Journal of Al-Qadisiyah for Computer Science and Mathematics 2022, pp Math . 19–32

Available online at www.qu.edu.iq/journalcm

JOURNAL OF AL-QADISIYAH FOR COMPUTER SCIENCE AND MATHEMATICS

ISSN:2521-3504(online) ISSN:2074-0204(print)

Sf-covering dimension and sf-local dimension

Ghazwan Farhan Shaker[1] and Raad Aziz Hussain Al-Abdulla [2] *

Department of Mathematics,""College of Science" "University of Al-Qadisiyah, Diwaniyah-Iraq .Email: ma20.post13@qu.edu.iq[1]

Department of Mathematics,""College of Science""University of Al-Qadisiyah, Diwaniyah-Iraq.Email: raad.hussain@qu.edu.iq[2]

ARTICLE INFO

Article history:
Received: 11 /06/2022
Rrevised form: 09 /07/2022
Accepted : 17 /07/2022
Available online: 12 /08/2022

Keywords: dim X , loc dim X ,sf-dim X , sf-loc dim X

ABSTRACT

The aim of this paper is to introduce new types of dimension functions which are sf-covering.dimension,sf-dim X, and sf-local dimension, sf-loc dim X , by using semi feebly open (sf-open) sets, with examples and theorems. The relationships between them and other concepts will be studied like sf-T_1-space, sf˝-regular space and sf`-normal space.

MSC..

https://doi.org/10.29304/jqcm.2022.14.3.992

1.Introduction:

The foundation of dimension theory is the "dimension function," It has the properties of d(X)=d(Y) if X and Y are homeomorphic and $d(R^n) = n$ for every positive integer n. It is a function defined on the class of topological spaces where d(X) is an integer or ∞ .The dimension functions taking topological spaces to the set {-1,0,1,…} .[1] studied the dimension functions ind,Ind,dim. Actually dimension functions $S-\mathrm{ind}X, S-\mathrm{Ind}X, S-\mathrm{dim}X$ were examined using S-open sets in [10], dimension functions $b-\mathrm{ind}X, b-\mathrm{Ind}X, b-\mathrm{dim}X$, were researched using b-open sets in [11], and dimension functions $f-\mathrm{ind}X, f-\mathrm{Ind}X, f-\mathrm{dim}X$, were studied using f-open sets in [5], [2] investigated the dimension functions $N-\mathrm{ind}X, N-\mathrm{Ind}X, N-\mathrm{dim}X$ utilizing N-open sets. We recall the definitions of dim, $loc\ dim$ from [1] , and then use $sf-$open sets to add the dimension functions $sf-dim$, $sf-\mathrm{loc\ dim}$ .Finally, certain connections between them are investigated, and some conclusions about these notions are established .

2.Preliminaries

In this section, we recall some of the basic definitions and theorems.

Definition (2.1): [7]

∗Corresponding author

Email addresses:

Communicated by 'sub etitor'

Let B be subset of a topological space X . If there is an open set D where $D \subseteq B \subseteq \overline{D}$, then B called semi-open (s-open) set in X

Remark(2.2): [6]

A set U is semi-open (s-open) set in X iff $U \subseteq \overline{U^{\circ}}$, equivalent $U = \overline{U^{\circ}}$, the complement semi-open set is semi-closed (s-closed) set.

If there is an closed set D where $D^{\circ} \subseteq U \subseteq D$, then U called semi-closed(s-closed) set in X .

Definition (2.3): [3]

Let D subset of a topological space X , define semi-closure (s-closure) of D as an the intersection of every semi-closed sets contained D in X and define semi-interior (s-interior) of D as an the union of semi open sub set of X contained D and are denoted by $\overline{D}^{s}$, $D^{\circ s}$respectfully

Proposition (2.4): [7]

let X be a topological space , if V is open subset of X and D is s-open set in X, then $D \cap V$ is an s-open set in X .

Proposition (2.5):[12]

If H is s-open in Y where Y is open subset of a topological space X,then H is s-open in X.

Proposition (2.6):[12]

If H is s-open of a topological space X and Y is open subset of X,then H is s-open in Y for $H \subseteq Y$.

Proposition (2.7):

If H is s-open set of a topological space X and W is open sub set of X , then $H \cap W$ is s-open in W

Proof:

By Proposition (2.4) and Proposition (2.6), then $H \cap W$ is s-open in W

Definition (2.8): [4]

If X be topological space and $E \subseteq X$ .Then E is called feebly open (f-open) set in X if there is an open set O like $O \subseteq E \subseteq \overline{O}^{s}$.

Remark (2.9): [6]

If D is subset of a topological space X , then D called feebly open set in X iff $D \subseteq \overline{D^{\circ}}^{\circ}$. Feebly closed(f-closed)isThe complement of feebly open set where $\overline{D^{\circ}} \subseteq D$.

Remark (2.10): [8]

Any open set is f-open set and any closed set is f-closed set

Definition(2.11): [9]

If B be subset of a topological space X ,then B called semi feebly open (sf −open) set in X ; if for any semi open set W where B⊆W we get $\overline{B}^{f} \subseteq W$. Semi feebly closed(sf-closed) is the counterpart of sf- open set that $W \subseteq B^{\circ f}$ where V semi closed set in X.

Remark (2.12): [9]

Any f-closed set and any closed set is sf-open set . however, the opposite is untrue.

Proposition (2.13): [9]

If $\{ A_\alpha : \alpha \in \Lambda \}$ is arbitrary collection of sf −open sets.Then $\bigcap_{\alpha\in\Lambda} A_\alpha$ is sf −open set.

Definition (2.14): [9]:

If the union of sf − open sets in topological space X is also sf − open sets, Then X is said to be satisfy conduct union

Definition (2.15): [9]

If X is a topological space and D ⊆X.Then sf-closure of D is the intersection of all sf-closed of X which Contains D and denoted by $\overline{D}^{sf}$, that means $\overline{D}^{sf} = \cap \{M : M \text{ is sf-closed in } X \text{ and } D \subseteq M\}$.

Definition (2.16): [9]

Let X be a topological space and D ⊆ X. Then sf-Interior of D is the union of all sf-open sets of X contained in D is and denoted by $D^{\circ sf}$, that means $D^{\circ sf} = \cup\{E : E \text{ is sf} - \text{open in } X \text{ and } E \subseteq D\}$.

Theorem (2.17):[12]

If Y is open subspace of a topological space X,then $\overline{D}^{fy} = \overline{D}^{f} \cap Y$ for each $D \subseteq Y$.

Definition (2.18): [9]

A topological space X is called sf-T_1-space if for any $x \neq y$ in X there is sf-open sets A and B where $x \in A, y \notin A$ and $y \in B,\ x \notin B$.

Proposition (2.19): [9]

Let X be a conduct union topological space, then $\{x\}$ is sf − closed set $\forall x \in X$ iff X is sf -T_1-space

Definition (2.20):[12]

A topological space X is called sf˝-regular space if any x in X and sf − closed subset M where $x \notin M$,there is disjoint sf-open sets A , B such that $x \in A, M \subseteq B$.

Proposition (2.21):[12]

Let X be a conduct union topological space, then X is sf˝-regular space iff $\forall$ x $\in$ X and $\forall$ sf-open set $A \ni x \in A$, $\exists$ sf-open set $V \ni x \in V \subseteq \overline{V}^{sf} \subseteq A$.

Definition (2.22): [9]

A topological space X is called sf`-normal space if for every disjoint sf-closed sets G_1 , G_2 there exists disjoint sf-open sets B_1 , B_2 such that $G_1 \subseteq B_1$, $G_2 \subseteq B_2$.

Proposition (2.23):[9]

A conduct union topological space X is sf`-normal space iff $\forall$ sf -closed set $E \subseteq X$, and $\forall$ sf -open set V in $X \ni E \subseteq V$, $\exists$ sf -open set $U \ni E \subseteq U \subseteq \overline{U}^{sf} \subseteq V$

Proposition (2.24):

Let Y open sub space of a topological space,then the subset D of Y is sf − open in Y if it is sf − open in X.

Proof:

Let D is sf-open in X.To show D is sf-open in Y ,by definition $D \subseteq H$ when H s −open in Y. Then H is s-open in X by proposition (2.5). Therefore $\overline{D}^{f} \subseteq H$, to prove $\overline{D}^{fy} \subseteq H$

$\overline{D}^{fy} = \overline{D}^{f} \cap Y$ by theorem (2.17)

$\subseteq \overline{D}^{f}$

$\subseteq H$, therefor D is sf − open in Y

Theorem(2.25):

Let Y is open and f-closed subset of a topological space X,then B is sf-open in X.If it is sf − open in Y

Proof:

To show B is sf-open in X, by definition $B \subseteq H$: H is s-open in X .Then $H \cap Y$ is s-open in Y by Proposition (2.7) . Since B is sf-open in Y and $B \subseteq H \cap Y$, then $\overline{B}^{f} \subseteq H \cap Y$

To prove $\overline{B}^f \subseteq H$

$\overline{B}^f = \overline{B \cap Y}^f$ (since B is sf-open in Y)

$\subseteq \overline{B}^f \cap \overline{Y}^f$

$= \overline{B}^f \cap Y$ (since Y is f-closed)

$= \overline{B}^{fy}$ by theorem (2.17)

$\subseteq H \cap Y$

$\subseteq H$

Therefor B is sf-open in X .

<u>Proposition (2.26):</u>

Let Y is open and f-closed subset of a topological space X,then $D \cap Y$ is sf-open in Y for any sf-open set D of X

Proof:

Since Y is f-closed in X, then Y is sf − open in X, then $D \cap Y$ is sf-open in X. Since $D \cap Y \subseteq Y$, then $D \cap Y$ is sf − open in Y by proposition (2.24)

<u>Theorem (2.27):</u>

Let Y is open and f-closed subset of a topological space X.then W is sf-open in Y iff $\exists$ V sf-open in X $\ni W = V \cap Y$.

Proof:

Let W is a sf-open in Y. Then W is sf-open in X by proposition (2.25) . Put W = V, we have $W = W \cap Y = V \cap Y$.

Conversely :

Let V be sf-open in X , then by proposition(2. 26) $V \cap Y$ is sf − open in Y .

therefor W is sf-open in Y.

<u>Definition(2.28)[1]:</u>

Let $\mathcal{H}$ be a family of subsets of a set X, we define an order of the family $\mathcal{H}$ as the greatest integer n where the family $\mathcal{H}$ have n+1 sets with a non-empty intersection ;we say the family $\mathcal{H}$ has order ∞ if there is no such integer. ord $\mathcal{H}$ denotes the order of a family $\mathcal{H}$.

Definition(2.29)[1]:

Let $\{G_\alpha\}_{\alpha\in\Lambda}$ be a cover of a set X,then $\{V_\gamma\}_{\gamma\in\Gamma}$ is said to be a refinement of $\{G_\lambda\}_{\lambda\in\Lambda}$ if it is a cover of X and for any $\gamma \in \Gamma$ there is $\alpha \in \Lambda$ such that $V_\gamma \subseteq G_\alpha$

Definition(2.30)[1]:

A reduction of the family $\{A_\lambda\}_{\lambda\in\Lambda}$ is any family $\{B_\lambda\}_{\lambda\in\Lambda}$ such that $B_\lambda \subset A_\lambda$ for each $\lambda \in \Lambda$ such that the union is the same. Then every reduction which covers X is a refinement

Definition (2.31))[1]:

Let $\{B_\alpha\}_{\alpha\in A}$ be a family of subsets of space X ,then$\{B_\alpha\}_{\alpha\in A}$ is called point finite if for any $x \in X$,the set $\{ \alpha \in A: x \in B_\alpha\}$ is finite .

3.The Main Results

Definition(3.1):

Let X be a topological space ,The sf −covering dimension ,sf − dimX, of X is the least integer n where each finite sf-open covering of X has an sf − open refinement of order $\leq$ n or is ∞ if no such integer exists .Thus $sf - \dim X = -1$ if and only if X is empty,and $sf - \dim X \leq n$ if each finite sf-open covering of X has sf − open refinement of order $\leq$ n. We have $sf - \dim X = n$ if it is true that $sf - \dim X \leq n$ but $sf - \dim X \leq n - 1$ is not true . Finally $sf - \dim X = \infty$ if for every integer n it is false that $sf - \dim X \leq n$.

Theorem (3.2):

Let X be a topological space with $sf - \dim X = 0$. Then X is $sf' -$ normal space.

Proof:

Let E_1 and E_2 be sf − closed sets of X with $E_1 \cap E_2 = \emptyset$. Then E= $\{X \setminus E_1 , X \setminus E_2\}$ is an sf − open covering of X, since $sf - \dim X = 0$, then there is U =$\{U_1 , U_2\}$ sf − open refinement of E which is covers X with order 0, hence $U_1 \cap U_2 = \emptyset$ and X= $U_1 \cup U_2$,then U_1 , U_2 are sf-open and $sf - closed$ sets,and $U_1 \subset X \setminus E_1$ and $U_2 \subset X \setminus E_2$.Thus

$E_1 \subseteq {U_1}^C = U_2 , E_2 \subseteq {U_2}^C = U_1$ and $U_1 \cap U_2 = \emptyset$,then X is sf −` normal space.

Theorem (3.3):

If X is a topological space and B is a clopn subset of X, then $sf - \dim B \leq sf - \dim X$.

Proof:

Let $sf-\dim X \leq n$, to prove $sf-\dim B \leq n$. Let $\{U_1, U_2, \dots, U_k\}$ be an finite $sf-$open covering of B, Then for any i, there is G_i is $sf-$open set in X such that $U_i = B \cap G_i$ by theorem(2.27).The finite $sf-$open covering $\{G_1, G_2, \dots, G_k, X \setminus B\}$ of X has an $sf-$open refinement $\mathcal{W}$ of order $\leq n$. Let $K = \{W \cap B : W \in \mathcal{W}\}$ where $W \cap B$ is an $sf-$open in B by proposition (2.26).There for K is an $sf-$open refinement of $\{U_1, U_2, \dots, U_k\}$ of order $\leq n$. There for $sf-\dim B \leq n$. ■

Proposition(3.4):

If X is $sf-T_1$-space and $sf-\dim X = 0$, then X is $sf''-$regular space .

Proof:

Let $x \in X$ and F $sf-$closed set of X with $x \notin F$.Then $x \in F^c$, where F^c is $sf-$open set in X .Since X is $sf-T_1$-space then $\{x\}$ is $sf-$closed by Proposition (2.19).Therefore $\{X-\{x\}, F^c\}$ be finite $sf-$open covering of X .Since $sf-\dim X = 0$,then $\{X-\{x\}, F^c\}$ has $sf-$open refinement$\{V,W\}$ which is cover of X with order zero. Then $V \subseteq X-\{x\}$, $W \subseteq F^c$ then $x \in W \subseteq F^c$ and $F \subseteq W^c = V$. Therefore there is V,W sf-open set where $F \subseteq V$ and $x \in W$ with $V \cap W = \emptyset$.Thus X is $sf''-$regular space

Proposition (3.5):

Let X be a conduct union topological space, then $sf-\dim X \leq n$ iff each finite $sf-$open cover of X can be reduced to an $sf-$open cover of order $\leq n$.

Proof:

If $\mathcal{G} = \{G_{\lambda_i}\}_{i=1}^{n}$ is finite $sf-$open covering ofX,since $sf-\dim X \leq n$ then $\mathcal{G}$ has $sf-$open refinement $\mathcal{W} = \{W_\alpha\}_{\alpha \in \Lambda}$ of order $\leq n$. If $W_\alpha \in \mathcal{W}$ then there exists $G_{\lambda_i} \in \mathcal{G}$ such that $W_\alpha \subset G_{\lambda_i}$ .Let $V_{\lambda_i} = \bigcup_{\alpha \in \Lambda} W_\alpha \subset G_{\lambda_i}$, then $\mathcal{V} = \{V_{\lambda_i}\}_{i=1}^{n}$ is $sf-$open reduction of $\mathcal{G}$ which is the X cover and it is order $\leq n$.

Conversely

Let $\mathcal{G} = \{G_{\lambda_i}\}_{i=1}^{n}$ be a finite $sf-$open covering of X then there exists a reduction $\mathcal{W} = \{W_\lambda\}_{\lambda \in \Lambda}$ which is $sf-$open covering of X . W is thus a refinement of $\mathcal{G}$ that is open of order $\leq n$. Hence $sf-\dim X \leq n$.

Theorem (3.6):

Let X be a conduct union topological space, then $sf-\dim X \leq n$ if, and only if, every $sf-$open cover of X with $n+2$ sets, has an $sf-$open reduction of $n+2$ sets with empty intersection.

Proof:

Suppose that $sf-\dim X \leq n$,then every finite $sf-$open covering of X has $sf-$open reduction of order $\leq n$ by Theorem (3.5).In particular if the number of elements of the cover is n+2 then the number of its reduction is also n+2 with order $\leq n$ that is the intersection is not empty for at most n+1 elements . Hence the intersection of all n+2 elements is empty

Conversely

Let $\mathcal{G}=\{G_1,G_2,\dots,G_k\}$ be a finite $sf-$open covering of X, suppose that the order of $\mathcal{G}>n$, then there exists $G_1,G_2,\dots,G_{n+2}\in\mathcal{G}$ such that $\bigcap_{i=1}^{n+2}G_i\neq\emptyset$ .Suppose that $G^*=G_{n+2}\cup\dots\cup G_k$ then $\{G_1,G_2,\dots,G_{n+1},G^*\}$ is $sf-$open covering of X with $n+2$ sets , which it has an $sf-$open reduction $\{V_1,V_2,\dots,V_{n+1},V^*\}$ with empty intersection. Then $\{V_1,V_2,\dots,V_{n+1},V^*\cap G_{n+2},\dots,V^*\cap G_k\}$ is $sf-$open reduction of $\mathcal{G}$ which cover X the non-empty intersection of the first $n+1$ sets may not be empty. But the intersection of all sets of the reduction is empty. Let $x\in X$ where $x\notin V_i, i=1,2,3,\dots n+1$ then $x\in V^*$, therefore $x\in\mathcal{G}_i$ for some $i=1,2,\dots,k$ hence $x\in V^*\cap\mathcal{G}_i$ for some $i=n+2,\dots,k$. By repeating the procedure finite we will get a reduction of $\mathcal{G}$ such that the intersection for each $n+2$ set is empty thus the order of reduction is $\leq n$,hence $sf-\dim X\leq n$. ■

Definition (3.7):

Let $\{A_\lambda\}_{\lambda\in\Lambda}$ be an $sf-$open cover of X, $\{A_\lambda\}_{\lambda\in\Lambda}$ is called $sf-$shrinkable if and only if there is an $sf-$open covering $\{B_\lambda\}_{\lambda\in\Lambda}$ where $B_\lambda\subseteq\overline{B_\lambda}^{sf}\subseteq A_\lambda$ for each $\lambda\in\Lambda$. In this case we say that $\{B_\lambda\}_{\lambda\in\Lambda}$ $sf-$shrinks $\{A_\lambda\}_{\lambda\in\Lambda}$, denoted by $B\ll_{sf}A$.

Theorem (3.8):

If X is an conduct union topological space ,the following claims are thus equivalent:

1) X is an $sf^{`}-$normal space.
2) every point-finite $sf-$open covering of X is $sf-$shrinkable .
3) every finite $sf-$open covering of X has $sf-$closed refinement.

Proof:1 → 2

Suppose that $\{G_\alpha\}_{\alpha\in A}$ be a point-finite $sf-$open covering of $sf-$normal space X and let A be well-ordered.We shall construct an $sf-$shrinkable of $\{G_\alpha\}_{\alpha\in A}$ by introduction of the transfinite. Let μ be an element of A and let for any $\alpha<\mu$ there are an $sf-$open set U_α where $\overline{U_\alpha}^{sf}\subset G_\alpha$ and for each $\gamma<\mu$, $\bigcup_{\alpha\leq\gamma}U_\lambda\cup\bigcup_{\alpha>\gamma}G_\alpha=X$.Let x be a point of X.Then since $\{G_\alpha\}_{\alpha\in A}$is a point finite there is a largest element Σ, say, of A where $x\in G_\Sigma$.If $\Sigma\geq\mu$ therefor $x\in\bigcup_{\alpha\geq\mu}G_\lambda$,whilst if $\Sigma<\mu$ then $x\in$

$\bigcup_{\alpha\leq\Sigma} U_\lambda \subset \bigcup_{\alpha<\mu} U_\alpha$. Hence $\bigcup_{\alpha<\mu} U_\alpha \cup \bigcup_{\alpha\geq\mu} G_\alpha = X$. Thus G_μ contains the complement of $\bigcup_{\alpha<\mu} U_\lambda \cup \bigcup_{\alpha>\mu} G_\alpha$.Since X is $sf^{`} -$ normal space,there exist an $sf -$ open set U_μ where:

$$X \setminus \left(\bigcup_{\alpha<\mu} U_\lambda \cup \bigcup_{\alpha\geq\mu} G_\alpha \right) \subset U_\mu \subset \overline{U_\mu}^{sf} \subset G_\mu$$

Thus $\overline{U_\mu}^{sf} \subset G_\mu$ and $\bigcup_{\alpha\leq\mu} U_\lambda \cup \bigcup_{\alpha>\mu} G_\lambda = X$.The construction of a sf-shrinking of $\{G_\alpha\}_{\alpha\in A}$ is completed by introduction of the transfinite.

$2 \longrightarrow 3$

Let $\{G_\alpha\}_{\alpha\in\Lambda}$ be a finite $sf -$ open covering of X then $\{G_\alpha\}_{\alpha\in\Lambda}$ is a point- finite $sf -$ open covering of X. Therefore there exists $\{U_\alpha\}_{\alpha\in\Lambda}$ an $sf -$ open family of covering of X such that $\overline{U_\alpha}^{sf} \subset G_\alpha$ for each $\alpha \in \Lambda$. Therefore $\left\{\overline{U_\alpha}^{sf}\right\}_{\alpha\in\Lambda}$ is a $sf -$ closed refinement of $\{G_\alpha\}_{\alpha\in\Lambda}$.

$3 \longrightarrow 1$

Let every finite $sf -$ open covering of X has a $sf -$ closed refinement and let M, N be disjoint $sf -$ closed sets of X. The covering $\{X \setminus M, X \setminus N\}$ of X has a $sf -$ closed refinement K.Let D be the union of the members of K disjoint from A and let H be the union of the members of K disjoint from N then D, H are $sf -$ closed sets and $D \cup H = X$. Then if $V = X \setminus D, W = X \setminus H$ then V, W are disjoint $sf -$ open sets $M \subseteq V, N \subseteq W$. Therefor X is $sf^{`} -$ normal space.

<u>Proposition (3.9):</u>

Let X is a conduct union topological space, the following claims are thus equivalent:

1) $sf - \dim X \leq n$.
2) For each finite $sf -$ open covering $\{A_1, A_2, \ldots, A_k\}$ of X there is an $sf -$ open covering $\{V_1, V_2, \ldots, V_k\}$ of order $\leq n$ such that $V_i \subset A_i$ for $i = 1,2, \ldots, k$.
3) If $\{A_1, A_2, \ldots, A_{n+2}\}$ is an $sf -$ open covering of X , there is an $sf -$ open covering $\{V_1, V_2, \ldots, V_{n+2}\}$ such that each $V_i \subset A_i$ and $\bigcap_{i=1}^{n+2} V_i = \emptyset$.

Proof: $1 \longrightarrow 2$

Suppose $sf - \dim X \leq n$ .The finite $sf -$ open covering $\{A_1, A_2, \ldots, A_k\}$ has an $sf -$ open refinement $\mathcal{W}$ of order $\leq n$. If $W \in \mathcal{W}$ then $W \subset A_i$ for some i. Let each w in $\mathcal{W}$ be associated with one of the sets A_i containing it and let V_i be the union of those members of $\mathcal{W}$ thus associated with A_i.Then V_i is $sf -$ open and $V_i \subset A_i$ and each point of X is in some member of $\mathcal{W}$ and hence in some V_i ,each point

$x \in X$ is in at most $n+1$ members of $\mathcal{W}$, each of which associated with a unique A_i, and hence x is in at most $n+1$ members of $\{V_i\}$. Thus $\{V_i\}$ is an sf − open covering of X of order $\leq n$.

2 ⟶ 1

Let $\mathcal{A} = \{A_1, A_2, \dots, A_k\}$ be a finite sf − open covering of X has sf − open covering $\mathcal{V} = \{V_1, V_2, \dots, V_k\}$ of order $\leq n$ such that $V_i \subset A_i$ for $i = 1,2,\dots,k$. Let $\mathcal{G} = \{G_1, G_2, \dots, G_k\}$ be sf − open covering of X such that $V_i = G_i$ for each i. Thus $\mathcal{G}$ be an sf − open refinement of X with order $\leq n$. Hence sf − dimX $\leq n$.

2 ⟶ 3

Let $\mathcal{A} = \{A_1, A_2, \dots, A_{n+2}\}$ be an sf − open covering of X then $\mathcal{U}$ has sf − open covering $\mathcal{V} = \{V_1, V_2, \dots, V_{n+2}\}$ of order $\leq n$ such that $V_i \subset A_i$. Since order$\mathcal{V}$ is $\leq n$ then $\bigcap_{i=1}^{n+2} V_i = \emptyset$.

3 ⟶ 2

Let X be a space satisfying (3) and let $\{A_1, A_2, \dots, A_k\}$ be a finite sf − open covering of X.We may assume that $k > n+1$. Let $G_i = A_i$ if $i \leq n+1$ and let $G_{n+2} = \bigcup_{i=n+2}^{k} A_i$. Then $\{G_1, G_2, \dots, G_{n+2}\}$ is an sf − open covering of X and hence by inference there exists an sf − open covering $\{H_1, H_2, \dots, H_{n+2}\}$ such that each $H_i \subset G_i$ and $\bigcap_{i=1}^{n+2} H_i = \emptyset$. Let $W_i = A_i$ if $n+1 \geq i$ and let $W_i = A_i \cap H_{n+2}$ if $n+1 < i$. Then $\mathcal{W} = \{W_1, W_2, \dots, W_{n+2}\}$ is an sf − open covering of X, each $W_i \subset A_i$ and $\bigcap_{i=1}^{n+2} W_i = \emptyset$. If there is a subset B of $\{1,2,\dots,k\}$ with elements such that $\bigcap_{i\in B} W_i \neq \emptyset$.Let's renumber the members of W to give a family $\mathcal{P} = \{P_1, P_2, \dots, P_k\}$ such that $\bigcap_{i=1}^{n+2} P_i \neq \emptyset$. By applying the above construction to $\mathcal{P}$, we obtain an sf − open covering $\mathcal{W}' = \{W_1', W_2', \dots, W_k'\}$ such that each $W_i' \subset P_i$ and $\bigcap_{i=1}^{n+2} W_i' = \emptyset$.Clearly if C is a subset of $\{1,2,\dots,k\}$ with $n+2$ elements where $\bigcap_{i\in C} P_i = \emptyset$ and $\bigcap_{i\in C} W_i' = \emptyset$. Thus by a finite number of repetitions of this process we obtain an sf − open covering $\{V_1, V_2, \dots, V_k\}$ of X of order $\leq n$ such that each $V_i \subset A_i$.

<u>Theorem (3.10):</u>

Let X be an sf` − normal space. Then the following statements are equivalents:

1. sf` − dimX $\leq n$.
2. Every finite sf − open covering is sf − shrinkable to an sf − open covering and the order of its sf − closure is $\leq n$.
3. Every finite sf − open covering can be reduced to sf − closed covering of order $\leq n$.
4. Every finite sf − open covering of $n+2$ sets can be reduced to a sf − closed with empty intersection.

Proof: 1 ⟶ 2

Suppose that $sf^{`}-\dim X \le n$ and let $\mathcal{U}=\{U_\lambda\}_{\lambda\in\Lambda}$ be a finite $sf-$open covering of X.Then $\mathcal{U}$ has an $sf-$open reduction $\mathcal{V}=\{V_\lambda\}_{\lambda\in\Lambda}$ of $sf-$open covering of order $\le n$ by Theorem(3.5).Since X is $sf^{`}-$normal space then $\mathcal{V}$ is $sf-$shrinkable to an $sf-$open covering $\mathcal{W}=\{W_\lambda\}_{\lambda\in\Lambda}$ hence $W_\lambda \subseteq \overline{W_\lambda}^{N} \subseteq V_\lambda$ therefore $\mathcal{W}$ is an $sf-$open covering of X and $sf-$shrinks $\mathcal{U}$. Since order$\mathcal{V}\le n$ then order$\overline{W}^{sf}\le n$ and $\bigcap_{i=1}^{n+2}\overline{W}^{sf} \subset \bigcap_{i=1}^{n+2} V_i=\emptyset$ hence order$\overline{W}^{sf}\le n$.

2 ⟶ 3

If U is a finite $sf-$open covering of X. Then U is $sf-$shrinkable to an $sf-$open covering W of X such that order$\overline{W}^{sf}\le n$. Then $\overline{W}^{sf}$ is $sf-$ closed reduction of U, order$\overline{W}^{sf}\le n$.

3 ⟶ 4

Let $\mathcal{U}=\{U_1,U_2,\dots,U_{n+2}\}$is afinite $sf-$open covering of X.Then $\mathcal{U}$ has $sf-$ closed reduction $\mathcal{F}=\{F_1,F_2,\dots,F_{n+2}\}$ of order $\le n$ which covers X.Since order$\mathcal{F}\le n$ then $\bigcap_{i=1}^{n+2}F_i=\emptyset$.

4 ⟶ 1

Let $\mathcal{U}=\{U_1,U_2,\dots,U_{n+2}\}$is afinite $sf-$open covering of X also let $\mathcal{F}=\{F_1,F_2,\dots,F_{n+2}\}$ be $sf-$ closed reduction $\mathcal{U}$ with empty intersection, for each i.

Let $G_i = X\setminus F_i$ then $\mathcal{G}=\{G_1,G_2,\dots,G_{n+2}\}$be $sf-$open covering of X.Since X is $sf^{`}-$normal space then $\mathcal{G}$ is $sf-$shrinkable to $sf-$open covering $\{V_1,V_2,\dots,V_{n+2}\}$ hence $V_i\subseteq \overline{V}_i i^{N}\subseteq G_i$. Let $W_i = U_i\setminus \overline{V}^{sf}\subset U_i$ for some i where $\mathcal{W}=\{W_1,W_2,\dots,W_{n+2}\}$ is $sf-$open reduction of U which covers X , then

$$\bigcap_{i=1}^{n+2}W_i=\bigcap_{i=1}^{n+2}\left(U_i\setminus\overline{V}_i^{sf}\right)=\bigcap_{i=1}^{n+2}\left(U_i\cap\overline{V}_i^{sf^C}\right)\subset\bigcap_{i=1}^{n+2}\left(\overline{V}_i^{sf^C}\right)\subset\bigcap_{i=1}^{n+2}\left(V_i^C\right)\subset\left(\bigcup_{i=1}^{n+2}V_i\right)^C$$
$$=X^C=\emptyset$$

Therefore by Theorem (3.6) then $sf-\dim X\le n$. ■

<u>Definition(3.11):</u>

Let X be a topological space ,then The $sf-$ local dimension, $sf-\mathrm{loc}\dim X$, of a space X has the following definition. If X is empty then $sf-\mathrm{loc}\dim X=-1$, otherwise $sf-\mathrm{loc}\dim X$ is the least integer n where for each point x of X there exists some $sf-$open set U containing x where $sf-\dim\overline{U}\le n$, or if there is no such integer then $sf-\mathrm{loc}\dim X=\infty$.

<u>Theorem(3.12):</u>

If(X,τ) be a topological space , thensf $-\mathrm{locdim}X\le sf-\dim X$

Proof:

Suppose that $sf-\dim X\le n$ and $x\in X$

hence X sf − open set X containing x,

then $sf-\dim\overline{X} = sf-\dim X \le n$

Proposition (3.13):

Let A be an closed set of a space X, then $sf-locdimA \le sf-locdimX$.

Proof:

Suppose that $sf-locdimX \le n$ and let $x\in A$,then there exists an sf − open set U of X where $x \in U$ and $sf-\dim\overline{U} \le n$. Then $U \cap A$ is an sf − open set in A such that $x \in U \cap A$ by Proposition (2.26).And the closure of $U \cap A$ in A is closed set of $\overline{U}$,therefor has sf-dimension $\le n$ by Proposition (3.3).Then $sf-locdimA \le n$, thus $sf-locdimA \le sf-locdimX$

Proposition (3.14):

Let Y be an open set of locally indiscrete sf″ − regular space X then, $sf-locdimY \le sf-locdimX$.

Proof:

Suppose that $sf-locdimX \le n$ and let $x\in Y$, then there is an s f − open set U of X such that $x \in U$ and $sf-\dim\overline{U} \le n$. Then $U \cap Y$ is an sf − open set in Y such that $x \in U \cap Y$ by Propositio (2.26). Since X is an sf″ − regular space, then there exists an sf − open set G such that $x \in G \subset \overline{G}^{sf} \subset U \cap Y$. Then G is an sf-open in Y , and $\overline{G}$ is closure of G in Y.Since $\overline{G}$ is closed subset of $\overline{U}$, it follows that $sf-\dim\overline{G} \le n$ by Proposition (3.3). Hence $sf-locdimY \le sf-locdimX$.

Theorem(3.15):

Let(X,τ) be a topological space , then $sf-locdimX \le n$ iff every sf-open covering of X has sf-open refinement $\{V_\lambda : \lambda \in \Lambda\}$ such that $sf-\dim\overline{V_\lambda} \le n$. $\forall \lambda \in \Lambda$

Proof: $(\rightarrow)$

let $sf-locdimX \le n$ and $\{U_\lambda : \lambda \in \Lambda\}$ be sf-open cover of X . Let $x \in X$, then

$x \in U_\lambda$ for some $\lambda \in \Lambda$, then

either $\exists\ V_\alpha$ sf-open such that $x \in V_\lambda \subset U_\lambda$

hence V_λ sf-open set such that $x \in V_\lambda \subset U_\lambda$ and $sf-\dim\overline{V_\lambda} \le n$

therefore the prove is finish .

Otherwise if $\exists\ W_\lambda$ sf-open set such that $x \in W_\lambda$ and $sf-\dim\overline{W_\lambda} \le n$

let $V_\lambda = U_\lambda \cap W_\lambda$, then $x \in V_\lambda \subset U_\lambda$.

since $\overline{V_\lambda}$ is closed in $\overline{W_\lambda}$,then $sf-\dim \overline{V_\lambda} \leq n$ by Proposition (3.3).

(←)

Let $x \in X$ and $\{U_\lambda : \lambda \in \Lambda\}$ is sf-open covering of X , then x∈U for some $\lambda \in \Lambda$

Then U has sf-open refinement $\{V_\lambda : \lambda \in \Lambda\}$

Hence V_λ is sf-open set such that $x \in V_\lambda$ some $\lambda \in \Lambda$ and $sf-\dim \overline{V} \leq n$

There for $sf-\text{locdim}X \leq n$

References

[1] A.P.Pears "On Dimension Theory of General Spaces" Cambridge university press, (1975).

[2] E. R. Ali "On Dimension Theory by Using N-Open Sets", M.Sc. thesis university of Al-Qadissiya college of mathematics and computer science, (2011).

[3] G, S.Grossly. and Hildebrand S. K. " semi closed set and semi continuity in Topological space "Txas J-sci (1971).

[4] Maheshwari S.N &Tapiu ."Feebly open sets" . Ann University . Timisaras, (1978).

[5] N.H.Hajee "On Dimension Theory by Using Feebly Open Set", M.Sc. thesis university of Al-Qadissiya college of mathematics and computer science, (2011).

[6] N.K. Ahmed "On some application of special sub sets of topological spaces "M.sc thesis, University of Salahaddin , College of science (1990).

[7] N.Levine "Semi open set and semi Continuity in topological space" . Amer .Math .Monthly (1963).

[8] N.M . Al Tabatbai "On New Types of Weekly open sets ", M .Sc thesis University of Baghdad (2004).

[9] R. A. H. Al-Abdulla and O. R.M. Al-Gharani "On paracompactness Via sf-open sets " University of Al-Qadissiya , college. of Mathematics and . computer science , (2020) .

[10] R.A.H.AL-Abdulla "On Dimension Theory", M.Sc. thesis university of Baghdad, college of science, (1992).

[11] S.K.Gaber "On b-Dimension Theory" M.Sc. thesis university of Al-Qadissiya , college of mathematics and computer science, (2010).

[12] Z.N.K. Hussain " On Countable Chain Condition by using sf-Open Sets " , M .Sc thesis University of Al-Qadissiya , college of science , (2021) .