On Negative Logarithm Semigroup

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\textbf{ABSTRACT}

In this work, we introduce certain type of semigroup, namely (Negative Logarithm semigroup), in the functional analytic study of differential equations. We construct a solution of the pde as the form:

\[
\frac{\partial u(t,x)}{\partial t} = -e^{-h(x)} \frac{\partial u(t,x)}{\partial x}, u(x,0) = \phi(x), \ h(0) = 0
\]

Properties of this semigroup is studied.

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\section{1. Introduction:}

A lot off scientists ([1],[2],[3]) worke in the area of operator semigroups and they introduce many types of semigroups. In particular, the progress is mad in asymptatic theory of strongly continuous semigroups. A semigroup on a Banach space is considered one of major results of this direction. In this work, we construction the solution of the following equations:

\[
\frac{\partial u(x,t)}{\partial t} = -e^{-h(x)} \frac{\partial u(x,t)}{h'(x)} , u(x,0) = \phi(x) , h(0) = 0
\]

By using the Negative \textbf{Logarithm semigroup}
2. Fundamental concepts

2.1 Definition [6]:

Let $E$ be a Banach space, then a function $\varphi(x)$ is continuous at point $x_0$ if $\|\varphi(x) - \varphi(x_0)\|_E \to 0$, at $x \to x_0$, continuous on interval $[a, b]$, if it is continuous for all elements in $[a, b]$.

2.2 Definition [9]:

If $x_n \in D(A), x_n \to x_0 \Rightarrow \|Ax_n - y_0\| \to 0$ and $Ax_0 = y_0$ then A is closed.

2.3 Definition [8]:

If $X$ is a Banach space. One parameter family $T(t), 0 \leq t < \infty$, of bounded linear operators from $X$ into $X$ is semigroup bounded linear operator if

i) $T(0) = I$, ($I$ is the identity operator on $X$)

ii) $T(t + s) = T(t)T(s)$ for every $t, s \geq 0$

A semigroup of bounded linear operators $T(t)$, is uniformly continuous if

$$\lim_{x \to 0} \|T(x)\| = I$$

The operators $A$ which is defined on the domain

$$D(A) = \left\{ x \in X: \left. \frac{dT(t)}{dt} \right|_{t=0} \text{exists} \right\}$$

Is considered generator of the semigroup $T(t), D(A)$ is the domain of $A$. 
2.4 Definition [6]:

If \((a, b)\) is interval and \(h(x)\) is a differentiable function and \(h(x) \to \infty\) where \(x \to b\), we introduce a space \(L_{p, \omega, h}\) by:

\[
L_{p, \omega, h} = \left\{ \phi: \|\phi\|_{p, \omega, h, g} = \left[ \int_a^b |\exp[\omega h(x)] g(x) \phi(x)|^p \, dh(x) \right]^{1/p}, \quad p \geq 1, \omega > 0, g(x) > 0, g'(x) > 0 \right\}
\]

III. Negative Logarithm semigroup

3.1 Definition:

Let \(t < 0\) and \(x \in (a, b) \subset \mathbb{R}\), \(h(t)\) is real functions and the domain \(D(h) = (a, b)\), continuous differentiable and strictly monotone \(h(x) \in D(h^{-1})\), where \(h^{-1}\) inverse function of \(h\), we define an one-parameter family of operators:

\[
T^h_{log}(t) \phi(x) = \phi \left( h^{-1} \left( \ln (e^{h(x)} - t) \right) \right).
\]  \hspace{1cm} (3.1)

3.2 Proposition:

\(T^h_{log}(t)\) is semigroup and its generator is

\[
A^h_{log} = \frac{-1}{e^{h(x)} h'(x)},
\]  \hspace{1cm} (3.2)
Proof:

1) $T_{log}^h(0)\varphi(x) = \varphi\left(h^{-1}\left(ln(e^h(x) - 0)\right)\right)$

$T_{log}^h(0)\varphi(x) = \varphi\left(h^{-1}\left(ln(e^h(x))\right)\right)$ $\rightarrow$ $T_{log}^h(0)\varphi(x) = \varphi\left(h^{-1}\left(h(x)\right)\right)$

$T_{log}^h(0)\varphi(x) = \varphi(x) \rightarrow$ Thus $T_{log}^h(0) = I$

2) $T_{log}^h(t_1)T_{log}^h(t_2)\varphi(x) = T_{log}^h(t_1)\varphi\left(h^{-1}\left(ln(e^h(x) - t_2)\right)\right)$

Let $T = h^{-1}\left(ln(e^h(x) - t_2)\right) \rightarrow h(T) = ln(e^h(x) - t_2)$

$T_{log}^h(t_1)T_{log}^h(t_2)\varphi(x) = T_{log}^h(t_1)\varphi(T)$

$T_{log}^h(t_1)T_{log}^h(t_2)\varphi(x) = \varphi\left(h^{-1}\left(ln(e^h(x) - t_1)\right)\right)$

$T_{log}^h(t_1)T_{log}^h(t_2)\varphi(x) = \varphi\left(h^{-1}\left(ln(e^{ln(e^h(x)+t_2)} - t_1)\right)\right)$

$T_{log}^h(t_1)T_{log}^h(t_2)\varphi(x) = T_{log}^h(t_1)\varphi\left(h^{-1}\left(ln(e^{h(x)} - t_1 - t_2)\right)\right)$

$T_{log}^h(t_1)T_{log}^h(t_2)\varphi(x) = T_{log}^h(t_1+t_2)\varphi(x)$

$A_{log}^h = \frac{dT_{log}^h(t)\varphi(x)}{dt}\bigg|_{t=0}$

$A_{log}^h = \frac{d}{dt}\varphi\left(h^{-1}\left(ln(e^h(x) - t)\right)\right)\bigg|_{t=0} \rightarrow A_{log}^h = \frac{d}{dt}\varphi(\tau)|_{t=0}$

$A_{log}^h = \varphi'(\tau)\frac{d\tau}{dt}|_{t=0}$

Let $\tau = h^{-1}\left(ln(e^h(x) - t)\right) \Rightarrow h(\tau) = ln(e^h(x) - t)$
\[ h'(\tau) \frac{d\tau}{dt} = \frac{-1}{e^{h(x)} - t} \rightarrow \frac{d\tau}{dt} = \frac{-1}{e^{h(x)} - t} \frac{d\tau}{dt} = \frac{-1}{e^{h(x)} - t} h'(h^{-1}(\ln(e^{h(x)} - t))) \]

\[ A_{log}^h = \varphi'(\tau) \frac{d\tau}{dt} \bigg|_{t=0} \rightarrow A_{log}^h = \varphi'(\tau) \frac{-1}{e^{h(x)} - t} h'(\varphi(h^{-1}(\ln(e^{h(x)} - t)))) \bigg|_{t=0} \]

\[ A_{log}^h = \frac{-1}{e^{h(x)} - 0} \rightarrow A_{log}^h = \frac{-1}{e^{h(x)}} \]

\[ A_{log}^h = \frac{-1}{e^{h(x)}} \rightarrow A_{log}^h = \frac{-1}{e^{h(x)}} \]

\[ A_{log}^h = \frac{-1}{e^{h(x)}} \frac{d\tau}{dt} \]

\[ A_{log}^h = \varphi'(\tau) \frac{d\tau}{dt} \]

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\[ 3.3 \text{ Remark:} \]

We say that the semigroup \( T_{log}^h(t) \) is a **Negative Logarithm semigroup**.

The following proposition shows that \( T_{log}^h(t) \) is bounded operator in the space \( L_{p,\omega,h} \), i.e. it is continuous operator to use this property to find the solution of certain types of pde.

\[ 3.4 \text{ Proposition:} \]

The operators \( T_{log}^h(t) \) is strongly continuous semigroup works on \( L_{p,\omega,h} \) – space and the following inequality holds:

\[ \| T_{log}^h(t) \varphi(x) \|_{L_{p,1,h}} \leq \| \varphi(x) \|_{L_{p,1,h}} \quad (3.3) \]
Proof:

\[
\|T_{\text{log}}^h(t)\phi(x)\|_{L^p_{1,h}}^p = \int_a^b e^{h(x)} |\varphi[h^{-1}(\ln(e^{h(x)}) - t)]|^p d(h(x))
\]

Let \( h^{-1}(\ln(e^{h(x)} - t)) = \tau \rightarrow h(\tau) = \ln(e^{h(x)} - t) \rightarrow e^{h(\tau)} = e^{\ln(e^{h(x)} - t)} \)

\[
e^{h(\tau) + t} = e^{h(x)}
\]

\[
\ln(e^{h(\tau) + t}) = \ln e^{h(x)}
\]

\[
h(x) = \ln(e^{h(\tau) + t})
\]

\[
d(h(x)) = \frac{e^{h(\tau)}d(h(\tau))}{e^{h(\tau) + t}}
\]

\[
\|T_{\text{log}}^h(t)\phi(x)\|_{L^p_{1,h}}^p \leq \int_a^b e^{h(x)} |\varphi(\tau)|^p e^{h(\tau)} d(h(\tau)) = \int_a^b e^{h(\tau)} |\varphi(\tau)|^p d(h(\tau))
\]

\[
\|T_{\text{log}}^h(t)\phi(x)\|_{L^p_{1,h}}^p \leq \|\phi(x)\|_{L^p_{1,h}}^p
\]

\[
\|T_{\text{log}}^h(t)\phi(x)\|_{L^p_{1,h}} \leq \|\phi(x)\|_{L^p_{1,h}}
\]

3.5 Proposition:

The solution of the following pde:

\[
\frac{\partial u(t, x)}{\partial t} = \frac{e^{-h(x)}}{h'(x)} \frac{\partial u(t, x)}{\partial x}, u(x, 0) = \phi(x), \quad h(0) = 0
\]

Is a function \( u(x, t) = T_{\text{log}}^h(t)\phi(x) \)
Proof:

To prove that the function $u(x,t) = T_{log}^h(t)\varphi(x)$ is solution, we must prove that $u(x,t)$ satisfies the equation.

$$\frac{\partial u(t,x)}{\partial t} = \frac{\partial T_{log}^h(t)\varphi(x)}{\partial t}, \quad \text{let } \tau = h^{-1}\left(ln\left(e^{h(x)} - t\right)\right) \Rightarrow h(\tau) = ln\left(e^{h(x)} - t\right)$$

$$\frac{\partial u(t,x)}{\partial t} = \varphi'(\tau) \frac{\partial \tau}{\partial t} \quad \text{since } h(\tau) = ln\left(e^{h(x)} - t\right) \text{ then } h'(\tau) \frac{\partial \tau}{\partial t} = \frac{-1}{e^{h(x)} - t} \rightarrow \frac{\partial \tau}{\partial t} = \frac{-1}{h'(\tau)},$$

thus we get

$$\frac{\partial u(t,x)}{\partial t} = \varphi'(\tau) \frac{-1}{e^{h(x)} - t} h'(\tau) \quad \text{(3.4)}$$

In other hand we can find $\frac{\partial u(t,x)}{\partial x}$ as following:

$$\frac{\partial u(t,x)}{\partial x} = \frac{\partial T_{log}^h(t)\varphi(x)}{\partial x}$$

Let $\tau = h^{-1}\left(ln\left(e^{h(x)} - t\right)\right) \Rightarrow h(\tau) = ln\left(e^{h(x)} - t\right)$

$$\frac{\partial u(t,x)}{\partial x} = \varphi'(\tau) \frac{\partial \tau}{\partial x}. \quad \text{Since } h(\tau) = ln\left(e^{h(x)} - t\right) \text{ then}$$

$$h'(\tau) \frac{\partial \tau}{\partial x} = \frac{1}{e^{h(x)} - t} e^{h(x)} h'(x) \rightarrow \frac{\partial \tau}{\partial x} = \frac{e^{h(x)} h'(x)}{e^{h(x)} - t} h'(\tau) \rightarrow , \quad \text{thus we get}$$

$$\frac{\partial u(t,x)}{\partial x} = \varphi'(\tau) \frac{e^{h(x)} h'(x)}{e^{h(x)} - t} h'(\tau) \quad \text{(3.5)}$$
By Eq.s(3.4 and 3.5) and cancelation law in $\varphi'(\tau)h'(\tau)$ we get:

$$\frac{\partial u(x, t)}{\partial x} \frac{1}{e^{h(x)}h'(x)} \frac{1}{e^{h(x)}-t} \partial u(x, t) \frac{-1}{e^{h(x)}-t}$$

$$\frac{\partial u(x, t)}{\partial t} = -e^{-h(x)} \frac{\partial u(x, t)}{h'(x)} \frac{\partial x}{\partial x}$$

4. Conclusion:

In this paper we found a solution of certain type of partial differential equation by using a canonical continuous semigroup namely, Negative Logarithm Semigroup.

References


