Journal of Al-Qadisiyah for Computer Science and Mathematics
Vol. 14(2) 2022 , pp Math. 46–53

Available online at www.qu.edu.iq/journalcm
JOURNAL OF AL-QADISIYAH FOR COMPUTER SCIENCE AND MATHEMATICS
ISSN:2521-3504(online) ISSN:2074-0204(print)

On $\mathcal{P}$-Clean Rings

***Esraa H. Ramzi*[b]** ***Akram S. Mohammed*[a]**

[ab]Tikrit University /College of Computer Science and Mathematics, Mathematics Department

asrhkmet@gmail.com
akr_tel@tu.edu.iq

ARTICLE INFO

Article history:
Received: 15 /04/2022
Rrevised form: 07 /06/2022
Accepted : 12 /06/2022
Available online: 14 /06/2022

Keywords: clean ring, r-clean ring , strongly r-clean ring

Each keyword to start on a new line

ABSTRACT

If all the elements of a ring $\mathcal{W}$ can be written as a sum of pure and idempotent elements", the ring is said to be $\mathcal{P}$-clean. In this paper, we introduce the $\mathcal{P}$-clean ring nation and look into some of its fundamental properties, examples, and relationship to the clean ring

MSC..

https://doi.org/10.29304/jqcm.2022.14.2.949

1-Introduction:

" $\mathcal{W}$ denotes an associative ring with identity throughout this paper, U($\mathcal{W}$), D($\mathcal{W}$), Pu($\mathcal{W}$), and J($\mathcal{W}$) denoting the units, idempotent, pure elements, and Jacobson radical of W ".

"An element t of a ring $\mathcal{W}$ is called P-clean if t=d+u, where d∈D($\mathcal{W}$) and u∈U($\mathcal{W}$), if each element of $\mathcal{W}$ is P-clean, then $\mathcal{W}$ is named P-clean ring. Nicholson[6] was the first to introduce clean rings.

∗Corresponding author

Email addresses: asrhkmet@gmail.com

Communicated by 'sub etitor'

Several mathematicians are interested in the topics of clean ring, r-clean ring, π -clean ring, and strongly r-clean" [1,2,3,4,5,7,8and9].

"If b∈ $\mathscr{W}$ exists, an element t∈ $\mathscr{W}$ "is said to be" pure. Such that t=tb, an element d∈ $\mathscr{W}$ is called idempotent if $d^2 = d$, and an element t∈ $\mathscr{W}$ is referred to as nilpotent if a positive integer m exists such that $a^m = 0$ ". [7]

"Obviously, every "clean ring" is a P-clean ring; however, We demonstrate a P-clean ring that is not a clean ring. In this work, we discuss the fundamental properties and applications of the P-clean ring ".

Definition 1. 1: If there are d ∈D($\mathscr{W}$) and p∈Pu($\mathscr{W}$) such that t = d + $\mathscr{P}$, "an element" t∈$\mathscr{W}$ is known as $\mathscr{P}$–clean.

Definition1. 2: Let $\mathscr{W}$ represent a ring. If each elements in $\mathscr{W}$ expresses "as the sum of an idempotent" and pure, then $\mathscr{W}$ is called a $\mathscr{P}$– clean ring.

Examples 1.3:

1. The ring ($\mathscr{W}$,+ , .) is a $\mathscr{P}$–clean ring .
2. The ring (Z ,+ , .) is a $\mathscr{P}$–clean ring .
3. The ring (Z_6 , $+_6$, $._6$) is a $\mathscr{P}$–clean ring .
4. every field is a $\mathscr{P}$–clean ring .

2. The main result

Proposition 2. 1: A clean ring $\mathscr{W}$ is a P – clean ring.

Proof : Allow $\mathscr{W}$ to be a clean ring and t∈ $\mathscr{W}$. Then t is equal to d + u. Where d∈ D($\mathscr{W}$) and u∈ U($\mathscr{W}$) are used. To demonstrate that t is a $\mathscr{P}$– clean element in $\mathscr{W}$, we need only show that u is a pure element, because u∈ U($\mathscr{W}$) implies that there is u^{-1} such that u u^{-1} = 1, and thus u u^{-1} u = u. Consider w =u^{-1}u, and then w∈ $\mathscr{W}$. As a result of u = u w, u is a pure element, and t is a $\mathscr{P}$– clean element. As a result, $\mathscr{W}$ is a $\mathscr{P}$–clean ring. The converse of above proposition is not true. ■

Example 2. 2: (Z, +, .) is a $\mathscr{P}$-clean ring. But it's not a clean ring.

Proposition 2.3: Let $\mathscr{W}$ be a ring , then

1. Each pure element of a ring $\mathscr{W}$ is a $\mathscr{P}$–clean element.
2. Every idempotent ring $\mathscr{W}$ element is a $\mathscr{P}$–clean.

Proof :

1. If p is any pure element of a ring $\mathscr{W}$, then p = 0 + p can be written. Where $0 \in D(\mathscr{W})$ and $p \in Pu(\mathscr{W})$ are used.
2. Assume that $d \in \mathscr{W}$ is such that $d = d^2$. Now d = (1 – d) + (2d – 1), and (1 – d) is clearly an idempotent element because $(1-d)^2 = 1-d-d+d^2 = 1-d-d+d = 1-d$. And (2d – 1) is a pure element because $(2d-1) = (2d-1)(2d-1)^2 = (2d-1)(4d^2-4d+1)(4d+4d+1) = (2d-1)(1) = (2d-1)$. ■

Proposition 2.4: Assume $\mathscr{W}$ is a $\mathscr{P}$–clean ring and $\mathscr{W}'$ is a ring. If $f: \mathscr{W} \to \mathscr{W}'$ denotes epimorphism, then $f(\mathscr{W})$ denotes a $\mathscr{P}$–clean ring.

Proof: Let $f: \mathscr{W} \to \mathscr{W}'$ is an epimorphism of $\mathscr{W}$ into $\mathscr{W}'$, and let $f(t) \in f(\mathscr{W})$ be such that $t \in \mathscr{W}$ and y = f (t). Since $\mathscr{W}$ is a $\mathscr{P}$–clean ring, t = d + p, where $d \in D(\mathscr{W})$ and $p \in Pu(\mathscr{W})$, so f(t) = f(d + p) = f(d) + f (p). Since $(f(d))^2 = f(d)$, f(d) is clearly an idempotent element. $(f(d))^2 = f(d)f(d) = f(d^2) = f(d)$, and f(p) = f(pw) =f (p). f(w) in order for f(p) to be a pure element in $f(\mathscr{W})$ As a result, y = f(t) = f(d) + f(p) implies that y is a $\mathscr{P}$– clean element in the ring $f(\mathscr{W})$. Hence $f(\mathscr{W})$ be a $\mathscr{P}$–clean ring . ■

Proposition 2.5 : Assume I is an ideal of a $\mathscr{P}$–clean ring $\mathscr{W}$. Then $\mathscr{W}/I$ is a $\mathscr{P}$– clean ring.

Proof : Assume that $t + I \in \mathscr{W}/I$, where $t \in \mathscr{W}$. Since $\mathscr{W}$ is a $\mathscr{P}$– clean ring, t = d + p, where $d \in D(\mathscr{W})$ and $p \in Pu(\mathscr{W})$, now t + I = (d + p) + I = (d + I) + (p + I), and (d + I) is an idempotent element in $\mathscr{W}/I$ because $(d + I)^2 = (d + I) + (d + I) = (d^2 + I) = (d + I)$, and (p + I) is a pure element in $\mathscr{W}/I$ because (p + I) = (pw + I) = (p + I) (w + I), so t + I is a $\mathscr{P}$ – clean element in $\mathscr{W}/I$, implying that $\mathscr{W}/I$ is a $\mathscr{P}$– clean ring. ■

Proposition 2.6: Let $\mathscr{W}_i$ be a $\mathscr{P}$–clean ring, (i = 1 ,2 , … , n). Then $\prod_{i=1}^{n} W_i$ is a $\mathscr{P}$–clean ring .

Proof : Let $(t_1, t_2, \dots, t_n) \in \prod_{i=1}^{n} W_i$. Then $t_i \in \mathscr{W}_i$, i = 1 ,2 , … , n . Since $\mathscr{R}_i$ is $\mathscr{P}$–clean ring ,there exists $e_i \in D(\mathscr{W}_i)$ and $p_i \in Pu(\mathscr{W}_i)$ such that $t_i = d_i + p_i \quad \forall i = 1, 2, \dots, n$. Hence $t = (t_i) = (t_1, t_2, \dots, t_n) = (d_1 + p_1, d_2 + p_2, \dots, d_n + p_n) = (d_1, d_2, \dots, d_n) + (p_1, p_2, \dots, p_n)$, you see $(d_1, d_2, \dots, d_n)$ is an idempotent element in $\prod_{i=1}^{n} W_i$ because $(d_1, d_2, \dots, d_n)^2 = (d_1, d_2, \dots, d_n) . (d_1, d_2, \dots, d_n) = (d_1^2, d_2^2, \dots, d_n^2) = (d_1, d_2, \dots, d_n)$ and $(p_1, p_2, \dots, p_n)$ is a pure element in $\prod_{i=1}^{n} W_i$ because $(p_1, p_2, \dots, p_n) = (p_1w_1, p_2w_2, \dots, p_nw_n) = (p_1, p_2, \dots, p_n).(w_1, w_2, \dots, w_n)$, which implies that x is $\mathscr{P}$–clean in $\prod_{i=1}^{n} W_i$. Hence $\prod_{i=1}^{n} W_i$ is a $\mathscr{P}$–clean ring. ■

Definition 2.7: "If vt = tv for all t∈$\mathscr{W}$, an element v∈$\mathscr{W}$ is called the central element". [6]

Definition 2. 8: "If each idempotent element in $\mathscr{W}$ is the central element, the ring $\mathscr{W}$ is said to be an abelian ring". [6]

Proposition 2.9: Let $\mathscr{W}$ be a central ring, t∈$\mathscr{W}$ be a $\mathscr{P}$–clean element, and d be an idempotent element in $\mathscr{W}$, if the element (-t) is a $\mathscr{P}$– clean element. Then (t + d) equals a $\mathscr{P}$– clean.

Proof : Assume that t is a $\mathscr{P}$– clean element in $\mathscr{W}$, and that we must prove that (1 – t) is a $\mathscr{P}$– clean element in $\mathscr{W}$, there exists an idempotent element d ∈$\mathscr{W}$ and a pure element p∈$\mathscr{W}$ such that t = d + p and 1–t =(1 – d) + (-p), where (1 – d)∈ D($\mathscr{W}$) and (-p)∈ Pu($\mathscr{W}$). Similarly, we can demonstrate that (-t) is a $\mathscr{P}$– clean if and only if (1 + t) is a $\mathscr{P}$– clean, implying that both a and (1 + t) are $\mathscr{P}$– clean in $\mathscr{W}$.

Let t = f + p ,where f ∈ D($\mathscr{W}$) and p ∈ Pu($\mathscr{W}$) , also let 1 + t = g + q , where g ∈ D($\mathscr{W}$) and q ∈ Pu($\mathscr{W}$). Now t + d = t + t d – t d + d = t d + t (1 – d) + d = (t +*1*) d + t(1 – d) = (g + q) d + (f + p) (1- d) = g d + q d + f(1– d) + p(1– d) = g d + f(1- d) + q d + p(1 – d), and we note that g d + f(1- d) ∈ D($\mathscr{W}$) since $[g d + f(1– d)]^2$ =(g d + f(1 – d)) . (g d + f(1– d))=$(g d)^2$ + g d f(1- d) + f(1– d)g d + $(f(1 – d))^2$ = g d + g d f - g d^2f +fg d – f d^2g + f(1 – d) = g d + g d f – g d f + fgd – fgd + f(1–d)=gd +f(1– d) . Also q d + p(1–p)∈Pu($\mathscr{W}$) since (q d + p(1– d)) (q^{-1} d +P^{-1}(1– d) (q d +p(1– d)) = q d +p(1– d)

(q d + p(1 – d)) . (d^2 + q^{-1} d p(1 – d) + p^{-1} (1 – d)q d + $(1 – d)^2$

(q d + p(1 – d)) . (d^2 + q^{-1} d p – q^{-1} d p + p^{-1} q d – p^{-1}q d + (1 – d)

(q d + p(1 – d)) . (d + 1 – d) = (q d + p(1 – d)) . (1) = q d + p(1 – d) Therefore , (t + d) is a $\mathscr{P}$–clean in $\mathscr{W}$. ■

Proposition 2. 10: If $\mathscr{W}$ is a ring, then t∈$\mathscr{W}$ is a $\mathscr{P}$–clean if and only if 1 – t is a $\mathscr{P}$–clean.

Proof : Allow t be $\mathscr{P}$–clean. Then write t = d + p, where d ∈D($\mathscr{W}$) and p∈ Pu($\mathscr{W}$) are the variables. As a result, 1–t=(1–d)+(-p), and being (1– d)∈ D($\mathscr{R}$), because (1– d)2 = (1 – d). Clearly, -p∈ Pu($\mathscr{W}$) because -p = -pw. As a result, 1 – t is a $\mathscr{P}$–clean.

Conversely : If 1 – t is $\mathscr{P}$–clean, write 1 – t = d + p, where d ∈ D($\mathscr{W}$) and p∈ Pu($\mathscr{W}$) are constants. Thus, t=(1–d)+(-p), as in the previous parts (1– d)∈ D($\mathscr{W}$) and -p ∈Pu($\mathscr{W}$). As a result, t is a $\mathscr{P}$–clean. ■

Theorem 2.11 : Let I = d$\mathscr{W}$ denote an ideal generated by the idempotent element d of a $\mathscr{P}$– clean ring $\mathscr{W}$. $\mathscr{W}$/ d$\mathscr{W}$ is thus a $\mathscr{P}$–clean ring.

Proof : Let t+d$\mathscr{W}$ ∈$\mathscr{W}$/d$\mathscr{W}$. Then, for t∈$\mathscr{W}$, since $\mathscr{W}$ is a $\mathscr{P}$–clean ring, ∃d*∈ D($\mathscr{W}$) and p∈ Pu($\mathscr{W}$) such that t = d * + p. Now (t + d$\mathscr{W}$) = (d* + p) + d$\mathscr{W}$= (d* + d$\mathscr{W}$) + (p + d$\mathscr{W}$), because $(d^* + d\mathscr{W})^2 = (d^* + d\mathscr{W})(d^* + d\mathscr{W}) = (d^{*2} + d\mathscr{W}) = (d^* + d\mathscr{W})$, thus (d* + d$\mathscr{W}$) is idempotent and it remains to prove (p + d $\mathscr{W}$) is a pure element in $\mathscr{W}$/d$\mathscr{W}$. Assume that q + d$\mathscr{W}$ ∈$\mathscr{W}$/d$\mathscr{W}$, where q∈$\mathscr{W}$ such that (p + d $\mathscr{W}$) (q + d$\mathscr{W}$)= pq + d$\mathscr{W}$, and x + d$\mathscr{W}$ is a $\mathscr{P}$– clean in $\mathscr{W}$/d$\mathscr{W}$. $\mathscr{W}$/d$\mathscr{W}$ is thus a $\mathscr{P}$–clean ring.■

Theorem 2. 12: Assume $\mathscr{W}$ is a commutative ring and t∈$\mathscr{W}$. $\mathscr{W}$ is a $\mathscr{P}$–clean ring if t$\mathscr{W}$= d$\mathscr{W}$ and d ∈ D($\mathscr{W}$).

Proof: If t∈$\mathscr{W}$, then t∈ t$\mathscr{W}$= d$\mathscr{W}$, and thus t = dt, where d ∈ D($\mathscr{W}$). Also, d ∈d$\mathscr{W}$= t$\mathscr{W}$, resulting in d =ts for some s∈$\mathscr{W}$. (d –1 + t) is now pure because (d –1+ t) (d –1+ ds) (d –1+ t)=(d –1+ t). As a result, d –1+ t = p ⟹ t =(1– d) + p, where(1– d)∈D($\mathscr{W}$) and p∈ Pu($\mathscr{W}$). As a result, $\mathscr{W}$ is a $\mathscr{P}$–clean ring. ■

Theorem 2.13: Let $\mathscr{W}$ be a commutative $\mathscr{P}$–clean ring and N ={t∈$\mathscr{W}$: $t^n = 0$, n∈z^+} be an ideal in $\mathscr{W}$. $\mathscr{W}$/N is therefore a $\mathscr{P}$–clean ring.

Proof : Let a + N ∈ $\mathscr{W}$/N, then a ∈$\mathscr{W}$ exists d ∈D($\mathscr{W}$) and p∈ Pu($\mathscr{W}$) such that a = d + p, and now a + N = (d + p) + N = (d + N) + (p + N), we must prove that (d + N) is an idempotent element in $\mathscr{W}$/N and (p + N) is a pure element in $\mathscr{W}$/N. Because $(d + N)^2 = (d+N)(d+N) = (d^2+N) = (d+N)$, (d + N) is an idempotent element in $\mathscr{W}$/ N. Because p∈ Pu($\mathscr{W}$), there is q∈ $\mathscr{W}$ such that p = pq. Now p + N = pq +N =(p + N)(q + N), implying that p+N is a pure in $\mathscr{W}$/N, and thus a + N, is "the sum of an idempotent and" pure. Hence $\mathscr{W}$/N is a $\mathscr{P}$–clean ring. ■

Lemma 2.14: If $\mathscr{W}$ is a ring and t∈ J($\mathscr{W}$), then t is a $\mathscr{P}$–clean.

Proof : If t∈ J($\mathscr{W}$), then (1 – t) is a unit element of $\mathscr{W}$, and thus (1 – t) is a pure element, because (1 – t) u = d ⇒ (1 – t) u (1 – t) = (1 – t). As a result of (1 – t) being a

$\mathscr{P}$– clean, t is also a $\mathscr{P}$– clean, according to proposition 2.10.■

Theorem 2. 15: Assume $\mathscr{W}$ is a ring. Then $\mathscr{W}$ is a $\mathscr{P}$–clean ring if and only if each element t∈$\mathscr{W}$ can be written as t = p – d, where p∈ Pu($\mathscr{W}$) and d ∈D($\mathscr{W}$) .

Proof : Let $\mathscr{W}$ be a ring and t∈$\mathscr{W}$, then as $\mathscr{W}$ is a $\mathscr{P}$– clean ring. So -t∈$\mathscr{W}$, thus -t= p + d, where p∈Pu($\mathscr{W}$) and d ∈D($\mathscr{W}$). As a result, t= -p– d, where (-p)∈ Pu($\mathscr{W}$) and d ∈D($\mathscr{W}$)

Conversely: Assume that every element t∈$\mathscr{W}$ can be written as t = p– d, where p∈Pu($\mathscr{W}$) and d ∈D($\mathscr{W}$), so for every element t∈$\mathscr{W}$, we can write –t=p– d, where p∈ Pu($\mathscr{W}$) and d ∈D($\mathscr{W}$). As a result, t=-p + d, where (-p)∈Pu($\mathscr{W}$) and d ∈D($\mathscr{W}$). As a result, $\mathscr{W}$ is a $\mathscr{P}$– clean ring.■

Proposition 2.16: Let $\mathscr{W}$ be a $\mathscr{P}$–clean ring with d as a central idempotent element. Then there's d$\mathscr{W}$d, which is $\mathscr{P}$–clean as well.

Proof : Because $\mathscr{W}$ is a $\mathscr{P}$–clean ring, then ∀ t ∈$\mathscr{W}$ can be written as t = d + p, where d ∈D($\mathscr{W}$) and p∈ Pu($\mathscr{W}$). And, because d is central, db = bd for all b∈$\mathscr{W}$, so d$\mathscr{W}$d is a homomorphic image of $\mathscr{W}$. As a result, (according to proposition 2.4), d$\mathscr{W}$d is a $\mathscr{P}$–clean. ■

References

[1] Andari, Ari. "The relationships between clean rings, r-clean rings, and f-clean rings." *AIP Conference Proceedings*. Vol. 2021. No. 1. AIP Publishing LLC, 2018.

[2] Ashrafi, Nahid, and Ebrahim Nasibi. "r-clean rings." *arXiv preprint arXiv:1104.2167* (2011).

[3] Ashrafi, N. A. H. I. D., and E. B. R. A. H. I. M. Nasibi. "Rings in which elements are the sum of an idempotent and a regular element." *Bulletin of the Iranian Mathematical Society* 39.3 (2013): 579-588.

[4] Chen, Weixing. "Notes on Clean Rings and Clean elements." Southeast Asian Bulletin of Mathematics 2998.32: 00-6.

[5] Mohammed, Akram S., Ibrahim S. Ahmed, and Samah H. Asaad. "Study of the rings in which each element express as the sum of an idempotent and pure." *International Journal of Nonlinear Analysis and Applications* 12.2 (2021): 1719-1724.

[6] Nicholson .W.K., "Lifting idempotent and exchange rings", Trans. Amer. Math. Soc., 229(1977), 269-278 .

[7] Sabet, SH A. Safari, M. Farmani, and M. Karami. "π-Clean and strongly π-clean rings." *Measurement* 109 (2017): 74-78.

[8] Sharma, Ajay, and Dhiren Kumar Basnet. "Weakly r-clean rings and weakly⋆-clean rings." *arXiv preprint arXiv:1702.06734* (2017).

[9] Sharma, Garima, and Amit B. Singh. "Strongly r-clean Rings." *International Journal of Mathematics and Computer Science* 13 (2018): 207-214.