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Fixed point theorems with various enriched contraction conditions in generalized Banach spaces

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ABSTRACT

In this paper we introduce some fixed point theorems type contractions on generalized Banach space and we introduce a class of enriched Chatterjea mapping, enriched Kannan contraction mappings, This section is repeated enriched Chatterjea contraction mapping and enriched Kannan and enriched Chatterjea contraction mapping. And we show that these mappings must have unique fixed points in generalized Banach space.

MSC.. 47H10, 54J25.

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1. Introduction:

We introduce enriched contractions, a big class of contractive mappings, a section that involves and several others contractive mappings, nonexpansive mappings and Picard–Banach contractions. We showed that each there is an unique fixed point in enriched contraction that can be approximated. using a Krasnoselskij iterative approach that is adequate, in the theory of fixed points proven to be either corollaries or ramifications of the primary findings of the most important outcomes of this section.

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The Picard–Banach fixed point theorem, throughout the last nine decades a substantial body of work has emerged. See, for example, the monographs [4,13,17], as well as the references to them, furthermore, the picard Banach fixed point theorems and several of it's expansions have shown to be extremely useful and adaptable in the solution of a variety of nonlinear problems: integral equations, differential equations, integro-differential equations, variational inequalities, optimization problems, etc, see [4,17,20] and [21].

Following that, Fixed points of enriched Banach contractions are studied for their existence and uniqueness, for the Kransnoselskij iteration, as well as a strong convergence theorem, It is used to find the fixed points of enriched Banach contractions, is proved. A local iteration of the Picard–Banach fixed point theorem is also discussed, after which apply the key findings to Banach contractions with asymptotic enriched.

Definition 1.1[16] :If M nonempty is a linear space having s ≥ 1, let $\|.\|$ dnotes a functon from linear space M into R that satisfies the following axioms:

1. for all $x \in M$ $\|x\| \geq 0$, $\|x\| = 0$ if and only if $x = 0$;

2. for all $x, y \in M$, $\|x + y\| \leq s\,[\|x\| + \|y\|]$;

3. for all $x \in M, \alpha \in R$, $\|\alpha x\| \leq |\alpha|\|x\|$;

(M, $\|.\|$)is called generalized normed linear space. If for s = 1, it reduces to standard normed linear space.

Definition 1.2[16] :A Banach space (M, $\|.\|$) is a normed vector space such that M is complete under the metric induced by the $\|.\|$.

Definition 1.3[16] :A linear generalized normed space in which every Cauchy sequence is convergent is called generalized Banach space.

Definition 1.4[16] Let (M, $\|.\|$) be a generalized normed space then the sequence $\{x_n\}$ in M is called

1. Cauchy sequence iff for each ε > 0, there exist n(ε) ∈ N such that for all m,n ≥ n(ε) we have $\|x_n - x_m\| < \varepsilon$.
2. Convergent sequence iff there exist x ∈ M such that for all ε > 0, there exist n(ε) ∈ N such that for every n ≥ n(ε) we have $\|x_n - x\| < \varepsilon$.

Lemma 1.1 [10]. Let $(H,\|.\|)$ be a generalized Banach space with a real number $s \geq 1$, and F self-mapping on H, suppose that $\{u_n\}$ is a sequence in H induced by $u_{n+1} = Fu_n$ if

$$\|u_n - u_{n+1}\| \leq \alpha \, \|u_{n-1} - u_n\|, \text{ for all, } n \in \mathbb{N}, \tag{1.1}$$

where $\alpha \in [\,0\,,1\,)$. Then $\{u_n\}$ is a Cauchy sequence.

Definition 1.5[7]. Let $(H, \|.\|)$ be a linear normed space. A mapping $F: H \to H$ is said to be a $(k\,,\alpha)$-enriched contraction if there exist $k \in [0\,,+\infty)$ and $\alpha \in [0\,,k+1)$ such that,

$$\|k(x-y) + Fx - Fy\| \leq \alpha\|x-y\|, \quad \text{for all } x\,,y \in H, \tag{1.2}$$

See [6,7,8] and [9], proved that $(k\,,\alpha)$-enriched contraction and $(k\,,b)$-enriched Kannan mapping have a unique fixed point.

See [9] Considering a self-mapping T on X, then for any $\lambda \in (0,1]$, the so-called averaged mapping T_λ given by

$$T_\lambda x = (1-\lambda)x + \lambda Tx, \text{ for all } x \in H, \tag{1.3}$$

has the property that $\mathrm{Fix}(T_\lambda) = \mathrm{Fix}(T)$.

2. Enriched contraction conditions in generalized Banach space

Definition (2.1): Let $(H,\|.\|)$ be a generalized Banach space and, $s \geq 1$. A mapping $F: H \to H$ is said to be an enriched Chatterjea mapping or call it $(k\,,\beta, s)$-enriched Chatterjea mapping, such that,

$$\|k(x-y) + Fx - Fy\| \leq \beta[\|(k+1)(x-y) + y - Fy\|$$
$$+ \|(k+1)(y-x) + x - Fx\|], \text{ for all } x\,,y \in H \tag{2.1}$$

where $\beta \in [0\,,\frac{1}{2})$, $(\beta s < \frac{1}{2})$ and $k \geq 0$.

Theorem (2.1): Let $(H,\|.\|)$ be a generalized Banach space, and, $F: H \to H$ a $(k\,,\beta, s)$-enriched Chatterjea mapping. Then F has a unique fixed point, and there exists $\lambda \in (0,1]$ such that,

$$u_{n+1} = (1-\lambda)u_n + \lambda Fu_n, \quad n \geq 0, \tag{2.2}$$

where $\{u_n\}$ Converges to u^*, for any $u_0 \in H$, $n = 0,1,2,3,\ldots$

Proof: Consider the averaged F_λ defined by (1.3) for $\lambda = \frac{1}{k+1}$, $0 < \lambda < 1$

we show that in this case k= $\frac{1}{\lambda} - 1$ and the contractive condition (2.1)

$$\left\|(\frac{1}{\lambda} - 1)(x-y) + Fx - Fy\right\| \le \beta[\|(k+1)(x-y) + y - Fy\|$$
$$+\|(k+1)(y-x) + x - Fx\|],$$

which can be written in the same way as,

$$\|F_\lambda x - F_\lambda y\| \le \beta(\|x - F_\lambda y\| + \|y - F_\lambda x\|), \quad \textit{for all } x, y \in H, \tag{1.3}$$

we show that F_λ is a Chatterjea mapping.

as stated by (9), the iterative $\{u_n\}_{n=0}^{\infty}$ define by (1.2) is the Picard iteration in relation to F_λ, that is,

$$u_{n+1} = F_\lambda u_n, \ n \ge 0.$$

let $x = u_n$ and $y = x_{n-1}$ in (1.3) to get

$$\|u_{n+1} - u_n\| \le \beta(\|u_n - u_n\| + \|u_{n-1} - u_{n+1}\|)$$
$$\|u_{n+1} - u_n\| \le \beta s(\|u_{n-1} - u_n\| + \|u_n - u_{n+1}\|),$$

we get,

$$\|u_{n+1} - u_n\| \le \frac{\beta s}{1-\beta s}\|u_{n-1} - u_n\| \text{ , where } \mu = \frac{\beta s}{1-\beta s} < 1 \ \ (\beta s < \frac{1}{2})$$

$$\|u_{n+1} - u_n\| \le \mu\|u_{n-1} - u_n\|$$

By Lemma (1.1) we can say that $\{u_n\}$ is a Cauchy sequence in $(H,\|.\|)$.
Since $(H,\|.\|)$ is a generalized Banach space, $\{u_n\}$ is a converges to some

$u^* \in H$ as $n \to +\infty$.

We will show that u^* is the fixed point of F.

$$\|u^* - F_\lambda u^*\| \le s[\|u^* - u_{n+1}\| + \|u_{n+1} - F_\lambda u^*\|]$$

$$= s[\|u^* - u_{n+1}\| + \|F_\lambda u_n - F_\lambda u^*\|]$$

$$\le s\|u^* - u_{n+1}\| + s[\beta(\|u_n - F_\lambda u^*\| + \|u^* - F_\lambda u_n\|)]$$

$$= s\|u^* - u_{n+1}\| + s[\beta(\|u_n - F_\lambda u^*\| + \|u^* - u_{n+1}\|)]$$

By taking $\lim_{n\to\infty}$ we get,

$$\|u^* - F_\lambda u^*\| \le s\beta\|u^* - F_\lambda u^*\| \quad \Longrightarrow \|u^* - F_\lambda u^*\| = 0 \ \text{ i.e. } F_\lambda u^* = u^*$$

we prove u^* is the fixed point of F_λ.

Now, we have to show that u^* is unique fixed point of F_λ.

Assume that v^* is deferent fixed point of F_λ then $F_\lambda v^* = v^*$,

and $\|u^* - v^*\| = \|F_\lambda u^* - F_\lambda v^*\| \leq \beta(\|u^* - F_\lambda v^*\| + \|v^* - F_\lambda u^*\|)$

$\|u^* - v^*\| \leq 2\beta \|u^* - v^*\|$, since $\beta < \frac{1}{2}$.

Which is a contradiction $\|u^* - v^*\| = 0 \Longrightarrow u^* = v^*$

As a result, u^* is the unique fixed point.

Definition (2.2): Let $(H, \|.\|)$ be a generalized Banach space and s ≥ 1. A mapping F : $H \to H$ is said to be an enriched Kannan contraction mapping or call it (k , θ , α, s)-enriched Kannan contraction mapping, such that,

$$\|k(x - y) + Fx - Fy\| \leq \theta\|x - y\| + \alpha(\|x - Fx\| + \|y - Fy\|), \tag{2.1}$$

for all $x, y \in H$,

where $\alpha \in [0, \frac{1}{2})$, $\theta \in [0, 1)$, $(\theta + 2\alpha < 1)$, $\alpha s < \frac{1}{2}$ and $k \geq 0$.

Theorem (2.2): Let (H,$\|.\|$) be a Banach space, and F : $H \to H$ a (k , θ , α , s)-enriched Kannan contraction mapping. Then F has a unique fixed point, and there exists $\lambda \in (0,1]$ such that,

$$u_{n+1} = (1 - \lambda)u_n + \lambda F u_n,\ n \geq 0, \tag{2.2}$$

where $\{u_n\}$ Converges to u*, for any $u_0 \in$ H, n= 0,1,2,3,…,

Proof: Consider the averaged F_λ defined by (1.3) for $\lambda = \frac{1}{k+1}$, $0 < \lambda < 1$

we show that in this case $k = \frac{1}{\lambda} - 1$ and the contractive condition (2.2),

$\left\|(\frac{1}{\lambda} - 1)(x - y) + Fx - Fy\right\| \leq \theta\|x - y\| + \alpha(\|x - Fx\| + \|y - Fy\|)$

which can be written in the same way as,

$\|F_\lambda x - F_\lambda y\| \leq \theta\lambda\|x - y\| + \alpha(\|x - F_\lambda x\| + \|y - F_\lambda y\|)$,

for all $x, y \in H$, since $\theta\lambda < \theta$,

we show that,

$$\|F_\lambda x - F_\lambda y\| \leq \theta\|x - y\| + \beta(\|x - F_\lambda x\| + \|y - F_\lambda y\|), \tag{2.3}$$

for all $x , y \in$ H,

We show that F_λ is a Kannan contraction mapping.

as stated by (1.3) , the iterative $\{u_n\}$ defined by (2.2) is the Picard iteration associated to F_λ, that is $u_{n+1} = F_\lambda u_n$, $n \geq 0$.
let $x = u_n$ and $y = x_{n-1}$ in (2.3) to get,

$$\|u_{n+1} - u_n\| \leq \|F_\lambda u_n - F_\lambda u_{n-1}\|$$
$$= \theta\|u_n - u_{n-1}\| + \alpha(\|u_n - u_{n+1}\| + \|u_{n-1} - u_n\|)$$

Now we get,

$$\|u_{n+1} - u_n\| \leq \frac{\theta+\alpha}{1-\alpha}\|u_{n-1} - u_n\| \text{ , where } \mu = \frac{\theta+\alpha}{1-\alpha} < 1 \quad (\theta + 2\alpha < 1)$$
$$\|u_{n+1} - u_n\| \leq \mu\|u_{n-1} - u_n\|.$$

By Lemma (1.1) we can say that $\{u_n\}$ is a Cauchy sequence in $(H, \|.\|)$.
Since $(H, \|.\|)$ is a generalized Banach space, $\{u_n\}$ is a converges to some $u^* \in H$ as $n \to \infty$.

We will show that u^* is the fixed point of F .

$$\|u^* - F_\lambda u^*\| \leq s\,[\,\|u^* - u_{n+1}\| + \|u_{n+1} - F_\lambda u^*\|\,]$$
$$= s\,[\,\|u^* - u_{n+1}\| + \|F_\lambda u_n - F_\lambda u^*\|\,]$$
$$\leq s\,\|u^* - u_{n+1}\| + s[\theta\|u_n - u^*\| + \alpha\,(\|u_n - F_\lambda u_n\| + \|u^* - F_\lambda u^*\|)]$$
$$= s\,\|u^* - u_{n+1}\| + s[\theta\|u_n - u^*\| + \alpha\,(\|u_n - u_{n+1}\| + \|u^* - F_\lambda u^*\|)]$$

By taking $\lim_{n\to\infty}$ we get,

$$\|u^* - F_\lambda u^*\| \leq \alpha s\,\|u^* - F_\lambda u^*\|, \alpha s < \frac{1}{2},$$
$$\Longrightarrow \|u^* - F_\lambda u^*\| = 0 \;\; i.e.\; F_\lambda u^* = u^*$$

we prove u^* is the fixed point of F_λ.

Now, we have to showed that u^* is unique fixed point of F_λ.

Assume that v^* is deferent fixed point of F_λ then,

$$F_\lambda v^* = v^* \text{ and } \|u^* - v^*\| = \|F_\lambda u^* - F_\lambda v^*\| \leq \theta\|u^* - v^*\| + \alpha\,(\|u^* - F_\lambda u^*\| + \|v^* - F_\lambda v^*\|)$$

$$\|u^* - v^*\| \leq \theta\;\|u^* - v^*\|, \text{ since } \theta < 1$$

Which is a contradiction $\|u^* - y^*\| = 0 \Longrightarrow u^* = y^*$

As a result, u^* is the unique fixed point.

Definition (2.3): Let $(H,\|.\|)$ be a generalized Banach space and s ≥ 1. A mapping F : $H \rightarrow H$ is said to be an enriched Chatterjea contraction mapping or call it (k , θ , β, s)-enriched Chatterjea contraction mapping, such that,

$$\|k(x-y)+Fx-Fy\| \leq \theta\|x-y\| + \beta[\|(k+1)(x-y)+y-Fy\| + \|(k+1)(y-x)+x-Fx\|], \quad (3.1)$$

for all $x , y \in H$,

where $\beta \in [0 , \frac{1}{2})$, $\theta \in [0 , 1)$, ($\theta + 2\,\beta s < 1$) and $k \geq 0$.

Theorem (2.3). Let $(H,\|.\|)$ be a Banach space, and F : $H \rightarrow H$ a (k , θ , β , s)-enriched Chatterjea contraction mapping. Then F has a unique fixed point, and there exists $\lambda \in (0,1]$ such that,

$$u_{n+1} = (1 - \lambda)u_n + \lambda F u_n, \quad n \geq 0 \quad (3.2)$$

Where $\{u_n\}$ Converges to u^*, for all $u_0 \in H$,

Proof: Consider the averaged F_λ defined by (1.3) for $\lambda = \frac{1}{k+1}$, $0 < \lambda < 1$
we showed that in this case $k = \frac{1}{\lambda} - 1$ and the contractive condition (3.1)

$$\left\|\left(\frac{1}{\lambda}-1\right)(x-y)+Fx-Fy\right\| \leq \theta\|x-y\| + \beta[\|(k+1)(x-y)+y-Fy\| + \|(k+1)(y-x)+x-Fx\|],$$

which can be written in the same way as,

$$\|F_\lambda x - F_\lambda y\| \leq \theta\lambda\|x-y\| + \beta(\|x-F_\lambda y\| + \|y-F_\lambda x\|),$$

for all $x , y \in H$, since $\theta\lambda < \theta$,
we show that,

$$\|F_\lambda x - F_\lambda y\| \leq \theta\|x-y\| + \beta(\|x-F_\lambda y\| + \|y-F_\lambda x\|), \quad (3.3)$$

for all $x , y \in H$,
We show that F_λ is a Chatterjea contraction mapping.
as stated by (1.3), the iterative $\{u_n\}_{n=0}^{\infty}$ defined by (3.2) is the Picard iteration associated to F_λ, that is,

$$u_{n+1} = F_\lambda u_n, \; n \geq 0.$$

let $x = u_n$ and $y = x_{n-1}$ in (3.3) to get,

$$\|u_{n+1} - u_n\| \leq \|F_\lambda u_n - F_\lambda u_{n-1}\|$$
$$= \theta\|u_n - u_{n-1}\| + \beta(\|u_n - u_n\| + \|u_{n-1} - u_{n+1}\|)$$

$\|u_{n+1} - u_n\| \leq \theta\|u_n - u_{n-1}\| + \beta s[\|u_{n-1} - u_n\| + \|u_n - u_{n+1}\|]$
we obtain
$\|u_{n+1} - u_n\| \leq \frac{\theta + \beta s}{1 - \beta s}\|u_{n-1} - u_n\|$, where $\mu = \frac{\theta + \beta s}{1 - \beta s} < 1 \quad (\theta + 2\beta s < 1)$
$\|u_{n+1} - u_n\| \leq \mu\|u_{n-1} - u_n\|$.
By Lemma (1.1) we can say that $\{u_n\}$ is a Cauchy sequence in $(H, \|.\|)$.
Since $(H, \|.\|)$ is a generalized Banach space, $\{u_n\}$ is a converges to some

$u^* \in H$ as $n \to \infty$.

we will showed that u^* is the fixed point of F.

$$\|u^* - F_\lambda u^*\| \leq s[\|u^* - u_{n+1}\| + \|u_{n+1} - F_\lambda u^*\|]$$

$$= s[\|u^* - u_{n+1}\| + \|F_\lambda u_n - F_\lambda u^*\|]$$

$$\leq s\ \|u^* - u_{n+1}\| + s[\theta\|u_n - u^*\| + \beta(\|u_n - F_\lambda u^*\| + \|u^* - F_\lambda u_n\|)]$$

$$= s\ \|u^* - u_{n+1}\| + s[\theta\|u_n - u^*\| + \beta(\|u_n - F_\lambda u^*\| + \|u^* - u_{n+1}\|)],$$

by taking $\lim_{n\to\infty}$ we get,
$\|u^* - F_\lambda u^*\| \leq \beta s\|u^* - F_\lambda u^*\|$, $\beta s < \frac{1}{2}$
$\Longrightarrow \|u^* - F_\lambda u^*\| = 0$ i.e. $F_\lambda u^* = u^*$.
We showed that u^* is the fixed point of F_λ.

Now, we have to show that u^* is unique fixed point of F_λ.

Assume that v^* is deferent fixed point of F_λ then,

$$F_\lambda v^* = v^* \text{ and } \|u^* - v^*\| = \|F_\lambda u^* - F_\lambda v^*\| \leq \theta\|u^* - v^*\| + \beta(\|u^* - F_\lambda v^*\| + \|v^* - F_\lambda u^*\|)$$

$\|u^* - v^*\| \leq (\theta + 2\beta)\|u^* - v^*\|$, since $\theta + 2\beta < 1$,

which is a contradiction $\|u^* - y^*\| = 0 \Longrightarrow u^* = y^*$.

As a result, u^* is the unique fixed point.

Definition (2.4): Let $(H, \|.\|)$ be a generalized Banach space and $s \geq 1$. A mapping $F : H \to H$ is said to be an enriched Kannan and Chatterjea contraction mapping or call it (k, α, β, s)-enriched contraction mapping, such that,

$\|k(x-y)+Fx-Fy\| \leq \alpha(\|x-Fx\|+\|y-Fy\|)$

$$+\beta[\|(k+1)(x-y)+y-Fy\|+\|(k+1)(y-x)+x-Fx\|], \qquad (4.1)$$

for all $x, y \in H$,

where $\beta \in [0, \frac{1}{2})$, $\alpha \in [0,1)$, $(s(\alpha+\beta) < \frac{1}{2})$ and $k \geq 0$.

Theorem (2.4): Let $(H, \|.\|)$ be a generalized Banach space, and, F : $H \to H$ a (k , α , β ,s)-enriched contraction mapping. Then F has a unique fixed point , and there exists $\lambda \in (0,1]$ such that

$$u_{n+1} = (1-\lambda)u_n + \lambda F u_n, \quad n \geq 0, \qquad (4.2)$$

where $\{u_n\}$ Converges to u^*, for all $u_0 \in H$,

Proof: Consider the averaged F_λ defined by (1.3) for $\lambda = \frac{1}{k+1}$, $0 < \lambda < 1$,
we showed that in this case $k = \frac{1}{\lambda} - 1$ and the contractive condition (4.1)
$\left\|(\frac{1}{\lambda}-1)(x-y)+Fx-Fy\right\| \leq \alpha(\|x-Fx\|+\|y-Fy\|$
$\beta[\|(k+1)(x-y)+y-Fy\|+\|(k+1)(y-x)+x-Fx\|]$,
which can be written in the same way as,

$$\|F_\lambda x - F_\lambda y\| \leq \alpha(\|x-F_\lambda x\|+\|y-F_\lambda y\|)+\beta(\|x-F_\lambda y\|+\|y-F_\lambda x\|), \qquad (4.3)$$

for all $x, y \in H$,
we showed that F_λ is a Kannan mapping.
According to (1.3), the iterative $\{u_n\}_{n=0}^{\infty}$ defined by (4.2) is the Picard iteration associated to F_λ, that is,

$$u_{n+1} = F_\lambda u_n, \; n \geq 0,$$

let $x = u_n$ and $y = x_{n-1}$ in (4.3) to get,
$\|u_{n+1}-u_n\| = \|F_\lambda u_n - F_\lambda u_{n-1}\|$
$\leq \alpha(\|u_n - F_\lambda u_n\|+\|u_{n-1}-F_\lambda u_{n-1}\|) + \beta(\|u_n - F_\lambda u_{n-1}\|+\|u_{n-1}-F_\lambda u_n\|)$
$= \alpha(\|u_n - u_{n+1}\|+\|u_{n-1}-u_n\|) + \beta(\|u_n - u_n\|+\|u_{n-1}-u_{n+1}\|)$
$\leq \alpha(\|u_n - u_{n+1}\|+\|u_{n-1}-u_n\|) + \beta s(\|u_{n-1}-u_n\|+\|u_n-u_{n+1}\|)$
now we get,
$\|u_{n+1}-u_n\| \leq \frac{\alpha+\beta s}{1-(\alpha+\beta s)}\|u_{n-1}-u_n\|$, where $\mu = \frac{\alpha+\beta s}{1-(\alpha+\beta s)} < 1$,
$(s(\alpha+\beta) < \frac{1}{2})$
$\|u_{n+1}-u_n\| \leq \mu\|u_{n-1}-u_n\|$
By Lemma (1.1) we can say that $\{u_n\}$ is a Cauchy sequence in $(H, \|.\|)$.

Since $(H, \|.\|)$ is a generalized Banach space, $\{u_n\}$ is a converges to some

$u^* \in H$ as $n \to \infty$.

we will show that u^* is the fixed point of F.

$$\|u^* - F_\lambda u^*\| \leq s\,[\,\|u^* - u_{n+1}\| + \|u_{n+1} - F_\lambda u^*\|\,]$$

$$= s\,[\,\|u^* - u_{n+1}\| + \|F_\lambda u_n - F_\lambda u^*\|\,]$$

$$\leq s\,\|u^* - u_{n+1}\| + s[\alpha\,(\|u_n - F_\lambda u_n\| + \|u^* - F_\lambda u^*\|) + \beta(\|u_n - F_\lambda u^*\| + \|u^* - F_\lambda u_n\|)\,]$$

$$= s\,\|u^* - u_{n+1}\| + s[\alpha\,(\|u_n - u_{n+1}\| + \|u^* - F_\lambda u^*\|) + \beta(\|u_n - F_\lambda u^*\| + \|u^* - u_{n+1}\|)\,].$$

By taking $\lim\limits_{n\to\infty}$ we get,

$$\|u^* - F_\lambda u^*\| \leq s(\alpha+\beta)\,\|u^* - F_\lambda u^*\|,\ s(\alpha+\beta) < \frac{1}{2}$$

$$\Longrightarrow \|u^* - F_\lambda u^*\| = 0\ \ i.e.\ F_\lambda u^* = u^*$$

we proved, u^* is the fixed point of F_λ.

Now, we have to showed that u^* is unique fixed point of F_λ.

Assume that, v^* is deferent fixed point of F_λ ,then $F_\lambda v^* = v^*$.

$$\|u^* - v^*\| = \|F_\lambda u^* - F_\lambda v^*\| \leq \alpha\,(\|u^* - F_\lambda u^*\| + \|v^* - F_\lambda v^*\|) + \beta\,(\|u^* - F_\lambda v^*\| + \|v^* - F_\lambda u^*\|)$$

$\|u^* - y^*\| \leq 2\beta\,\|u^* - y^*\|$, since, $2\beta < 1$, which is a contradiction

$$\|u^* - y^*\| = 0 \Longrightarrow u^* = y^*$$

i.e. u^* is the unique fixed point.

Remark (2.1). By selecting:

- s = 1 in definition (2.1), we get definition (1) of [6].
- s = 1and $\theta = 0$ in definition (2.2), we get definition (2.1) of [5].
- s = 1and $\alpha = 0$ in definition (2.2), we get definition (2.1) of [7].
- s = 1 in definition (2.2), we get definition (2.3) of [9].

- s = 1and θ = 0 in definition (2.3), we get definition (1) of [6].
- s = 1and β = 0 in definition (2.3), we get definition (2.1) of [7].
- s = 1 and β = 0, in definition (2.4), we get definition (2.1) of [5].
- s = 1 and α= 0 in definition (2.4), we get definition (1) of [6].

Conclusion:

1. We have demonstrated that each enriched contraction has a single fixed point that may be approximated using Kransnoselskij iterations. Specifically, we get the traditional Banach contraction principle in the case of a Banach space using the fixed point techniques presented in this study.
2. It's worth noting that enriched contractions retain a basic quality of Picard–Banach contractions, namely that each enriched contraction has a single fixed point and is continuous (as the definition shows).

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