + h_2t^2 + h_1t + h_0$,

then $h_5 \leq h_4 \leq h_3 \leq h_2 \geq h_1 \geq h_0$ is unimodality h-vector

4.3. The Product of Two Brikhoff Polytopes

Now, the product for two Birkhoff polytopes with the same and different dimensions are introduced with forms that discussed the results

Example 4.9 For the product if we take Q and P are Birkhoff polytopes with $n = 2$,$B_2 = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$ with unimodal h-vector and the Ehrhart polynomial of B_2 is $t + 1$, depending on the definition of product the following results are obtained, that is $B_2 \times B_2 = L_{B2}(t) \times L_{B2}(t) = t^2 + 2t + 1$

$f(t) = t^2 + 2t + 1$, then by using theorem (2.3) to get

$$\sum_{t\in Z\geq 0} f(t) = \sum_{t\in Z\geq 0} t^2x^t + 2\sum_{t\in Z\geq 0} tx^t + \sum_{t\in Z\geq 0} x^t = (x\frac{d}{dx})^2\sum_{t\in Z\geq 0} x^t + 2(x\frac{d}{dx})\sum_{t\in Z\geq 0} x^t + \sum_{t\in Z\geq 0} x^t$$

Therefore, the result is that get it with simple step $= \frac{1+x}{(1-x)^3}$

Similarly, the following is obtained, that is for $B_3 = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}$, the Ehrhart polynomial of B_3 is $\frac{1}{8}t^4 + \frac{3}{4}t^3 + \frac{15}{8}t^2 + \frac{9}{4}t + 1$, and $B_3 \times B_3 = L_{B3}(t) \times L_{B3}(t)$

$$= \frac{1}{64}t^8 + \frac{3}{16}t^7 + \frac{33}{32}t^6 + \frac{27}{8}t^5 + \frac{457}{64}t^4 + \frac{159}{16}t^3 + \frac{141}{16}t^2 + \frac{9}{2}t + 1$$

$f(t) = \frac{1}{64}t^8 + \frac{3}{16}t^7 + \frac{33}{32}t^6 + \frac{27}{8}t^5 + \frac{457}{64}t^4 + \frac{159}{16}t^3 + \frac{141}{16}t^2 + \frac{9}{2}t + 1$, then by using theorem (2.3) to get

$$\sum_{t\in Z\geq 0} f(t) = \frac{1}{64}\sum_{t\in Z\geq 0} t^8x^t + \frac{3}{16}\sum_{t\in Z\geq 0} t^7x^t + \frac{33}{32}\sum_{t\in Z\geq 0} t^6x^t + \frac{27}{8}\sum_{t\in Z\geq 0} t^5x^t + \frac{457}{64}\sum_{t\in Z\geq 0} t^4x^t + \frac{159}{16}\sum_{t\in Z\geq 0} t^3x^t + \frac{141}{16}\sum_{t\in Z\geq 0} t^2x^t + \frac{9}{2}\sum_{t\in Z\geq 0} tx^t + \sum_{t\in Z\geq 0} x^t$$

After simple computation the following is obtained

$$= \frac{(x^6+27x^5+153x^4+268x^3+153x^2+27x+1)}{(1-x)^9}$$

From the properties of Birkhoff polytope $h_i = h_{d-i}$ for all i, $h_0 = h_6, h_1 = h_5, h_2 = h_4, h_3 = h_3$ This means that $h_0t^d + h_1t^{d-1} + h_2t^{d-2} + \cdots h_3t^3 + h_2t^2 + h_1t + h_0$

then $h_6 \leq h_5 \leq h_4 \leq h_3 \geq h_2 \geq h_1 \geq h_0$ is unimodality h-vector.

Now the product of different dimensions of Birkhoff polytopes is given

Example 4.10. if Q and P are Birkhoff polytopes that is $B_2 = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$, $B_3 = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}$ be the Brikhoff polytope with unimodal h-vector and the Ehrhart polynomial

$B_3 \times B_2 = L_{B3}(t) \times L_{B2}(t)$
$= (\frac{1}{8}t^4 + \frac{3}{4}t^3 + \frac{15}{8}t^2 + \frac{9}{4}t + 1) \times (t+1)$
$= \frac{1}{8}t^5 + \frac{7}{8}t^4 + \frac{21}{8}t^3 + \frac{33}{8}t^2 + \frac{13}{4}t + 1$, then by using theorem (2.3) to get

$$\sum_{t\in Z\geq 0} f(t) = \frac{1}{8}\sum_{t\in Z\geq 0} t^5x^t + \frac{7}{8}\sum_{t\in Z\geq 0} t^4x^t + \frac{21}{8}\sum_{t\in Z\geq 0} t^3x^t + \frac{33}{8}\sum_{t\in Z\geq 0} t^2x^t + \frac{13}{4}\sum_{t\in Z\geq 0} tx^t + \sum_{t\in Z\geq 0} x^t$$

$$= \frac{1}{8}\left(x\frac{d}{dx}\right)^5 \sum_{t\in Z\geq 0} x^t + \frac{7}{8}\left(x\frac{d}{dx}\right)^4 + \frac{21}{8}\left(x\frac{d}{dx}\right)^3 \sum_{t\in Z\geq 0} x^t + \frac{33}{8}\left(x\frac{d}{dx}\right)^2 \sum_{t\in Z\geq 0} x^t + \frac{13}{4}x\frac{d}{dx}\sum_{t\in Z\geq 0} x^t + \sum_{t\in Z\geq 0} x^t$$

After simple computation the following is obtained $= \frac{2x^3+6x^2+6x+1}{(1-x)^6}$

The unimodality does not satisfied for the product of Birkhoff polytope with different dimension

An algorithm that gives the results of the computation is given below.

4. ALGORITHM

Input: the vertices of P and Q;

Output: The h-vector of product and the free sum of two integral polytopes.

Step 1. To find the Earhart polynomial for both P and Q, we need to compute the polynomial coefficients

i. compute the volume of the polytopes P and Q;

ii. compute the surface area of the polytopes P and Q;

iii. compute the number of lattice points of the polytopes P and Q;

Step 2 . Compute the Ehrhart series by the theorem(2.3)

Step 3. Find the $Ehr_{P\oplus Q}$(x) with d_P+ $d_Q$dimensions;

Step 4. Find $L_{P(t)}$ × $L_{P(t)}$ with d_P+ $d_Q$dimensions;

Step 5. Find the Ehrhart series for ($L_{P(t)}$ × $L_{P(t)}$)

5. CONCLUSION

As previously stated, polytopes have a wide range of applications in several fields; for additional information. Unimadility notions have also emerged in recent years for various reasons, particularly for reflexive polytope. Reflexive polytopes provide a wealth of unanswered questions and educated guesses. The Birkhoff polytopes are special cases of transportation polytopes. The study of this class of polytopes, which are usually just as intriguing as the Birkhoff polytopes, was prompted by challenges in linear programming; see, for example, [27] and [28] for combinatorial features. Because reflexive and Birkhoff polytopes are also necessary for the production of massive h-vectors, the main goal of this search is to establish the unimodality of the free sum and product for two reflexive polytope and product for Birkhoff polytopes, and a theorem proving the unimodality is given.

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