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Technique for Solving the Bagley-Torvik Equation via Integer-Order Differential Equations

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ABSTRACT

This paper proposes an analytical technique for solving the Bagley-Torvik equation (BTE) in the Caputo sense. The main idea of this technique is based on reformulating the considered problem as a system of linear FDEs of half-order. Then the resulting system is transformed into a set of integer-order differential equations. In such a transformation, the singularity terms are removed from the FDE system. So, the solution of the BTE can be obtained via solving this system. Finally, two examples are given to demonstrate the efficiency of the proposed technique.

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1. Introduction

Fractional calculus (FC) is the branch of mathematics that studies the properties of non-integer order integrals and derivatives. Furthermore, the concepts and techniques of solving differential equations involving fractional derivatives of unknown functions are studied in-depth in this discipline. Differential equations with fractional derivatives are always called fractional differential equations (FDEs) and are widely used to model a variety of real-

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world problems, like viscoelastic polymers [16], Photovoltaic Cell [1], thermal modeling [17], signal processing [28], calculus of variations [13], control systems [20], fluid flow [31], diffusive transport [24], optics [3], networks [22], porous medium [38], rheology [25], electrochemistry [26], and many other applications. Moreover, FDEs are also an excellent tool for describing the hereditary characteristics of materials and processes, where the behavior of memory terms in such models not only brings into account the process's history, but also its impact on the process's present and future. FDEs are also considered as an alternative model to nonlinear differential equations [9]. As a result, the subject of FDEs is getting interesting, as several leading researchers have contributed to the theory of FC, including Holmgren, Abel, Riemann, Caputo, Fourier, Hadamard, Letnikov, Hardy, Grünwald, Heaviside, Laplace, Leibniz, Euler, Riesz, Lioville, Weyl, and etc (see e.g., [19] [29]).

The Bagley-Torvik equation (BTE) has recently become one of the most widely used mathematical models in the in various branches of Applied Mathematics and Mechanics. In their work in 1984 [37], R.L.Bagley and P.J.Torvik proposed modeling the viscoelastic behavior of geologic minerals, glasses, and strata using FDE, demonstrating that this approach can be used to describe structures with viscoelastic behavior. In addition, many scholars have attempted to solve this problem analytically and numerically since then. In 1999 Podlubny [29] proposed a numerical technique for the inhomogeneous BTE, as well as a fractional $G_3(t)$ function to solve the constant coefficient of the BTE analytically. Afterward, many authors worked on the numerical solution of the BTE. Because analytical solutions to the FC are difficult and quite cumbersome, a lot of numerical approaches have been developed to solve this equation since its appearance. For example, Legendre-collocation [14], hybridizable discontinuous Galerkin [18], the generalized Taylor collocation [12], Adomian decomposition [33], Haar wavelet [32], homotopic perturbation [2], the Bessel collocation [40], hybrid of block-pulse functions [23], Gegenbauer wavelet [36], and also many other remarkable works in the following papers (see e.g., [4], [5], [6], [7], [34], [35], [39]). The motions of real physical systems such as an immersed plate in a Newtonian fluid and a gas in a fluid are described by this equation with a $3/2$-order or $1/2$-order derivative.

In this work, a different technique for solving BTE is proposed. Its exact solution is found by solving the system of integer-order differential equations using the Laplace transform.

2. Elementary definitions and concepts

In this section, we give a brief overview of some fundamental FC definitions and properties that will be used in this study.

Definition 2.1 [19] Let be a continuous function for all , then the fractional integral of a type left-sided Riemann-Liouville (R-L) for is defined as follows:

$$ {}_0I_t^{\alpha}\Psi(t)=\begin{cases}\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t\frac{\Psi(\eta)}{(t-\eta)^{1-\alpha}}\,d\eta, & \alpha>0,\\ \Psi(t), & \alpha=0,\end{cases} \tag{1}$$

here and in the following the Euler gamma function is denoted by $\Gamma(\cdot)$.

The features of Riemann-Liouville integral [27], [29]

(1) ${}_0I_t^{\alpha_1}\,{}_0I_t^{\alpha_2}\Psi(t) = {}_0I_t^{\alpha_1+\alpha_2}\Psi(t);$

(2) ${}_0I_t^{\alpha_1}\,{}_0I_t^{\alpha_2}\Psi(t) = {}_0I_t^{\alpha_2}\,{}_0I_t^{\alpha_1}\Psi(t);$

(3) ${}_0I_t^{\alpha}t^{\kappa} = \dfrac{\Gamma(\kappa+1)}{\Gamma(\kappa+\alpha+1)}t^{\alpha+\kappa}, \kappa > -1;$

for the constant α_1, α_2 and κ.

Definition 2.2 [10] Let $\Psi:[0,\infty) \to R$ be a continuous and differentiable function for all $t>0$, then the fractional derivative of a type Caputo for Ψ is defined as follows:

$$
{}_C\mathbf{D}_{0,t}^{\alpha}\Psi(t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)}\displaystyle\int_0^t \dfrac{\Psi^{(n)}(\eta)}{(t-\eta)^{1-n+\alpha}}d\eta, & \alpha \in (n-1,n), \\ \dfrac{d^n}{dt^n}\Psi(t), & \alpha = n, \end{cases} \tag{2}
$$

Here, and in the following $[\alpha]$ is the integer part of α, and $n=[\alpha]+1 \in N$. The Caputo derivative is the most common definition of physical problems because the initial-value conditions for FDEs in the Caputo sense are the same as for an integer-order differential equation. These initial-value conditions have clear physical meanings in applications. Also, the following are some of the major Caputo derivative features that will be used in this article [8].

(i) ${}_C\mathbf{D}_{0,t}^{\alpha}\left(\Psi_1(t)+\Psi_2(t)\right) = {}_C\mathbf{D}_{0,t}^{\alpha}\Psi_1(t) + {}_C\mathbf{D}_{0,t}^{\alpha}\Psi_2(t);$

(ii) ${}_C\mathbf{D}_{0,t}^{\alpha}\,{}_0I_t^{\alpha}\Psi(t) = \Psi(t);$

(iii) ${}_0I_t^{\alpha}\,{}_C\mathbf{D}_{0,t}^{\alpha}\Psi(t) = \Psi(t) - \sum_{\kappa=0}^{n-1}\Psi^{(\kappa)}(0)\dfrac{t^{\kappa}}{\kappa!};$

(iv) ${}_C\mathbf{D}_{0,t}^{\alpha}\kappa = 0$, where κ is constant.

Definition 2.3 [11] For the power function $\Psi(t)=t^{\lambda}, \lambda \in R,$ in the sense of Caputo, the fractional derivative is obtained as:

$$ {}_{C}\mathbf{D}_{0,t}^{\alpha}\Psi(t)=\begin{cases}0, & \text{if}\lambda\in\{0,1,\ldots,[\alpha]-1\},\\ \dfrac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-\alpha)}t^{\lambda-\alpha}, & \text{if}\lambda\in\square \text{ and}\lambda\geq[\alpha]\text{or},\\ & \text{if}\lambda\notin\square \text{ and}\lambda>[\alpha]-1,\end{cases} \tag{3} $$

Of this definition, it is easy to observe that if $\Psi(t)=t^{\beta}$,

$$ \lim_{t\to 0}{}_{C}\mathbf{D}_{0,t}^{\gamma}\Psi(t)=\begin{cases}0, & \gamma<\beta,\\ \Gamma(\beta+1), & \gamma=\beta,\\ \Psi(0), & \gamma=0,\end{cases} \tag{4} $$

Where γ and β are all rational numbers.

Definition 2.4 [27] The Laplace transform is defined as follows:

$$ \mathrm{L}\{\Psi(t)\}=\bar{\Psi}(s)=\int_{0}^{\infty}e^{-st}\Psi(t)dt, \qquad s>0, \tag{5} $$

$$ \mathrm{L}\{t^{\alpha}\}=\frac{\Gamma(\alpha+1)}{s^{\alpha}}, \tag{6} $$

If (5) is a convergent integral, the Laplace transform of the function $y(t)$ is said to exist, whereas one last necessary property of the Laplace transform of a derivative of the integer-order n of a function $\Psi(t)$ as follows:

$$ \mathrm{L}\{\Psi^{n}(t)\}=s^{n}\bar{\Psi}(s)-\sum_{\kappa=0}^{n-1}s^{n-\kappa-1}\Psi^{\kappa}(0). \tag{7} $$

Lemma 2.1 Let $\Psi(t)\in C^{n}[0,\infty)$ and $\alpha\in(n-1,n)$, then

$$ {}_{C}\mathbf{D}_{0,t}^{\alpha_1}{}_{C}\mathbf{D}_{0,t}^{\alpha_2}\Psi(t)={}_{C}\mathbf{D}_{0,t}^{\alpha_1+\alpha_2}\Psi(t) \tag{8} $$

under the assumption that $\Psi(t)$ is differentiable in order $\alpha_1+\alpha_2$.

Theorem 2.1 [21] If $\Psi(t)\in C^{1}[0,\infty)$, then

$$ {}_{C}\mathbf{D}_{0,t}^{\alpha_1}\,{}_{C}\mathbf{D}_{0,t}^{\alpha_2}\Psi(t)={}_{C}\mathbf{D}_{0,t}^{\alpha_2}\,{}_{C}\mathbf{D}_{0,t}^{\alpha_1}\Psi(t)={}_{C}\mathbf{D}_{0,t}^{\alpha_2+\alpha_1}\Psi(t), \tag{9} $$

where $\alpha_1,\alpha_2\in\square^{+}$ and $\alpha_1+\alpha_2\in(0,1]$.

Theorem 2.2 [21] If $\Psi(t) \in C^n[0,\infty)$, then

$$ {}_C\mathbf{D}_{0,t}^{\alpha}\Psi(t) = {}_C\mathbf{D}_{0,t}^{\alpha_m}\, {}_C\mathbf{D}_{0,t}^{\alpha_{m-1}} \cdots {}_C\mathbf{D}_{0,t}^{\alpha_2}\, {}_C\mathbf{D}_{0,t}^{\alpha_1}\Psi(t), \tag{10} $$

in which $\alpha = \sum_{j=1}^{m} \alpha_j$, $\alpha_j \in (0,1]$, $\alpha \in [n-1, n)$, and there exist $j_\kappa < m$ s.t $\sum_{i=1}^{j_\kappa} \alpha_i = \kappa$ and $\kappa = 1, 2, \cdots n-1$.

3. The technique of solution Bagley-Torvik equation

In this section, a technique for solving BTE will be constructed, and it has the general form:

$$ Ay''(t) + B_{1c}\mathbf{D}_{0,t}^{\alpha} y(t) + B_2 y(t) = f(t), \tag{11} $$

subject to initial-value conditions (ICs):

$$ y(0) = y_0, y'(0) = y_1, \tag{12} $$

Here, $f(t)$ is a continuous real-valued function, $y \in C^2(0,\infty)$, and the constants B_1, B_2, and $A \in R$.

Now, we find it convenient to rewrite the original BTE (11) when $\alpha = 3/2$ as a system of FDE of $1/2$-order as follows:

$$ \begin{aligned} {}_C\mathbf{D}_{0,t}^{\frac{1}{2}} y_1(t) &= y_2(t), \\ {}_C\mathbf{D}_{0,t}^{\frac{1}{2}} y_2(t) &= y_3(t), \\ {}_C\mathbf{D}_{0,t}^{\frac{1}{2}} y_3(t) &= y_4(t), \\ {}_C\mathbf{D}_{0,t}^{\frac{1}{2}} y_4(t) &= -kB_2 y_1(t) - kB_2 y_4(t) + kf(t), \end{aligned} \tag{13} $$

where $y_1(t) = y(t)$. However, using a new column vector $\Psi = [y_1, \ldots, y_4]^T \in C^2(0,\infty)$ of dimension 4 for comfort and $k = A^{-1}$, we can reformulation the above equation as:

$$ {}_C\mathbf{D}_{0,t}^{\frac{1}{2}} \Psi(t) = \mathrm{M}\Psi(t) + \mathrm{F}(t), \tag{14} $$

subject to

$$ \begin{cases} y_1(0) = y_0, \\ y_2(0) = 0, \\ y_3(0) = y_1, \\ y_4(0) = 0, \end{cases} \tag{15} $$

Here, $\mathrm{F}=[0,\ldots,kf(t)]^T$ is vector function of dimension 4, and also M is a square matrix with a dimension 4×4 that is given as:

$$\mathrm{M}=\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -kC & 0 & 0 & -kB \end{pmatrix} \tag{16}$$

In the following sense, we show that the BTE (11) equipped with ICs (12) is equivalent to the system of Linear FDEs (13) with ICs (15):

(i) If Ψ^T with $y\in C^2(0,\infty)$ is a solution of the system (14) with ICs (15), then $y(t)$ solves the BTE (11) and satisfies its corresponding ICs (12).

(ii) If $y\in C^2(0,\infty)$ is a solution of the BTE (11), with ICs (12), the column vector Ψ^T satisfies the system (14) with ICs (15).

Repeating the same procedure can be done above when $\alpha=1/2\cdot$ On the other hand, to deduce an integer-order differential equation from equations (14)-(15), we must eliminate the singularity terms in (14) by using the properties of the Caputo derivative and the following transformation:

$$\Phi(t)=\Psi(t)-\bar{\lambda}t^{\frac{1}{2}} \tag{17}$$

Further, $\Phi^T=[z_1,\ldots,z_4]^T$ and we can obtain $\Phi(0)=\Psi(0)$.

First, by multiplying the differential operator ${}_C\mathbf{D}_{0,t}^{\frac{1}{2}}$ on both sides of (17) and taking (14) into account, one can get the following

$${}_C\mathbf{D}_{0,t}^{\frac{1}{2}}\Phi(t)=\mathrm{M}\Phi(t)+\mathrm{F}(t)+\bar{\lambda}\mathrm{M}\,t^{\frac{1}{2}}-\bar{\lambda}{}_C\mathbf{D}_{0,t}^{\frac{1}{2}}t^{\frac{1}{2}} \tag{18}$$

Then, under the assumption that there is no singularity of RHS of (18) to zero at $t=0$, we seek to compute the constant $\bar{\lambda}$ and the following is from (19):

$$\bar{\lambda}=\frac{2}{\sqrt{\pi}}[\mathrm{M}\Phi(0)+\mathrm{F}(0)] \tag{20}$$

Once more, by applying the differential operator ${}_C\mathbf{D}_{0,t}^{\frac{1}{2}}$ to (18), we can obtain the following:

$$\Phi'(t)=M^2\Phi(t)+\frac{2}{\sqrt{\pi}}[M^3\Phi(0)+M^2F(0)]t^{\frac{1}{2}}+MF(t)+{}_C\mathbf{D}_{0,t}^{\frac{1}{2}}F(t) \tag{21}$$

It is clear, there is no fractional derivative for the unknown function $\Phi(t)$. Accordingly, this equation can present us with an explicit and exact state-space illustration, as long ${}_C\mathbf{D}_{0,t}^{\frac{1}{2}}F(t)$ is can be explicitly expressed.

To confirm the preceding result, we state and prove the next theorem.

Theorem 3.1 The system (14) has the following solution:

$$\Psi(t)=\Phi(t)+\bar{\lambda}t^{\frac{1}{2}} \tag{22}$$

where, $\Phi(t)$, is an exact solution to a system of integer-order differential equations

$$\Phi'(t)=M^2\Phi(t)+R(t) \tag{23}$$

$$\Phi(0)=\Psi(0) \tag{24}$$

with

$$R(t)=\frac{2}{\sqrt{\pi}}[M^3\Phi(0)+M^2F(0)]t^{\frac{1}{2}}+MF(t)+{}_C\mathbf{D}_{0,t}^{\frac{1}{2}}F(t). \tag{25}$$

Proof: From the Definition (26), we have

$$\frac{1}{\Gamma(\frac{1}{2})}t^{\frac{-1}{2}}*\Psi'(t)=M\Psi(t)+F(t) \tag{27}$$

Next, by using the Laplace transform to (27), resulting in

$$(s^{\frac{1}{2}}I-M)\bar{\Psi}(s)=s^{-\frac{1}{2}}\Psi(0)+\bar{F}(s) \tag{28}$$

By multiplying (28) by $s^{\frac{1}{2}}I+M$, we can obtain the following:

$$(sI-M^2)\bar{\Psi}(s)=\sum_{k=0}^{1}M^k s^{\frac{-k}{2}}\Psi(0)+\sum_{k=0}^{1}M^k s^{\frac{1-k}{2}}\bar{F}(s) \tag{29}$$

Using Laplace transform for (22) and substitute it in equation (29), one can have

$$(sI-M^2)\bar{\Phi}(s)=\sum_{k=0}^{1}M^k s^{\frac{-k}{2}}\Phi(0)-(sM-M^3)\Phi(0)+\sum_{k=0}^{1}M^k s^{\frac{1-k}{2}}\bar{F}(s)\text{-}sF(0)+M^3 \tag{30}$$

If we apply the inverse Laplace transform to the last equation (30), we have

$$\Phi'(t) = \mathrm{M}^2\Phi(t) + \frac{2}{\sqrt{\pi}}[\mathrm{M}^3\Phi(0) + \mathrm{M}^2\mathrm{F}(0)]t^{\frac{1}{2}} + \mathrm{MF}(t) +_C \mathbf{D}_{0,t}^{\frac{1}{2}}\mathrm{F}(t) \tag{31}$$

Now, the proof is completed. □

4. Illustrative example

In this section, we will give some examples to verify the efficacy of the proposed technique. To solve the examples, we used Maple.

Example (1): Consider the following BTE [15]:

$$ {}_C\mathbf{D}_{0,t}^{\frac{3}{2}}y(t) = -y(t) + \frac{2t^{\frac{1}{2}}}{\Gamma(\frac{3}{2})} + t^2 - t, t \geq 0, \tag{29}$$

subject to

$$y(0) = 0, \quad \text{and} \quad y'(0) = -1. \tag{30}$$

Applying theorem (2.2), we reduce (29) to the subsequent system of equations with its corresponding ICs,

$$\begin{cases} {}_C\mathbf{D}_{0,t}^{\frac{1}{2}}y_1(t) = y_2(t), \\ {}_C\mathbf{D}_{0,t}^{\frac{1}{2}}y_2(t) = y_3(t), \\ {}_C\mathbf{D}_{0,t}^{\frac{1}{2}}y_3(t) = -y_1(t) + \dfrac{2t^{\frac{1}{2}}}{\Gamma(\frac{3}{2})} + t^2 - t, \end{cases} \tag{31}$$

where $y_1(t) = y(t)$ and ICs $y_1(0) = y_2(0) = 0$, and $y_3(0) = -1$. Next, from theorem (3.1), system (31), and the initial-value conditions, we have

$$\begin{cases} z'_1(t) = z_3(t), \\ z'_2(t) = -z_1(t) + \dfrac{2t^{\frac{1}{2}}}{\Gamma(\frac{3}{2})} + t^2 - t, \\ z'_3(t) = z_2(t) + \dfrac{8}{3}\dfrac{t^{\frac{3}{2}}}{\sqrt{\pi}} + 2, \end{cases} \tag{32}$$

where $z(t) = z_1(t)$ and ICs $z_1(0) = z_2(0) = 0$, and $z_3(0) = -1$. Now, by using the Laplace transform of the system (32), we get

$$z(t) = t^2 - t \tag{33}$$

According to theorem (3.1), the analytical solution to (29), as follows:

$$y(t) = t^2 - t. \tag{34}$$

Example (2): Consider the fractional BTE [15], [30]:

$$y'(t) + {}_C\mathbf{D}_{0,t}^{\frac{1}{2}} y(t) + y(t) = t^3 + 6t + \frac{16}{5}\frac{t^{\frac{5}{2}}}{\sqrt{\pi}},\ t \geq 0, \tag{35}$$

subject to

$$y(0) = 0, \quad \text{and} \quad y'(0) = 0. \tag{36}$$

Applying theorem (2.2), we reduce (35) to the subsequent system of equations with its corresponding ICs,

$$\begin{cases} {}_C\mathbf{D}_{0,t}^{\frac{1}{2}} y_1(t) = y_2(t), \\ {}_C\mathbf{D}_{0,t}^{\frac{1}{2}} y_2(t) = y_3(t), \\ {}_C\mathbf{D}_{0,t}^{\frac{1}{2}} y_3(t) = y_4(t), \\ {}_C\mathbf{D}_{0,t}^{\frac{1}{2}} y_4(t) = t^3 + 6t + \frac{16}{5}\frac{t^{\frac{5}{2}}}{\sqrt{\pi}} - y_2(t) - y_1(t), \end{cases} \tag{37}$$

where $y_1(t) = y(t)$ and ICs $y_1(0) = y_2(0) = y_3(0) = y_4(0) = 0$. Next, from theorem (3.1), system (37), and the initial-value conditions, we get

$$\begin{cases} z'_1(t) = z_3(t), \\ z'_2(t) = z_4(t), \\ z'_3(t) = -z_1(t) - z_2(t) + t^3 + 6t + \frac{16}{5}\frac{t^{\frac{5}{2}}}{\sqrt{\pi}}, \\ z'_4(t) = -z_2(t) - z_3(t) + \frac{16}{5}\frac{t^{\frac{5}{2}}}{\sqrt{\pi}} + \frac{12t^{\frac{1}{2}}}{\sqrt{\pi}} + 3t^2, \end{cases} \tag{38}$$

where $z(t) = z_1(t)$ and ICs $z_1(0) = z_2(0) = z_3(0) = z_4(0) = 0$. Now, by using the Laplace transform of the system (38), we have

$$z(t) = t^3 \tag{39}$$

According to theorem (3.1), the analytical solution to (35), as follows:

$$y(t) = t^3. \tag{40}$$

5. Conclusions

In this article, we have presented a technique for solving BTE by converting it to a system of linear differential equations of integer order. The main advantage of this technique is that it is being able to write the exact and explicit solutions of the BTE, unlike most of the different techniques that give the solution of BTE in terms of Mittag-Leffler (infinite string function) functions. To demonstrate the ability and effectiveness of this idea, some examples are given.

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