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On WNQP – Submodules

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ABSTRACT

Our goals in this paper are to introduce many results of a WNQP submodules such as triple zero of WNQP submodules. Moreover, several characterization of WNQP Submodule in some types of modules such as (multiplication, content, and finitely generated) modules. Furthermore WNQP radical of submodule are discuses.

MSC..

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1. Introduction

In this paper, unless otherwise established, all rings are commutative with identity, and all modules are unitary left modules. Let R be a ring and $\mathfrak{I}$ be an R-module. The concept of WNQP submodule was introduced recently in [1], where a proper submodule E of an R- module $\mathfrak{I}$ is said to be WNQP submodule if for all $0\neq abt\in E$, for $a, b\in R$, $t\in\mathfrak{I}$, implies that either $at\in E+J(\mathfrak{I})$ or $bt\in E+J(\mathfrak{I})$, and an ideal I of a ring R is WNQP ideal if I is WNQP R-submodule for an R-module R [1], where $J(\mathfrak{I})$ is the Jacobson radical for $\mathfrak{I}$ defined to be the intersection of all maximal submodules of $\mathfrak{I}$ [2], as new generalization of prime submodule, where a proper submodule E of an R- module $\mathfrak{I}$ is said to be prime submodule if for all $at\in E$, for $a\in R$, $t\in\mathfrak{I}$, implies that either $t\in E$ or $a\in[E:_R\mathfrak{I}]$ [3]. The residual of E by $\mathfrak{I}$ denoted by $[E:_R\mathfrak{I}] = \{ r \in R: r\mathfrak{I} \subseteq E\}$ which is an ideal of R [4], in particular the ideal $[0:_R\mathfrak{I}]$ is called annihilator of $\mathfrak{I}$ and is denoted by $Ann_R(\mathfrak{I})$ [5]. An R-module $\mathfrak{I}$ is called a multiplication module provided that for every submodule E of $\mathfrak{I}$ there exists an ideal I of R so that $E=I\mathfrak{I}$ [6], equivalently $E=[E:_R\mathfrak{I}]\mathfrak{I}$ [7]. For each submodule E, B of a multiplication R- module $\mathfrak{I}$ with $E = I_1\mathfrak{I}$, $B = I_2\mathfrak{I}$ for some ideals I_1, I_2 in R define $EB= I_1I_2\mathfrak{I}$ and $EB=I_1B$. In particular $E\mathfrak{I}=I_1\mathfrak{I}\mathfrak{I}=I_1\mathfrak{I}=E$ [8]. If $\mathfrak{I}$ is a multiplication R-module and $t_1, t_2\in\mathfrak{I}$, by t_1t_2 means the product of two submodules Rt_1, Rt_2 that is $t_1t_2=Rt_1Rt_2$ is a submodule of $\mathfrak{I}$ [9]. Recall that an R-module $\mathfrak{I}$ is content module if $(\bigcap_{i\in I} E_i)\ \mathfrak{I} = \bigcap_{i\in I} E_i\mathfrak{I}$ for each family of ideals E_i in R [10]. A submodule E of $\mathfrak{I}$ is called completely irreducible if for each submodules K, L of $\mathfrak{I}$ with $K\cap L\subseteq E$ then either $K\subseteq E$ or $L\subseteq E$ [11].

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2. Triple zero of WNQP Submodules

In this part we introduced the definition of triple zero of WNQP submodules, with some properties.

Definition 2.1 Let E be a WNQP submodule for an R-modules $\mathfrak{J}$ and r, s∈R, t∈$\mathfrak{J}$ with rst=0 and rt ∉E+J($\mathfrak{J}$) and st∉E+J($\mathfrak{J}$), then we say that (r , s , t) is nearly quasi triple zero of E.

Proposition 2.2 Let E be WNQP submodule for $\mathfrak{J}$ with rsk ⊆ E for some r, s∈ R , and some submodule k of $\mathfrak{J}$. If (r , s , t) is not nearly quasi triple zero of E for every t∈$\mathfrak{J}$, then rk ⊆ E + J($\mathfrak{J}$) or sk ⊆ E + J($\mathfrak{J}$).

Proof For some r, s ∈ R, t ∈ $\mathfrak{J}$, let (r , s , t) is not nearly quasi triple zero of E for any t ∈ K , and assume rK ⊄ E + J($\mathfrak{J}$) and sK ⊊ E + J($\mathfrak{J}$), it follows that rt_1 ∉ E + J($\mathfrak{J}$) and st_2 ∉ E + J($\mathfrak{J}$) for some t_1, t_2 ∈ K. If 0≠ rst_1∈E with rt_1∉E+ J($\mathfrak{J}$) and E is a WNQP submodule for $\mathfrak{J}$ then sm_1∈ E + J($\mathfrak{J}$). Since (r , s , t) is not nearly quasi triple zero of E, then st∈E + J($\mathfrak{J}$). In similar way since (r , s , t) is not nearly quasi triple zero of E and st_2∈E + J($\mathfrak{J}$) then rt_2∈E + J($\mathfrak{J}$), thus rs(t_1 + t_2)∈E and (r, s, t_1 + t_2) is not nearly quasi triple zero of E, then either r(t_1 + t_2) ∈ E + J($\mathfrak{J}$) or s(t_1 + t_2) ∈E + J($\mathfrak{J}$). If r(t_1 +t_2)=rt_1 + rt_2∈E + J($\mathfrak{J}$) and since rt_2∈E + J($\mathfrak{J}$) we have rt_1 ∈E + J($\mathfrak{J}$) which is a contradiction. If s(t_1 + t_2)=st_1 + st_2∈E + J($\mathfrak{J}$) and since st_1∈E + J($\mathfrak{J}$), then st_2∈E + J($\mathfrak{J}$) which is a contradiction. Thus rK⊆E + J($\mathfrak{J}$) or sK⊆E + J($\mathfrak{J}$) .

Definition 2.3 Let E be a WNQP submodule of $\mathfrak{J}$ with IJK ⊆ E for some ideals I, J in R and some submodule K for $\mathfrak{J}$, we say that E is a free nearly quasi triple zero with respect to IJK. If (r , s , t) is not nearly quasi triple zero of E for any r, s ∈ R and t ∈ K, that is either rt ∈ E + J($\mathfrak{J}$) or st ∈ E + J($\mathfrak{J}$).

Proposition 2.4 Let A be a WNQP submodule for $\mathfrak{J}$ with IJK⊆E for some ideals I, J in R and some submodule K for $\mathfrak{J}$. If E is a free nearly quasi triple zero with respect IJK, then either IK ⊆ E + J($\mathfrak{J}$) or JK ⊆ E + J($\mathfrak{J}$).

Proof Suppose that E is a free nearly quasi triple zero with respect to IJK and IK⊈E + J($\mathfrak{J}$) and JK ⊈E + J($\mathfrak{J}$), implies that rK⊈E + J($\mathfrak{J}$) and sK ⊈E + J($\mathfrak{J}$) for some r ∈ I, s ∈ J. Since rsK ⊆ E and E is a free nearly quasi triple zero with respect to IJK, then rK⊆ E + J($\mathfrak{J}$) or sK⊆ E + J($\mathfrak{J}$) which is a contradiction. Thus IK ⊆ E + J($\mathfrak{J}$) or JK ⊆ E + J($\mathfrak{J}$).

The following propositions give some properties of nearly quasi triple zero.

Proposition 2.5 Let E be a WNQP submodule for $\mathfrak{J}$ with (r , s , t) is a nearly quasi triple zero of E for some r, s ∈ R, t ∈ $\mathfrak{J}$. Then rsE=(0).

Proof Suppose that rsE ≠ (0), then rsa ≠ 0 for some a∈E. But (r , s , t) is a nearly quasi triple zero of E, implies that rst = 0, rt ∉ E + J($\mathfrak{J}$) and st ∉ E + J($\mathfrak{J}$). But 0 ≠ rsa ∈ E and E is a WNQP submodule for $\mathfrak{J}$, then either ra ∈ E + J($\mathfrak{J}$) or sa ∈ E + J($\mathfrak{J}$). Since 0≠rs(t + a)=rst + rsa =rsa∈E, implies that r(t + a) = rt+ ra∈ E + J($\mathfrak{J}$) or s(t + a) = st + sa ∈ E + J($\mathfrak{J}$). If rt + ra∈ E + J($\mathfrak{J}$) and ra∈ E + J($\mathfrak{J}$) then rt ∈ E + J($\mathfrak{J}$) which is a contradiction. If sm + sa ∈ E + J($\mathfrak{J}$) and sa ∈ E + J($\mathfrak{J}$), then st ∈ E + J($\mathfrak{J}$) which is a contradiction. Hence rsA = (0).

Proposition 2.6 Let E be WNQP submodule for $\mathfrak{J}$ with (r , s , t) is a nearly quasi triple zero of E for some r, s ∈ R, t ∈ $\mathfrak{J}$. Then r[E:$_R\mathfrak{J}$]t = (0).

Proof Suppose that r[E:$_R\mathfrak{J}$]t ≠ (0), it follows that rct ≠ 0 for some c ∈ [E:$_R\mathfrak{J}$]. But (r , s , t) is a nearly quasi triple zero of E and rst = 0, rt ∉ E + J($\mathfrak{J}$) and st ∉ E + J($\mathfrak{J}$). Since 0 ≠ rct ∈ E and E is a WNQP submodule for $\mathfrak{J}$, then either rt ∈ E + J($\mathfrak{J}$) or ct ∈ E + J($\mathfrak{J}$). Now 0 ≠ r(s + c)t = rst + rct ∈E, implies that either rt ∈ E + J($\mathfrak{J}$) or (s + c)t = st + ct ∈ E + J($\mathfrak{J}$). But ct ∈ E + J($\mathfrak{J}$), implies that st ∈E+ J($\mathfrak{J}$) which is a contradiction. Thus r[E:$_R\mathfrak{J}$]t = (0).

Proposition 2.7 Let E be a WNQP submodule for $\mathfrak{J}$ such that (r , s , t) is a nearly quasi triple zero of E for some r, s ∈ R, t ∈ $\mathfrak{J}$. Then s[E:$_R\mathfrak{J}$]t = (0).

Proof Similar way of propasition 2.6.

Proposition 2.8 Let E be a WNQP submodule for $\mathfrak{I}$ such that (r , s , t) is a nearly quasi triple zero of E for some r, s ∈ R, t ∈ $\mathfrak{I}$. Then $[E:_R\mathfrak{I}][E:_R\mathfrak{I}]t = (0)$.

Proof Assume that $[E:_R\mathfrak{I}]\,[E:_R\mathfrak{I}]t \neq (0)$, then there exists $r_1, r_2 \in [E:_R\mathfrak{I}]$ such that $r_1r_2t \neq 0$, then by proposition (2.6) and proposition (2.7) we have $(r+r_1)(s+r_2)t=rst+r\,r_2t + r_1st + r_1r_2t = r_1r_2t \neq 0$, implies that $0 \neq (r + r_1)(s + r_2)t \in E$. Since E is a WNQP submodule of $\mathfrak{I}$, it follows that either $(r + r_1)t \in E + J(\mathfrak{I})$ or $(s +r_2)t \in E + J(\mathfrak{I})$. Thus $rt \in E + J(\mathfrak{I})$ or $st \in E + J(\mathfrak{I})$ which is a contradiction. Thus $[E:_R\mathfrak{I}]\,[E:_R\mathfrak{I}]t = (0)$.

Corollary 2.9 Let E be a WNQP submodule for $\mathfrak{I}$ such that (r , s , t) is a nearly quasi triple zero of E for some r, s ∈ R, t ∈ $\mathfrak{I}$. Then $r[E:_R\mathfrak{I}]\,E = (0)$

Proof Follows by propositions 2.5 and 2.6.

Corollary 2.10 Let E be a WNQP submodule for $\mathfrak{I}$ such that (r , s , t) is a nearly quasi triple zero of E for some r, s ∈ R, t ∈ $\mathfrak{I}$. Then $s[E:_R\mathfrak{I}]E = (0)$.

Proof Follows by propositions 2.5 and 2.6.

3. Characterizations of WNQP Submodule in Multiplication Modules.

In this part we introduced some characterizations of WNQP submodule in class of multiplication modules.

Proposition 3.1 Let $\mathfrak{I}$ be multiplication R- module and $E \subsetneq \mathfrak{I}$. Then A is a WNQP for $\mathfrak{I}$ if and only if for all $(0) \neq BCD \subseteq E$, for some submodules B, C and D in $\mathfrak{I}$, implies that either $BD \subseteq E + J(\mathfrak{I})$ or $CD \subseteq E + J(\mathfrak{I})$.

Proof (⟹) Suppose $(0)\neq BCD\subseteq E$, for some submodules B, C and D in $\mathfrak{I}$. Since $\mathfrak{I}$ is a multiplication, then $B = I_1\mathfrak{I}$, $C = I_2\mathfrak{I}$, $D = I_3\mathfrak{I}$ for some ideals I_1 , I_2 , and I_3 in R, hence $(0)\neq I_1 I_2 I_3 \mathfrak{I} \subseteq E$. But E is WNQP submodule for $\mathfrak{I}$ then by [1,prop. (2.8)], we have either $I_1 I_3 \mathfrak{I} \subseteq E + J(\mathfrak{I})$ or $I_2 I_3 \mathfrak{I} \subseteq E + J(\mathfrak{I})$, hence either $BD \subseteq E + J(\mathfrak{I})$ or $CD \subseteq E + J(\mathfrak{I})$.

(⟸) Assume $(0) \neq I_1 I_2 L \subseteq E$, for some ideals I_1 ,I_2 in R, and L is a submodule of $\mathfrak{I}$. But $\mathfrak{I}$ is a multiplication, so $L = I_3 \mathfrak{I}$ for some ideal I_3 for R, that is $(0) \neq I_1 I_2 I_3 \mathfrak{I} \subseteq E$ implies that $BCL \subseteq E$ for $B = I_1 \mathfrak{I}$, $C = I_2 \mathfrak{I}$ by hypothesis either $BL \subseteq E + J(\mathfrak{I})$ or $CL \subseteq E + J(\mathfrak{I})$, that is $I_1 L \subseteq E + J(\mathfrak{I})$ or $I_2 L \subseteq E + J(\mathfrak{I})$. Hence by [1, prop. 2.8] E is a WNQP submodule for $\mathfrak{I}$.

As a direct application of proposition 3.1 we have these results.

Corollary 3.2 Let $\mathfrak{I}$ be multiplication R- module and $E \subsetneq \mathfrak{I}$. Then E is a WNQP if and only if for all $(0) \neq BCt \subseteq E$, for some submodules B , C in $\mathfrak{I}$, and $t \in \mathfrak{I}$, implies that either $Bt \subseteq E + J(\mathfrak{I})$ or $Ct \subseteq E + J(\mathfrak{I})$.

Corollary 3.3 Let $\mathfrak{I}$ be a multiplication R- module and $E \subsetneq \mathfrak{I}$. Then E is a WNQP submodule of $\mathfrak{I}$ if and only if for all $(0) \neq t_1 B t_2 \subseteq E$, for some $t_1, t_2 \in \mathfrak{I}$, and B is a submodule for $\mathfrak{I}$ implies that either $t_1t_2 \subseteq E + J(\mathfrak{I})$ or $Bt_2 \subseteq E + J(\mathfrak{I})$.

Corollary 3.4 Let $\mathfrak{I}$ be a multiplication R- module and $E \subsetneq \mathfrak{I}$. Then E is a WNQP submodule of $\mathfrak{I}$ if and only if whenever $t_1t_2t_3 \subseteq E + J(\mathfrak{I})$ for t_1 , t_2 , t_3 of $\mathfrak{I}$, implies that either $t_1t_3 \subseteq E + J(\mathfrak{I})$ or $t_2t_3 \subseteq E + J(\mathfrak{I})$.

We need to recall this lemma

Lemma 3.5 [12, prop. 1.11] If be content module then $J(\mathfrak{I}) = J(R)\,\mathfrak{I}$.

Proposition 3.6 Let $\mathfrak{I}$ be a multiplication R- module and $E \subsetneq \mathfrak{I}$. Then E is a WNQP submodule for $\mathfrak{I}$ if and only if $[E:_R \mathfrak{I}]$ is a WNQP for R.

Proof (⟹) Suppose $(0) \neq I_1 I_2 I_3 \subseteq [E:_R \mathfrak{I}]$ for I_1, I_2, I_3 are ideals in R, implies that $(0) \neq I_1 I_2 (I , \mathfrak{I}) \subseteq E$. But $\mathfrak{I}$ is a multiplication, then $(0) \neq BCD \subseteq E$, Where $B = I_1 \mathfrak{I}$, $C = I_2 \mathfrak{I}$ and $D = I_3 \mathfrak{I}$. Since E is a WNQP then by proposition (3.1) either $BD \subseteq E + J(\mathfrak{I})$ or $CD \subseteq E + J(\mathfrak{I})$, that is either $I_1I_3\mathfrak{I} \subseteq E + J(\mathfrak{I})$ or $I_2I_3\mathfrak{I} \subseteq E + J(\mathfrak{I})$. But $\mathfrak{I}$ is contain and multiplication, then by lemma (3.5) $J(\mathfrak{I}) = J(R)\,\mathfrak{I}$ and $E = [E :_R \mathfrak{I}]\,\mathfrak{I}$. Thus either $I_1 I_3 \mathfrak{I} \subseteq [E :_R \mathfrak{I}]\,\mathfrak{I} + J(R)\,\mathfrak{I}$ or $I_2I_3\mathfrak{I}$

⊆ [E :$_R$ 𝔍] 𝔍 + J(R) 𝔍, it follows that $I_1 I_3$ ⊆ [E:$_R$ 𝔍] + J(R) or $I_2 I_3$ ⊆ [E :$_R$ 𝔍] + J(R). Hence by [1, prop. (2.8)] [E :$_R$ 𝔍] is a WNQP ideal for R.

(⟸) Assume (0) ≠ rsB ⊆ E, for r, s ∈ R, and B is a submodule for 𝔍. Since 𝔍 is a multiplication so B = I for some ideal I for R, hence (0) ≠ rsI𝔍 ⊆ E, it follows that rsI ⊆ [E:$_R$ 𝔍]. Since [E:$_R$ 𝔍] is a WNQP then by [1 , prop. (2.7)] either rI ⊆ [E:$_R$ 𝔍] + J(R) or sI ⊆ [E :$_R$ 𝔍] + J(R) implies that either rI𝔍 ⊆ [E:$_R$ 𝔍] 𝔍 + J(R) 𝔍 or sI𝔍 ⊆ [E:$_R$ 𝔍] 𝔍 + J(R) 𝔍. But 𝔍 is content and multiplication, then J(R) 𝔍 = J(𝔍) and [E :$_R$ 𝔍] 𝔍 = E. Thus either rB ⊆ E + J(𝔍) or sB ⊆ E +J(𝔍). Hence by [1, prop. 2.7] E is a WNQP submodule for 𝔍.

Lemma 3.7 [13, Coro. of Theo. 9] "Let 𝔍 be a finitely generated multiplication R-module and I, J are ideals in R. Then I 𝔍 ⊆J 𝔍 if and only if I⊆ J+ann$_R$(𝔍)".

Proposition 3.8 Let 𝔍 be a finitely generated multiplication content R-module and I is a proper ideal of R with ann$_R$(𝔍) ⊆ I. Then I is a WNQP ideal of R if and only if I𝔍 is a WNQP submodule of 𝔍.

Proof (⟹) Suppose (0) ≠ rst ∈ I𝔍, for r, s ∈ R, t ∈ 𝔍. Since 𝔍 is a multiplication, so t = Rt=At for some ideal A in R, hence (0) ≠ rsA𝔍 ⊆ I𝔍, implies that by lemma (2.7) (0) ≠ rsA ⊆ I + ann$_R$(𝔍). But ann$_R$(𝔍) ⊆ I, so I + ann$_R$(𝔍) = I, that is (0) ≠ rsA ⊆ A. Since I is a WNQP ideal for R, then by [1 , prop.2.2] either rA⊆ I + J(R) or sA ⊆ I + J(R), imples that either rA𝔍 ⊆ I 𝔍 + J(R) 𝔍 or sA𝔍 ⊆ I 𝔍 + J(R) 𝔍. But is content module, then by lemma (3.5) J(R) 𝔍 = J(𝔍). Thus, either rt ∈ I𝔍 + J(𝔍) or st ∈ I𝔍 + J(𝔍).

(⟸) Assume 0 ≠ rsL ⊆ I for r, s ∈ R, L is an ideal for R, then 0 ≠ rsL𝔍 ⊆ I 𝔍. Since I is WNQP submodule for 𝔍, hence by [1, prop. (2.7)] either rL𝔍 ⊆ L 𝔍 + J(𝔍) or sL𝔍 ⊆ L 𝔍 + J(𝔍). But 𝔍 is content module, then J(𝔍) = J(R) 𝔍, it follows that either rL𝔍 ⊆ L 𝔍 + J(R) 𝔍 or sL𝔍 ⊆ L 𝔍 + J(R) 𝔍, that is either rL ⊆ I + J(R) or sL ⊆ I + J(R). Thus I is a WNQP ideal for R.

It is well known that cyclic module is multiplication [14], and cyclic module is finitely generated [2] we get the following result.

Corollary 3.9 Let 𝔍 be a cyclic content R- module and I is a proper ideal of R with ann$_R$(𝔍) ⊆ I. Then I is a WNQP ideal of R if and only if I𝔍 is a WNQP submodule of 𝔍.

"Recall that an R-module 𝔍 is finitely generated if $\mathfrak{J} = Rx_1 + Rx_2 + \cdots + Rx_n$ where $x_1, x_2, \ldots, x_n \in \mathfrak{J}$" [15].

From proposition 2.6 and proposition 2.8 we get the following results.

Corollary 3.10 Let 𝔍 be a finitely generated multiplication content R-module and E⊊𝔍. Then the following statements are equivalent:

1- E is a WNQP submodule of 𝔍.

2- [E : 𝔍] is WNQP ideal of R.

3- E = I for some WNQP ideal I of R with ann$_R$(𝔍)⊆I.

Corollary 3.11 Let 𝔍 be cyclic content R- module and E⊊𝔍. Then the following statements are equivalent:

1- E is a WNQP submodule of 𝔍.

2- [E : 𝔍] is WNQP ideal of R.

3- E = I for some WNQP ideal I of R with ann$_R$(𝔍)⊆I.

4. WNQP Radical Submodules.

Definition 4.1 Let 𝔍 be an R-module and E be a submodule for 𝔍. If there is a WNQP submodule for 𝔍 which contain E, then the intersection of all WNQP submodules for which containing E is called WNQP radical for E, denoted by

WNQP rad(E). If there is no WNQP submodule for $\mathfrak{I}$ containing E, then WNQP rad(E) = $\mathfrak{I}$. If $\mathfrak{I}$=R and I is an ideal in R, then WNQP rad(I) is the intersection of all WNQP ideals for R containing I.

The following proposition gives basic properties of WNQP radical of submodule.

Proposition 4.2 Let E, B be submodule of $\mathfrak{I}$. Then the following are satisfy

1- E ⊆ WNQP rad(E).

2- If E ⊆ B, then WNQP rad(E) ⊆ WNQP rad(B).

3- WNQP rad(WNQP rad(E)) = WNQP rad(E).

4- WNQP rad(E + B) = WNQP rad(WNQP rad(E) + WNQP rad(B)).

Proof (1) Hold from definition of WNQP rad(E).

(2) Suppose that E⊆B and K be a WNQP submodule of $\mathfrak{I}$ with B⊆K, then E⊆B⊆K implies that E⊆K. Thus WNQP rad(E) ⊆ WNQP rad(B).

(3) From part (1) and part (2) we have WNQP rad(E) ⊆ WNQP rad(WNQP rad(E)). But from definition of WNQP rad(E) we have WNQP rad(WNQP rad(E)) is the intersection of all WNQP submodule K of $\mathfrak{I}$ with WNQP rad(E) ⊆ K. Again by part (1) E ⊆ WNQP rad(E), implies that WNQP rad(WNQP rad(E)) ⊆ WNQP rad(E). Thus WNQP rad(WNQP rad(E)) = WNQP rad(E).

(4) Since E + B ⊆ WNQP rad(E) + WNQP rad(B). Hence by part (2) we have WNQP rad(E + B) ⊆ WNQP rad(WNQP rad(E) + WNQP rad(B). Let K be a WNQP submodule of $\mathfrak{I}$ with E + B ⊆ K. We must show that WNQP rad(E) + WNQP rad(B) ⊆ K. Since E + B ⊆ K and E ⊆ E + B, B ⊆ E + B then E⊆K, B⊆K and WNQP rad(E)⊆K and WNQP rad(B)⊆K and WNQP rad(B) ⊆ B, so WNQP rad(E)+WNQP rad(B) ⊆ K, thus we have WNQP rad (WNQP rad(E)+WNQP rad(B)) ⊆ WNQP rad(E+B). Hence WNQP rad(WNQP rad(E)+ WNQP rad(B)) ⊆ WNQP rad(E + B).

Proposition 4.3 Let E, B be submodule of $\mathfrak{I}$ with every WNQP submodule of $\mathfrak{I}$ which contain E∩B is completely irreducible submodule of $\mathfrak{I}$. Then WNQP rad(E ∩ B) =WNQP rad(E) ∩ WNQP rad(B).

Proof Since E ∩ B ⊆ B, and E ∩ B ⊆B, then by part (2) of proposition 4.2 we have WNQP rad(E ∩ B) ⊆ WNQP rad(E) and WNQP rad(E∩B) ⊆ WNQP rad(B) , implies that WNQP rad(E ∩ B) ⊆ (WNQP rad(E) + WNQP rad(B)). If WNQP rad(E ∩ B) = $\mathfrak{I}$, then WNQP rad(B) = WNQP rad(E) = $\mathfrak{I}$, implies that WNQP rad(E ∩ B) = WNQP rad(E) ∩ WNQP rad(B). If WNQP rad(E ∩ B) ≠ $\mathfrak{I}$, then there exists a WNQP submodule K of $\mathfrak{I}$ such that E ∩ B ⊆ K , implies that either E ⊆ K or B ⊆ K, that is either WNQP rad(E) ⊆ K or WNQP rad(B) ⊆ K. But every WNQP submodule of $\mathfrak{I}$ containing E∩B is completely irreducible, then either WNQP rad(E) ⊆ WNQP rad(E ∩ B) or WNQP rad(B) ⊆ WNQP rad(E ∩ B). Hence WNQP rad(E) ∩ WNQP rad(B) ⊆ WNQP rad(E ∩ B), that is WNQP rad(E ∩ B) = WNQP rad(E) ∩ WNQP rad(B).

"Recall that a submodule E of R-module $\mathfrak{I}$ is called maximal submodule if E⊊K, then K=R; K is submodule of $\mathfrak{I}$ [16]".

Proposition 4.4 Let $\mathfrak{I}$ be finitely generated R-module and E is a submodule for $\mathfrak{I}$. Then WNQP rad(E) = $\mathfrak{I}$ if and only if E = $\mathfrak{I}$.

Proof (⟹) Assume E ≠ $\mathfrak{I}$, and for $\mathfrak{I}$ is finitely generated , then there is maximal submodule L for $\mathfrak{I}$ such that E ⊆ L, it follows that L is a WNQP submodule of $\mathfrak{I}$. Thus WNQP rad(E)⊆L which is a contradiction. Thus E = $\mathfrak{I}$.

(⟸) If E = $\mathfrak{I}$, implies that WNQP rad(E) = WNQP rad($\mathfrak{I}$) = $\mathfrak{I}$.

"Recall that a proper ideal I of a ring R is called a prime ideal, if whenever ab∈I, for a, b∈R implies that either a∈I or b∈I [17]".

Proposition 4.5 Let $\mathfrak{J}$ be multiplication R-module with $ann_R(\mathfrak{J})$ is a prime ideal for R and E, B are submodule for $\mathfrak{J}$. Then E + B = $\mathfrak{J}$ if and only if WNQP rad(E) + WNQP rad(B) = $\mathfrak{J}$.

Proof (⟹) Since $\mathfrak{J}$ is a multiplication with $ann_R(\mathfrak{J})$ is a prime ideal then by [18, Coro. 3.6] $\mathfrak{J}$ is finitely generated. Thus by proposition 4.4 WNQP rad(E + B)=$\mathfrak{J}$, it follows by proposition 4.2(4) WNQP rad(WNQP rad(E) + WNQP rad(B)) = $\mathfrak{J}$. Again by proposition 4.4 we get WNQP rad(E)+WNQP rad(B)= $\mathfrak{J}$.

(⟸) Since WNQP rad(E) + WNQP rad(B) = $\mathfrak{J}$, then by proposition 4.4 we get WNQP rad(WNQP rad(E + WNQP rad(B))=$\mathfrak{J}$, that is by proposition 4.2(4) we get WNQP rad(E + B) = $\mathfrak{J}$. Thus E + B =$\mathfrak{J}$.

Proposition 4.6 If WNQP rad(E) = $\mathfrak{J}$ then WNQP rad($[E:_R \mathfrak{J}]\, \mathfrak{J}$) ⊆ WNQP rad(E). Now, let B be a WNQP submodule for $\mathfrak{J}$ contain E, then $[E:_R \mathfrak{J}] \subseteq [B:_R \mathfrak{J}]$. But B is WNQP submodule for $\mathfrak{J}$ with $J(\mathfrak{J}) \subseteq B$, we prove that $[B :_R \mathfrak{J}]$ is a WNQP ideal for R. Let $0 \neq abt \in [B:_R \mathfrak{J}]$, for a, b∈R implies that $(0) \neq ab(t\mathfrak{J}) \subseteq B$. Since B is a WNQP then either $at\mathfrak{J} \subseteq B + J(\mathfrak{J})$ or $bt\mathfrak{J} \subseteq B + J(\mathfrak{J})$. But $J(\mathfrak{J}) \subseteq B$, implies that $B + J(\mathfrak{J}) = B$. Thus either $at\mathfrak{J} \subseteq B$ or $bt\mathfrak{J} \subseteq B$, then either $at \in [B:_R \mathfrak{J}] \subseteq [B:_R \mathfrak{J}] + J(R)$ or $bt \in [B:_R \mathfrak{J}] \subseteq [B:_R \mathfrak{J}] + J(R)$. Hence $[B:_R \mathfrak{J}]$ is a WNQP ideal of R. Therefore WNQP rad($[E:_R \mathfrak{J}]\, \mathfrak{J}$) $\subseteq [B:_R \mathfrak{J}]\, \mathfrak{J} \subseteq B$. That is WNQP rad($[E:_R \mathfrak{J}]\, \mathfrak{J}$) ⊆ WNQP rad(E).

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