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# On Negative Logarithm Semigroup

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**ABSTRACT**

In this work, we introduce certain type of semigroup, namely (**Negative Logarithm semigroup**) ,in the functional analytic study of differential equations. We construct a solution of the pde as the form:

$$\frac{\partial u(t,x)}{\partial t} = \frac{-e^{-h(x)}}{h'(x)}\frac{\partial u(t,x)}{\partial x}, u(x,0) = \varphi(x), \quad h(0) = 0$$

Properties of this semigroup is studied.

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## 1. Introduction:

A lot off scientists ([1],[2],[3]) worke in the area of operator semigroups and they introduce many types of semigroups. In particular, the progress is mad in asymptatic theory of strongly continuous semigroups. A semigroup on a Banach space is considered one of major results of this direction. In this work , we constraction the solution of the following equations:

$$\frac{\partial u(x,t)}{\partial t} = \frac{-e^{-h(x)}}{h'(x)}\frac{\partial u(x,t)}{\partial x} \quad ,u(x,0) = \varphi(x)\,,h(0) = 0$$

By using the Negative **Logarithm semigroup**

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## 2. Fundamental concepts

### 2.1 Definition [6]:

Let E be a Banch space ,then a function $\varphi(x)$ is continuous at point $x_0$ if $\|\varphi(x)-\varphi(x_0)\|_E \longrightarrow 0$, at $x \longrightarrow x_0$, continouos on interval $[a,b]$, if it is continuous for all elements in [a,b].

### 2.2 Definition [9]:

If $x_n \in D\ (A)\ , x_n \rightarrow x_0\ \Rightarrow \|Ax_n - y_0\| \longrightarrow 0$ and $Ax_0 = y_0$ then A is closed

### 2.3 Defnition [8]:

If X is a Banach space. One parameter family $T(t),\ 0 \leq t < \infty$ ,of bounded linear oprators from $X$ into $X$ is semigroup bonded linear operator if

$i)\ T(0) = I, (I\ is\ the\ identity\ operator\ on\ X)$

$ii)\ T(t+s) = T(t)T(s) for\ every\ t,s \geq 0$

A semigroup of buonded linear opertors $T(t)$ , is uniformly continous if

$$\lim_{x\to 0}\|T(x)\| = I$$

The operators $A$ which is defined on the domain

$$D(A) = \left\{x \in X: \left.\frac{dT(t)}{dt}\right|_{t=0} exists\right\}$$

Is considered generator of the semigroup $T(t), D(A)$ is the domain of $A$.

## 2.4 Definition [6]:

If $(a,b)$ is interval and h(x) is a differentiable function and $h(x) \to \infty$ *where* $x \to b$ , we introduce a space $L_{p,\omega,h}$ by :

$$L_{p,\omega,h} = \left\{ \phi : \ \|\phi\|_{p,\omega,h,g} = \left[ \int_a^b |\exp[\omega h(x)]\, g(x)\phi(x)|^p dh(x) \right]^{\frac{1}{p}}, p \geq 1 , \omega > 0 , g(x) > 0, g'(x) > 0 \right\}$$

## III. Negative Logarithm semigroup

### 3.1 Defnition:

Let $t < 0$ *and* $x \in (a,b) \subset R$, $h(t)$ is rael functions and the domain $D(h) = (a,b)$, continuous differentiable and strictly montone $h(x) \in \mathrm{D}\,(h^{-1})$, where $h^{-1}$ inverse function of h, we define an oneparameter family of operators:

$$T_{log}^{h}(t)\varphi(x) = \varphi\left(h^{-1}\left(ln\left(e^{h(x)} - t\right)\right)\right) . \qquad (3.1)$$

### 3.2 Proposition:

$T_{log}^{h}(t)$ is semigroup and its generator is

$$A_{log}^{h} = \frac{-1}{e^{h(x)}\ h'(x)} \quad , \qquad (3.2)$$

**Proof**:

1) $T_{log}^{h}(0)\varphi(x) = \varphi\left(h^{-1}\left(ln\left(e^{h(x)} - 0\right)\right)\right)$

$T_{log}^{h}(0)\varphi(x) = \varphi\left(h^{-1}\left(ln\left(e^{h(x)}\right)\right)\right) \rightarrow T_{log}^{h}(0)\varphi(x) = \varphi\left(h^{-1}\left(h(x)\right)\right)$

$T_{log}^{h}(0)\varphi(x) = \varphi(x)$ →Thus $T_{log}^{h}(0) = I$

2) $T_{log}^{h}(t_1)T_{log}^{h}(t_2)\varphi(x) = T_{log}^{h}(t_1)\varphi\left(h^{-1}\left(ln\left(e^{h(x)} - t_2\right)\right)\right)$

Let $\mathcal{T} = h^{-1}\left(ln\left(e^{h(x)} - t_2\right)\right) \rightarrow h(\mathcal{T}) = ln\left(e^{h(x)} - t_2\right)$

$T_{log}^{h}(t_1)T_{log}^{h}(t_2)\varphi(x) = T_{log}^{h}(t_1)\varphi(\mathcal{T})$

$T_{log}^{h}(t_1)T_{log}^{h}(t_2)\varphi(x) = \varphi\left(h^{-1}\left(ln\left(e^{h(\mathcal{T})} - t_1\right)\right)\right)$

$T_{log}^{h}(t_1)T_{log}^{h}(t_2)\varphi(x) = \varphi\left(h^{-1}\left(ln\left(e^{ln\left(e^{h(x)}+t_2\right)} - t_1\right)\right)\right)$

$T_{log}^{h}(t_1)T_{log}^{h}(t_2)\varphi(x) = T_{log}^{h}(t_1)\varphi\left(h^{-1}\left(ln\left(e^{h(x)} - t_1 - t_2\right)\right)\right)$

$T_{log}^{h}(t_1)T_{log}^{h}(t_2)\varphi(x) = T_{log}^{h}(t_{1+}t_2)\varphi(x)$

$$A_{log}^{h} = \left.\frac{dT_{log}^{h}(t)\varphi(x)}{dt}\right|_{t=0}$$

$$A_{log}^{h} = \left.\frac{d}{dt}\varphi\left(h^{-1}\left(ln\left(e^{h(x)} - t\right)\right)\right)\right|_{t=0} \rightarrow A_{log}^{h} = \frac{d}{dt}\varphi(\tau)|_{t=0}$$

$$A_{log}^{h} = \varphi'(\tau)\left.\frac{d\tau}{dt}\right|_{t=0}$$

Let $\tau = h^{-1}\left(ln\left(e^{h(x)} - t\right)\right) \Longrightarrow h(\tau) = ln\left(e^{h(x)} - t\right)$

$$h'(\tau)\frac{d\tau}{dt}=\frac{-1}{e^{h(x)}-t}\rightarrow\frac{d\tau}{dt}=\frac{\frac{-1}{e^{h(x)}-t}}{h'(\tau)}\rightarrow\frac{d\tau}{dt}=\frac{\frac{-1}{e^{h(x)}-t}}{h'\left(h^{-1}\left(ln(e^{h(x)}-t)\right)\right)}$$

$$A_{log}^{h}=\varphi'(\tau)\frac{d\tau}{dt}\Big|_{t=0}\rightarrow A_{log}^{h}=\varphi'(\tau)\frac{\frac{-1}{e^{h(x)}-t}}{h'\left(h^{-1}\left(ln(e^{h(x)}-t)\right)\right)}\Bigg|_{t=0}$$

$$A_{log}^{h}=\frac{\frac{-1}{e^{h(x)}-0}}{h'\left(h^{-1}\left(ln(e^{h(x)}-0)\right)\right)}\rightarrow A_{log}^{h}=\frac{\frac{-1}{e^{h(x)}}}{h'\left(h^{-1}\left(ln(e^{h(x)})\right)\right)}$$

$$A_{log}^{h}=\frac{\frac{-1}{e^{h(x)}}}{h'\left(h^{-1}\left(h(x)\right)\right)}\rightarrow A_{log}^{h}=\frac{\frac{-1}{e^{h(x)}}}{h'(x)}$$

$$A_{log}^{h}=\frac{-1}{e^{h(x)}\,h'(x)}$$

**3.3 Remark:**

We say that the semigroup $T_{log}^{h}(t)$ is a **Negative Logarithm semigroup**.

The following proposition shows that $T_{log}^{h}(t)$ is bounded operator in the space $L_{p,\omega,h}$ , i.e. it is continuous operator to use this property to find the solution of certain types of pde .

**3.4 Proposition:**

The operators $T_{log}^{h}(t)$ is strongly continuous semigroup works on $L_{p,\omega,h}-space$ and the following inequality holds:

$$\left\|T_{log}^{h}(t)\varphi(x)\right\|_{L_{p,1,h}}\leq\|\varphi(x)\|_{L_{p,1,h}}\qquad(3.3)$$

**Proof**:

$$\left\|T_{log}^{h}(t)\varphi(x)\right\|_{L_{p,1,h}}^{p} = \int_{a}^{b} e^{h(x)}\left|\varphi\left[h^{-1}\left(\ln\left(e^{h(x)}\right)-t\right)\right]\right|^{p} d\left(h(x)\right)$$

Let $h^{-1}\left(\ln\left(e^{h(x)}-t\right)\right)=\tau \to h(\tau)=\ln\left(e^{h(x)}-t\right) \quad \to e^{h(\tau)}=e^{\ln\left(e^{h(x)}-t\right)}$

$$e^{h(\tau)}+t=e^{h(x)}$$

$$ln\left(e^{h(\tau)}+t\right)=ln e^{h(x)}$$

$$h(x)= ln\left(e^{h(\tau)}+t\right)$$

$$d\left(h(x)\right)=\frac{e^{h(\tau)}d\left(h(\tau)\right)}{e^{h(\tau)}+t}$$

$$\left\|T_{log}^{h}(t)\varphi(x)\right\|_{L_{p,1,h}}^{p} \le \int_{a}^{b} e^{h(x)}|\varphi(\tau)|^{p}\frac{e^{h(\tau)}}{e^{h(x)}}d\left(h(\tau)\right)=\int_{a}^{b} e^{h(\tau)}|\varphi(\tau)|^{p}\, d\left(h(\tau)\right)$$

$$\left\|T_{log}^{h}(t)\varphi(x)\right\|_{L_{p,1,h}}^{p} \le \|\varphi(x)\|_{L_{p,1,h}}^{p}$$

$$\left\|T_{log}^{h}(t)\varphi(x)\right\|_{L_{p,1,h}} \le \|\varphi(x)\|_{L_{p,1,h}}$$

## 3.5 Proposition:

The solution of the following pde :

$$\frac{\partial u(t,x)}{\partial t}=\frac{e^{-h(x)}}{h'(x)}\frac{\partial u(t,x)}{\partial x}, u(x,0)=\varphi(x), \quad h(0)=0$$

Is a function $u(x,t)=T_{log}^{h}(t)\varphi(x)$

**Proof:**

To prove that the function $u(x,t) = T_{log}^{h}(t)\varphi(x)$ is solution , we must prove that u(x,t) satisfies the equation.

$\frac{\partial u(t,x)}{\partial t} = \frac{\partial T_{log}^{h}(t)\varphi(x)}{\partial t}$, let $\tau = h^{-1}\left(ln\left(e^{h(x)} - t\right)\right) \Longrightarrow h(\tau) = ln\left(e^{h(x)} - t\right)$

$\frac{\partial u(t,x)}{\partial t} = \varphi'(\tau)\frac{\partial \tau}{\partial t}$ since $h(\tau) = ln\left(e^{h(x)} - t\right)$ then $h'(\tau)\frac{\partial \tau}{\partial t} = \frac{-1}{e^{h(x)}-t} \rightarrow \frac{\partial \tau}{\partial t} = \frac{\frac{-1}{e^{h(x)}-t}}{h\prime(\tau)}$, thus we get

$$\frac{\partial u(t,x)}{\partial t} = \varphi'(\tau)\frac{\frac{-1}{e^{h(x)}-t}}{h'(\tau)} \tag{3.4}$$

In other hand we can find $\frac{\partial u(t,x)}{\partial x}$ as following:

$$\frac{\partial u(t,x)}{\partial x} = \frac{\partial T_{log}^{h}(t)\varphi(x)}{\partial x}$$

Let $\tau = h^{-1}\left(ln\left(e^{h(x)} - t\right)\right) \Longrightarrow h(\tau) = ln\left(e^{h(x)} - t\right)$

$\frac{\partial u(t,x)}{\partial x} = \varphi'(\tau)\frac{\partial \tau}{\partial x}$. Since $h(\tau) = ln\left(e^{h(x)} - t\right)$ then

$h'(\tau)\frac{\partial \tau}{\partial x} = \frac{1}{e^{h(x)}-t}e^{h(x)}h'(x) \rightarrow \frac{\partial \tau}{\partial x} = \frac{\frac{e^{h(x)}h'(x)}{e^{h(x)}-t}}{h\prime(\tau)} \rightarrow$, thus we get

$$\frac{\partial u(t,x)}{\partial x} = \varphi'(\tau)\frac{\frac{e^{h(x)}h'(x)}{e^{h(x)}-t}}{h'(\tau)} \tag{3.5}$$

By Eq.s(3.4 and 3.5) and cancelation law in $\varphi'(\tau)h'(\tau)$ we get:

$$\frac{\partial u(x,t)}{\partial x}\frac{1}{\frac{e^{h(x)}h'(x)}{e^{h(x)}-t}}=\frac{\partial u(x,t)}{\partial t}\frac{1}{\frac{-1}{e^{h(x)}-t}}$$

$$\frac{\partial u(x,t)}{\partial t}=\frac{-e^{-h(x)}}{h'(x)}\frac{\partial u(x,t)}{\partial x}$$

## 4.Conclusion:

In this paper we found a solution of certain type of partial differential equation by using a canonical continuous semigroup namely , Negative Logarithm Semigroup.

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