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New Results on Soft Spectral in Soft Banach Algebra

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Abstract:
In this paper, concepts of soft character, soft division algebra, soft ideal, soft maximal ideals are introduced .Soft Spectral Radius Formula are introduced and proved, some properties of Soft gelfand algebra are proved.

Keywords: soft spectrum, soft spectral radius, soft ideal, soft maximal ideal, soft character, soft regular.

Mathematics Subject Classification: 46S40.

1. Introduction

The soft set theory was first introduced by Russian researcher Molodtsov [1]. Maji, K., Biswas, R. and Roy, R.in [12] introduced many of new concepts of this as an inclusion relation between the soft set, Cagman N. and Enginoglu S. in [3] introduced a new study of a Soft set theory and uni-int decision making European J. Oper. Researcher, The application of the soft sets provided a natural framework for generalizing many concepts of topology which is called the soft topological space as initiated by [4] and [11].

Banach Algebra is an important field of functional analysis; therefore Banach algebra structure of soft background was introduced by Thakur R. and Smanta S. in [9]. After that Petroudi S., Sadati S. and Yaghobi A., in [10] obtained a series of new results on Soft banach algebra.

In this paper, concepts of soft character, soft division algebra, soft ideal, soft maximal ideals are introduced .Soft Spectral Radius Formula are introduced and proved, some properties of Soft gelfand algebra are proved.

2. Preliminaries

Definition (2.1) [1]:
Let X be a universe set and E be a set of parameters, $P(X)$ the power set of X and $A \subseteq E$. A pair $(F,A)$is called soft set over X with respect to A and F is a mapping given by $F: A \to P(X)$,
$(F,A) = \{F(e) \in P(X): e \in A\}$.

Definition (2.2) [2]:
$(\boldsymbol{i})$ A soft set (F,A) over X is called null soft set, denoted by $\widetilde{\emptyset}_A$, if for all $e \in A$, we have $F(e) = \emptyset$.
$(\boldsymbol{ii})$ A soft set (F,A) over X is called absolute soft set and it's denoted by $\tilde{X}_A$, if for all $e \in A$, we have $F(e) = X$.

Definition (2.3) [2, 3]:
Let (F,A) and (G,B) be two soft sets over$X$, we say that (F,A) is a soft subset of (G,B) and denoted by $(F,A) \tilde{\subseteq} (G,B)$, if:
$(\boldsymbol{i})$ $A \subseteq B$.
$(\boldsymbol{ii})$ $F(e) \subseteq G(e)$, $\forall e \in A$.
Also, we say that (F,A) and (G,B) are soft equal is denoted by $(F,A) = (G,B)$, if $(F,A) \tilde{\subseteq} (G,B)$ and $(G,B) \tilde{\subseteq} (F,A)$.
It is clear that:
$(\boldsymbol{a})$ $\widetilde{\emptyset}_A$ is a soft subset of any soft set (F,A).
$(\boldsymbol{b})$ Any soft set (F,A) is a soft subset of $\tilde{X}_A$.

Definition (2.4) [2, 4]:
$(\boldsymbol{i})$ The intersection of two soft sets (F,A) and (G,B) over a universe X is the soft set $(H,C) = (F,A)\tilde{\cap}(G,B)$, where $C = A \cap B$ and for all $e \in C$, write $(H,C) = (F,A)\tilde{\cap}(G,B)$ such that
$H(e) = F(e) \cap G(e)$.
$(\boldsymbol{ii})$ The union of two soft sets (F,A) and (G,B) over X is the soft set (G,B) , where $C = A \cup B$ and $\forall\, e \in A$ we write
$(H,C) = (F,A)\tilde{\cup}(G,B)$ Such that

$$H(e) = \begin{cases} F(e) & , \text{if } e \in A \backslash B \\ G(e) & , \text{if } e \in B \backslash A \\ F(e) \cup G(e) & , \text{if } e \in A \cap B \end{cases} .$$

$(\boldsymbol{iii})$ The difference of two soft sets (F,A) and (G,A) over X , denoted by
$(H,C) = (F,A)\tilde{\backslash}(G,A)$ is defined as
$H(e) = F(e) \backslash G(e)$, for all $e \in A$.

Definition (2.5) [2, 5]:
The soft complement of a soft set (F,A) over a universe X is denoted by $(F,A)^c$ and it is defined by $(F,A)^c = (F^c, A)$, where F^ca mapping is given by $F^c: A \to P(X)$,
$F^c(e) = X \backslash F(e)$, for all $e \in A$.
i.e. $(F,A)^c = \{(e, X\backslash F(e)): \forall\, e \in A\}$.
It is clear that $\widetilde{\emptyset}_A{}^c = \tilde{X}_A$; $\tilde{X}_A{}^c = \widetilde{\emptyset}_A$.

Definition (2.6) [6]:
Let X be non-empty set and E be non- empty parameter set. Then a function $\varepsilon: E \to X$ is said to be a soft element of X. A soft element ε of X is said to belong to soft set A of X, which is denoted by $\varepsilon \tilde{\in} A$, if $\varepsilon(e) \in A(e)$,
$\forall e \in E$. Thus for a soft set A of X with respect to the index set E, we have
$A(e) = \{\varepsilon(e): \varepsilon(e) \tilde{\in} A\}, e \in E\}$.

Definition (2.7) [7]:
Let $\mathbb{C}$ be the set of all complex numbers and $\wp(\mathbb{C})$ be the collection of all non-empty bounded subset of $\mathbb{C}$ and E be a set of parameters.
Then a mapping $F: E \to \wp(\mathbb{C})$ is called a soft complex set.
It is denoted by (F,E). If in particular (F,E) is a singleton soft set then after identifying (F,E) with the corresponding soft element, it well be called a soft complex number. The soft complex numbers is denoted by $\mathbb{C}(E)$.

Definition (2.8) [7]:
The inverse of any soft complex number $\tilde{r}$, denoted by $\tilde{r}^{-1}$ and defined as
$\tilde{r}^{-1}(e) = (\tilde{r}(e))^{-1}$, for all $e \in A$.

Definition (2.9) [8]:
Let $\tilde{X}$ be the absolute soft vector space i.e., $\tilde{X}(\lambda) = X, \forall \lambda \in E$. Then a mapping
$\|.\| : SE(\tilde{X}) \to R(E)^*$ is said to be soft norm on the soft vector space $\tilde{X}$ if $\|.\|$ satisfies the following conditions:
(***i***) $\|\tilde{x}^e\| \geqq \bar{0}$, for all $\tilde{x}^e \tilde{\in} \tilde{X}$.
(***ii***) $\|\tilde{x}^e\| = \bar{0}$ if and only if $\tilde{x}^e = \tilde{\Theta}^e$.
(***iii***) $\|\tilde{\lambda}\tilde{x}^e\| \geqq |\tilde{\lambda}|\|\tilde{x}^e\|$, for all $\tilde{x}^e \tilde{\in} \tilde{X}$ and for every soft scalar $\tilde{\lambda}$.
(***iv***) $\|\tilde{x}^e + \tilde{y}^e\| \leqq \|\tilde{x}^e\| + \|\tilde{y}^e\|$, for all $\tilde{x}^e, \tilde{y}^e \tilde{\in} \tilde{X}$.
The soft vector space $\tilde{X}$ with a soft norm $\|.\|$ on $\tilde{X}$ is said to be a soft normed linear space and is denoted by $(\tilde{X}, \|.\|)$.

Definition (2.10) [٩]:
A sequence of soft elements $\{\tilde{x}^e{}_n\}$ in a soft normed linear space $(\tilde{X}, \|.\|)$. is said to be convergent and converges to a soft element $\tilde{x}^e$ if $\|\tilde{x}^e{}_n - \tilde{x}^e\| \to \bar{0}$ as $n \to \infty$ This means for every $\tilde{\epsilon} \geqq \bar{0}$, chosen arbitrary, there exists a natural number $= N(\epsilon)$, such that $\bar{0} \leqq \|\tilde{x}^e{}_n - \tilde{x}^e\| \leqq \tilde{\epsilon}$, whenever $n > N$. We denote this by $\tilde{x}^e{}_n \to \tilde{x}^e$ as $n \to \infty$ or $\lim_{n\to\infty} \tilde{x}^e{}_n = \tilde{x}^e$. $\tilde{x}^e$ is said to be the limit of the sequence $\tilde{x}^e{}_n$ as $n \to \infty$.

Definition (2.11) [9]:
Let $T: SE(\tilde{X}) \to SE(\tilde{Y})$ be an operator. Then T is said to be soft linear if
$T(\tilde{\lambda}_1\tilde{x}^e{}_1 + \tilde{\lambda}_2\tilde{x}^e{}_2) = \tilde{\lambda}_1 T(\tilde{x}^e{}_1) + \tilde{\lambda}_2 T(\tilde{x}^e{}_2)$
for every soft scalar $\tilde{\lambda}$ and $\tilde{x}^e{}_1, \tilde{x}^e{}_2 \tilde{\in} \tilde{X}$.

Definition (2.12) [11]:
Let (F, E), (G, E) are soft sets in $S(X)_E$, a soft mapping
f: (F,E) ⇾ (G,E) is said to be (soft injective)if each soft element in (F,E) is related to a different in (G,E).
More formally f $(\tilde{x}_1)$ = f $(\tilde{x}_2)$ implies $\tilde{x}_1 = \tilde{x}_2$.

Definition (2.13) [11]:
Let (F, E), (G, E) are soft sets in $S(X)_E$, a soft mapping
f : (F,E) ⇾ (G,E) is said to be (soft surjective) if for every soft element $\tilde{y}$ in (G,E), there is a soft element $\tilde{x}$ in (F,E) such that f$(\tilde{x}) = \tilde{y}$,
[i.e. when the soft range equals to the soft image f((F,E)) = (G,E)] .

Definition (2.14 [9]:
Let V be algebra over a scalar field $\mathbb{R}$ and let E be the parameter set and F_E be a soft set over V. Now, F_E is called soft algebra (in short SA) of V over $K(E)$ if $F(e)$ is a sub algebra of V for all $e \in E$. It is very easy to see that in SA. the soft elements satisfy the properties:
(***i***) $(\tilde{x}^e\tilde{y}^e)\tilde{z}^e = \tilde{x}^e(\tilde{y}^e\tilde{z}^e)$.
(***ii***) $\tilde{x}^e(\tilde{y}^e \tilde{\mp} \tilde{z}^e) = \tilde{x}^e\tilde{y}^e \tilde{\mp} \tilde{x}^e\tilde{z}^e$; $(\tilde{y}^e \tilde{\mp} \tilde{z}^e)\tilde{x}^e = \tilde{y}^e\tilde{x}^e \tilde{\mp} \tilde{z}^e\tilde{x}^e$.
(***iii***) $\tilde{\alpha}(\tilde{x}^e\tilde{y}^e) = (\tilde{\alpha}\tilde{x}^e)\tilde{y}^e = \tilde{x}^e(\tilde{\alpha}\tilde{y}^e)$, where for all $\tilde{x}^e, \tilde{y}^e, \tilde{z}^e \tilde{\in} F_E$ and for any soft scalar $\tilde{\alpha}$. If F_E is also soft Banach space w.r.t. a soft norm that satisfies the inequality $\|\tilde{x}^e\tilde{y}^e\| \leq \|\tilde{x}^e\|\|\tilde{y}^e\|$ and F_E is contains the unitary element $\tilde{u}^e$ such that $\tilde{x}^e\tilde{u}^e = \tilde{u}^e\tilde{x}^e = \tilde{x}^e$ with $\|\tilde{u}^e\| = \bar{1}$, then is called soft Banach algebra (in short SBA).

Definition (2.15) [10]:
Let $\mathfrak{A}$ be a (SBA) with $\tilde{u}^e$ and $\tilde{x}^e \tilde{\in} \mathfrak{A}$. Then the soft set $\delta(\tilde{x}^e) = \{\tilde{\lambda} \tilde{\in} \mathbb{R}(E) : (\tilde{x}^e \tilde{-} \tilde{\lambda}\tilde{u}^e)$ is soft singular$\}$ is called soft spectrum of $\tilde{x}^e \tilde{\in} \mathfrak{A}$.

Definition (2.16): [10]
We denote the soft spectral radius of $\tilde{x}^e$ by $r(\tilde{x}^e)$ and define it by
$r(\tilde{x}^e) = \text{Sup}\{|\tilde{\alpha}| : \tilde{\alpha} \tilde{\in} \delta(\tilde{x}^e)\}$.

3. Results

Theorem (3.1):
Let $\mathfrak{A}$ be (SBA) with identity element $(\bar{1})$ and $\tilde{x}^e \tilde{\in} \mathfrak{A}$, then $\delta(\tilde{x}^{e^n}) = (\delta(\tilde{x}^e))^n$.

Proof:
Let $\tilde{\alpha} \tilde{\in} \delta(\tilde{x}^e)$, then$(\tilde{x}^e - \tilde{\alpha}.\bar{1}) \tilde{\in} S_E$
Consider
$\tilde{x}^{e^n} - \tilde{\alpha}. = (\tilde{x}^e - \tilde{\alpha}_1.\bar{1}), \ldots\ldots, (\tilde{x}^e - \tilde{\alpha}_n.\bar{1})$
Where $\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots\ldots, \tilde{\alpha}_n$ are roots of $\tilde{x}^{e^n} - \tilde{\alpha}.\bar{1}$
Let $\tilde{\alpha} \tilde{\in} \delta(\tilde{x}^{e^n})$, i.e. $\tilde{x}^{e^n} - \tilde{\alpha}.\bar{1} \tilde{\in} S_E$
then there is at least one i such that
$(\tilde{x}^e - \tilde{\alpha}_i.\bar{1}) \tilde{\in} S_E$, hence $\tilde{\alpha}_i \tilde{\in} \delta(\tilde{x}^e)$
So $\tilde{\alpha}_i{}^n \tilde{\in} (\delta(\tilde{x}^e))^n$, then $\tilde{\alpha} \tilde{\in} (\delta(\tilde{x}^e))^n$
Hence $\delta(\tilde{x}^{e^n}) \tilde{\subseteq} (\delta(\tilde{x}^e))^n$ ……… (1)
Let $\tilde{\alpha}^n \tilde{\in} (\delta(\tilde{x}^e))^n$, where $\tilde{\alpha} \tilde{\in} \delta(\tilde{x}^e)$.
Note that $(\tilde{x}^e - \tilde{\alpha}.\bar{1})$ is soft factor of
$\tilde{x}^{e^n} - \tilde{\alpha}^n.\bar{1}$, hence$(\tilde{x}^e - \tilde{\alpha}.\bar{1}) \tilde{\in} S_E$
$\tilde{x}^{e^n} - \tilde{\alpha}^n.\bar{1} \tilde{\in} S_E$ So $\tilde{\alpha}^n \tilde{\in} \delta(\tilde{x}^{e^n})$.
Then $(\delta(\tilde{x}^e))^n \tilde{\subseteq} \delta(\tilde{x}^{e^n})$ ……… (2)
by (1) and (2),we have $\delta(\tilde{x}^{e^n}) = (\delta(\tilde{x}^e))^n$.

Theorem (3.2):
Let $\mathfrak{A}$ be (SBA) with identity element $(\bar{1})$ and $\tilde{x}^e \tilde{\in} \mathfrak{A}$, then $r(\tilde{x}^{e^n}) = (r(\tilde{x}^e))^n$.

Proof:
Since $r(\tilde{x}^e) = \text{Sup}\{|\tilde{\alpha}| : \tilde{\alpha} \tilde{\in} \delta(\tilde{x}^e)\}$.
Then $(r(\tilde{x}^e))^n = (\text{Sup}\{|\tilde{\alpha}| : \tilde{\alpha} \tilde{\in} \delta(\tilde{x}^e)\})^n$.
Hence $(r(\tilde{x}^e))^n = \mathit{Sup}\{|\tilde{\alpha}|^n : \tilde{\alpha} \tilde{\in} \delta(\tilde{x}^e)\}$.
Now
$r(\tilde{x}^{e^n}) = \text{Sup}\{|\tilde{\alpha}| : \tilde{\alpha} \tilde{\in} \delta(\tilde{x}^{e^n})\}$
$= \text{Sup}\{|\tilde{\alpha}| : \tilde{\alpha} \tilde{\in} (\delta(\tilde{x}^e))^n\}$
$= \text{Sup}\{|\tilde{\beta}^n| : \tilde{\beta} \tilde{\in} \delta(\tilde{x}^e)\}$, for some $\tilde{\beta} \tilde{\in} \delta(\tilde{x}^e)$
$= (\text{Sup}\{|\tilde{\beta}| : \tilde{\beta} \tilde{\in} \delta(\tilde{x}^e)\})^n$
$= (r(\tilde{x}^e))^n$, as $|\tilde{\beta}^n| = |\tilde{\beta}|^n$, $\tilde{\beta} \tilde{\in} \mathbb{C}(E)$.

Theorem (3.3) [10]:
Let $\mathfrak{A}$ be a (SBA) with $\tilde{u}^e$.
Then every $\tilde{x}^e \tilde{\in} \mathfrak{A}$ for $\|\tilde{u}^e \tilde{-} \tilde{x}^e\| \leqq \bar{1}$ is soft regular and $\tilde{x}^{e^{-1}} = \sum_{n=0}^{\infty} (\tilde{u}^e \tilde{-} \tilde{x}^e)^n$.

Theorem (3.4): (Soft Spectral Radius Formula)
Let $\mathfrak{A}$ be (SBA) with soft identity element and $\tilde{x}^e \tilde{\in} \mathfrak{A}$, then $\lim_{n\to\infty}\|\tilde{x}^{e^n}\|^{\frac{1}{n}}$ exists and $r(\tilde{x}^e) = \lim_{n\to\infty}\|\tilde{x}^{e^n}\|^{\frac{1}{n}}$.

Proof:

Let $g(\tilde{x}^e) = \lim_{n\to\infty}\|\tilde{x}^{e^n}\|^{\frac{1}{n}}$ .For $\epsilon > 0$ then there is k Such that

$\|\tilde{x}^{e^k}\|^{\frac{1}{k}} \lesssim g(\tilde{x}^e) \tilde{+} \epsilon \ldots\ldots\ldots\ldots (*)$

for any $n \in \mathbb{N}$ we have

$n = \alpha k + \beta, 0 \le \beta < k, k < n, \alpha, \beta \in \mathbb{Z}^+$

$\bar{1} = \alpha\frac{k}{n} + \frac{\beta}{n}$, as $k < n \Rightarrow \frac{k}{n} \to 0, \frac{\beta}{n} \to 0$ as $n \to \infty \Rightarrow k\frac{\alpha}{n} \to \bar{1} \Rightarrow \frac{\alpha}{n} \to \frac{\bar{1}}{k}$.

Now

$\|\tilde{x}^{e^n}\|^{\frac{1}{n}} = \|(\tilde{x}^{e^k})^{\alpha} \tilde{\cdot}\, \tilde{x}^{e^\beta}\|^{\frac{1}{n}}$

$\lesssim \|(\tilde{x}^{e^k})^{\alpha}\|^{\frac{1}{n}} \tilde{\cdot}\, \|\tilde{x}^{e^\beta}\|^{\frac{1}{n}} \lesssim \|\tilde{x}^{e^k}\|^{\frac{\alpha}{n}} \tilde{\cdot}\, \|\tilde{x}^e\|^{\frac{\beta}{n}}$

Since $\|\tilde{x}^{e^k}\|^{\frac{\alpha}{n}} \to \|\tilde{x}^{e^k}\|^{\frac{1}{k}}$ and $\|\tilde{x}^e\|^{\frac{\beta}{n}} \to \bar{1}$

$\Rightarrow \|\tilde{x}^{e^n}\|^{\frac{1}{n}} \lesssim \|\tilde{x}^{e^k}\|^{\frac{1}{k}}$

$\Rightarrow \lim_{n\to\infty}\|\tilde{x}^{e^n}\|^{\frac{1}{n}} \lesssim \|\tilde{x}^{e^k}\|^{\frac{1}{k}}$

Then from $(*)$,we have

$\|\tilde{x}^{e^n}\|^{\frac{1}{n}} \lesssim g(\tilde{x}^e) \tilde{+} \epsilon$,$\forall$ n

Note that $g(\tilde{x}^e) \lesssim \|\tilde{x}^{e^n}\|^{\frac{1}{n}}$, $\forall$ n

$\Rightarrow g(\tilde{x}^e) \lesssim \|\tilde{x}^{e^n}\|^{\frac{1}{n}} \lesssim g(\tilde{x}^e) \tilde{+} \epsilon$

$\Rightarrow \lim_{n\to\infty}\|\tilde{x}^{e^n}\|^{\frac{1}{n}} = g(\tilde{x}^e)$

Hence the limit exists.

Now to show that $r(\tilde{x}^e) = \lim_{n\to\infty}\|\tilde{x}^{e^n}\|^{\frac{1}{n}}$, since $r(\tilde{x}^e) = \text{Sup}\{|\tilde{\alpha}|: \tilde{\alpha} \tilde{\in} \delta(\tilde{x}^e)\}$ and $|\tilde{\alpha}| \lesssim \|\tilde{x}^e\|$, for $\tilde{\alpha} \tilde{\in} \delta(\tilde{x}^e) \Rightarrow |\tilde{\alpha}^n| \lesssim \|\tilde{x}^{e^n}\|$

$\Rightarrow |\tilde{\alpha}|^n \lesssim \|\tilde{x}^{e^n}\|$ by $(\delta(\tilde{x}^{e^n})) = (\delta(\tilde{x}^e))^n$

$\Rightarrow |\tilde{\alpha}| \lesssim \|\tilde{x}^{e^n}\|^{\frac{1}{n}}$, $\forall$ n

$\therefore \text{Sup}\{|\tilde{\alpha}|: \tilde{\alpha} \tilde{\in} \delta(\tilde{x}^e)\} \lesssim \|\tilde{x}^{e^n}\|^{\frac{1}{n}}$, $\forall$ n

$\Rightarrow r(\tilde{x}^e) \lesssim g(\tilde{x}^e)$............(1).

Let $\tilde{\alpha} \tilde{\in} \mathbb{C}(E)$ with $|\tilde{\alpha}| \gtrsim \|\tilde{x}^e\|$.

i.e $\tilde{\alpha} \tilde{\notin} \delta(\tilde{x}^e) \Rightarrow \left\|\frac{\tilde{x}^e}{\tilde{\alpha}}\right\| \lesssim \bar{1}$

$\Rightarrow \left\|\tilde{u}^e \tilde{-} (\tilde{u}^e \tilde{-} \frac{\tilde{x}^e}{\tilde{\alpha}})\right\| \lesssim \bar{1}$ and by theorem (3.3), we get $(\tilde{u}^e \tilde{-} \frac{\tilde{x}^e}{\tilde{\alpha}})$ is soft regular and $\left(\tilde{u}^e \tilde{-} \frac{\tilde{x}^e}{\tilde{\alpha}}\right)^{-1} = \sum_{n=0}^{\infty}\left(\frac{\tilde{x}^e}{\tilde{\alpha}}\right)^n$

$\Rightarrow \left(\frac{\tilde{\alpha} \tilde{-} \tilde{x}^e}{\tilde{\alpha}}\right)^{-1} = \sum_{n=0}^{\infty}\left(\frac{\tilde{x}^e}{\tilde{\alpha}}\right)^n$

$\Rightarrow (\tilde{x}^e \tilde{-} \tilde{\alpha})^{-1} = -\sum_{n=0}^{\infty}\frac{\tilde{x}^{e^n}}{\tilde{\alpha}^{n+1}}$ ....$(*)$

$= \tilde{x}^e(\tilde{\alpha})$. This series $(*)$ is convergent because it is absolutely convergent.

Now, let $\mathbb{f} \tilde{\in} A^* =$ set of all soft bounded linear functional

$\Rightarrow \mathbb{f}(\tilde{x}^e(\tilde{\alpha})) = \sum_{n=0}^{\infty}\frac{\mathbb{f}(\tilde{x}^{e^n})}{\tilde{\alpha}^{n+1}}$ and again $\sum_{n=0}^{\infty}\frac{\mathbb{f}(\tilde{x}^{e^n})}{\tilde{\alpha}^{n+1}}$ is also soft convergent for $|\tilde{\alpha}| \gtrsim \|\tilde{x}^e\| \gtrsim r(\tilde{x}^e) \Rightarrow \sum_{n=0}^{\infty}\frac{|\mathbb{f}(\tilde{x}^{e^n})|}{\tilde{\alpha}^{n+1}} \lesssim \sum_{n=0}^{\infty}\frac{|\mathbb{f}(\tilde{x}^e)|^n}{\tilde{\alpha}^{n+1}}$ and since soft bounded i.e. $|\mathbb{f}(\tilde{x}^{e^n})| \lesssim \tilde{B} \tilde{\cdot} \|\tilde{x}^e\| \Rightarrow \sum_{n=0}^{\infty}\frac{|\mathbb{f}(\tilde{x}^{e^n})|}{\tilde{\alpha}^{n+1}} \lesssim \sum_{n=0}^{\infty}\frac{\tilde{B}^n \tilde{\cdot} \|\tilde{x}^e\|^n}{\tilde{\alpha}^{n+1}}$

We know that for all soft bounded linear functional

$\Rightarrow \mathbb{f}(\tilde{x}^e(\tilde{\alpha}))$ is analytic for

$|\tilde{\alpha}| \gtrsim r(\tilde{x}^e)$

$\therefore$ It has a series representation

$\Rightarrow \mathbb{f}(\tilde{x}^e(\tilde{\alpha})) = \sum_{n=0}^{\infty}\frac{\mathbb{f}(\tilde{x}^{e^n})}{\tilde{\alpha}^{n+1}}$ exists

$\Rightarrow \frac{\mathbb{f}(\tilde{x}^{e^n})}{\tilde{\alpha}^{n+1}} \to \theta$ as $n \to \infty$ i.e $\{\frac{\mathbb{f}(\tilde{x}^{e^n})}{\tilde{\alpha}^{n+1}}\} \to \theta$ as $n \to \infty$.

the soft set $\tilde{E} = \{\frac{\tilde{x}^{e^n}}{\tilde{\alpha}^{n+1}}\}$ is soft bounded $\Rightarrow$i.e $\left\|\frac{\tilde{x}^{e^n}}{\tilde{\alpha}^{n+1}}\right\| \lesssim \bar{K}$,$\forall$n $\Rightarrow \|\tilde{x}^{e^n}\| \lesssim \bar{K} \tilde{\cdot} |\tilde{\alpha}^{n+1}| = \bar{K} \tilde{\cdot} |\tilde{\alpha}| \tilde{\cdot} |\tilde{\alpha}|^n \Rightarrow \|\tilde{x}^{e^n}\|^{\frac{1}{n}} \lesssim (\bar{K} \tilde{\cdot} |\tilde{\alpha}|)^{\frac{1}{n}} \tilde{\cdot} |\tilde{\alpha}|$

$\Rightarrow \lim_{n\to\infty}\|\tilde{x}^{e^n}\|^{\frac{1}{n}} \lesssim \lim_{n\to\infty}(\bar{K} \tilde{\cdot} |\tilde{\alpha}|)^{\frac{1}{n}} \tilde{\cdot} |\tilde{\alpha}| \Rightarrow \lim_{n\to\infty}\|\tilde{x}^{e^n}\|^{\frac{1}{n}} \lesssim |\tilde{\alpha}|$ where $|\tilde{\alpha}| \gtrsim r(\tilde{x}^e)$, hence $r(\tilde{x}^e) \gtrsim g(\tilde{x}^e)$ ....(2)

From (1) and (2)

We get $r(\tilde{x}^e) = \lim_{n\to\infty}\|\tilde{x}^{e^n}\|^{\frac{1}{n}}$.

Definition (3.5):
Let $\mathfrak{A}$ be (S.B.A) over $\mathbb{K}(E)$. A soft character (in short S.char) of $\mathfrak{A}$ is a nontrivial Soft algebra soft homomorphism $\tilde{\mathfrak{I}}: \mathfrak{A} \to \mathbb{K}(E)$, which means that:

(i) $\tilde{\mathfrak{I}}$ is soft linear.

(ii) $\tilde{\mathfrak{I}}$ is soft multiplicative.

Remark (3.6):
If $\mathfrak{A}$ has $\tilde{u}^e$, then the fact that $\tilde{\mathfrak{I}}$ is a nontrivial force the equality $\tilde{\mathfrak{I}}(\tilde{u}^e) = \bar{1}$. we then define $S.\text{char}(\mathfrak{A}) = \{\tilde{\mathfrak{I}}: \mathfrak{A} \to \mathbb{K}(E): \tilde{\mathfrak{I}} \text{ soft character}\}$.

Definition (3.7) [9]: Let $\mathfrak{A}$ is a (SBA) with $\tilde{u}^e$. Then $\tilde{x}^e \tilde{\in} \mathfrak{A}$ is said to be soft regular, if $\tilde{x}^e$ is invertible (i.e. there a soft element $\tilde{x}^{e-1}$ called the inverse of $\tilde{x}^e$, such that $\tilde{x}^e(\tilde{x}^e)^{-1} = (\tilde{x}^e)^{-1}\tilde{x}^e = \tilde{u}^e$).

Definition (3.8)
A soft algebra $\mathfrak{A}$ is said to be soft division algebra if $\tilde{x}^e \tilde{\in} \mathfrak{A}$ and $\tilde{x}^e \neq \Theta^e$is soft regular.

Theorem (3.9) (Soft gelfand algebra)**:**
Every SBA and commutative with $\tilde{u}^e$over $\mathbb{C}(E)$ which is soft division algebra is soft isomorphic with $\mathbb{C}(E)$.

Definition (3.10):
(i) A soft set I(E) of soft elements of $\mathfrak{A}$ (S.BA) is called soft left ideal if it is no null soft set and has the following properties:
(a)If $\tilde{x}^e, \tilde{y}^e \tilde{\in}$ I(E) then $\tilde{x}^e \tilde{\mp} \tilde{y}^e \tilde{\in}$ I(E);
(b)If $\tilde{x}^e \tilde{\in}$ I(E) and $\tilde{\alpha} \tilde{\in} \mathbb{C}(E)$ then $\tilde{\alpha} \tilde{\cdot} \tilde{x}^e \tilde{\in}$ I(E);
(c) If $\tilde{x}^e \tilde{\in}$ I(E) and $\tilde{r}^e \tilde{\in} \mathfrak{A}$ then $\tilde{r}^e \tilde{\cdot} \tilde{x}^e \tilde{\in}$ I(E);
(ii) A soft set I(E) of soft elements of $\mathfrak{A}$ (SBA) is called soft right ideal if it is no null soft set and has the following properties:
(a)If $\tilde{x}^e, \tilde{y}^e \tilde{\in}$ I(E) then $\tilde{x}^e \tilde{\mp} \tilde{y}^e \tilde{\in}$ I(E);
(b) If $\tilde{x}^e \tilde{\in}$ I(E) and $\tilde{\alpha} \tilde{\in} \mathbb{C}(E)$ then $\tilde{x}^e \tilde{\cdot} \tilde{\alpha} \tilde{\in}$ I(E);
(c)If $\tilde{x}^e \tilde{\in}$ I(E) and $\tilde{r}^e \tilde{\in} \mathfrak{A}$ then $\tilde{x}^e \tilde{\cdot} \tilde{r}^e \tilde{\in}$ I(E);
(iii) I(E) is called soft two sided ideal in $\mathfrak{A}$ if I(E) is both soft left ideal and soft right ideal .

Definition (3.11):
A soft ideal I(E) of $\mathfrak{A}$ (SBA) is said to be proper if I(E) $\neq \mathfrak{A}$.

Definition(3.12):
Let $\mathfrak{A}$ is a (S.B.A) , a soft proper ideal $\mathcal{M}$(E) of $\mathfrak{A}$ is said to be soft maximal left (or right, or two sided) ideal if there is no soft left (or right, or two sided) ideal $\mathcal{I}$(E) in $\mathfrak{A}$ with $\mathcal{M}(E) \tilde{\subsetneq} \mathcal{I}(E) \tilde{\subsetneq} \mathfrak{A}$.

Definition (3.13):
The soft set of all soft maximal ideals $\mathcal{M}$of given soft commutative banach algebra $\mathfrak{A}$ with $\tilde{u}^e$ by $\mathfrak{M}$. Every soft ideal $\mathcal{M} \tilde{\in} \mathfrak{M}$ generates a soft homomorphism of $\mathfrak{A}$ onto $\mathbb{C}(E)$, we denote the soft number corresponding to the soft element $\tilde{x}^e \tilde{\in} \mathfrak{A}$ under this soft homomorphism by $\tilde{x}^e(\mathcal{M})$. For fixed $\tilde{x}^e \tilde{\in} \mathfrak{A}$ we obtain in this way soft function $\tilde{x}^e(\mathcal{M})$ on the soft set $\mathfrak{M}$. Consequently, we obtain a correspondence $\tilde{x}^e \tilde{\rightarrow} \tilde{x}^e(\mathcal{M})$ between the soft element $\tilde{x}^e$of $\mathfrak{A}$ and soft functions $\tilde{x}^e(\mathcal{M})$ on the soft set $\mathfrak{M}$.
I.e. the natural soft homomorphism
$\vartheta: \mathfrak{A} \rightarrow \frac{\mathfrak{A}}{\mathcal{M}} = \mathbb{C}(E)$ Defined by
$\vartheta(\tilde{x}^e) = \tilde{x}^e \tilde{\mp} \mathcal{M}$ assign to each $\tilde{x}^e \tilde{\in} \mathfrak{A}$, a soft complex number $\tilde{x}^e(\mathcal{M})$ defined by
$\tilde{x}^e(\mathcal{M}) = \tilde{x}^e \tilde{\mp} \mathcal{M}$.

Theorem (3.14):
The correspondence $\tilde{x}^e \tilde{\rightarrow} \tilde{x}^e(\mathcal{M})$ has the following properties:
(i) $(\tilde{x}^e \tilde{\mp} \tilde{y}^e)\mathcal{M} = \tilde{x}^e(\mathcal{M}) \tilde{\mp} \tilde{y}^e(\mathcal{M})$.
(ii)For $\tilde{\alpha} \tilde{\in} \mathbb{C}(E)$, then
$(\tilde{\alpha}\tilde{x}^e)(\mathcal{M}) = \tilde{\alpha}(\tilde{x}^e(\mathcal{M}))$.
(iii)$(\tilde{x}^e \tilde{\cdot} \tilde{y}^e)\mathcal{M} = \tilde{x}^e(\mathcal{M}) \tilde{\cdot} \tilde{y}^e(\mathcal{M})$.
(iv) $\tilde{u}^e(\mathcal{M}) = \overline{\tilde{u}^e}$.
(v) $\tilde{x}^e(\mathcal{M}) = \overline{\Theta^e}$ if and only if $\tilde{x}^e \tilde{\in} \mathcal{M}$.
(vi) $|\tilde{x}^e(\mathcal{M})| \tilde{\leq} \|\tilde{x}^e\|$, for all $\tilde{x}^e \tilde{\in} \mathfrak{A}$.

Proof:
(i)
$(\tilde{x}^e \tilde{\mp} \tilde{y}^e)\mathcal{M} = \tilde{x}^e \tilde{\mp} \tilde{y}^e \tilde{\mp} \mathcal{M} = (\tilde{x}^e \tilde{\mp} \mathcal{M}) \tilde{\mp} (\tilde{y}^e \tilde{\mp} \mathcal{M}) = \tilde{x}^e(\mathcal{M}) \tilde{\mp} \tilde{y}^e(\mathcal{M})$.
(ii) - for $\tilde{\alpha} \tilde{\in} \mathbb{C}(E)$, $(\tilde{\alpha}\tilde{x}^e)(\mathcal{M}) = \tilde{\alpha}\tilde{x}^e \tilde{\mp} \mathcal{M}$
$= \tilde{\alpha}(\tilde{x}^e \tilde{\mp} \mathcal{M}) = \tilde{\alpha}(\tilde{x}^e(\mathcal{M}))$.
(iii) $(\tilde{x}^e \tilde{\cdot} \tilde{y}^e)\mathcal{M} = (\tilde{x}^e \tilde{\cdot} \tilde{y}^e) \tilde{\mp} \mathcal{M}$
$= (\tilde{x}^e \tilde{\mp} \mathcal{M}) \tilde{\cdot} (\tilde{y}^e \tilde{\mp} \mathcal{M}) = \tilde{x}^e(\mathcal{M}) \tilde{\cdot} \tilde{y}^e(\mathcal{M})$.
(iv) $\tilde{u}^e(\mathcal{M}) = \tilde{u}^e \tilde{\mp} \mathcal{M} = \overline{\tilde{u}^e}$.
(v) $\tilde{x}^e(\mathcal{M}) = \overline{\Theta^e}$ iff $\tilde{x}^e \tilde{\mp} \mathcal{M} = \mathcal{M}$ iff $\tilde{x}^e \tilde{\in} \mathcal{M}$.
(vi) $|\tilde{x}^e(\mathcal{M})| = |\tilde{x}^e \tilde{\mp} \mathcal{M}| = \inf\{\|\tilde{x}^e \tilde{\mp} \mathcal{M}\| : \tilde{x}^e \tilde{\in} \mathfrak{A}\} \tilde{\leq} \|\tilde{x}^e\|$, for all $\tilde{x}^e \tilde{\in} \mathfrak{A}$.

Definition (3.15):
Let $\mathfrak{A}$ be Soft gelfand algebra and $\mathcal{M}$be the soft set of all soft maximal ideals of $\mathfrak{A}$. Now for all $\tilde{x}^e \tilde{\in} \mathfrak{A}$, define $\widehat{\tilde{x}^e}: \mathcal{M} \rightarrow \mathbb{C}(E)$ by $\widehat{\tilde{x}^e}(M) = \tilde{\alpha}$ for all $M \tilde{\in} \mathcal{M}$ where
$\widehat{\tilde{x}^e}(M) = \tilde{x}^e(M) = \tilde{x}^e \tilde{\mp} M = \tilde{\alpha}$ also
$\mathbb{C}(E) = \frac{\mathfrak{A}}{\mathcal{M}} = \{\tilde{x}^e \tilde{\mp} \mathcal{M} : \tilde{x}^e \tilde{\in} \mathfrak{A}\}$
$= \{\tilde{\alpha} : \tilde{\alpha} \tilde{\in} \mathbb{C}(E)\}$. Denoted $\widehat{\mathfrak{A}} = \{\widehat{\tilde{x}^e} : \tilde{x}^e \tilde{\in} \mathfrak{A}\}$.

Theorem (3.16):
Let $\mathfrak{A}$ be (S.B.A) with $\tilde{u}^e$and let $\mathbb{f}_1, \mathbb{f}_2$ be two soft multiplicative functions on $\mathfrak{A}$ and $\ker\mathbb{f}_1 = \ker\mathbb{f}_2$ then $\mathbb{f}_1 = \mathbb{f}_2$.

Proof:
Suppose that $\ker\mathbb{f}_1 = \ker\mathbb{f}_2 = \mathcal{M}$.
Since $\mathbb{f}_1, \mathbb{f}_2$are soft multiplicative
$\Rightarrow \mathbb{f}_1 \neq \Theta^e, \mathbb{f}_2 \neq \Theta^e \Rightarrow \exists\, \tilde{x}^e{}_0 \tilde{\in} \mathfrak{A}$ Such that $\mathbb{f}_1(\tilde{x}^e{}_0) \neq \Theta^e$
$\Rightarrow \tilde{x}^e{}_0 \tilde{\notin} \ker\mathbb{f}_1 = \mathcal{M}$. Now for all $\tilde{x}^e \tilde{\in} \mathfrak{A}$.
$\tilde{x}^e = \tilde{x}^e \tilde{-} \frac{\mathbb{f}_2(\tilde{x}^e)}{\mathbb{f}_2(\tilde{x}^e{}_0)} \tilde{\cdot} \tilde{x}^e{}_0 \tilde{\mp} \frac{\mathbb{f}_2(\tilde{x}^e)}{\mathbb{f}_2(\tilde{x}^e{}_0)} \tilde{\cdot} \tilde{x}^e{}_0$.
Now, $\mathbb{f}_2(\tilde{x}^e \tilde{-} \frac{\mathbb{f}_2(\tilde{x}^e)}{\mathbb{f}_2(\tilde{x}^e{}_0)} \tilde{\cdot} \tilde{x}^e{}_0)\, \mathbb{f}_2(\tilde{x}^e) \tilde{-} \frac{\mathbb{f}_2(\tilde{x}^e)}{\mathbb{f}_2(\tilde{x}^e{}_0)} \tilde{\cdot} \mathbb{f}_2(\tilde{x}^e{}_0) = \Theta^e$.
$\Rightarrow \tilde{x}^e \tilde{-} \frac{\mathbb{f}_2(\tilde{x}^e)}{\mathbb{f}_2(\tilde{x}^e{}_0)} \tilde{\cdot} \tilde{x}^e{}_0 \tilde{\in} \ker\mathbb{f}_2 = \mathcal{M}$
$\Rightarrow \tilde{x}^e \tilde{-} \frac{\mathbb{f}_2(\tilde{x}^e)}{\mathbb{f}_2(\tilde{x}^e{}_0)} \tilde{\cdot} \tilde{x}^e{}_0 = m$, for some
$m \in \mathcal{M} \Rightarrow \tilde{x}^e = m \tilde{\mp} \frac{\mathbb{f}_2(\tilde{x}^e)}{\mathbb{f}_2(\tilde{x}^e{}_0)} \tilde{\cdot} \tilde{x}^e{}_0$
$\Rightarrow \tilde{x}^e = m \tilde{\mp} \beta \tilde{\cdot} \tilde{x}^e{}_0$ where $\beta = \frac{\mathbb{f}_2(\tilde{x}^e)}{\mathbb{f}_2(\tilde{x}^e{}_0)} \Rightarrow \mathbb{f}_1(\tilde{x}^e) = \mathbb{f}_1(m) \tilde{\mp} \beta \mathbb{f}_1(\tilde{x}^e{}_0)$
$= \Theta^e \tilde{\mp} \frac{\mathbb{f}_2(\tilde{x}^e)}{\mathbb{f}_2(\tilde{x}^e{}_0)} \tilde{\cdot} \mathbb{f}_1(\tilde{x}^e{}_0)$
$\Rightarrow \mathbb{f}_1(\tilde{x}^e) = \alpha \tilde{\cdot} \mathbb{f}_2(\tilde{x}^e)$ where $\alpha = \frac{\mathbb{f}_1(\tilde{x}^e{}_0)}{\mathbb{f}_2(\tilde{x}^e{}_0)}$.
Two show that $\alpha = \tilde{u}^e$, consider

$$\alpha \tilde{\cdot} (\mathbb{f}_2(\tilde{x}^e))^2 = \alpha \tilde{\cdot} \mathbb{f}_2(\tilde{x}^e) \tilde{\cdot} \mathbb{f}_2(\tilde{x}^e)$$
$$= \alpha \tilde{\cdot} \mathbb{f}_2(\tilde{x}^{e2}) = \mathbb{f}_1(\tilde{x}^{e2})$$
$$= (\alpha \tilde{\cdot} \mathbb{f}_2(\tilde{x}^e))^2 = \alpha^2 \tilde{\cdot} (\mathbb{f}_2(\tilde{x}^e))^2$$

$\Rightarrow \alpha \tilde{\cdot} (\mathbb{f}_2(\tilde{x}^e))^2 = \alpha^2 \tilde{\cdot} (\mathbb{f}_2(\tilde{x}^e))^2$
$\Rightarrow (\alpha^2 \tilde{-} \alpha) \tilde{\cdot} (\mathbb{f}_2(\tilde{x}^e))^2 = \Theta^e$
$\Rightarrow \alpha^2 \tilde{-} \alpha = \Theta^e$ as $\mathbb{f}_2(\tilde{x}^e) \neq \Theta^e \Rightarrow \alpha = \Theta^e$or $\alpha = \tilde{u}^e$, if $\alpha = \Theta^e$
$\Rightarrow \mathbb{f}_1(\tilde{x}^e) = \Theta^e$
This is contradiction, then $\alpha = \tilde{u}^e$.

Theorem (3.17):
There is a soft bijective corresponding between the soft set of all soft maximal ideals of $\mathfrak{A}$ and the soft set of all soft characters of $\mathfrak{A}$.

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Proof:

Let $\mathcal{M}$ be soft set of all maximal ideals of $\mathfrak{A}$ and $\mathcal{C}$ be soft set of all soft characters of $\mathfrak{A}$. Define $\mathfrak{Q}:\mathcal{M}\to\mathcal{C}$ by $\mathfrak{Q}(\mathcal{M})=\mathbb{f}_{\mathcal{M}}$ such that $\mathbb{f}_{\mathcal{M}}(\tilde{x}^e)=\tilde{x}^e(\mathcal{M})=\tilde{x}^e\mp\mathcal{M}$. Let $\mathcal{M}_1=\mathcal{M}_2$, $\mathcal{M}_1$ and $\mathcal{M}_2\tilde{\in}\mathcal{M}$

$\Rightarrow\tilde{x}^e\mp\mathcal{M}_1=\tilde{x}^e\mp\mathcal{M}_2\Rightarrow\tilde{x}^e(\mathcal{M}_1)=\tilde{x}^e(\mathcal{M}_2)\Rightarrow\mathbb{f}_{\mathcal{M}_1}=\mathbb{f}_{\mathcal{M}_2}$

$\Rightarrow\mathfrak{Q}(\mathcal{M}_1)=\mathfrak{Q}(\mathcal{M}_2)\Rightarrow\mathfrak{Q}$ is well define

Let $\mathfrak{Q}(\mathcal{M}_1)=\mathfrak{Q}(\mathcal{M}_2)\Rightarrow\mathbb{f}_{\mathcal{M}_1}=\mathbb{f}_{\mathcal{M}_2}\Rightarrow\tilde{x}^e(\mathcal{M}_1)=\tilde{x}^e(\mathcal{M}_2)$

$\Rightarrow\tilde{x}^e\mp\mathcal{M}_1=\tilde{x}^e\mp\mathcal{M}_2\Rightarrow\mathcal{M}_1=\mathcal{M}_2\Rightarrow\mathfrak{Q}$ is soft injective.

To show that $\mathfrak{Q}$ is soft Surjective, for $\mathbb{f}\tilde{\in}\mathcal{C}$, to show $\mathbb{f}=\mathbb{f}_{\mathcal{M}_0}$ for some $\mathcal{M}_0\tilde{\in}\mathcal{M}$. Let

$\mathcal{M}_0=\ker\mathbb{f}=\{\tilde{x}^e\tilde{\in}\mathfrak{A}:\mathbb{f}(\tilde{x}^e)=\Theta^e\}$, now $\mathcal{M}_0$ is soft proper ideal of $\mathfrak{A}$. If not $\Rightarrow\tilde{u}^e\tilde{\in}\mathcal{M}_0\Rightarrow\mathbb{f}(\tilde{u}^e)=\Theta^e$ this contradiction

To show that $\mathcal{M}_0$ is soft maximal.

Let $\mathcal{M}_0\subsetneq\mathcal{J}$ to show that $\mathcal{J}=\mathfrak{A}$, now $\mathcal{J}\tilde{\subseteq}\mathfrak{A}$ and let $\tilde{x}^e\tilde{\in}\mathfrak{A}$, since $\mathcal{M}_0\subsetneq\mathcal{J}$

$\Rightarrow\exists\tilde{x}^e{}_0\tilde{\in}\mathcal{J}$ and $\tilde{x}^e{}_0\tilde{\notin}\mathcal{M}_0$, now $\tilde{x}^e$

$=\tilde{x}^e\simeq\frac{\mathbb{f}(\tilde{x}^e)}{\mathbb{f}(\tilde{x}^e{}_0)}\tilde{\cdot}\tilde{x}^e{}_0\mp\frac{\mathbb{f}(\tilde{x}^e)}{\mathbb{f}(\tilde{x}^e{}_0)}\tilde{\cdot}\tilde{x}^e{}_0$

$\Rightarrow\mathbb{f}\left(\tilde{x}^e\simeq\frac{\mathbb{f}(\tilde{x}^e)}{\mathbb{f}(\tilde{x}^e{}_0)}\tilde{\cdot}\tilde{x}^e{}_0\right)$

$=\mathbb{f}(\tilde{x}^e)\simeq\frac{\mathbb{f}(\tilde{x}^e)}{\mathbb{f}(\tilde{x}^e{}_0)}\tilde{\cdot}\,\mathbb{f}(\tilde{x}^e{}_0)=\Theta^e$

$\Rightarrow\tilde{x}^e\simeq\frac{\mathbb{f}(\tilde{x}^e)}{\mathbb{f}(\tilde{x}^e{}_0)}\tilde{\cdot}\tilde{x}^e{}_0\tilde{\in}\ker\mathbb{f}=\mathcal{M}_0$

$\Rightarrow\tilde{x}^e\simeq\frac{\mathbb{f}(\tilde{x}^e)}{\mathbb{f}(\tilde{x}^e{}_0)}\tilde{\cdot}\tilde{x}^e{}_0=\mathrm{m}$, for some $\mathrm{m}\tilde{\in}\mathcal{M}_0$

$\Rightarrow\tilde{x}^e=\mathrm{m}\mp\alpha\tilde{\cdot}\tilde{x}^e{}_0$, where $\alpha\tilde{\in}\mathbb{C}(E)$ and

$\alpha=\frac{\mathbb{f}(\tilde{x}^e)}{\mathbb{f}(\tilde{x}^e{}_0)}$. Now $\tilde{x}^e{}_0\tilde{\in}\mathcal{J}$ and $\alpha\tilde{\in}\mathbb{C}€$

$\Rightarrow\alpha\tilde{\cdot}\tilde{x}^e{}_0\tilde{\in}\mathcal{J}$ also $\mathrm{m}\tilde{\in}\mathcal{M}_0\Rightarrow\tilde{x}^e\tilde{\in}\mathcal{J}$

$\Rightarrow\mathcal{J}=\mathfrak{A}\Rightarrow\mathcal{M}_0$ is soft maximal.

To show that $\ker\mathbb{f}=\ker\mathbb{f}_{\mathcal{M}_0}$

$\ker\mathbb{f}_{\mathcal{M}_0}=\{\tilde{x}^e\tilde{\in}\mathfrak{A}:\mathbb{f}_{\mathcal{M}_0}(\tilde{x}^e)=\overline{\Theta^e}\}=\{\tilde{x}^e\tilde{\in}\mathfrak{A}:\tilde{x}^e(\mathcal{M}_0)=\overline{\Theta^e}\}$

$=\{\tilde{x}^e\tilde{\in}\mathfrak{A}:\tilde{x}^e\mp\mathcal{M}_0=\mathcal{M}_0\}$

$=\{\tilde{x}^e\tilde{\in}\mathfrak{A}:\tilde{x}^e\tilde{\in}\mathcal{M}_0\}=\ker\mathbb{f}$

$\Rightarrow\mathbb{f}=\mathbb{f}_{\mathcal{M}_0}$ (by theorem3) $\Rightarrow\mathfrak{Q}$ is soft surjective and hence $\mathfrak{Q}$ is a soft bijective.

Theorem (3.18):

Let $\mathfrak{A}$ be Soft gelfand algebra then for all $\tilde{x}^e\tilde{\in}\mathfrak{A}$

(*i*) $\|\tilde{x}^{e^2}\|=\|\tilde{x}^e\|^2$.

(*ii*) $r(\tilde{x}^e)=\|\tilde{x}^e\|$.

(*iii*) $\|\widehat{\tilde{x}^e}\|=\|\tilde{x}^e\|$.

Proof:

(*i*) $\Rightarrow$ (*ii*)

Suppose that $\|\tilde{x}^{e^2}\|=\|\tilde{x}^e\|^2$, for all $\tilde{x}^e\tilde{\in}\mathfrak{A}$, then in general we have

$\|\tilde{x}^{e^{2k}}\|=\|\tilde{x}^e\|^{2k}$, $k=0,1,2,\dots$.

Since $r(\tilde{x}^e)=\lim_{n\to\infty}\|\tilde{x}^{e^n}\|^{\frac{1}{n}}$ (soft spectral radius formula), and $\{\|\tilde{x}^{e^{2k}}\|=\|\tilde{x}^e\|^{2k}\}$ is sub sequence of the sequence $\left\{\|\tilde{x}^{e^n}\|^{\frac{1}{n}}\right\}$, we have

$r(\tilde{x}^e)=\lim_{k\to\infty}\|\tilde{x}^{e^{2k}}\|^{\frac{1}{2k}}=\lim_{k\to\infty}(\|\tilde{x}^e\|^{2k})^{\frac{1}{2k}}=\|\tilde{x}^e\|$.

(*ii*) $\Rightarrow$ (*i*)

Suppose that $r(\tilde{x}^e)=\|\tilde{x}^e\|$

$\Rightarrow r(\tilde{x}^{e^2})=\|\tilde{x}^{e^2}\|$, and by theorem (3.2)

$r(\tilde{x}^{e^2})=(r(\tilde{x}^e))^2\Rightarrow(r(\tilde{x}^e))^2=\|\tilde{x}^{e^2}\|$ and also $r(\tilde{x}^e)=\|\tilde{x}^e\|\Rightarrow(r(\tilde{x}^e))^2=\|\tilde{x}^e\|^2$.

(*ii*) $\Leftrightarrow$ (*iii*)

Suppose that $r(\tilde{x}^e)=\|\tilde{x}^e\|$, iff

$r(\tilde{x}^e)=\mathrm{Sup}\{|\tilde{\alpha}|:\tilde{\alpha}\tilde{\in}\delta(\tilde{x}^e)\}$

$=\mathrm{Sup}\{|\widehat{\tilde{x}^e}(M)|:M\tilde{\in}\mathcal{M}\}=\|\widehat{\tilde{x}^e}\|$.

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نتائج جديدة حول الطيف الواهن في بناخ الجبرا الواهنة

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المستخلص:

في هذا البحث، مفاهيم الحرف الواهن ، تقسيم الجبرا الواهن ، المثالي الاكبر الواهن ، قد قدمت . صيغة نصف القطر الطيف الواهن قد قدمت وبرهنت ، بعض خواص كيلفند الجبرا الواهنه قد برهنت

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On Differential Sandwich Theorems of Meromorphic Univalent Functions

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Abstract

By using of linear operator, we obtain some Subordinations and superordinations results for certain normalized meromorphic univalent analytic functions in the in the punctured open unit disk U^*. Also we derive some sandwich theorems .

Keywords :Analytic Function, Differential Subordination, Hadamard Product, Meromorphic Univalent Function.

Mathematics Subject Classification :30C45

1. Introduction

Let $\mathcal{H}$ be the Linear space of all analytic functions in U. For a positive integer number n and $a \in \mathbb{C}$, we let
$\mathcal{H}[a,n] = \{f \in \mathcal{H}: f(z) = a + a_n z^n + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \cdots\}$.
For two functions f and g analytic in U .We say that the function g is subordinate to f in U and write $g(z) \prec f(z)$, if there exists a Schwarz function ω, which is analytic in U with $\omega(0) = 0$ and $|\omega(z)| < 1$ $(z \in U)$, such that $g(z) = f(\omega(z)), (z \in U)$.

If the function $f(z)$ is if the function f is univalent in U, then we have

$g(z) \prec f(z) \Leftrightarrow g(0) = f(0)$ and $g(U) \subset f(U)$,

$$f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} a_k z^k \ , \qquad (1.1)$$

which are analytic and meromorphic univalent function in the punctured open unit disk $U^* = \{z: z \in \mathbb{C}$ and $0 < |z| < 1\}$.

Let $p, h \in \mathcal{H}$, and $\emptyset(r,s,t;z): \mathbb{C}^3 \times U \to \mathbb{C}$.

If p and $\emptyset(p(z), zp'(z), z^2p''(z); z)$ are univalent functions in U and if p satisfies the second- order superordination

$$h(z) \prec \emptyset(p(z), zp'(z), z^2p''(z); z), \ (z \in U), \qquad (1.2)$$

then p is called a solution of the differential superordination (1.2) , (if f subordinate to g, then g is superordinate to f).

An analytic function q is called a subordinate of the differential superordination if $q \prec p$ for all p satisfying (1.2). A univalent subordinate $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinates q of (1.2)is said to be the best subordinate. Recently Miller and Mocnu [3] obtained sufficient conditions on the functions h , p and $\emptyset$ for which the following implication holds :

$$h(z) \prec \emptyset(p(z), zp'(z), z^2p''(z); z) \Rightarrow q(z) \prec p(z), (z \in U). \qquad (1.2)$$

If $f \in W$ is given by (1.1) and $g \in W$ given by

$$g(z) = \frac{1}{z} + \sum_{k=0}^{\infty} b_k z^k.$$

The Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) = \frac{1}{z} + \sum_{k=0}^{\infty} a_k b_k z^k = (g * f)(z).$$

Using the results, Bulboacă [4] considered certain classes of first order differential superordinations as well as superordination preserving integral operator [1]. Ali et al. [5], have used the results of Bulboacă [4] to obtain sufficient conditions for normalized analytic functions to satisfy:

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z),$$

where q_1 and q_2 are given univalent functions in U with $q_1(0) = q_2(0) = 1$. Also, Tuneski [6] obtained a sufficient conditions for starlikeness of f in terms of the quantity

$$\frac{f''(z)f(z)}{(f'(z))^2}.$$

Recently, Shanmugam et al. [7,8] and Goyal et al. [9] also obtained sandwich results for certain classes of analytic functions.

Ali et al. [10] introduced and investigated the linear operator

$I_1(n,\lambda): W \to W$

which is defined as follows:

$$I_1(n,\lambda)f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} \left(\frac{k+\lambda}{\lambda-1}\right)^n a_k z^k,$$

$$(z \in U^*, \lambda > 1). \qquad (1.4)$$

The general Hurwitz- lerch zeta function

$$\Phi(z,s,r) = \sum_{k=0}^{\infty} \frac{z^k}{(r+k)^s} \ , r \in \mathbb{C} \setminus \mathbb{Z}_0^-, s \in \mathbb{C}$$

when $0 < |z| < 1$.

Definition 1.1. Let $f \in W, z \in U^*, r \in \mathbb{C}\backslash\mathbb{Z}_0^-$, $s \in \mathbb{C}$ and $\lambda > 1$, we define the operator $\mathcal{J}_{s,r,1}(n,\lambda)f(z): W \longrightarrow W$, where

$$\mathcal{J}_{s,r,1}(n,\lambda)f(z) = \frac{\Phi(z,s,r)}{zr^{-s}} * I_1(n,\lambda)f(z)$$

$$= \frac{1}{z} + \sum_{k=0}^{\infty}\left(\frac{r}{1+k+r}\right)^s\left(\frac{k+\lambda}{\lambda-1}\right)^n a_k z^k \qquad (1.5)$$

We note from (1.5) that, we have

$$\lambda\, \mathcal{J}_{s,r,1}(n,\lambda)f(z) = z\left(\mathcal{J}_{s,r,1}(n,\lambda)f(z)\right)' - (\lambda-1)\,\mathcal{J}_{s,r,1}(n+1,\lambda)f(z), \qquad (1.6)$$

$\mathcal{J}_{0,r,1}(n,\lambda)f(z) = I_1(n,\lambda)f(z)$

and $\mathcal{J}_{0,r,1}(0,\lambda)f(z) = f(z)$.

The main object of this idea is to find sufficient conditions for certain normalized analytic functions f to satisfy:

$$q_1(z) \prec \left(\frac{(1-\beta)\, z\mathcal{J}_{s,r,1}(n,\lambda)f(z) + \beta z\, \mathcal{J}_{s,r,1}(n+1,\lambda)f(z)}{\beta+1}\right)^{\delta} \prec q_2(z),$$

and

$$q_1(z) \prec \left(z\mathcal{J}_{s,r,1}(n,\lambda)f(z)\right)^{\delta} \prec q_2(z),$$

where $q_1(z)$ and $q_2(z)$ are given uninvent functions in U with $q_1(0) = q_2(0) = 1$.

2. Preliminaries

In order to prove our subordinations and superordinations results, we need the following definition and lemmas.

Definition 2.1. [2]: Denote by Q the set of all functions q that are analytic and injective on $\overline{U} \setminus E(q)$, where $\overline{U} = U \cup \{z \in \partial U\}$, and

$$\mathrm{E}(q) = \left\{\zeta \in \partial U: \lim_{z\to\zeta} q(z) = \infty\right\} \qquad (1.7)$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial U \backslash E(q)$. Further, let the subclass of Q for which $q(0) = a$ be denoted by $Q(a)$, $Q(0) \equiv Q_0$ and $Q(1) \equiv Q_1$.

Lemma 2.1. [5] Let $q(z)$ be convex univalent function in U, let $\alpha \in \mathbb{C}, \beta \in \mathbb{C}\backslash\{0\}$ and suppose that

$$\mathrm{Re}\left(1 + \frac{zq''(z)}{q'(z)}\right) > \max\left\{0, -\mathrm{Re}\left(\frac{\alpha}{\beta}\right)\right\}.$$

If $p(z)$ is analytic in U and

$\alpha p(z) + \beta z p'(z) \prec \alpha q(z) + \beta z q'(z)$,

then $p(z) \prec q(z)$ and q is the best dominant.

Lemma 2.2. [1]

Let q be univalent in U and let $\emptyset$ and θ be analytic in the domain D containing $q(U)$ with $\emptyset(w) \neq 0$, when $w \in q(U)$. Set

$Q(z) = zq'(z)\emptyset(q(z))$ and $h(z) = \theta(q(z)) + Q(z)$,

suppose that

1 − Q is starlike univalent in U,

2 − $\mathrm{Re}\left(\frac{zh'(z)}{Q(z)}\right) > 0, z \in U$.

If p is analytic in U with $p(0) = q(0)$, $p(U) \subseteq D$ and

$$\emptyset(p(z)) + zp'(z)\emptyset(p(z)) \prec \emptyset(q(z)) + zq'(z)\emptyset(q(z)),$$

then $p \prec q$, and q is the best dominant.

Lemma 2.3. [3] Let q(z) be convex univalent in the unit disk U and let θ and ϕ be analytic in a domain D containing $q(U)$. Suppose that

1 − $\mathrm{Re}\left\{\frac{\theta'(q(z))}{\phi(q(z))}\right\} > 0$ for $z \in U$,

2 − $zq'(z)\phi(q(z))$ is starlike univalent in $z \in U$.

If $p \in \mathcal{H}[q(0),1] \cap Q$, with $p(U) \subseteq D$, and $\theta(p(z)) + zp'(z)\phi(p(z))$ is univalent in U, and

$$\theta(q(z)) + zq'(z)\phi(q(z)) \prec \theta(p(z)) + zp'(z)\phi(p(z)), \qquad (1.8)$$

then $q \prec p$, and q is the best subordinant

Lemma 2.4. [3]: Let $q(z)$ be convex univalent in U and $q(0) = 1$. Let $\beta \in \mathbb{C}$, that $\mathrm{Re}\{\beta\} > 0$. If $p(z) \in \mathcal{H}[q(0),1] \cap Q$ and $p(z) + \beta z p'(z)$ is univalent in U, then

$q(z)+\beta z q'(z) \prec p(z)+\beta z p'(z),$

which implies that $q(z) \prec p(z)$ and $q(z)$ is the best subordinant.

3. Subordination Results

Theorem 3.1. Let $q(z)$ be convex univalent in U with $q(0)=1, \eta, \delta \in \mathbb{C}\backslash\{0\}$. Suppose that

$$\operatorname{Re}\left(1+\frac{zq''(z)}{q'(z)}\right) > \max\left\{0, \operatorname{Re}\left(\frac{\delta}{\eta}\right)\right\}. \qquad (3.1)$$

If $f \in W$ is satisfies the Subordination

$$G(z) \prec q(z)+\frac{\eta}{\delta} zq'(z), \qquad (3.2)$$

where

$$G(z)=\left(\frac{(1-\beta)\, z\mathcal{J}_{s,r,1}(n,\lambda)f(z)+\beta z\, \mathcal{J}_{s,r,1}(n+1,\lambda)f(z)}{\beta+1}\right)^{\delta} \times (1+\eta \times \left(\frac{(\beta\lambda-\lambda+1-\beta)\mathcal{J}_{s,r,1}(n,\lambda)f(z)+(\lambda-1-2\lambda\beta+2\beta)\mathcal{J}_{s,r,1}(n+1,\lambda)f(z)+(\beta\lambda-\beta)\mathcal{J}_{s,r,1}(n+2,\lambda)f(z)}{(1-\beta)\mathcal{J}_{s,r,1}(n,\lambda)f(z)+\beta\mathcal{J}_{s,r,1}(n+1,\lambda)f(z)}\right)), \qquad (3.3)$$

then

$$\left(\frac{(1-\beta)\, z\mathcal{J}_{s,r,1}(n,\lambda)f(z)+\beta z\, \mathcal{J}_{s,r,1}(n+1,\lambda)f(z)}{\beta+1}\right)^{\delta} \prec q(z), \qquad (3.4)$$

and $q(z)$ is the best dominant.

Proof. Define a function $g(z)$ by $g(z)=$

$$\left(\frac{(1-\beta)\, z\mathcal{J}_{s,r,1}(n,\lambda)f(z)+\beta z\, \mathcal{J}_{s,r,1}(n+1,\lambda)f(z)}{\beta+1}\right)^{\delta}, \qquad (3.5)$$

then the function $g(z)$ is analytic in U and q(0)=1,therefore, differentiating (3.5) logarithmically with respect to z and using the identity (1.6) in the resulting equation,

$$G(z)=\left(\frac{(1-\beta)\, z\mathcal{J}_{s,r,1}(n,\lambda)f(z)+\beta z\, \mathcal{J}_{s,r,1}(n+1,\lambda)f(z)}{\beta+1}\right)^{\delta} \times (1+\eta \times \left(\frac{(\beta\lambda-\lambda+1-\beta)\mathcal{J}_{s,r,1}(n,\lambda)f(z)+(\lambda-1-2\lambda\beta+2\beta)\mathcal{J}_{s,r,1}(n+1,\lambda)f(z)+(\beta\lambda-\beta)\mathcal{J}_{s,r,1}(n+2,\lambda)f(z)}{(1-\beta)\mathcal{J}_{s,r,1}(n,\lambda)f(z)+\beta\mathcal{J}_{s,r,1}(n+1,\lambda)f(z)}\right))$$

$$=g(z)+\frac{\eta}{\delta} zg'(z).$$

Thus the subordination (3.2) is equivalent to

$$g(z)+\frac{\eta}{\delta} zg'(z) \prec q(z)+\frac{\eta}{\delta} zq'(z).$$

An application of Lemma (2.1) with $\beta=\frac{\eta}{\delta}$ and $\alpha=1$, we obtain (3.4).

Taking $q(z)=\frac{1+Az}{1+Bz} (-1 \le B < A \le 1),$ in Theorem (3.1), we obtain the following Corollary.

Corollary3.2. Let $\eta, \delta \in \mathbb{C}\backslash\{0\}$ and $(-1 \le B < A \le 1)$. Suppose that

$$\operatorname{Re}\left(\frac{1-Bz}{1+Bz}\right) > \max\left\{0, -\operatorname{Re}\left(\frac{\delta}{\eta}\right)\right\}.$$

If $f \in W$ is satisfy the following Subordination condition :

$$G(z) \prec \frac{1+Az}{1+Bz}+\frac{\eta}{\delta}\frac{(A-B)z}{(1+Bz)^2},$$

where $G(z)$ given by (3.3), then

$$\left(\frac{(1-\beta)\, z\mathcal{J}_{s,r,1}(n,\lambda)f(z)+\beta z\, \mathcal{J}_{s,r,1}(n+1,\lambda)f(z)}{\beta+1}\right)^{\delta} \prec \frac{1+Az}{1+Bz},$$

and $\frac{1+Az}{1+Bz}$ is best dominant .

Taking $A=1$ and $B=-1$ in Corollary (3.2), we get following result.

Corollary 3.3. Let $\eta, \delta \in \mathbb{C}\backslash\{0\}$ and suppose that

$$\operatorname{Re}\left(\frac{1+z}{1-z}\right) > \max\left\{0, -\operatorname{Re}\left(\frac{\delta}{\eta}\right)\right\}.$$

If $f \in W$ is satisfy the following Subordination

$$G(z) \prec \frac{1+z}{1-z}+\frac{\eta}{\delta}\frac{2z}{(1-z)^2},$$

where $G(z)$ given by (3.3), then

$$\left(\frac{(1-\beta)\, z\mathcal{J}_{s,r,1}(n,\lambda)f(z)+\beta z\, \mathcal{J}_{s,r,1}(n+1,\lambda)f(z)}{\beta+1}\right)^{\delta} \prec \frac{1+z}{1-z},$$

and $\frac{1+z}{1-z}$ is best dominant .

Theorem 3.4. Let $q(z)$ be convex univalent in unit disk U with $q(0)=1$, let $\varsigma\, \eta, \delta \in \mathbb{C}\backslash\{0\}, \alpha, t, \mu, \tau \in \mathbb{C}, f \in W$ and suppose that f and q satisfy the following conditions $\operatorname{Re}\left\{\frac{\mu}{\varsigma} q(z)+\frac{2\tau\alpha}{\varsigma} q^2(z)+1+z\frac{q''(z)}{q'(z)}-z\frac{q'(z)}{q(z)}\right\}>0, \qquad (3.6)$

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and

$$z\mathcal{J}_{s,r,1}(n,\lambda)f(z) \neq 0. \qquad (3.7)$$

If

$$r(z) \prec t + \mu q(z) + \tau\alpha q^2(z) + \varsigma\frac{zq'(z)}{q(z)}, \qquad (3.8)$$

where

$$r(z) = \left(z\mathcal{J}_{s,r,1}(n,\lambda)f(z)\right)^{\delta} \times \left(\left(\mu + \tau\alpha\left(\left(z\mathcal{J}_{s,r,1}(n,\lambda)f(z)\right)^{\delta}\right) + t + \varsigma\delta(\lambda-1)\left(\frac{\mathcal{J}_{s,r,1}(n+1,\lambda)f(z)}{\mathcal{J}_{s,r,1}(n,\lambda)f(z)} - 1\right)\right.\right., \qquad (3.9)$$

then

$\left(z\mathcal{J}_{s,r,1}(n,\lambda)f(z)\right)^{\delta} \prec q(z)$, and $q(z)$ is best dominant.

Proof . Define analytic function $g(z)$ by

$$g(z) = \left(z\mathcal{J}_{s,r,1}(n,\lambda)f(z)\right)^{\delta}. \qquad (3.10)$$

Then the function $g(z)$ is analytic in U and $g(0) = 1$, differentiating (3.10) logarithmically with respect to z, we get

$$\frac{zg'(z)}{g(z)} = \delta(\lambda-1)\left(\frac{\mathcal{J}_{s,r,1}(n+1,\lambda)f(z)}{\mathcal{J}_{s,r,1}(n,\lambda)f(z)} - 1\right). \quad (3.11)$$

By setting $\theta(w) = t + \mu w + \tau\alpha w^2$ and $\emptyset(w) = \frac{\varsigma}{w}$, it can be easily observed that $\theta(w)$ is analytic in $\mathbb{C}$, $\emptyset(w)$is analytic in $\mathbb{C}\backslash\{0\}$ and that $\emptyset(w) \neq 0, w \in \mathbb{C}\backslash\{0\}$.

Also, if we let

$Q(z) = zq'(z)\emptyset(z) = \varsigma\frac{zq'(z)}{q(z)}$ and $h(z) = \theta(q(z)) + Q(z)$

$$= t + \mu q(z) + \tau\alpha q^2(z) + \varsigma\frac{zq'(z)}{q(z)},$$

we find that $Q(z)$ is starlike univalent in U ,we have

$$h'(z) = \mu q'(z) + 2\tau\alpha q(z)q'(z) + \varsigma\frac{q'(z)}{q(z)} + \varsigma z\frac{q''(z)}{q(z)} - \varsigma z\left(\frac{q'(z)}{q(z)}\right)^2,$$

and

$$\frac{zh'(z)}{Q(z)} = \frac{\mu}{\varsigma}q(z) + \frac{2\tau\alpha}{\varsigma}q^2(z) + 1 + z\frac{q''(z)}{q'(z)} - z\frac{q'(z)}{q(z)},$$

hence that

$$\mathrm{Re}\left(\frac{zh'(z)}{Q(z)}\right) = \mathrm{Re}\left(\frac{\mu}{\varsigma}q(z) + \frac{2\tau\alpha}{\varsigma}q^2(z) + 1 + z\frac{q''(z)}{q'(z)} - z\frac{q'(z)}{q(z)}\right) > 0.$$

By using (3.11), we obtain

$$\mu g(z) + \tau\alpha g^2(z) + \frac{zg'(z)}{g(z)} = \left(z\mathcal{J}_{s,r,1}(n,\lambda)f(z)\right)^{\delta} \times \left(\mu + \tau\alpha\left(z\mathcal{J}_{s,r,1}(n,\lambda)f(z)\right)^{\delta}\right) + t + \varsigma\delta(\lambda-1)\left(\frac{\mathcal{J}_{s,r,1}(n+1,\lambda)f(z)}{\mathcal{J}_{s,r,1}(n+1,\lambda)f(z)} - 1\right).$$

By using (3.8), we have

$$\mu g(z) + \tau\alpha g^2(z) + \varsigma\frac{zg'(z)}{g(z)} \prec \mu q(z) + \tau\alpha q^2(z) + s\frac{zq'(z)}{q(z)}$$

and by using Lemma (2.2), we deduce that subordination (3.8) implies that $g(z) \prec q(z)$ and the function $q(z)$ is the best dominant.

Taking the function $q(z) = \frac{1+Az}{1+Bz}$ $(-1 \leq B < A \leq 1)$, in Theorem (3.4),the condition (3.6) becoms

$$\mathrm{Re}\left(\frac{\mu}{\varsigma}\frac{1+Az}{1+Bz} + \frac{2\tau\alpha}{\varsigma}\left(\frac{1+Az}{1+Bz}\right)^2 + 1 + \frac{(A-B)z}{(1+Bz)(1+Az)} - \frac{2Bz}{1+Bz}\right) > 0 (s \in \mathbb{C}\backslash\{0\}), \qquad (3.12)$$

hence, we have the following Corollary.

Corollary 3.5. Let $(-1 \leq B < A \leq 1), s, \delta \in \mathbb{C}\backslash\{0\}, \alpha, t, \tau, \mu \in \mathbb{C}$. Assume that (3.12) holds.

If $f \in W$ and

$$r(z) \prec t + \mu\frac{1+Az}{1+Bz} + \tau\alpha\left(\frac{1+Az}{1+Bz}\right)^2 + \varsigma\frac{(A-B)z}{(1+Bz)(1+Az)},$$

where $r(z)$ is defined in (3.9), then

$\left(z\mathcal{J}_{s,r,1}(n,\lambda)f(z)\right)^{\delta} \prec \frac{1+Az}{1+Bz}$,and $\frac{1+Az}{1+Bz}$ is best dominant.

Taking the function $q(z) = \left(\frac{1+z}{1-z}\right)^{\rho}$ $(0 < \rho \leq 1)$, in Theorem (3.4),the condition(3.6) becoms

$$\mathrm{Re}\left\{\frac{\mu}{\varsigma}\left(\frac{1+z}{1-z}\right)^{\rho} + \frac{2\tau\alpha}{\varsigma}\left(\frac{1+z}{1-z}\right)^{2\rho} + \frac{2z^2}{1-z^2}\right\} > 0, \quad (\varsigma \in \mathbb{C}\backslash\{0\}), \qquad (3.13)$$

hence, we have the following Corollary.

Corollary 3.6. Let $0 < \rho \le 1, \varsigma, \delta \in \mathbb{C}\backslash\{0\}, \alpha, t, \tau, \mu \in \mathbb{C}$. Assume that (3.13) holds.

If $f \in W$ and

$$r(z) \prec t + \mu\left(\frac{1+z}{1-z}\right)^{\rho} + \tau\alpha\left(\frac{1+z}{1-z}\right)^{2\rho} + \varsigma\frac{2\rho z}{1-z^2},$$

where $r(z)$ is defined in (3.9), then $\left(z\mathcal{J}_{s,r,1}(n,\lambda)f(z)\right)^{\delta} \prec \left(\frac{1+z}{1-z}\right)^{\rho}$,and $\left(\frac{1+z}{1-z}\right)^{\rho}$ is best dominant.

4. Superordination Results

Theorem 4.1. Let $q(z)$ be convex univalent in U with $q(0) = 1, ,\delta \in \mathbb{C}\backslash\{0\}$,Re$\{\eta\} > 0$, if $f \in W$, such that

$$\frac{(1-\beta)\, z\mathcal{J}_{s,r,1}(n,\lambda)f(z) + \beta z\, \mathcal{J}_{s,r,1}(n+1,\lambda)f(z)}{\beta+1} \neq 0$$

and

$$\left(\frac{(1-\beta)\, z\mathcal{J}_{s,r,1}(n,\lambda)f(z) + \beta z\, \mathcal{J}_{s,r,1}(n+1,\lambda)f(z)}{\beta+1}\right)^{\delta} \in \mathcal{H}[q(0),1] \cap Q. \qquad (4.1)$$

If the function $G(z)$ defined by (3.3) is univalent and the following superordination condition:

$$q(z) + \frac{\eta}{\delta} zq'(z) \prec \mathrm{G}(z), \qquad (4.2)$$

holds, then

$$q(z) \prec \left(\frac{(1-\beta)\, z\mathcal{J}_{s,r,1}(n,\lambda)f(z) + \beta z\, \mathcal{J}_{s,r,1}(n+1,\lambda)f(z)}{\beta+1}\right)^{\delta} \qquad (4.3)$$

and $q(z)$ is the best subordinant .

Proof . Define a function $g(z)$ by

$$g(z) = \left(\frac{(1-\beta)\, z\mathcal{J}_{s,r,1}(n,\lambda)f(z) + \beta z\, \mathcal{J}_{s,r,1}(n+1,\lambda)f(z)}{\beta+1}\right)^{\delta}. \qquad (4.4)$$

Differentiating (4.4) with respect to z logarithmically ,we get

$$\frac{zg'(z)}{g(z)} = \delta\left(\frac{(1-\beta)z(\mathcal{J}_{s,r,1}(n,\lambda)f(z))' + \beta z(\mathcal{J}_{s,r,1}(n+1,\lambda)f(z))'}{(1-\beta)\mathcal{J}_{s,r,1}(n,\lambda)f(z) + \beta\mathcal{J}_{s,r,1}(n+1,\lambda)f(z)}\right). \qquad (4.5)$$

A simple computation and using (1.6), from (4.5) ,we get

$$\mathrm{G}(z) = \left(\frac{(1-\beta)\, z\mathcal{J}_{s,r,1}(n,\lambda)f(z) + \beta z\, \mathcal{J}_{s,r,1}(n+1,\lambda)f(z)}{\beta+1}\right)^{\delta} \times \left((1+\eta\left(\frac{(\beta\lambda-\lambda+1-\beta)\mathcal{J}_{s,r,1}(n,\lambda)f(z) + (\lambda-1-2\lambda\beta+2\beta)\mathcal{J}_{s,r,1}(n+1,\lambda)f(z) + (\beta\lambda-\beta)\mathcal{J}_{s,r,1}(n+2,\lambda)f(z)}{(1-\beta)\mathcal{J}_{s,r,1}(n,\lambda)f(z) + \beta\mathcal{J}_{s,r,1}(n+1,\lambda)f(z)}\right)\right),$$

$$= g(z) + \frac{\eta}{\delta} zg'(z),$$

now, by using Lemma(2.4), we get the desired result .

Taking $q(z) = \frac{1+Az}{1+Bz}$ $(-1 \le B < A \le 1)$, in Theorem (4.1), we get the following Corollary.

Corollary 4.2. Let Re$\{\eta\} > 0, \delta \in \mathbb{C}\backslash\{0\}$ and $-1 \le B < A \le 1$, such that

$$\left(\frac{(1-\beta)\, z\mathcal{J}_{s,r,1}(n,\lambda)f(z) + \beta z\, \mathcal{J}_{s,r,1}(n+1,\lambda)f(z)}{\beta+1}\right)^{\delta} \in \mathcal{H}[q(0),1] \cap Q.$$

If the function $G(z)$ given by (3.3) is univalent in U and $f \in W$ satisfies the following superordination condition :

$$\frac{1+Az}{1+Bz} + \frac{\eta}{\delta}\frac{(A-B)z}{(1+Bz)^2} \prec G(z),$$

then

$$\frac{1+Az}{1+Bz} \prec \left(\frac{(1-\beta)\, z\mathcal{J}_{s,r,1}(n,\lambda)f(z) + \beta z\, \mathcal{J}_{s,r,1}(n+1,\lambda)f(z)}{\beta+1}\right)^{\delta},$$

and the function $\frac{1+Az}{1+Bz}$ is the best subordinant.

Theorem 4.3. Let $q(z)$ be convex univalent in unit disk U, let $\varsigma, \delta \in \mathbb{C}\backslash\{0\}\alpha, t, \mu, \tau \in \mathbb{C}, , q(z) \neq 0$, and $f \in W$. Suppose that Re$\left\{\frac{q(z)}{s}(2\tau\alpha q(z) + \mu)\right\}q'(z) > 0$, and satisfies the next conditions

$$\left(z\mathcal{J}_{s,r,1}(n,\lambda)f(z)\right)^{\delta} \in \mathcal{H}[q(0),1] \cap Q, \qquad (4.6)$$

and

$$z\mathcal{J}_{s,r,1}(n,\lambda)f(z) \neq 0,$$

If the function $r(z)$ is given by (3.9) is univalent in U ,

$$\mathrm{t} + \mu q(z) + \tau\alpha q^2(z) + \varsigma\frac{zq'(z)}{q(z)} \prec r(z), \qquad (4.7)$$

implies

$q(z) \prec \left(z\mathcal{J}_{s,r,1}(n,\lambda)f(z)\right)^{\delta}$, and $q(z)$ is the best subordinant.

Proof . Let the function $g(z)$ defined on U by (3.14) . Then a computation show that

$$\frac{zg'(z)}{g(z)} = \delta(\lambda-1)\left(\frac{\mathcal{J}_{s,r,1}(n+1,\lambda)f(z)}{\mathcal{J}_{s,r,1}(n,\lambda)f(z)}-1\right), \qquad (4.8)$$

By setting $\theta(w) = t + \mu w + \tau\alpha w^2$ and $\emptyset(w) = \frac{\varsigma}{w}$, it can be easily observed that $\theta(w)$ is analytic in $\mathbb{C}$, $\emptyset(w)$is analytic in $\mathbb{C}\backslash\{0\}$ and that $\emptyset(w) \neq 0$ $(w \in \mathbb{C}\backslash\{0\})$. Also, we get

$$Q(z) = zq'(z)\emptyset(q(z)) = \varsigma\,\frac{zq'(z)}{q(z)},$$

it observed that $Q(z)$ is starlike univalent in U.

Since $q(z)$ is convex , it follows that

$$\operatorname{Re}\left(\frac{z\theta'(q(z))}{\emptyset(q(z))}\right) = \operatorname{Re}\left\{\frac{q(z)}{\varsigma}(2\tau\alpha q(z)+\mu)\right\}q'(z) > 0.$$

By making use of (4.8) the hypothesis (4.7) can be equivalently written as

$$\theta(q(z) + zq'(z)\emptyset(q(z))) = \theta(g(z) + zg'(z)\emptyset(g(z))),$$

thus, by applying Lemma (2.3), the proof is completed .

5 . Sandwich Results

Combining Theorem (3.1) with Theorem (4.1), we obtain the following sandwich theorem .

Theorem 5.1. Let q_1 and $q_2$be convex univalent in U with $q_1(0) = q_2(0) = 1$ and q_2 satisfies (3.1) . Suppose that $\operatorname{Re}\{\eta\} > 0,\ \eta,\delta \in \mathbb{C}\backslash\{0\}$.

If $f \in W$, such that

$\left(\frac{(1-\beta)\, z\mathcal{J}_{s,r,1}(n,\lambda)f(z)+\beta z\, \mathcal{J}_{s,r,1}(n+1,\lambda)f(z)}{\beta+1}\right)^{\delta} \in \mathcal{H}[q(0),1] \cap Q$,

and the function $G(z)$ defined by (3.3) is univalent and satisfies

$$q_1(z) + \frac{\eta}{\delta}\, z\, q_1'(z) \prec \mathrm{G}(z) \prec q_2(z) + \frac{\eta}{\delta}\, z\, q_2'(z), \qquad (5.1)$$

then

$$q_1(z) \prec \left(\frac{(1-\beta)\, z\mathcal{J}_{s,r,1}(n,\lambda)f(z)+\beta z\, \mathcal{J}_{s,r,1}(n+1,\lambda)f(z)}{\beta+1}\right)^{\delta} \prec q_2(z),$$

where $q_1$and q_2 are, respectively ,the subordinant and the best dominant of (5.1).

Combining Theorem (3.4) with Theorem (4.3), we obtain the following sandwich theorem .

Theorem 5.2. Let q_i be two convex univalent functions in U,such that $q_i(0) = 1$, $q_i(0) \neq 0 (i = 1,2)$ .Suppose that $q_1$and q_2 satisfies (4.8) and (3.8), respectively.

If $f \in W$ and suppose that f satisfies the next conditions :

$\left(z\mathcal{J}_{s,r,1}(n,\lambda)f(z)\right)^{\delta} \in \mathcal{H}[q(0),1] \cap Q$,

and

$z\mathcal{J}_{s,r,1}(n,\lambda)f(z) \neq 0$,

and $r(z)$ is univalent in U, then

$$\mathrm{t} + \mu\, q_1(z) + \tau\alpha\, q_1^2(z) + \varsigma\frac{zq_1'(z)}{q_1(z)} \prec r(z) \prec \mathrm{t} + \mu\, q_1(z) + \tau\alpha\, q_1^2(z) + \varsigma\frac{zq_1'(z)}{q_1(z)}, \qquad (5.2)$$

implies

$q_1(z) \prec \left(z\mathcal{J}_{s,r,1}(n,\lambda)f(z)\right)^{\delta} \prec q_2(z)$,

and q_1 and $q_2$are the best subordinant and the best dominant respectively and $r(z)$ is given by (3.9).

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حول مبرهنات الساندوج التفاضلية لدوال احادية التكافؤ الميرومورفيه

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المستخلص :
باستخدام المؤثر الخطي، حصلنا على بعض النتائج للتبعية التفاضلية والتبعية التفاضلية العليا للدوال التحليلية الاحادية التكافؤ الاكيدة في قرص الوحدة المثقوب U^*. ايضا اشتقينا بعض مبرهنات الساندوج.

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Proof of Stronger Goldbach Conjecture

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Abstract

The stronger (binary) Goldbach conjecture expresses that **"every even integer greater than or equal to 4 can be written as a sum of two odd prime numbers"**. The introduce paper demonstrates this conjecture by proven that there exists a positive integer a for each integer number $n \geq 4$, such that $n - a$ and $n + a$ are simultaneously primes.

Keywords: Number theory; Primes and their distribution; Goldbach conjecture.

Mathematics Subject Classification : 11P32

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1.Introduction.

The issue under thought had its beginning in a letter composed by Goldbach to Euler in 1742 [1]. He expounded on his thought to the celebrated mathematician Euler, who at initially addressed the letter with some neglect, viewing the outcome as minor. That wasn't very wise of Euler: the "Goldbach conjecture", as it's turned out to be known, remains unproven 'til today which has been verified and it is currently known to be align to $4\cdot 10^{14}$, (see [2]). This guess suggests the guess that "all odd numbers greater than 7 are the sum of three odd primes", which is alluded to today differently as the odd Goldbach guess, or the ternary Goldbach guess. While the feeble Goldbach guess seems to have been at last demonstrated in 2013 by Helfgott [3], [4].

We should attempt with a few cases, for instance:

$4 = 2 + 2,$
$6 = 3 + 3,$
$10 = 3 + 7,$
$12 = 5 + 7,$
$14 = 7 + 7,$
$16 = 3 + 13,$
$18 = 7 + 11.$

See that the cases are not one of a kind, for example $14 = 3 + 11 = 7 + 7$.

2.The proof of stronger Goldbach conjecture.

The proof is very simple after we prove the following theorem :

Theorem (2.1) For each $n \in \mathbb{Z}, n \geq 4$, $\exists a \in \mathbb{Z}^+$, such that $(n - a)$ and $(n + a)$ are simultaneously primes.

Example (2.2) We first verify that there exists such a positive integer number a, for $4 \leq n \leq 71$, and the results were recorded as table 1.

n	a	n	a
4	1	38	9,15,21,23,33,35
5	2	39	2,8,20,22,28,32,34
6	1	40	3,21,27,33
7	4	41	12,18,30,38
8	3,5	42	1,5,11,19,25,29,31,37
9	2,4	43	24,30,36,40
10	3,7	44	3,15,27,39
11	6,8	45	2,8,14,16,22,26,28, 34,38
12	1,5,7	46	15,33,43
13	6,10	47	6,24,36,42
14	3,9	48	5,11,19,25,31,35,41
15	2,4,8	49	12,18,30
16	3,13	50	3,9,21,33,39,47
17	6,12,14	51	8,10,20,22,28,32,38, 46
18	1,5,11,13	52	9,15,21,45,49
19	12	53	6,30,36,48,50
20	3,9,17	54	7,13,17,25,35,43,47,51
21	2,8,10,16	55	12,18,24,42,48,52
22	9,15,19	56	3,15,27,33,45,51,53
23	6,18	57	4,10,14,16,26,40,44,46, 50,52
24	5,7,13,17,19	58	15,21,39,45,51,55
25	6,12,18,22	59	12,30,42,48,54
26	3,15	60	1,7,13,19,23,29,37,41,43, 47,49,53
27	4,10,14,16,20	61	18,42,48
28	9,15,25	62	9,21,39,45,51
29	12,18,24	63	10,16,20,26,34,40,44,46, 50
30	1,7,11,13,17,23	64	3,33,45
31	12,28	65	6,18,24,36,42,48
32	9,15,21,27,29	66	5,7,13,23,35,37,43,47,61
33	4,10,14,20,26,28	67	6,30,36,60,64
34	3,27	68	15,21,39,45,63
35	6,12,18,24,32	69	2,10,28,32,38,40,58,62
36	5,7,11,17,23,25,31	70	3,9,27,33,39,57,67
37	6,24,30,34	71	12,18,30,42,60,66,68

Table1. Checking conjecture 1, $4 \leq n \leq 71$.

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Remark (2.3) From table 1, we observe that notes:

i. If n is even (odd) then a is odd (even). This is always true since we want $n+a$ odd prime number.

ii. Since we want $n-a$ prime number, we have $n-a>0$. Therefor, always we have $a<n$.

iii. For every even integer no. n: a can takes the values $n-1$, $n-3, n-5, \ldots, n-(n-1)$. Also for every odd integer no. n: a can takes the values $n-1$, $n-3, n-5, \ldots, n-(n-2)$.

iv. we get for every integer $n \geq 4$, $a = n-p$, where p is prime number, $3 \leq p \leq (n-1)$ when n is even, and $3 \leq p \leq (n-2)$ when n is odd, $n-a = n-(n-p) = p$.

v. Let $S = \{1,3,5,7, \ldots, n-1\}$. And $T = \{2,4,6,8, \ldots, n-1\}$. It is easy to show that two cases:
   1. Let $a \in S$. For every positive even integer n , $n+a$ is odd number such that $2n-(n-1) \leq n+a \leq 2n-1$.
   2. Let $a \in T$. For every positive odd integer n, $n+a$ is odd number Such that $2n-(n-2) \leq n+a \leq 2n-1$, in both cases there exists a such that $n+a$ is prime.

vi. From the above notes, we observe that for every integer $n \geq 4$, there exists positive integers a such that $n+a$ is prime and $n-a$ is prime. Now we are looking for an integer a makes $n+a$ and $n-a$ are Simultaneously primes. If $n-a=p$ then $n+a = n+n-p = 2n-p$, where p is prime number such that $3 \leq p \leq (n-1)$, if n is even and $3 \leq p \leq (n-2)$, if n is odd. Therefor we want to show that $2n-p$ is prime for some p.

Example (2.4)

Consider the following example, when n odd number. We define a function $f(x): A \to B$, such that
$A = \{1,3,5, \ldots, n-2\}$,
$B = \{2n-1, 2n-3, 2n-5, \ldots, n+2\}$ by
$f(x) = 2n-x$ one to one and onto function, see for example when $n=19$,

$$19+2=21 \leftrightarrow 19-2=17$$
$$19+4=23 \leftrightarrow 19-4=15$$
$$19+6=25 \leftrightarrow 19-6=13$$
$$19+8=27 \leftrightarrow 19-8=11$$
$$19+10=29 \leftrightarrow 19-10=9$$
$$19+12=\mathbf{31} \leftrightarrow 19-12=\mathbf{7}$$
$$19+14=33 \leftrightarrow 19-14=5$$
$$19+16=35 \leftrightarrow 19-16=3$$
$$19+18=37 \leftrightarrow 19-18=1$$

$A = \{1,3,5, \ldots, 17\}$, $B = \{37,35,33, \ldots, 21\}$,
$f: A \to B$ defined by $f(x) = 2n-x$.
If $a=12$ then $19+12$ and $19-12$ are simultaneously primes . Similarly when n even number, we can define the one to one and onto function from the set
$C = \{1,3,5, \ldots, n-1\}$ to the set
$D = \{2n-1, 2n-3, 2n-5, \ldots, n+1\}$ by
$f(x) = 2n-x$.

Proof of theorem (2.1)

For every even integer $n \geq 4$, there exists a positive integer $a = n-p$ where p is a prime number such that $3 \leq p \leq (n-1)$, makes $n-a$ is prime for all p, as well
$n+a = n+n-p = 2n-p$. We want to show that $2n-p$ is prime for some p, such that $3 \leq p \leq (n-1)$. Now let
$C = \{1,3,5, \ldots, n-1\}$, the set of all possible $n-a$,
$D = \{2n-1, 2n-3, 2n-5, \ldots, n+1\}$, the set of all possible $n+a$, and $f: C \to D$ defined by $f(x) = 2n-x$. In the both sets C and D, there are primes numbers, we show by contradiction that $2n-x$ can be prime number for some prime no. x. Assume that $2n-x$ is not prime number $\forall x = p$. That is $2n-p \neq \bar{p}$, $\forall(\bar{p}$ prime number in the set $D)$.
Therefor $f^{-1}(2n-p) \neq f^{-1}(\bar{p}), \forall \bar{p}$.
So that $p \neq 2n-\bar{p}$, $\forall \bar{p}$. Similarly for odd integer . ■

Theorem (2.5)

The necessary and sufficient condition of stronger Goldbach conjecture is theorem (2.1).

Proof.

Let $n \geq 2$, $2n = p + q$ (p and q are prime numbers).

We want to show that: $\exists a \in \mathbb{Z}^+,\ s.t, (n - a)$ and $(n + a)$ are simultaneously primes. We have $2n - p = q$, there exists a positive integer $a = n - p$ such that $n - a = p$ and $n + a = q$.

Conversely, $2n = n + n$, (if n is prime number then $a = 0$). Then by theorem (2.1), $\exists a \in \mathbb{Z}^+,\ s.t,$ $(n - a)$ and $(n + a)$ are simultaneously primes.

Therefore

$2n = n + n = (n - a) + (n + a)$.■

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اثبات لمخمنة غولدباخ الأقوى (الثنائية)

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المستخلص :

يعبر عن مخمنة غولدباخ الأقوى (الثنائية) بأنها " كل عدد صحيح زوجي أكبر من أو يساوي 4 يمكن كتابته كمجموع لعددين أوليين ". والتي لم يتم اثبات صحتها منذ 1742 ولغاية هذا اليوم حيث تعد من المسائل المفتوحة في الرياضيات . في هذا البحث قدمنا الاثبات على صحة هذه المخمنة من خلال إثبات وجود عدد صحيح موجب a لكل عدد صحيح n ، أكبر من أو يساوي 4 بحيث يجعل كل من n − a و n + a اعداد اولية في ذات الوقت.

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Coefficient Estimates for Subclasses of Bi-Univalent Functions

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Abstract

In the present paper, we introduce two new subclasses of the class $\sum$ consisting of analytic and bi-univalent functions in the open unit disk U. Also, we obtain the estimates on the Taylor-Maclurin coefficients $|a_2|$ and $|a_3|$ for functions in these subclasses. We obtain new special cases for our results.

Keywords : Analytic function , Univalent function , Bi-univalent function , Coefficient estimates .

Mathematics Subject Classification : 30C45.

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1. Introduction

Let $\mathcal{H}$ be the class of the functions of the form :

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j \;, \qquad (z \in U), \tag{1.1}$$

which are analytic in the open unit disk $U = \{z: z \in \mathbb{C} \text{ and } |z| < 1\}$. Also, let S denoted the class of all functions in $\mathcal{H}$ which are univalent and normalized by the conditions $f(0) = 0 = f'(0) - 1$ in U [1]. It is well known that every univalent function f has inverse f^{-1} satisfying:

$$f^{-1}(f(z)) = z \ (z \in U),$$

and

$$f(f^{-1}(w)) = w \ (|w| < r_0(f);\ r_0(f) \geq \tfrac{1}{4}),$$

where

$$f^{-1}(w) = g(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \cdots. \tag{1.2}$$

A function $f \in \mathcal{H}$ is said to be bi-univalent in U if both $f(z)$ and $f^{-1}(z)$ are univalent in U. Let $\sum$ denote the class of bi-univalent functions defined in the unit disk U given by (1.1). For a brief history and interesting example in the class $\sum$ (see [2]). However , the familiar Koebe function is not bi-univalent. The class $\sum$ of bi-univalent functions was first inverstigated by Lewin [3] and it was shown that $|a_2| < 1.51$. Brannan and Clunie [4] improved Lewin's result and conjectured that $|a_2| \leq \sqrt{2}$. Later , Netanyahu [5], showed that if $f \in \sum$, then $\max|a_2| = \frac{4}{3}$.

Recently, Srivastava et al. [6], Frasin and Aouf [7], BansaL and Sokol [8] and Srivastava and BansaL [2] are also introduced and investigated the various subclasses of bi- univalent functions and obtained bounds for the initial coefficients $|a_2|$ and $|a_3|$.

The coefficient estimate problem involving the bound of $|a_n|$ $(n \in N\backslash\{1,2\}\,; N = \{1,2,3,\ldots\})$ for each $f \in \sum$ given by (1.1) is still an open problem.

The object of this work is to find estimates on the Taylor –Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions in this subclasses $S_{\sum}(\tau,\gamma,\delta;\alpha)$ and $S_{\sum}(\tau,\gamma,\delta;\beta)$ of the functions class $\sum$. Several related classes are also considered and connections to earlier known results are made.

In order to prove in our main results, we require the following lemma.

Lemma 1.1. [1] If $h \in p$ the $|c_k| \leq 2$ for each k , where p is the family of all functions h analytic in U for which $\mathrm{Re}(h(z)) > 0$

$$h(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots. \quad \text{for } z \in U$$

2. Coefficient Bounds for Function class $S_{\sum}(\tau,\gamma,\delta;\alpha)$

To prove our main results , we need to introduce the following definition .

Definition 2.1. A function $f(z)$ given by (1.1) is said to be in the class $S_{\sum}(\tau,\gamma,\delta;\alpha)(\tau \in \mathbb{C}\backslash\{0\}, 0 \leq \gamma \leq 1, 0 \leq \delta < 1, 0 < \alpha \leq)$ if the following conditions are satisfied :

$$f \in \sum, \left|\arg\left(1 + \frac{1}{\tau}\left[\frac{z(f'(z)+\gamma z f''(z))}{\gamma z(f'(z)+\delta z f''(z))+(1-\gamma)(\delta z f'(z)+(1-\delta) f(z))} - 1\right]\right)\right| < \frac{\alpha\pi}{2} \quad (z \in U) \tag{2.1}$$

and

$$g \in \sum, \left|\arg\left(1 + \frac{1}{\tau}\left[\frac{w(g'(w)+\gamma w\, g''(w))}{w\gamma(g'(w)+\delta g''(w))+(1-\gamma)(\delta w g'(w)+(1-\delta)\, g(w))} - 1\right]\right)\right| < \frac{\alpha\pi}{2} \quad (w \in U) \tag{2.2}$$

where the function $g(w)$ is given by (1.2).

Theorem 2.2. Let $f(z)$ given by (1.1) be in the class $S_{\sum}(\tau,\gamma,\delta;\alpha)(\tau \in \mathbb{C}\backslash\{0\}, 0 \leq \gamma \leq 1, 0 \leq \delta < 1, 0 < \alpha \leq 1)$. Then

$$|a_2| \leq \frac{2\alpha|\tau|}{\sqrt{|2\tau\alpha\left((2-2\delta+4\gamma-4\delta\gamma)-(1+2\gamma+\gamma^2-2\delta^2\gamma-\delta^2-\delta^2\gamma^2)\right)+(1-\alpha)(1-\delta+\gamma-\delta\gamma)^2|}} \,. \tag{2.3}$$

and

$$|a_3| \le \frac{2|\tau|\alpha}{|(2-2\delta+4\gamma-4\delta\gamma)|} + \frac{4|\tau|^2\alpha^2}{(1-\delta+\gamma-\delta\gamma)^2}. \qquad (2.4)$$

Proof: Let $f(z) \in S_{\Sigma}\ (\tau,\gamma,\delta;\alpha)$.Then

$$1+\frac{1}{\tau}\left[\frac{z(f'(z)+\gamma z f''(z))}{\gamma z(f'(z)+\delta z f''(z))+(1-\gamma)(\delta z f'(z)+(1-\delta)f(z))}-1\right] = [r(z)]^{\alpha} \qquad (2.5)$$

and

$$1+\frac{1}{\tau}\left[\frac{w(g'(w)+\gamma w\, g''(w))}{w\gamma(g'(w)+\delta g''(w))+(1-\gamma)(\delta w g'(w)+(1-\delta)\, g(w))}-1\right] = [h(w)]^{\alpha}. \qquad (2.6)$$

Where $r(z)$ and $h(w)$ are in p and have the following series representations :

$$r(z) = 1 + r_1 z + r_2 z^2 + r_3 z^3 + \cdots \qquad (2.7)$$

and

$$h(w) = 1 + h_1 w + h_2 w^2 + h_3 w^3 + \cdots. \qquad (2.8)$$

Since

$$1+\frac{1}{\tau}\left[\frac{z(f'(z)+\gamma z f''(z))}{\gamma z(f'(z)+\delta z f''(z))+(1-\gamma)(\delta z f'(z)+(1-\delta)f(z))}-1\right] = 1+\frac{1}{\tau}(1-\delta+\gamma-\delta\gamma)a_2 z+\frac{1}{\tau}((2-2\delta+4\gamma-4\delta\gamma)a_3 -(1+2\gamma+\gamma^2-2\delta^2\gamma-\delta^2-\delta^2\gamma^2)\, a_2^2 z^2 + \cdots, \qquad (2.9)$$

and

$$1+\frac{1}{\tau}\left[\frac{w(g'(w)+\gamma w\, g''(w))}{w\gamma(g'(w)+\delta g''(w))+(1-\gamma)(\delta w g'(w)+(1-\delta)\, g(w))}-1\right] = 1-\frac{1}{\tau}(1-\delta+\gamma-\delta\gamma)a_2 w+\frac{1}{\tau}((2-2\delta+4\gamma-4\delta\gamma)(2a_2^2-a_3)-(1+2\gamma+\gamma^2-2\delta^2\gamma-\delta^2-\delta^2\gamma^2)a_2^2 w^2+\cdots. \qquad (2.10)$$

Now , equating the coefficients in (2.5) and (2.6), we get

$$\frac{1}{\tau}\ (1-\delta+\gamma-\delta\gamma)a_2 = \alpha r_1\ , \qquad (2.11)$$

$$\frac{1}{\tau}((2-2\delta+4\gamma-4\delta\gamma)a_3-(1+2\gamma+\gamma^2-2\delta^2\gamma-\delta^2 -\delta^2\gamma^2)a_2^2) = \alpha r_2 + r_1^2\,\frac{\alpha(\alpha-1)}{2}, \qquad (2.12)$$

$$-\frac{1}{\tau}(1-\delta+\gamma-\delta\gamma)a_2 = \alpha h_1\ , \qquad (2.13)$$

and

$$\frac{1}{\tau}((2-2\delta+4\gamma-4\delta\gamma)(2a_2^2-a_3)-(1+2\gamma+\gamma^2-2\delta^2\gamma -\delta^2-\delta^2\gamma^2)a_2^2 = \alpha h_2 + h_1^2\,\frac{\alpha(\alpha-1)}{2}\ . \qquad (2.14)$$

From (2.11) and (2.13) , we find

$$r_1 = -h_1 \qquad (2.15)$$

and

$$\frac{2}{\tau^2}(1-\delta+\gamma-\delta\gamma)^2 a_2^2 = \alpha^2(r_1^2+h_1^2). \qquad (2.16)$$

Also, from (2.12), (2.14) and (2.16), we find that

$$\frac{2}{\tau}((2-2\delta+4\gamma-4\delta\gamma)a_2^2 - (1+2\gamma+\gamma^2-2\delta^2\gamma-\delta^2-\delta^2\gamma^2)a_2^2) = \alpha(\ r_2+h_2)+\frac{\alpha(\alpha-1)}{2}(r_1^2+h_1^2) = \alpha(\ r_2+h_2)+\frac{(\alpha-1)}{\alpha\tau^2}\ (1-\delta+\gamma-\delta\gamma)^2\ a_2^2\ . \qquad (2.17)$$

Therefore ,we obtain

$$a_2^2 = \frac{\alpha^2\tau^2(r_2+h_2)}{2\alpha\tau((2-2\delta+4\gamma-4\delta\gamma)-(1+2\gamma+\gamma^2-2\delta^2\gamma-\delta^2-\delta^2\gamma^2))+(1-\alpha)(1-\delta+\gamma-\delta\gamma)^2}.$$

Applying Lemma (1.1) for the coefficients r_2 and h_2, we readily get

$$|a_2| \le \frac{2\alpha|\ \tau|}{\sqrt{|2\tau\alpha((2-2\delta+4\gamma-4\delta\gamma)-(1+2\gamma+\gamma^2-2\delta^2\gamma-\delta^2-\delta^2\gamma^2))+(1-\alpha)(1-\delta+\gamma-\delta\gamma)^2|}}.$$

The last inequality gives the desired estimate on $|a_2|$ given in (2.3).

Next, in order to find the bound on $|a_3|$, by subtracting (2.12) and (2.14), we get

$$\frac{1}{\tau}(2-2\delta+4\gamma-4\delta\gamma)(2a_3-2a_2^2) = (\alpha r_2+r_1^2\ \frac{\alpha(\alpha-1)}{2}) -(\ \alpha h_2+h_1^2\frac{\alpha(\alpha-1)}{2}). \qquad (2.18)$$

It follows from (2.15) , (2.16) and (2.18), that

$$a_3 = \frac{\alpha\tau(r_2-h_2)}{2(2-2\delta+4\gamma-4\delta\gamma)} + \frac{\alpha^2\tau^2(r_1^2-h_1^2)}{2(1-\delta+\gamma-\delta\gamma)^2}\ .$$

Applying Lemma (1.1) once again for the coefficients r_1 , r_2, h_1 and h_2 , we immediately

$$|a_3| \leq |\frac{2\alpha|\tau|}{(2-2\delta+4\gamma-4\delta\gamma)} + \frac{4|\tau|^2\alpha^2}{(1-\delta+\gamma-\delta\gamma)^2}.$$

This complete the proof of Theorem (2.2) .

3. Coefficient Bounds for Function class $S_\Sigma(\tau,\gamma,\delta;\beta)$

To prove our main results , we need to introduce the following definition .

Definition 3.1. A function $f(z)$ given by (1.1) is said to be in the class $S_\Sigma(\tau,\gamma,\delta;\beta)(\tau \in \mathbb{C}\backslash\{0\}, 0 \leq \gamma \leq 1, 0 \leq \delta < 1, 0 \leq \beta < 1)$ if the following conditions are satisfied :

$$f \in \Sigma, \mathrm{Re}\left(1+\frac{1}{\tau}\left[\frac{z(f'(z)+\gamma z f''(z))}{\gamma z(f'(z)+\delta z f''(z))+(1-\gamma)(\delta z f'(z)+(1-\delta)f(z))} - 1\right]\right) > \beta \quad (z \in U) \qquad (3.1)$$

and

$$g \in \Sigma, \mathrm{Re}\left(1+\frac{1}{\tau}\left[\frac{w(g'(w)+\gamma w g''(w))}{w\gamma(g'(w)+\delta g''(w))+(1-\gamma)(\delta w g'(w)+(1-\delta)g(w))} - 1\right]\right) > \beta \quad (w \in U), \qquad (3.2)$$

where the function $g(w)$ is given by (1.2).

Theorem 3.2. Let $f(z)$ given by (1.1) be in the class $S_\Sigma(\tau,\gamma,\delta;\beta)$ $(\tau \in \mathbb{C}\backslash\{0\}, 0 \leq \gamma \leq 1, 0 \leq \delta < 1, 0 \leq \beta < 1)$. Then

$$|a_2| = \sqrt{\frac{2|\tau|(1-\beta)}{|(1+2\gamma-\gamma^2-2\delta-4\delta\gamma+2\delta^2\gamma+\delta^2+\delta^2\gamma^2)|}} \qquad (3.3)$$

and

$$|a_3| \leq \frac{|\tau|(1-\beta)}{|1-\delta+2\gamma-2\delta\gamma|} + \frac{4|\tau|^2(1-\beta)^2}{|(1-\delta+\gamma-\delta\gamma)^2|}. \qquad (3.4)$$

Proof : Let $f(z) \in S_\Sigma(\tau,\gamma,\delta;\beta)$. Then

$$1+\frac{1}{\tau}\left[\frac{z(f'(z)+\gamma z f''(z))}{\gamma z(f'(z)+\delta z f''(z))+(1-\gamma)(\delta z f'(z)+(1-\delta)f(z))} - 1\right] = \beta+(1-\beta)r(z) \qquad (3.5)$$

and

$$1+\frac{1}{\tau}\left[\frac{w(g'(w)+\gamma w g''(w))}{w\gamma(g'(w)+\delta g''(w))+(1-\gamma)(\delta w g'(w)+(1-\delta)g(w))} - 1\right] = \beta+(1-\beta)h(w), \qquad (3.6)$$

where $g(w) = f^{-1}(w), r(z)$ and $h(w)$ have form (2.7)and (2.8), respectively.

Now, equating the coefficients in (3.5) and (3.6) , we get

$$\frac{1}{\tau}(1-\delta+\gamma-\delta\gamma)a_2 = (1-\beta)r_1, \qquad (3.7)$$

$$\frac{1}{\tau}\left((2-2\delta+4\gamma-4\delta\gamma)a_3 - (1+2\gamma+\gamma^2-2\delta^2\gamma-\delta^2-\delta^2\gamma^2)a_2^2\right) = (1-\beta)r_2, \qquad (3.8)$$

$$-\frac{1}{\tau}(1-\delta+\gamma-\delta\gamma)a_2 = (1-\beta)h_1, \qquad (3.9)$$

and

$$\frac{1}{\tau}\left((2-2\delta+4\gamma-4\delta\gamma)(2a_2^2-a_3) - (1+2\gamma+\gamma^2-2\delta^2\gamma-\delta^2-\delta^2\gamma^2)a_2^2\right) = (1-\beta)h_2. \qquad (3.10)$$

From (3.7) and (3.9), we obtain

$$r_1 = -h_1 \qquad (3.11)$$

and

$$\frac{2}{\tau^2}(1-\delta+\gamma-\delta\gamma)^2a_2^2 = (1-\beta)^2(r_1^2+h_1^2). \qquad (3.12)$$

Also , from (3.8) and (3.10) , we have

$$\frac{2}{\tau}\left((2-2\delta+4\gamma-4\delta\gamma) - (1+2\gamma+\gamma^2-2\delta^2\gamma-\delta^2-\delta^2\gamma^2)\right)a_2^2 = (1-\beta)(r_2+h_2). \qquad (3.13)$$

Therefore , we get

$$a_2^2 = \frac{\tau(1-\beta)(r_2+h_2)}{2((2-2\delta+4\gamma-4\delta\gamma)-(1+2\gamma+\gamma^2-2\delta^2\gamma-\delta^2-\delta^2\gamma^2))}. \qquad (3.14)$$

Applying Lemma (1.1) for coefficients r_2 and h_2 , we obtain

$$|a_2| \leq \sqrt{\frac{2|\tau|(1-\beta)}{|(1+2\gamma-\gamma^2-2\delta-4\delta\gamma+2\delta^2\gamma+\delta^2+\delta^2\gamma^2)|}}. \qquad (3.15)$$

This gives the bound on $|a_2|$ as asserted in (3.3).

Next in order to find the bound on $|a_3|$, by subtracting (3.8) and (3.10) , we thus get

$$\frac{1}{\tau}(2-2\delta+4\gamma-4\delta\gamma)(2a_3-2a_2^2)=(1-\beta)\,(r_2-h_2) \quad (3.16)$$

or, equivalently,

$$a_3=\frac{\tau(1-\beta)(r_2-h_2)}{4(1-\delta+2\gamma-2\delta\gamma)}+a_2^2 \quad (3.17)$$

It follows from (3.12) and (3.17) , that

$$a_3=\frac{\tau(1-\beta)(r_2-h_2)}{4(1-\delta+2\gamma-2\delta\gamma)}+\frac{\tau^2(1-\beta)^2(r_1^2+h_1^2)}{2(1-\delta+\gamma-\delta\gamma)^2}.$$

Applying Lemma (1.1) once again for the coefficients r_1 , r_2 , h_1 and h_2 , we obtain

$$|a_3|\leq\frac{|\tau|(1-\beta)}{|(1-\delta+2\gamma-2\delta\gamma)|}+\frac{4|\tau|^2(1-\beta)^2}{|(1-\delta+\gamma-\delta\gamma)^2|}.$$

This completes the prove of Theorem (3.2) .

4. Corollaries and Consequence

This section is devoted to the presentation of some special cases of the main results .

These results are given in the form of corollaries :

If we set τ=1 and δ=0 in Theorems (2.2) and (3.2) , then ,we get following results due to Keerthi and Raja [9] :

Corollary 4.1. Let $f(z)$ given by (1.1) be in the class $\mathcal{B}_\Sigma\,(\gamma;\alpha)(0\leq\gamma\leq 1, 0<\alpha\leq 1)$. Then

$$|a_2|\leq\frac{2\alpha}{\sqrt{|4\alpha(1+2\gamma)+(1-3\alpha)(\gamma+1)^2|}}$$

and

$$|a_3|\leq\frac{4\alpha^2}{(1+\gamma)^2}+\frac{\alpha}{1+2\gamma}.$$

Corollary 4.2. Let $f(z)$ given by (1.1) be in the class $\mathcal{B}_\Sigma\,(\gamma\,;\beta)\,(0\leq\gamma\leq 1, 0\leq\beta<1)$. Then

$$|a_2|\leq\sqrt{\frac{2(1-\beta)}{|1+2\gamma-\gamma^2|}}$$

and

$$|a_3|\leq\frac{4(1-\beta)^2}{(1+\gamma)^2}+\frac{1-\beta}{1+2\gamma}.$$

The classes $\mathcal{B}_\Sigma\,(\gamma;\alpha)$ and $\mathcal{B}_\Sigma\,(\gamma\,;\beta)$ are respectively defined as follows:

Definition 4.3. A function $f(z)$ given by (1.1) is said to be in the class $\mathcal{B}_\Sigma\,(\gamma;\alpha)$ $(0\leq\gamma\leq 1, 0<\alpha\leq 1)$ if the following conditions are satisfied:

$$f\in\Sigma,\left|\arg\left(\frac{z(f'(z)+\gamma zf''(z))}{(1-\gamma)f(z)+\gamma zf'(z)}\right)\right|<\frac{\alpha\pi}{2}\quad(z\in U)$$

and

$$g\in\Sigma,\left|\arg\left(\frac{w(g'(w)+\gamma wg''(w))}{(1-\gamma)g(w)+\gamma wg'(w)}\right)\right|<\frac{\alpha\pi}{2},\quad(w\in U)$$

where the function $g(w)$ is given by (1.2) .

Definition 4.4. A function $f(z)$ given by (1.1) is said to be in the class $\mathcal{B}_\Sigma(\beta;\gamma)(0\leq\gamma\leq 1, 0\leq\beta<1)$if the following conditions are satisfied:

$$f\in\Sigma\,,\mathrm{Re}\left(\frac{z(zf'(z)+\gamma zf''(z))}{(1-\gamma)f(z)+\gamma zf'(z)}\right)>\beta\quad(z\in U)$$

and

$$g\in\Sigma\,,\mathrm{Re}\left(\frac{w(g'(w)+\gamma wg''(w))}{(1-\gamma)g(w)+\gamma zg'(w)}\right)>\beta\,,\quad(w\in U)$$

where the function $g(w)$ is given by (1.2).

If we set $\tau=1$ and $\gamma=0$ in Theorems (2.2) and (3.2), then the classes $S_\Sigma\,(\tau,\gamma,\delta;\alpha)$ and $S_\Sigma\,(\tau,\gamma,\delta;\beta)$ reduce to the classes $\mathcal{G}_\Sigma\,(\delta;\alpha)$ and $\mathcal{G}_\Sigma\,(\delta;\beta)$ investigated by Murugusundaramoorthy et al. [10] ,which are defined as follows :

Definition 4.5. *A* function $f(z)$ given by (1.1) is said to be in the class $\mathcal{G}_\Sigma\,(\delta;\alpha)(0<\alpha\leq 1, 0\leq\delta<1)$ if the following conditions are satisfied:

$$f\in\Sigma,\left|\arg\left(\frac{zf'(z)}{(1-\delta)f(z)+\delta f'(z)}\right)\right|<\frac{\alpha\pi}{2}\quad(z\in U)$$

and

$$g\in\Sigma,\left|\arg\left(\frac{wg'(w)}{(1-\delta)g(w)+\delta g'(w)}\right)\right|<\frac{\alpha\pi}{2},\quad(w\in U)$$

where the function $g(w)$ is given by (1.2).

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Definition 4.6. A function $f(z)$ given by (1.1) is said to be in the class $\mathcal{G}_{\Sigma}(\delta;\beta)(0 \le \beta < 1, 0 \le \delta < 1)$ if the following conditions are satisfied:

$$f \in \Sigma, \mathrm{Re}\left(\frac{zf'(z)}{(1-\delta)f(z)+\delta f'(z)}\right) > \beta \qquad (z \in U)$$

where and

$$g \in \Sigma, \mathrm{Re}\left(\frac{wg'(w)}{(1-\delta)g(w)+\delta g'(w)}\right) > \beta, \quad (w \in U)$$

the function $g(w)$ is is given by (1.2).

In this case .Theorems (2.2) and (3.2) reduce to the following:

Corollary 4.7. Let $f(z)$ given by (1.1)be in the class $\mathcal{G}_{\Sigma}(\delta;\alpha)(0 < \alpha \le 1, 0 \le \delta < 1)$.Then

$$|a_2| \le \frac{2\alpha}{(1-\delta)\sqrt{(\alpha+1)}}$$

and

$$|a_3| \le \frac{4\alpha^2}{(1-\delta)^2} + \frac{\alpha}{1-\delta}.$$

Corollary 4.8. Let $f(z)$ given by (1.1) be in the class $\mathcal{G}_{\Sigma}(\delta;\alpha)(0 \le \beta < 1, 0 \le \delta < 1)$.Then

$$|a_2| \le \frac{\sqrt{2(1-\beta)}}{(1-\delta)}$$

and

$$|a_2| \le \frac{4(1-\beta)^2}{(1-\delta)^2} + \frac{1-\beta}{(1-\delta)}.$$

Letting $\tau = 1$ and $\gamma = 1$ in Theorems (2.2) and (3.2) gives the following corollaries:

Corollary 4.9. Let $f(z)$ given by (1.1) be in the class $\mathcal{D}_{\Sigma}(\delta;\alpha)(0 < \alpha \le 1, 0 \le \delta < 1)$.Then

$$|a_2| \le \frac{2\alpha}{\sqrt{|2\alpha(6-6\delta)-(4-4\delta^2)) + (1-\alpha)(2-2\delta)^2|}}$$

and

$$|a_3| \le \frac{\alpha}{3(1-\delta)} + \frac{\alpha^2}{(1-\delta)^2}.$$

Corollary 4.10. Let $f(z)$ given by (1.1) be in the class $\mathcal{D}_{\Sigma}(\delta;\beta)(0 \le \beta < 1, 0 \le \delta < 1)$ .Then

$$|a_2| \le \sqrt{\frac{(1-\beta)}{|2-3\delta+2\delta^2|}}$$

and

$$|a_3| \le \frac{(1-\beta)}{|3-3\delta|} + \frac{(1-\beta)^2}{(1-\delta)^2}.$$

The classes $\mathcal{D}_{\Sigma}(\delta;\alpha)$and $\mathcal{D}_{\Sigma}(\tau,\delta;\beta)$ are given explicitly in the next definitions .

Definition 4.11. A function $f(z)$ given by (1.1) is said to be in the class $\mathcal{D}_{\Sigma}(\delta;\alpha)$ $(0 < \alpha \le 1, 0 \le \delta < 1)$ if the following conditions are satisfied :

$$f \in \Sigma, \left|\arg\left(\frac{f'(z)+zf''(z)}{f'(z)+\delta zf''(z)}\right)\right| < \frac{\alpha\pi}{2} \qquad (z \in U)$$

and

$$g \in \Sigma, \left|\arg\left(\frac{g'(w)+w\,g''(w)}{g'(w)+\delta wg''(w)}\right)\right| < \frac{\alpha\pi}{2}, \quad (w \in U)$$

where the function $g(w)$ is given by (1.2).

Definition 4.12. A function $f(z)$ given by (1.1) is said to be in the class $\mathcal{D}_{\Sigma}(\delta;\beta)(0 \le \beta < 1, 0 \le \delta < 1)$ if the following conditions are satisfied :

$$f \in \Sigma, \mathrm{Re}\left(\frac{f'(z)+zf''(z)}{f'(z)+\delta zf''(z)}\right) > \beta \qquad (z \in U)$$

and

$$g \in \Sigma, \mathrm{Re}\left(\frac{g'(w)+w\,g''(w)}{g'(w)+\delta g''\,g(w)}\right) > \beta, \quad (w \in U)$$

where the function $g(w)$ is given by (1.2).

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مخمنات المعامل لاصناف جزئيه من الدوال الثنائية التكافؤ

وقاص غالب عطشان رجاء علي هريس

قسم الرياضيات / كلية علوم الحاسوب وتكنولوجيا المعلومات / جامعة القادسية

المستخلص :

في هذ البحث قدمنا صفيين جزئيين جديدين من الصنف Σ متكون من الدوال ثنائية التكافؤ التحليلية في قرص الوحدة المفتوح U و حصلنا على مخمنات حول معاملات تايلر – ماكلورين $|a_2|$ و $|a_3|$ للدوال في هذه الاصناف الجزئية. حصلنا ايضا على حالات خاصة جديده لنتائجنا

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Properties of Compact fuzzy Normed Space

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Abstract

In this paper we recall the definition of fuzzy norm then basic properties of fuzzy normed space is recalled after that we introduced the definition of compact fuzzy normed space. Then basic properties of compact fuzzy normed space is proved.

KeyWords: Fuzzy normed space, fuzzy continuous operator, Uniform fuzzy continuous operator, Compact fuzzy normed Space.

Mathematics Subject Classification: **46S40**.

1.Introduction

Through his studying the notion of fuzzy topological vector spaces Katsaras in 1984 [1], was the first researcher who introduced the notion of fuzzy norm on a linear vector space. A fuzzy metric space was introduced by Kaleva and Seikkala in 1984 [2]. The notion of fuzzy norm on a linear space was introduced by Felbin in 1992 [3] in such a way that the corresponding fuzzy metric is of Kaleva and Seikkala type. Another type of fuzzy metric space was introduced by Kramosil and Michalek in [4]. The notion fuzzy norm on a linear space was introduce by Cheng and Mordeson in 1994 [5] so that the corresponding fuzzy metric is of Kramosil and Michalek type.

A finite dimensional fuzzy normed linear spaces was studied by Bag and Samanta [6] in 2003. Some results on fuzzy complete fuzzy normed spaces was studied by Saadati and Vaezpour in 2005 [7]. Fuzzy bounded linear operators on a fuzzy normed space was studied by Bag and Samanta in 2005 [8]. The fixed point theorems on fuzzy normed linear spaces of Cheng and Mordeson type was proved by Bag and Samanta in 2006, 2007 [9], [10]. The fuzzy normed linear space and its fuzzy topological structure of Cheng and Mordeson type was studied by Sadeqi and Kia in 2009 [11]. Properties of fuzzy continuous mapping on a fuzzy normed linear spaces of Cheng and Mordeson type was studied by Nadaban in 2015 [12].

2. Properties of Fuzzy normed space

In this section we recall basic properties of fuzzy normed space

Definition 2.1:[1]

Suppose that U is any set, a fuzzy set $\widetilde{A}$ in U is equipped with a membership function, $\mu_{\widetilde{A}}(u)$: U→ [0,1]. Then $\widetilde{A}$ is represented by $\widetilde{A}=\{(u,\mu_{\widetilde{A}}(u)): u\in U, 0\le \mu_{\widetilde{A}}(u)\le 1\}$.

Definition 2.2: [7]

Let $*$: $[0,1]\times[0,1]\to[0,1]$ be a binary operation then $*$ is called a continuous **t -norm** (or **triangular norm**) if for all $\alpha,\beta,\gamma,\delta\in[0,1]$ it has the following properties

(1)$\alpha*\beta=\beta*\alpha$, (2)$\alpha*1=\alpha$, (3)$(\alpha*\beta)*\gamma=\alpha*(\beta*\gamma)$

(4)If $\alpha\le\beta$ and $\gamma\le\delta$ then $\alpha*\gamma\le\beta*\delta$

Remark 2.3:[8]

(1)If $\alpha>\beta$ then there is γ such that $\alpha*\gamma\ge\beta$

(2)There is δ such that $\delta*\delta\ge\sigma$ where $\alpha,\beta,\gamma,\delta,\sigma\in[0,1]$

Definition 2.4 : [8]

The triple $(V,L,*)$ is said to be a **fuzzy normed space** if V is a vector space over the field $\mathbb{F}$, $*$ is a t-norm and $L: V\times[0,\infty)\to[0,1]$ is a fuzzy set has the following properties for all a, b$\in$ V and $\alpha,\beta>0$.

1-$L(a,\alpha)>0$

2-$L(a,\alpha)=1 \Leftrightarrow a=0$

3-$L(ca,\alpha)=L\left(a,\frac{\alpha}{|c|}\right)$ for all $c\neq 0\in\mathbb{F}$

4-$L(a,\alpha)*L(b,\beta)\le L(a+b,\alpha+\beta)$

5-$L(a,.):[0,\infty)\to[0,1]$ is continuous function of α.

6-$\lim_{\alpha\to\infty}L(a,\alpha)=1$

Remark 2.5 : [13]

Assume that $(V,L,*)$ is a fuzzy normed space and let $a\in V, t>0, 0<q<1$. If $L(a,t)>(1-q)$ then there is s with $0<s<t$ such that $L(a,s)>(1-q)$.

Definition 2.6:[6]

Suppose that $(V,L,*)$ is a fuzzy normed space. Put

$FB(a,p,t)=\{b\in X: L(a-b,t)>(1-p)\}$

$FB[a,p,t]=\{b\in X: L(a-b,t)\ge(1-p)\}$

Then $FB(a,p,t)$ and $FB[a,p,t]$ is called **open and closed fuzzy ball** with the center a $\in$V and radius p, with p>0.

Lemma 2.7 :[7]

Suppose that $(V,L,*)$ is a fuzzy normed space then $L(x-y,t)=L(y-x,t)$ for all $x,y\in V$ and $t>0$

Definition 2.8: [6]

Assume that $(V,L,*)$ is a fuzzy normed space. W $\subseteq$V is called **fuzzy bounded** if we can find $t>0$ and $0<q<1$ such that $L(w,t)>(1-q)$ for each $w\in W$.

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Definition 2.9 :[6]

A sequence (v_n) in a fuzzy normed space $(V, L,*)$ is called **converges to** $v \in V$ if for each $q > 0$ and $t > 0$ we can find N with $L[v_n - v, t] > (1 - q)$ for all $n \geq N$. Or in other word $\lim_{n\to\infty} v_n = v$ or simply represented by $v_n \to v$, v is known the limit of (v_n) or $\lim_{n\to\infty} L[v_n - v, t] = 1$.

Definition 2.10 :[8]

A sequence (v_n) in a fuzzy normed space $(V, L,*)$ is said to be a **Cauchy sequence** if for all $0 < q < 1$, $t > 0$ there is a number N with $L[v_m - v_n, t] > (1 - q)$ for all

$m, n \geq N$.

Definition 2.11:[4]

Suppose that $(V, L,*)$ is a fuzzy normed space and let W be a subset of V. Then the **closure of W** is written by $\overline{W}$ or $CL(W)$ and which is $\overline{W} = \cap\{W \subseteq B$: B is closed in V$\}$.

Lemma 2.12:[13]

Assume that $(V, L,*)$ is a fuzzy normed space and suppose that W is a subset of V. Then $y \in \overline{W}$ if and only if there is a sequence (w_n) in W with (w_n) converges to y.

Definition 2.13:[13]

Suppose that $(V, L,*)$ is a fuzzy normed space and $W \subseteq V$. Then W is called **dense** in V when $\overline{W} = V$.

Theorem 2.14:[13]

Suppose that $(V, L,*)$ is a fuzzy normed space and assume that W is a subset of V. Then W is dense in V if and only if for every $x \in V$ there is $w \in W$ such that

$L[x - w, t] > (1 - \varepsilon)$ for some $0 < \varepsilon < 1$ and $t > 0$.

Definition 2.15:[10]

A fuzzy normed space (V, L, ∗) is said to be **complete** if every Cauchy sequence in V converges to a point in V.

Definition 2.16: [8]

Suppose that $(V, L_V,*)$ and $(W, L_W,*)$ are two fuzzy normed spaces .The operator $S: V \to W$ is said to be **fuzzy continuous at v_0** $\in V$ if for all $t > 0$ and for all $0 < \alpha < 1$ there is s[depends on t , α and v_0] and there is β [depends on t, α and v_0] with, $L_V[v - v_0, s] > (1 - p)$ we have $L_W[S(v) - S(v_0), t] > (1 - \alpha)$ for all $v \in V$.

Theorem 2.17:[13]

Suppose that $(V, L_V,*)$ and $(U, L_U,*)$ are two fuzzy normed spaces.The operator $T: V \to U$ is fuzzy continuous at $a \in X$ if and only if $a_n \to a$ in V implies $T(a_n) \to T(a)$ in U.

Definition 2.18:[13]

Suppose that $(V, L_V,*)$ and $(W, L_W,*)$ are two fuzzy normed spaces. Let $S: V \to W$ be an operator S is said to be **uniformly fuzzy continuous** if for $t > 0$ and for every $0 < \alpha < 1$ there exists β [depends on t and α] and there exists $s > 0$ [depends on t and α] such that $L_W[S(x) - S(y), t] > (1 - \alpha)$ whenever $L_V[x - y, s] > (1 - \beta)$ for all $x, y \in V$

3. Compact fuzzy normed space

Definition 3.1:

Suppose that $(V, L,*)$ is a fuzzy normed space and W is a subset of V. Assume that $\Psi = \{ A \subseteq V$: A is open sets in V $\}$ where $W \subseteq \cup_{A\in\Psi} A$. Then Ψ is said to be an **open cover** or open covering of W. If $\Psi = \{A_1, A_2, \dots, A_k\}$ with $W = \cup_{i=1}^{k} A_i$ then Ψ is known as a finite **sub covering** of W.

Definition 3.2:

A fuzzy normed space $(V, L,*)$ is called **compact** if $V = \cup_{A\in\Psi} A$ where Ψ is an open covering then we can find $\{A_1, A_2, \dots, A_n\} \subset \Psi$ with $V = \cup_{i=1}^{k} A_i$.

Example 3.3:

The interval (0, 1) in the fuzzy normed space $(\mathbb{R}, L_{|.|},*)$ where $L_{|.|}(x,t) = \frac{t}{t+|x|}$ and $a * b = a \cdot b$ for all $a, b \in [0,1]$ is not compact since the collection $A_n=\{(0,\frac{1}{n})$: n=2, 3. …} form an open covering for $(0,1)$ but has no finite sub covering for (0,1).

Remark 3.4:

When W is a finite subset of the fuzzy normed space $(V, L,*)$ then W is compact

Definition 3.5:

Suppose that $(V, L,*)$ is a fuzzy normed space and $W \subseteq V$ then it is said to be totally fuzzy bounded if for any $\sigma \in (0, 1)$, $t > 0$ we can find $W_\sigma=\{a_1, a_2, \dots, a_n\}$ in W with any $v \in V$ there is some $a_i \in \{a_1, a_2, \dots, a_k\}$ with $L(v - a_i, t) > (1 - \sigma)$. Then W_σ is called σ -fuzzy net.

Proposition 3.6:

Let $(V, L,*)$ be a fuzzy normed space if V is totally fuzzy bounded then V is fuzzy bounded.

Proof:

Suppose that V is a totally fuzzy bounded and let $0 < \varepsilon < 1$ so we can find a finite ε-fuzzy net for V say S. Now put $L\left[s,\frac{t}{2}\right] = \min\left\{L\left(a,\frac{t}{2}\right) : a \in S\right\}$. Let $v\in V$ so we can find $a \in S$ with $L[v - a, t] > (1 - \varepsilon)$. Now we can find $\sigma \in (0, 1)$ with

$(1 - \varepsilon) * L[s, t] > (1 - \sigma)$, it follows that

$$L[v,t] = L[v - a + a, t] \geq L\left[v - a,\frac{t}{2}\right] * L\left[a,\frac{t}{2}\right]$$
$$\geq (1 - \varepsilon) * L\left[s,\frac{t}{2}\right] > (1 - \sigma).$$

Hence V is fuzzy bounded.

Theorem 3.7:

Suppose that $(V, L,*)$ is a fuzzy normed space and assume that $W \subseteq V$. Then W is totally fuzzy bounded if and only if every sequence in W contains a Cauchy subsequence.

Proof:

Let W be totally fuzzy bounded. Suppose that $(w_n) \in W$. Choose a finite 0.5- fuzzy net in W then we can find a fuzzy open ball of radius 0.5 its center in the 0.5- fuzzy net contains infinite members of (w_n). Let $\left(w_n^{(1)}\right)$ denote this subsequence. Choose finite 0.25- fuzzy net in W. So we can find a fuzzy balls of radius 0.25- where its center in the finite 0.25 fuzzy net contains infinite members of $\left(w_n^{(1)}\right)$. Let $\left(w_n^{(2)}\right)$ denote this subsequence. Continue in this process we get a sequence of sequences each is a subsequence of proceeding one, so that $\left(w_n^{(j)}\right)$ lies in the fuzzy ball of radius $\frac{1}{2^j}$ with center in the $\frac{1}{2^j}$ fuzzy net. Now $\left(w_n^{(n)}\right) \subseteq (w_n)$. Now when $0 < \varepsilon < 1$ be given and $t > 0$, let $\left(1 - \frac{1}{2^j}\right) * \left(1 - \frac{1}{2^{j+1}}\right) * \dots * \left(1 - \frac{1}{2^{k-1}}\right) > (1 - \varepsilon)$. then for all $k > j \geq N$ where N is positive number, we have $L\left[w_j^{(j)} - w_k^{(k)}, t\right] \geq L\left[w_j^{(j)} - w_{j+1}^{(j+1)}, \frac{t}{k-j}\right] * L\left[w_{j+1}^{(j+1)} - w_{j+2}^{(j+2)}, \frac{t}{k-j}\right] * \dots * L\left[w_{k-1}^{(k-1)} - w_k^{(k)}, \frac{t}{k-j}\right] \geq \left(1 - \frac{1}{2^j}\right) * \left(1 - \frac{1}{2^{j+1}}\right) * \dots * \left(1 - \frac{1}{2^{k-1}}\right) > (1 - \varepsilon)$.

Hence $\left(w_j^{(j)}\right)$ is a Cauchy.

Conversely, suppose that every sequence in W has a Cauchy subsequence. Let $0 < \varepsilon < 1$ be given and $t > 0$. Let $w_1 \in W$. if $W - FB(w_1, \varepsilon, t) = \emptyset$, we find an ε- fuzzy net, namely $\{w_1\}$ otherwise choose $w_2 \in W - FB(w_1, \varepsilon, t)$. if $W - [FB(w_1, \varepsilon, t) \cup FB(w_2, \varepsilon, t)] = \emptyset$. we found an ε -fuzzy net namely $\{w_1, w_2\}$. After finite steps this process will stops.

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If it does not stop, we will get $(w_n) \in W$ with $L[w_n - w_m, t] \leq (1-\varepsilon), n \neq m$. That is (w_n) has no Cauchy subsequence, which is contradiction.

Proposition 3.8:

If the fuzzy normed space $(V, L, *)$ is compact then it is totally fuzzy bounded.

Proof:

For any given $0 < \varepsilon < 1$ and $t > 0$ the collection of all fuzzy balls $FB(v, \varepsilon, t)$ is an open cover for V. But V is compact hence this cover contains a finite sub cover say $\{FB(v_1, \varepsilon, t), FB(v_2, \varepsilon, t), \dots, FB(v_k, \varepsilon, t)\}$ thus the finite set $\{v_1, v_2, \dots, v_k\}$ is ε-fuzzy net for V. Hence V is totally fuzzy bounded.

Proposition 3.9:

If $(V, L, *)$ is compact fuzzy normed space then it is complete.

Proof:

Suppose that $(V, L, *)$ is not complete then there exists a Cauchy sequence (v_n) in V has no limit in V. Let $v \in V$ and since (v_n) is not converge to v so we can find $\sigma \in (0, 1)$, $t > 0$ with $L[v_n - v, t] \leq (1-\sigma)$ for infinite members. But (v_n) is Cauchy so we can find $N \in \mathbb{N}$ with $L[v_n - v_m, t] > (1-\sigma)$ for all $n, m \geq N$. Choose $m \geq N$ for which $L[v_m - v, t] > (1-\sigma)$ so the open fuzzy ball $FB(v, \sigma, t)$ contains finite members of v_n. In this manner, so for any $v \in V$ we can find a fuzzy open ball $FB(v, \sigma(v), t)$, with $\sigma(v) \in (0, 1)$ depends on v and an open fuzzy ball $FB(v, \sigma(v), t)$ which contains finite of v_n. Now $V = \cup_{v \in V} FB(v, \sigma(v), t)$ that is $\{FB(v, \sigma(v), t) : v \in V\}$ is an open covering for V using V is compact we have

$V = \cup_{j=1}^{k} FB\left(v^{(j)}, \sigma\left(v^{(j)}\right), t\right)$ but any $FB\left(v^{(j)}, \sigma\left(v^{(j)}\right), t\right)$ contains finite of v_n. This means that V must contain finite of v_n. But this is impossible hence V must be complete.

Lemma 3.10:

Suppose that $(V, L, *)$ is fuzzy normed space and $W \subset V$. If V is totally fuzzy bounded then so is W.

Proof:

Let $S = \{v_1, v_2, \dots, v_k\}$ be ε -fuzzy net for V then for any $v \in V$

$L[v - v_j, t] > (1-\varepsilon)$ for $t > 0$ and some $v_j \in S$. now let $S_1 = \{e_1, e_2, \dots, e_m\} \subset W$. Then $L[e_j - v_n, t] > (1-\varepsilon)$ for each $1 < j < m$

and for some $v_n \in S$. Now

$$L[v - e_j, t] = L[v - v_n + v_n - e_j, t]$$
$$\geq L\left[v - v_n, \frac{t}{2}\right] * L\left[v_n - e_j, \frac{t}{2}\right]$$
$$\geq (1-\varepsilon) * (1-\varepsilon) > (1-r)$$

For some $0 < r < 1$ hence W is totally fuzzy bounded

Theorem 3.11:

If $(V, L, *)$ is totally fuzzy bounded and complete fuzzy normed space then V is compact.

Proof:

Suppose V is not compact then $V = \cup_{\lambda \in \Lambda} G_\lambda$ and $V \neq \cup_{i=1}^{n} G_i$. But V is totally fuzzy bounded it is fuzzy bounded by proposition (3.6), hence for some $\sigma \in (0, 1)$ and some $v \in V, t > 0$, we have $V \subseteq FB(v, \sigma, t)$ which implies that $V = FB(v, \sigma, t)$. let $\varepsilon_n = \frac{\sigma}{2^n}$ since V is totally fuzzy bounded so it can be covered by finite many fuzzy balls of radius ε_1 but by our assumption there is $FB(v_1, \varepsilon_1, t) \neq \cup_{i=1}^{n} G_i$. But $FB(v_1, \varepsilon_1, t)$ is it self totally fuzzy bounded by Lemma (3.10), so we can find $v_2 \in FB(v_1, \varepsilon_1, t)$ such that $FB(v_2, \varepsilon_2, t) \neq \cup_{i=1}^{n} G_i$. Thus there is a sequence $(v_n) \in V$ with $FB(v_n, \varepsilon_n, t) \neq \cup_{i=1}^{n} G_i$ and $v_{n+1} \in FB(v_n, \varepsilon_n, t)$. Since $v_{n+1} \in FB(v_n, \varepsilon_n, t)$ it follows that

$L[v_n - v_{n+1}, t] > (1-\varepsilon_n)$, let $0 < \varepsilon < 1$ be given with

$(1-\varepsilon_n)*(1-\varepsilon_{n+1})*\ldots*(1-\varepsilon_m)>(1-\varepsilon)$.

Hence for $m>n$ $L[v_n-v_m,t] \geq L\left[v_n-v_{n+1},\frac{t}{m-n}\right]*$

$$L\left[v_{n+1}-v_{n+2},\frac{t}{m-n}\right]*\ldots*L\left[v_{m-1}-v_m,\frac{t}{m-n}\right]$$

$\geq(1-\varepsilon_n)*(1-\varepsilon_{n+1})*\ldots*(1-\varepsilon_m)>(1-\varepsilon)$

So (v_n) is a Cauchy sequence in V but V is complete so $v_n\to y$ since $y\in V$ there is $\lambda_0\in\Lambda$ such that $y\in G_{\lambda_0}$. since G_{λ_0} is open it contains $FB(y,\delta,t)$ for some $0<\delta<1$. Choose a positive number N such that $L[v_n-y,t]>(1-\delta)$ for all $n\geq N$ and $(1-\varepsilon_n)>(1-\delta)$ then for any $v\in V$ with $L\left[v-v_n,\frac{t}{2}\right]>(1-\varepsilon_n)$. So

$$L[v-y,t]\geq L\left[v-v_n,\frac{t}{2}\right]*L\left[v_n-y,\frac{t}{2}\right]$$

$$\geq(1-\delta)*(1-\delta)>(1-r)$$

for some $0<r<1$. So that $FB(v_n,\varepsilon_n,t)\subseteq FB(y,r,t)$. Therefore $FB(v_n,\varepsilon_n,t)$ has a finite sub covering namely the set $G_{\lambda_\circ}$. This contradicts that $V\neq\cup_{i=1}^{n}G_i$.

Proposition 3.12:

Suppose that $(V,L,*)$ is a fuzzy normed space. Then for any set S={v_n : 1≤n< ∞} in V has at least one limit point v in V if and only if every (v_n) in V contains (v_{n_k}) with $v_{n_k}\to v$.

Proof:

Let $(v_n)\in V$ when S={$v_1,v_2,\ldots v_k$} then choose $v_j\in S$. Thus $(v_j,v_j,\ldots)\in(v_n)$ and converges to v_j. Suppose that the set S is infinite. Then by our assumption it has at least one limit point $v\in V$. Let $n_1\in\mathbb{N}$ with

$L[v_{n_1}-v,t]>0$. let $n_{k+1}\in\mathbb{N}$ with $n_{k+1}>n_k$ and

$L[v_{n_{k+1}}-v,t]>\left(1-\frac{1}{(k+1)}\right)$. Then $v_{n_k}\to v$.

Conversely let S={v_n : 1≤n< ∞} ⊂ V. Then we can find $(v_n)\in V$ with $v_i\neq v_j$ so by our assumption (v_n) has a subsequence (v_{n_k}) of distinct with $v_{n_k}\to v\in V$. Thus any FB(v,σ,t) contains an infinite members of (v_{n_k}). Hence any FB(v,σ,t) contains infinite members of S. This means that $v\in V$ is a limit point of S.

Theorem 3.13:

The fuzzy normed space $(V,L,*)$ is compact if and only if for any (v_n) in V contains (v_{n_k}) with $v_{n_k}\to v$.

Proof:

Let V be compact then V is totally fuzzy bounded and complete by proposition (3.8) and proposition (3.9). Suppose that $(v_n)\in V$ since V is totally fuzzy bounded using theorem (3.7) we have (v_n) contains a Cauchy (v_{n_k}). So $v_{n_k}\to v\in V$ since V is complete. Hence every (v_n) in V contains (v_{n_k}) with $v_{n_k}\to v$.

To prove the converse let every (v_n) in V contains (v_{n_k}) with $v_{n_k}\to v$.

Now by using theorem (3.7) we have V is totally fuzzy bounded. To prove that V is complete. Let (v_n) be a Cauchy sequence in V so (v_n) contains (v_{n_k}) with $v_{n_k}\to v\in V$. We now prove that $v_n\to v$. Let $0<\varepsilon<1$ be given and $t>0$ by remark (2.3) there is $0<r<1$ such that $(1-r)*(1-r)>(1-\varepsilon)$. Now $v_{n_k}\to v$ there is N_1 such that $L\left[v_{n_k}-v,\frac{t}{2}\right]>(1-r)$ for all $n_k>N_1$. But (v_n) is Cauchy there is N_2 with $L\left[v_n-v_m,\frac{t}{2}\right]>(1-r)$ for any $m,n>N_2$. Now let $N=N_1\wedge N_2$ then for all $n\geq N$, $L[v_n-v,t]\geq L\left[v_n-v_{n_k},\frac{t}{2}\right]*L\left[v_{n_k}-v,\frac{t}{2}\right]>(1-r)*(1-r)>(1-\varepsilon)$

hence (v_n) converges to $v\in V$.

Corollary 3.14:

Suppose that $(V, L,*)$ a compact fuzzy normed space and $W \subset V$. If W is closed then W is compact

Proof:

Assume that $(w_n) \in W$ then $(w_n) \in V$ so (w_n) has a subsequence (w_{n_k}) converges to $w \in W$. Then $w \in W$ since W is closed. Hence W is compact by theorem (3.13)

Proposition 3.15:

Suppose that $(V, L,*)$ a fuzzy normed space and $W \subset V$. If W is compact then W is closed

Proof:

Assume that $v \in V$ be a limit point of W then there is $(w_n) \in W$ with $w_n \to v$ so (w_n) is Cauchy sequence in W. Since W is complete by proposition (3.9) so (w_n) converges to $w \in W$. Therefore v= $w \in W$ this implies that W has all it limit points. Hence W is closed.

Theorem 3.16:

Suppose that $(V, L_V,*)$ and $(U, L_U,*)$ are two fuzzy normed spaces and $T: V \to U$ be fuzzy continuous operator. If V is compact then T(V) is compact

Proof:

Assume that $(T(v_n)) \in T(V)$ then $(v_n) \in V$. So $v_{n_k} \to v$ since V is compact. Hence by Theorem 2.17 $Tv_{n_k} \to T(v) \in T(V)$ since T is continuous. Thus by Theorem 3.13 T(V) is compact.

Theorem 3.17:

Suppose that $(V, L_V,*)$ is a compact fuzzy normed space and assume that $(U, L_U,*)$ is a fuzzy normed space. Suppose that $T: V \to U$ is a fuzzy continuous operator. Then T is uniformly fuzzy continuous that is for each $0 < \varepsilon < 1$ and $t > 0$ there exists $\delta, 0 < \delta < 1$ and $s > 0$ [δ depending on ε only] such that $T(FB(v, \delta, s)) \subset FB(T(v), \varepsilon, t)$ for all $v \in V$.

Proof:

Let $\sigma \in (0, 1)$ with $(1 - \sigma) * (1 - \sigma) > (1 - \varepsilon)$ for some $0 < \varepsilon < 1$ and $t > 0$. then the collection of fuzzy balls $\{FB(u, \sigma, t): u \in U\}$ from an open cover for U. since T is fuzzy continuous then the set $\{T^{-1}[FB(u, \sigma, t): u \in U]\}$ from an open cover for V but V is compact so the set $\{T^{-1}[FB(u_1, \sigma_1, t)], T^{-1}[FB(u_2, \sigma_2, t)], \dots, T^{-1}[FB(u_k, \sigma_k, t)]\}$ cover V that is $V = \bigcup_{j=1}^{k} T^{-1}[FB(u_j, \sigma_j, t)]$. Now let $0 < \delta < 1$ be $\delta > \sigma_j$ for some $1 < j < k$. Thus for each $v \in V$ the fuzzy ball $FB(v, \delta, s)$ lies in $T^{-1}[FB(u_j, \sigma_j, t)]$ so $T[FB(v, \delta, s)] \subseteq FB(u, \sigma, t)$ for some $u \in U$. Since $T(v) \in FB(u, r, t)$ we can find $z \in B(v, \delta, s)$ with

$$L_U[T(z) - T(v), t] \geq L_U\left[T(z) - u, \frac{t}{2}\right] * L_U\left[u - T(v), \frac{t}{2}\right] \geq (1 - \sigma) * (1 - \sigma) > (1 - \varepsilon)$$

Thus $T[FB(v, \delta, s)] \subseteq FB(T(v), \varepsilon, t)$

Conclusion

The principle goal of this research is to continue the study of fuzzy normed space and introduce more notions or results. In this paper the notion compact fuzzy normed is introduced and basic results properties of this space is proved.

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خواص فضاء القياس الضبابي المتراص

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المستخلص
في هذا البحث تم اعادة استخدام تعريف القياس الضبابي ثم تم استعراض الخواص الاساسية لفضاء القياس الضبابي بعد ذلك عرفنا الفضاء القياسي الضبابي المتراص. وتم برهان الخواص الاساسية لفضاء القياس الضبابي المتراص.

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Topology Fuzzy Normed Algebra

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Abstract:
In this paper, we deal with the basic concepts in topology on fuzzy normed algebra, such as the balls open and balls closed. Next, we study their properties. Furthermore, the concept of derived and closure are discussed.

Keywords: Fuzzy field, fuzzy vector space, fuzzy normed space, fuzzy normed algebra .

Mathematics Subject Classification: **46S40 .**

1. Introduction

The Theory of Fuzzy sets is introduced by Lotfi Zadeh [1], and the fuzzy topology is defined by Chang [2]. Many mathematicians have tried to extend to fuzzy set theory the main notations of topologies, algebras and groups see ([3]-[4]) and others as in ([5]-[6]-[7]-[8]-[9]-[10]) .

The apprehensible of fuzzy fields and fuzzy vector spaces was defined first by S.Nanda [11] and redefined by R.Biswas [12] . Also introduced the apprehensible of fuzzy algebra over fuzzy field was defined first S,Nanda [13] and redefined by Gu and Lu [14]. Gu Wenxiang and Lu Tu [15] introduced the notions of fuzzy vector spaces. In this papers, we will present new definitions in topology is called a fuzzy normed algebra over fuzzy field. Moreover, some of their characteristics are given in this work.

2. Preliminaries:

In this section, we will recall some definitions which are needed in this work.

Definition (2.1): [15]

Let F be a field . A fuzzy set S of F is called a fuzzy field of F. If the subsequent prerequisites are gratifying :

(1) $S(a+b) \geq \min\{S(a), S(b)\}$, $\forall a, b \in F$.

(2) $S(-a) \geq S(a), \forall a \in F$.

(3) $S(ab) \geq \min\{S(a), S(b)\}$, $\forall a, b \in F$.

(4) $S(a^{-1}) \geq S(a), \forall a (\neq 0) \in F$.

We dented by (S, F).

Definition (2.2): [15]

Let (S, F) be a fuzzy field in F. A fuzzy set A in a vector space X over F is called fuzzy vector space in X and denoted by (A, X). If the subsequent properties gratifying :

(1) $A(a+b) \geq \min\{A(a), A(b)\}$, $\forall a, b \in X$.

(2) $A(-a) \geq A(a), \forall a \in X$.

(3) $A(\alpha a) \geq \min\{S(\alpha), A(a)\}$, $\forall a \in X$, and $\alpha \in F$.

If S is an usual field then prerequisite (3) above will be fungible by the subsequent axiom :

$A(\alpha a) \geq A(a), \forall \alpha \in F$ and $\forall a \in X$.

Definition (2.3): [14]

Let (S, F) be a fuzzy field in F. A fuzzy set A in algebra Xover F is called a fuzzy algebra (A, X)over fuzzy field (S, F). If the subsequent prerequisites materialized :

(1) $A(a+b) \geq \min\{A(a), A(b)\}$, $\forall a, b \in X$.

(2) $A(\alpha a) \geq \min\{S(\alpha), A(a)\}, \forall \alpha \in F$ and $a \in X$.

(3) $A(ab) \geq \min\{A(a), A(b)\}$, $\forall a, b \in X$.

(4) $S(1) \geq A(a), \forall a \in X$.

Definition (2.4): [16]

Let (S, F) be a fuzzy field in F, X be vector space over F, and let (A, X) be a fuzzy vector space over (S, F). A norm on (A, X) is a function, $\|\cdot\|: X \longrightarrow F$ gratifying the subsequent prerequisites:

(1) $S(\| a \|) \geq A(a)$ for all $a \in X$.

(2) $\| a \| \geq 0$ for all $a \in X$.

(3) $\| a \| = 0$ if and only if $a = 0$.

(4) $\| \alpha a \| = |\alpha| \, \| a \|$ for all $\alpha \in F$ and $\in X$.

(5) $\| a + b \| \leq \| a \| + \| b \|$ for all $a, b \in X$.

The tuple $(A, X, \|\cdot\|)$ is called a fuzzy normed vector space .

3. Fuzzy Normed Algebra

In this section, we will introduced and study the concept of fuzzy normed algebra

Definition (3.1):

Let(S, F) be a fuzzy field in F, and let A be a fuzzy set in algebra X over F. $(A, X, \|\cdot\|)$ is said to be a fuzzy normed algebra over fuzzy field (S, F) if:

(1) (A, X) is a fuzzy algebra .

(2) $\|\cdot\|$ is a norm on (A, X) .

(3) $\| ab \| \leq \| a \| \| b \|$ for all $a, b \in X$.

Definition (3.2):

Let $(A, X, \|\cdot\|)$ be a fuzzy normed algebra and for each $a_0 \in X$, $0 < r$. the open ball $\beta_r(a_0)$ in X of radius r and amidst at a_0 is defined by

$\beta_r(a_0) = \{a \in X : \| a - a_0 \| < r,$
$S \| a - a_0 \| \geq \min\{A(a), A(a_0)\}$

and closed ball $\overline{\beta_r}(a_0)$ in X of radius r and amidst at x_0 is defined by

$\overline{\beta_r}(a_0) = \{a \in X : \| a - a_0 \| \le r,$

$S \| a - a_0 \| \ge \min\{A(a), A(a_0)\}$

Theorem (3.3):

Every open and closed balls in fuzzy normed space are convex .

Proof:

Let $(A, X, \|\cdot\|)$ be a fuzzy normed algebra .

(1) Let a , $b \in \beta_r(a_0)$ and $0 \le \alpha \le 1$

$\Rightarrow \| a - a_0 \| < r,\ \| b - a_0 \| < r$.

We must to prove

$\alpha a + (1-\alpha)b \in \beta_r(a_0)$

$\alpha a + (1-\alpha)b - a_0$

$= \alpha(a - a_0) + (1-\alpha)(b - a_0)$

$\| \alpha a + (1-\alpha)b - a_0 \|$

$= \| \alpha(a - a_0) + (1-\alpha)(b - a_0) \|$

$\le |\alpha| \| a - a_0 \| + |1-\alpha| \| b - a_0 \|$

$< \alpha r + (1-\alpha)r = r$.

Inasmuch$|1-\alpha| = 1-\alpha, |\alpha| = \alpha$ because $\alpha, 1-\alpha \ge 0$.

And $S \| \alpha a + (1-\alpha)b - a_0 \|$

$\ge A(\alpha a + (1-\alpha)b - a_0)$

$= A(\alpha a + (1-\alpha)b + (-a_0))$

$\ge \min\{A(\alpha a + (1-\alpha)b), A(-a_0)\}$

$\ge \min\{A(\alpha a + (1-\alpha)b), A(a_0)\}$

(since $A(-a_0) \ge A(a_0)$)

$\Rightarrow \alpha a + (1-\alpha)b \in \beta_r(a_0)$

$\Rightarrow \beta_r(a_0)$ is a convex .

(2) Now to prove $\overline{\beta_r}(a_0)$ is a convex .

Let $a, b \in \overline{\beta_r}(a_0)$ and $0 \le \alpha \le 1$

$\Rightarrow \| a - a_0 \| \le r,\ \| b - a_0 \| \le r$.

We must to prove

$\alpha a + (1-\alpha)b \in \overline{\beta_r}(a_0)$

$\alpha a + (1-\alpha)b - a_0$

$= \alpha(a - a_0) + (1-\alpha)(b - a_0)$

$\| \alpha a (1-\alpha)b - a_0 \|$

$= \| \alpha(b - a_0) + (1-\alpha)(b - a_0) \|$

$\le |\alpha| \| a - a_0 \| + |1-\alpha| \| b - a_0 \|$

$\le \alpha r + (1-\alpha)r = r$.

Inasmuch $|1-\alpha| = 1-\alpha, |\alpha| = \alpha$ because $, 1-\alpha \ge 0$.

And $S \| \alpha a + (1-\alpha)b - a_0 \|$

$\ge A(\alpha a + (1-\alpha)b - a_0)$

$= A(\alpha a + (1-\alpha)b + (-a_0))$

$\ge \min\{A(\alpha a + (1-\alpha)b), A(-a_0)\}$

$\ge \min\{A(\alpha a + (1-\alpha)b), A(a_0)\}$

(since $A(-a_0) \ge A(a_0)$)

$\Rightarrow \alpha a + (1-\alpha)b \in \overline{\beta_r}(a_0)$

$\Rightarrow \overline{\beta_r}(a_0)$ is a convex .

Definition (3.4):

Let $(A, X, \|\cdot\|)$ be a fuzzy normed algebra and $B \subseteq X$. B is said to be an open set in X if for any $a \in B$ there exists $r > 0$ such that $\beta_r(a) \subset B$. And B is called a closed set in X if B^c is an open set in X .

Theorem (3.5):

(1) Each open ball will be an open set.

(2) Each closed ball will be a closed set.

Proof:

(1)Let $(A, X, \|\cdot\|)$ be a fuzzy normed algebra and let $a_0 \in X$, $r > 0$ (1) We must to prove $\beta_r(a_0)$ is an open set .

Let $a \in \beta_r(a_0) \Rightarrow \| a - a_0 \| < r$

$\Rightarrow r - \| a - a_0 \| > 0.$

But $r_1 = r - \| a - a_0 \| \Rightarrow r_1 > 0,$

we must to prove $\beta_{r_1}(a_0) \subset \beta_r(a_0)$.

Let $b \in \beta_{r_1}(a_0) \Rightarrow \| b - a \| < r_1$

$\Rightarrow \| b - a \| < r - \| a - a_0 \|$

$\Rightarrow \| b - a \| + \| a - a_0 \| < r$.

Inasmuch $\| b - a_0 \|$

$\le \| b - a \| + \| a - a_0 \|$

$\Rightarrow \| b - a_0 \| < r$.

And $S \| b - a_0 \| \ge A(b - a_0)$

$\ge \min\{A(b), A(a_0)\}$

$\Rightarrow b \in \beta_r(a_0)$

$\beta_r(a_0)$ is an open set .

(2) We must to prove $\overline{\beta_r}(a_0)$ is a closed set . And let $B = (\overline{\beta_r}(a_0))^c$

Inasmuch

$\overline{\beta_r}(a_0) = \{a \in X : \| a - a_0 \| \leq r,$

$S \| a - a_0 \| \geq \min\{A(a), A(a_0)\}\}$

$\Rightarrow B = \{a \in X : \| a - a_0 \| > r,$

$S \| a - a_0 \| < \min\{A(a), A(a_0)\}\}$

let $a \in B \Rightarrow \| a - a_0 \| > r.$

But $r_2 = \| a - a_0 \| - r \Rightarrow r_2 > 0,$

we must to prove $\beta_{r_2}(a_0) \subset B$.

Let $b \in \beta_{r_2}(a_0) \Rightarrow \| b - a \| < r_2$

$\Rightarrow \| b - a \| < \| a - a_0 \| - r$

$\Rightarrow \| a - a_0 \| - \| b - a \| > r$.

Inasmuch $\| a - a_0 \|$

$\leq \| a - b \| + \| b - a_0 \|$

$\Rightarrow \| a - a_0 \| - \| b - a \| \leq \| b - a_0 \|$

$\Rightarrow \| b - a_0 \| > r.$

And $S \| b - a_0 \| \geq A(b - a_0)$

$\geq \min\{A(b), A(a_0)\}.$

$\Rightarrow b \in B \Rightarrow B_{r_2}(a_0) \subset B$

$\Rightarrow B$ is an open set .

Hence $B^c = \overline{\beta_r}(a_0)$ is a closed set.

Theorem (3.6):

In any a fuzzy normed algebra $(A, X, \|\cdot\|)$ each single set is a closed and hence finite set is a closed .

proof:

Let B be a single set.

Suppose $B = \{b\}$, we must to prove B is a closed.

Let $a \in B^c \Rightarrow a \neq b$

$\Rightarrow \| b - a \| > 0$.

But $r = \| b - a \| \Rightarrow r > 0.$

Since $\| b - a \| \geq r$

$\Rightarrow b \in \beta_r(a) \Rightarrow \beta_r(a) \cap B = \emptyset$

$\Rightarrow \beta_r(a) \subset B^c \Rightarrow B^c$ is an open set

$\Rightarrow B$ is a closed set .

Now to prove each finite set is a closed .

Let C be a finite subset of X if $C = \emptyset$ the proof ends .

Either if $\neq \emptyset$.

Suppose $C = \{c_1, c_2, \dots, c_n\}$

inasmuch $\{c_i\}$ is a closed for each

$i = 1,2, \dots, n$

$\Rightarrow C = \cup_{i=1}^{n} \{c_i\}$ is a closed set in X .

Definition (3.7):

Let $(A, X, \|\cdot\|)$ be a fuzzy normed algebra and $\subseteq X$.

(1) The point$a \in X$ is called a limit point to set B if for every $r > 0$ there exists

$b \in B$ such that $a \neq b$ and if

$\| a - b \| < r,$

$S \| a - b \| \geq \min\{A(a), A(b)\}$. Set all limit point of set B is called (Derived) of set B and denotes by B'

$B' = \{a \in X : \forall r > 0, \exists b \in B \ni b \neq a$, if $\| a - b \| < r,$

$S \| a - b \| \geq \min\{A(a), A(b)\}\}.$

(2)The point $x \in X$ is called a closure point to set B if for all $r > 0$ there exist

$b \in B$ such that $\| a - b \| < r,$

$S \| a - b \| \geq \min\{A(a), A(b)\}$. The set whose elements all point closure of set B is called (Closure) of set B and denotes by $\bar{B}$

$\bar{B} = \{a \in X : \forall r > 0, \exists b \in B$

$\ni \| a - b \| < r,$

$S \| a - b \| \geq \min\{A(a), A(b)\}\}.$

Theorem (3.8):

Let $(A, X, \|\cdot\|)$ a fuzzy normed algebra and let $B \subseteq X.$

(1) $B' \subset \bar{B}$.

(2) $\bar{B} = B \cup B'$.

Proof :

(1)Let $a \in B'$ $\Longrightarrow$for all $r > 0$ there exists

$b \in B$ such that $b \neq a$ and

$\| a - b \| < r$, $S \| a - b \| \geq A(a - b)$

$\geq \min\{A(a), A(b)\}$

$\Longrightarrow$for all $r > 0$ there exists $b \in B$ such that $b \neq a$ and $\| a - b \| < r$,

$S \| a - b \| \geq A(a - b)$

$\geq \min\{A(a), A(b)\}$

$\Longrightarrow a \in \bar{B} \Longrightarrow B' \subset \bar{B}$

(2) From (1) we consider that $B' \subset \bar{B}$. This implies that $\cup B' \subset B \cup \bar{B}$. But

$B \cup \bar{B} = \bar{B}$ (since $B \subset \bar{B}$) and hence $B \cup B' \subset \bar{B}$.

Conversely, suppose $a \in \bar{B}$ there are two possibilities

(a) If $a \in B \Longrightarrow a \in B \cup B'$

$\Longrightarrow \bar{B} \subset B \cup B'$

(b) If $a \notin B$ inasmuch $a \in \bar{B} \Longrightarrow$ for all $r > 0$ there exists $b \in B$ such that $b \neq a$ and

$\| a - b \| < r$, $S \| a - b \| \geq A(a - b)$

$\geq \min\{A(a), A(b)\}$

inasmuch $b \in B \Longrightarrow b \neq a \Longrightarrow a \in B'$

$\Longrightarrow a \in B \cup B' \Longrightarrow \bar{B} \subset B \cup B'$.

Hence $\bar{B} = B \cup B'$.

Theorem (3.9):

Let B convex set in fuzzy normed algebra $(A, X, \|\cdot\|)$, then $\bar{B}$ convex set .

Proof :

Let , $b \in \bar{B}$, $0 \leq \alpha \leq 1$.

We must to prove that

$\alpha a + (1 - \alpha)b \in \bar{B}$.

Inasmuch $a, b \in \bar{B} \Longrightarrow a \in \bar{B}$

there exists sequence $\{a_n\}$ in B such that

$a_n \longrightarrow a$

and $b \in \bar{B}$ there exists sequence $\{b_n\}$ in B such that $b_n \longrightarrow b$.

Let $z_n = \alpha a_n + (1 - \alpha)b_n$.

Inasmuch $a_n, b_n \in M$ for all n

$\Longrightarrow \alpha a_n + (1 - \alpha)b_n \in M$

and $a_n \longrightarrow a$, $b_n \longrightarrow b$

$\Longrightarrow \alpha a_n + (1 - \alpha)b_n \longrightarrow \alpha a + (1 - \alpha)b$

$\Longrightarrow \alpha a + (1 - \alpha)b \in \bar{B}$

$\Longrightarrow \bar{B}$ is a convex set .

Theorem (3.10):

Let M subalgebra of fuzzy normed algebra $(A, X, \|\cdot\|)$, then $\bar{M}$ subalgebra in (A, X).

Proof:

Let , $b \in \bar{M}$ and $\mu, \beta \in F$.

We must to prove that $\mu a + \beta b \in \bar{M}$.

Inasmuch $a, b \in \bar{M} \Longrightarrow a \in \bar{M}$ there exists sequence $\{a_n\}$ in M such that

$a_n \longrightarrow a$ and $b \in \bar{M}$ there exists Sequence $\{b_n\}$ in MSuch that

$b_n \longrightarrow b$.

Let $z_n = \mu a_n + \beta b_n$.

Inasmuch $a_n, b_n \in M$ for all n

$\Longrightarrow \mu a_n + \beta b_n \in M$

and $a_n \longrightarrow a$, $b_n \longrightarrow b$

$\Longrightarrow \mu a_n + \beta b_n \longrightarrow \mu a + \beta b$.

Hence $a + \beta b \in \bar{M}$, then $\bar{M}$ subalgebra

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تبولوجيا جبر المعياري الضبابي

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المستخلص:
في هذا البحث تناولنا المفاهيم الاساسية في التبولوجيا على جبر المعياري الضبابي ،مثل الكرة المفتوحة والكرة المغلقة. بعد ذلك، درسنا خصائصها. وعلاوة على ذلك، ناقشنا مفهوم الانغلاق والاشتقاق

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Generalizing of Finite Difference Method for Certain Fractional Order Parabolic PDE's

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Abstract

Space-time fractional differential equation with integral term (S-TFDE) has been considered. The finite difference method (implicit and explicit) combined with the trapezoidal integration formula has been used to find special formula to solve this equation. The stability and convergence have been discussed. The effect of adding an integral term to the common classical equation has been considered. Graphical representation of the calculate solutions (obtained by the explicit and the implicit methods) for three numerical examples with their exact solution, are considered. All the calculations and graphs are designed with the help of MATLAB.

Keywords: fractional order PDE, fractional itegro-differential equation, fractional Parabolic

Mathematics Subject Classification: **ASMC-204.**

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1 – Introduction

Fractional order differential equations have excited, in recent years, a considerable interest both in mathematics and in applications. They were used in modeling of many physical, chemical processes and engineering. A physical mathematical approach to anomalous partial differential equations (PDE), may be based on generalized (PDE) containing derivatives of fractional order in one only (space or time), or in together space and time. It is well known that the differential equations represent local interactions in the mathematical models, while the representation of integral equation represent the global interactions of the phenomenon, see for examples [1, 2,3, 4and 5]. Many researchers used different methods to solve different models of the fractional order equations. Meerschaert and Tadiran [6] used the finite difference method to solve the space-fractional advection dispersion. R. Gorenflo, F. Mainardi [7] used Laplace transform to solve Fractional Order linear Integral and Differential Equations. J.P. Roop, [8], considered boundary value problems in R2 with the finite element method. Our main objective is studying the following fractional order equation:

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = \frac{\partial^{\beta} u(x,t)}{\partial x^{\beta}} + \int_{o}^{t} u(x,s)ds + q(x,t) \qquad (1)$$

where: $0 < \alpha \le 1$; $1< \beta \le 2$; $0\le x \le 1$; $0 \le t \le T$ with initial and boundary conditions given respectively:

$u(x,0) = f(x) \qquad 0\le x \le 1$

$u(0, t) = u(1, t) = 0$

Corresponding to the classical integro-differential parabolic form:

$$\frac{\partial\ u(x,t)}{\partial t} = \frac{\partial^{2} u(x,t)}{\partial x^{2}} + \int_{o}^{t} u(x,s)ds + q(x,t)$$

Considered by [**9**]. The effect of the integral term will be studied in both, implicit and explicit methods, when solving the class of initial boundary value space-time fractional equation (1).

2-material and method

The numerical treatment of fractional order partial differential equations has its importance because the limited use of the analytical methods In many cases there is no analytical treatment for different reasons concerning the domain under consideration or the regularity of the boundary or even the equation itself. Many authors have considered the numerical treatment of space or time fractional partial differential equations. Zhuang and Liu [10], implicit difference approximation for the time fractional diffusion equation has been considered. Also they analyzed the stability and convergence. S. Shen and F. Liu [11] proposed an explicit difference approximation for the space fractional diffusion equation and gave an error analysis. M. Meerschaert and C. Tadjeran [12] proposed finite difference approximation for fractional advection dispersion flow equations. Mainardi [13] the fundamental solution of the space-time fractional diffusion equation was discussed, he deals with the Cauchy problem for the space-time fractional diffusion equation. Gorenflo [14], a discrete random walk model for space-time fractional diffusion was proposed .Diego A. Murio[15],developed an implicit unconditionally stable finite difference scheme to solve the linear one-dimensional diffusion equation with fractional time derivatives. F. Liu, S. Shen, V. Anh and I. Turner[18], an explicit finite difference scheme for time fractional differential equation is presented. Discrete models of a non-Markovian random walk are generated for simulating random processes whose spatial probability density evolves in time according to this fractional diffusion equation. In this work proposed fractional order implicit and explicit finite difference approximation for space-time fractional heat equation with integral term (1), (S-T FDE). Riemann-Liouvill fractional derivative of order $1< \beta \le 2$, Caputo fractional derivative of order $0< \alpha \le 1$, are using, trapezoidal method has been used to approximate the integral term, studying of stability and convergence of both methods, that will be given through studying of different examples.

3-Theory and basic definitions

Riemann, Caputo and Grunewald, fractional integral and fractional derivatives that be used for approximating derivatives, will be given. Also, trapezoidal rule will be used to approximate integral term, For more detail, see [15,16,17].

3-1 Riemann-Liouville

fractional Integral of order $\beta > 0$ given by the form [1-18],

$$J_t^{\beta} f(t) = \begin{cases} \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s)ds \\ J_t^0 = I \end{cases}$$

(2) $J_t^{\beta}\ J_t^{\alpha} = J_t^{\alpha}\ J_t^{\beta} = J_t^{\alpha+\beta}$ Where $\alpha \ge 0, \beta \ge 0$ (3)

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3-2 Riemann-Liouville fractional derivative of order

let m denotes a positive integer such that m-1<β≤m, then fractional order derivative Riemann-Liouville of order β will be given by the form:

$$ {}^{R}D_t^{\beta} = D_t^m \; J_t^{m-\beta} f(t) \quad \text{This mean} $$

$$ {}^{R}D_t^{\beta} f(t) = \begin{cases} \dfrac{d^m}{dt^m}\left[\dfrac{1}{\Gamma(m-\beta)}\int_0^t (t-s)^{m-\beta-1} f(s)ds\right] & m-1<\beta<m \\ \dfrac{d^m}{dt^m} f(t) & \beta = m \end{cases} \tag{4} $$

$$ D_t^m \; J_t^{\beta} = I, \quad \beta > 0 \quad where\; D_t^0 = I \tag{5} $$

3-3 Caputo fractional derivate

Let m denotes a positive integerm-1< α ≤m, then the **Caputo's** fractional derivative of order α given by:

$$ {}^{c}D_t^{\alpha} = J_t^{m-\alpha} D_t^{m} f(t) \quad \text{This mean:} $$

$$ {}^{c}D_t^{\alpha} f(t) = \begin{cases} \left[\dfrac{1}{\Gamma(m-\alpha)}\int_0^t (t-s)^{m-\alpha-1} f^{(m)}(s)ds\right] & m-1<\alpha<m \\ \dfrac{d^m}{dt^m} f(t) & \alpha = m \end{cases} \tag{6} $$

Some properties of fractional derivatives:

$$ D_t^{\alpha}\, t^k = \begin{cases} \dfrac{\Gamma(k+1)}{\Gamma(k-\alpha+1)} t^{k-\alpha} & k \ge \alpha \\ 0 & k < \alpha \end{cases} $$

Since $e^{at} = \sum_{k=0}^{\infty} \dfrac{(at)^k}{\Gamma(k+1)}$ and using first property, with the linearity of operator D_t^{α}, then

$D_t^{\alpha} e^{at} = a^{\alpha} \sum_{k=0}^{\infty} \dfrac{(at)^{k-\alpha}}{\Gamma(k-\alpha+1)}$, and since all terms of these infinite series equal zero if (k < α), let s= k- α then:

$$ D_t^{\alpha} e^{-t} = a^{\alpha} \sum_{s=0}^{\infty} \frac{(at)^s}{\Gamma(s+1)} = a^{\alpha} e^{-t} = a^{\alpha} E_{\alpha}^{at} = e^{i\pi\alpha} E_{\alpha}^{at} $$

,

where a= -1; for a=1then $D_t^{\alpha} e^t = e^t$ by the same way

$D_t^{\alpha} \sin(bt) = (b)^{\alpha} \sin(bt + \dfrac{\alpha\pi}{2})$ .where b is constant.

3-4 Grünwald formula

The fractional derivative can be written with the help of Grünwald formula as:

$$ \frac{d^{\beta}}{dx^{\beta}} f(x) = \lim_{m\to\infty} \frac{1}{h^{\beta}} \sum_{k=o}^{m} g_k \; f(x-kh) \tag{7} $$

Where the normalized **Grünwald's weights** function will be defined as:

$$ g_0 = 1; g_1 = -\beta; g_k = \frac{\beta(\beta-1)(\beta-2)\cdots(\beta-k+1)}{k!}. \tag{8} $$

Note: that these normalized weights depend only on the order β and the index **k**.

M.M. Meerschaert, J. Mortensen and H.P, Scheffler, [18] developed an extension of the Grünwald formula for vector fractional derivatives. And use this result for numerical solution of fractional partial differential equations where the space variable is a vector.

3-5 The trapezoidal rule

To approximate the integral term appear in equation (1), trapezoidal rule will be used as.

$$ \int_a^b f(x)dx \cong T(f,a,b) + E_T(f,a,b), $$

Where, $T(f,a,b) = \dfrac{(b-a)}{2}\big(f(a) + f(b)\big)$

$$ E_T(f,a,b) = \frac{(b-a)^3 f^{(2)}(\xi)}{12}, \quad \xi \in (a,b) $$

To preserve the accuracy of the overall approximation of the finite difference representation of equations (1) we use the composite form of the trapezoidal rule, suppose that the interval [a,b] is subdivided into m subintervals $[x_{i-1}, x_i],\; i = 1, 2, \cdots, m$ of width $h = \dfrac{b-a}{m}$; so that $x_i = a + i\,h$, the composite rule takes the form

$$ \int_a^b f(x)dx \cong T(f,h) + E_T(f,h) $$

Where:

$$ T(f,h) = \frac{h}{2}(f(a) + f(b)) + h\sum_{i=1}^{m-1} f(x_i), $$

$E_T(f,h) = O(h^2) = -\dfrac{(b-a) f^{(2)}(\xi)}{12} h^2$, Now for u(x, t); t ∊ [0,T], divide [0,T], into m subintervals $[t_{k-1}, t_k]$ of width $\tau = \dfrac{T-0}{n}$, let t_k=k τ

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where (k=0,1,2,…,n), $n\in N^{+}$, then

$$\int_0^T u(x,s)ds \cong \frac{\tau}{2}[u(x,0)+u(x,T)]+\tau\sum_{k=1}^{n-1}u(x,t_k)+O(\tau^2)$$

(9)

4- Numerical Solution:

Consider equation (1) with the initial and boundary conditions, where the time fractional derivative is understood in the sense of Caputo and the space derivative appearing in the right hand side is understood in the sense of Riemann-Liouville.

Let $u_i^k = u(x_i, t_k)$ for all i, k, let $x_i = ih$, $h=\frac{1}{m}$ and $t_k = k\tau$; $\tau=\frac{T}{n}$ where $i=0,1,2,\cdots,m$; $k=0,1,2,\cdots,n$.

Replace the terms in equation (1) by its approximation to obtain an algebraic relations which are satisfied some accuracy at each point. in these algebraic equations, The approximation will classify as explicit or implicit according to the appearance of the unknowns in each equation. The algebraic system or the approximation is termed explicit, if the system can be arranged , where that every equation contains only one unknown otherwise it is implicit.

Let $u_i^k = u(x_i,t_k)$; (i=0,1,…,m; k=0,1,…,n) be the exact solution of equation (1) at the mesh points (x_i,t_k).

Let U_i^k be the numerical approximation to exact solution at the same mesh points (x_i,t_k).

4-1 Explicit Method

Explicit finite difference method will be used in this section, to find approximation-solution of equation (1).

Using the following approximations:

The approximation of Caputo's fractional derivative of order α given as:

$$\frac{\partial^\alpha u(x_i,t_{k+1})}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{k} \frac{u(x_i,t_{j+1})-u(x_i,t_j)}{\tau}\int_{j\tau}^{(j+1)\tau}\frac{dz}{(t_{k+1}-z)^\alpha}+o(\tau)$$

(10)

Let $s=(t_{k+1}-z)$ then equation (10) becomes:

$$\frac{\partial^\alpha u(x_i,t_{k+1})}{\partial t^\alpha} \cong \frac{\tau}{\Gamma(1-\alpha)} \sum_{j=0}^{k}[u(x_i,t_{k-j+1})-u(x_i,t_{k-j})]\int_{j\tau}^{(j+1)\tau}\frac{dz}{s^\alpha}+o(\tau)$$

(11)

$$\frac{\partial^\alpha u(x_i,t_{k+1})}{\partial t^\alpha} \cong \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=0}^{k}[u(x_i,t_{k-j+1})-u(x_i,t_{k-j})][(j+1)^\alpha-(j)^{1-\alpha}]+o(\tau)$$

(12)

Let $b_j=[(j+1)^\alpha-(j)^{1-\alpha}]$; j=0, 1, 2,… (13)

$$\frac{\partial^\alpha u(x_i,t_{k+1})}{\partial t^\alpha} \cong \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=0}^{k} b_j[u(x_i,t_{k-j+1})-u(x_i,t_{k-j})]+o(\tau) \quad ;0<\alpha\le 1$$

(14)

Now. Grünwald formula used to approximate Riemann-Liouville fractional derivative of order $1\le\beta\le 2$:

$$\frac{\partial^\beta u(x_i,t_k)}{\partial x^\beta} = D_x^\beta u(x_i,t_k) = \frac{1}{h^\beta}\sum_{j=0}^{i+1} g_j\, u(x_j-(i-1)h,t_k)+O(\tau+h)$$

(15)

Where for i=0, 1, 2, …; $0<\beta\le 2$;

$$g_0=1; g_1=-\beta; g_j=(-1)^j\frac{\beta(\beta-1)(\beta-2)\ldots(\beta-j+1)}{j!}$$

(16)

Let $x_i=ih$, $h=\frac{1}{m}$ and $t_k=k\tau$; $\tau=\frac{T}{n}$

Where $i=0,1,2,\cdots,m$; $k=0,1,2,\cdots,n$.

$$\omega=\tau^\alpha\Gamma(2-\alpha);\ r=\frac{\omega}{h^\beta};\ c=\frac{\tau\omega}{2}$$

Now putting equations (9, 14 and 15) in equation (1), with some simple algebraic operations, the general system of equations has been written as:

$$\sum_{j=0}^{k} b_j(u_i^{k-j+1}-u_i^{k-j}) = r\sum_{j=0}^{k} g_j u_{i-j+1}^{k+1} + \sum_{j=0}^{k} c(u_i^{k-j+1}-u_i^{k-j})+q_i^k+o(\tau+h)$$

(17)

This system of equations (17) has the forms at (k= 0 and k ≥1) respectively:

$$u_i^1=(1-\beta r)u_1^0+r\sum_{\substack{j=0\\ j\neq 1}}^{i+1} g_1 u_{i-j+1}^0 \qquad \text{For } k=0$$

(18)

$$u_i^{k+1}=(1+c-\beta r-b_1)u_i^k+r\sum_{\substack{j=0\\ j\neq 1}}^{i+1} g_j u_{i-j+1}^{k+1}+(1-b_1)u_1^0+\sum_{j=1}^{k-1}(2c+b_j-b_{j+1})u_i^{k-1}+\omega q_i^k \qquad k\ge 1$$

(19)

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By using matrix formula this system will be written as: $U^{k+1} = A\,U^k$ where

$$\begin{cases} U^1 = AU^0 \\ U^{k+1} = AU^k + (c+b_k)U^0 + \omega G_i^k + \\ \quad + \sum_{j=1}^{k-1}(2c+b_j-b_{j+1})u_i^{k-1} \\ U^0 = f \qquad \text{theinitialvalue} \end{cases}$$

(20)

Where A =[A_{ij}] is the matrix of coefficient, has form:

$$A_{ij} = \begin{cases} r & j=i+1 \\ 1-\beta r & j=i=1 \\ 1+c-\beta r-b_1 & j=i=2,3,..,m \\ r\,g_2 & j=i-1 \\ r\,g_{i-j+1} & j\le i-2 \\ 0 & \text{otherwise} \end{cases}$$

(21)

$$\left.\begin{array}{l} U_k = [U_1^k, U_2^k, \ldots, U_{m-1}^k]^T; \\ \text{Where } f = [f(x_1), f(x_2), f(x_3), \ldots, f(x_{m-1})]^T \\ Q_k = [q_1^k, q_2^k, \ldots, q_{m-1}^k]^T; \end{array}\right\}$$

4-2 Implicit Method:

By using the same approximation in section 4-1 to approximate the fractional derivatives in implicit formula one will get:

$$\frac{\partial^\alpha u(x_i,t_{k+1})}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{k} \frac{u(x_i,t_{j+1})-u(x_i,t_j)}{\tau} \int_{j\tau}^{(j+1)\tau} \frac{dz}{(t_{k-1}-z)^\alpha} + o(\tau)$$

(22)

Let s= $(t_{k+1}-z)$ we have:

$$\frac{\partial^\alpha u(x_i,t_{k+1})}{\partial t^\alpha} \cong \frac{\tau}{\Gamma(1-\alpha)} \sum_{j=0}^{k} [u(x_i,t_{k-j+1})-u(x_i,t_{k-j})] \int_{j\tau}^{(j+1)\tau} \frac{dz}{s^\alpha} + o(\tau)$$

(23)

$$\frac{\partial^\alpha u(x_i,t_{k+1})}{\partial t^\alpha} \cong \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=0}^{k} [u(x_i,t_{k-j+1})-u(x_i,t_{k-j})][(j+1)^\alpha-(j)^{1-\alpha}] + o(\tau)$$

(24)

Let $b_j = [(j+1)^\alpha - (j)^{1-\alpha}]$; j=0, 1, 2,…

(25)

$$\frac{\partial^\alpha u(x_i,t_{k+1})}{\partial t^\alpha} \cong \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=0}^{k} b_j[u(x_i,t_{k-j+1})-u(x_i,t_{k-j})] + o(\tau) \quad ;0<\alpha\le 1$$

(26)

Define this operator:

$$L_{h,\tau}^\alpha u(x_i,t_{k+1}) = \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=0}^{k} [u(x_i,t_{k-j+1})-u(x_i,t_{k-j})] \quad (27)$$

Let c and c_1 are two constants, then:

$$\left|\frac{\partial^\alpha u(x_i,t_{k+1})}{\partial t^\alpha} - L_{h,\tau}^\alpha u(x_i,t_{k+1})\right| \le c_1\tau \int_0^{t_{k+1}} \frac{ds}{(t_{k+1}-s)^\alpha} \le c\tau$$

(28)

Now, shifted Grünwald formula used to approximate Riemann-Liouville of order $1\le\beta\le 2$:

$$\frac{\partial^\beta u(x,t)}{\partial x^\beta} = D_x^\beta u(x_i,t_{k+1}) = \frac{1}{h^\beta}\sum_{j=0}^{i+1} g_j\, u(x_j-(i-1)h, t_{k+1}) + O(\tau+h)$$

(29)

Where for i=0,1,2,… ; $0<\beta\le 2$

$$g_0 = 1; g_1 = -\beta; g_j = (-1)^j \frac{\beta(\beta-1)(\beta-2)\ldots(\beta-j+1)}{j!}$$

(30)

Put equations (9, 26 and 29) in equation (1), yield

$$\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=0}^{k} b_j(u_i^{k-j+1}-u_i^{k-j}) = \frac{1}{h^\beta}\sum_{j=0}^{i+1} g_j u_{i-j+1}^{k+1} + \sum_{j=1}^{k} \frac{\tau}{2}(u_i^{k-j+1}-u_i^{k-j}) + q_i^{k-1} + o(\tau+h)$$

(31)

Let $\omega = \tau^\alpha\Gamma(2-\alpha); r = \frac{\omega}{h^\beta}; c = \frac{\tau\omega}{2}$ then:

$$(u_i^{k+1}-u_i^k) - r\sum_{j=1}^{i+1} g_j u_{i-j+1}^{k+1} = \sum_{j=0}^{k} c(u_i^{k-j+1}-u_i^{k-j}) - \sum_{j=1}^{k} b_j(u_i^{k-j+1}-u_i^{k-j}) + q_i^{k-1} + o(\tau+h)$$

(32)

$$(1+\beta r-c)u_i^{k+1} - r\sum_{\substack{j=0\\ j\ne 1}}^{i+1} g_j u_{i-j+1}^{k+1} = (1+b_1)u_i^k + (c+b_k)u_i^0 + \sum_{j=1}^{k-1}(2c+b_j-b_{j+1})u_i^{k-1} + \omega q_i^{k+1}$$

(33)

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Where i=1,2,…,m-1; k=1,2,…,n-1 further, the system of equations(33) written at k=0, k=1and k >1 respectively:

$$(1+\beta r-c)u_i^1-r\sum_{\substack{j=0\\j\neq 1}}^{i+1} g_j u_{i-j+1}^1=(1+c)u_i^0+\omega q_i^1$$

(34)

$$(1+\beta r-c)u_i^2-r\sum_{\substack{j=0\\j\neq 1}}^{i+1} g_j u_{i-j+1}^2=(1+2c-b_1)u_i^1+ (c+b_1)u_i^0+\omega q_i^1$$

(35)

$$(1+\beta r-c)u_i^{k+1}-r\sum_{\substack{j=0\\j\neq 1}}^{i+1} g_j u_{i-j+1}^{k+1}=(1+2c-b_1)u_i^k+ (c+b_k)u_i^0+\omega q_i^{k+1}+\sum_{j=1}^{k-1}(2c+b_j-b_{j+1})u_i^{k-1}$$

(36)

System of equations (34, 35 and 36) will be written by matrix formula as: $AU^{k+1}=U^k$

$$\begin{cases} AU^1=(1+c)U^0+\omega G^1 & k=0\\ AU^2=(1+2c-b_1)U^1+(c+b_1)U^0+\omega G^2 & k=1\\ AU^{k+1}=(1+2c-b_1)U^k+(c+b_k)U^0+\omega G^{k+1}+ & \\ \sum_{j=1}^{k=i}(2c+b_j-b_{j+1})U^{k+1} & k>2\end{cases}$$

(37)

Where $A=[A_{ij}]$ is the matrix of coefficient, it has the form:

$$A_{ij}=\begin{cases} -r & j=i+1\\ 1-c+\beta r & j=i\\ -r\,g_2 & j=i-1\\ -r\,g_{i-j+1} & j<i-1\\ 0 & otherwise\end{cases}$$

(38)

$$\left.\begin{aligned} U_k&=[U_1^k,U_2^k,\ldots,U_{m-1}^k]^T;\\ \text{Where } f&=[f(x_1),f(x_2),f(x_3),\ldots,f(x_{m-1})]^T\\ Q_k&=[q_1^k,q_2^k,\ldots,q_{m-1}^k]^T;\end{aligned}\right\}$$

(39)

5-Stability and Convergence:

There are three fundamental properties (consistency, convergence and stability), that every approximation of partial differential equations by finite differences, should possess it. The (Peter Lax theory), below, Will be shown the relation between these three properties.

consistency

implies that the finite difference equation is a good approximation of the partial differential equation,

convergence

implies that the solution of the difference equation approaches the solution of the partial differential equation as the computational mesh is refined.

Stability

implies that the solution of the difference equation is not too sensitive to small perturbations (say, initial data), These properties are often difficult to verify for realistic problems, but they can be explained and illustrated quite easily using difference schemes for some simple model problems. Peter Lax, has made major contributions in areas including mathematical physics, in areas of numerical analysis. He gaves important theory, in this theory, to prove convergence one can work with the discrete scheme alone, providing it is consistent.

5-1 Stability and Convergence of explicit finite differe-nce method, equation (19).

Theorem1 (Lax Equivalence Theorem)

If the finite difference method $U^{n+1}=BU^n+kf^n$ is stable, then $\|U_n-u_n\|\leq CT\max_{m=0,\cdots,n-1}\|T_m\|$ for all n such that

$nk=T$. Where:

1- U_n ,u_n denotes the vector of approximate and exact solutions(x_j, t_n) at mesh points(x_j, t_n) respectively, T_m dented a vector of local truncation errors $T(x_j, t_m)$.

2- So provided the method is consistent, the convergence rate is determined by how quickly the maximum over all local truncation errors (up to t = T) approaches 0 as

k→0. So "**consistency** + **stability**⟹ **convergence**". For more detail of proof, see [20,12].

Theorem2 (Gerschgorin's Theorem):

Let A be a coefficients matrix $A=(a_{ij})$, and let $x=(x_1, x_2,\ldots,x_n)$, be an eigenvector of A corresponding to the Eigen value λ. Then for some i we have $|x_i|\geq|x_j|$ for all j≠i, and since x is an eigen-value, then

$|x_i|\geq 0$ and $Ax=\lambda x$ or $(\lambda I-A)x=0$, Which represents n simultaneous equations for the i^th equation as:

$$(\lambda-a_{ij})x_i-\sum_{j\neq i}a_{ij}x_j=0 \text{ Then } \lambda=a_{ii}-\sum_{j\neq i}a_{ij}\frac{x_j}{x_i}=0$$

These eigenvalue lies in one circles $|\lambda-a_{ii}|\leq\sum_{j\neq i}a_{ij}$

This means there are n circles corresponding to i=1,2,…,n.

Suppose that B(r), $0\leq r\leq 1$ is the (n by n) matrix given by $b_{ii}=a_{ii}$ then $b_{ij}=ra_{ij}$;i≠ j then eigenvalues of b(r) lie in the circles $|t-a_{ii}|\leq t\sum_{j\neq i}|a_{ij}|$.

Since in this method a Grunewald formula is using to approximate Riemann fractional derivative and approximate Caputo fractional derivative, then the consistency proof for this case are facilitated by assuming zero Dirichlet boundary conditions, So that the solution may be zero-extended beyond the interval $0 \le x \le L$. thus the Riemann,Grünwald and Caputo definitions for the discratization have been shown to be O (Δx) for $1\le \beta \le 2$ and O (Δt) for $0\le \alpha \le 1$. See [14-15-16].

In view of **Lax's equivalence theorem** these methods **converge if and only if these** are stable. Since the system ofequation of explicit written by the matrix form as: $U^{k+1} = AU^k + \omega F^k$ Where

$U_k = [u_1^k, u_2^k, \ldots, u_{m-1}^k]^T$; $F^k = [f_0^k, f_1^k, f_2^k, \ldots, f_{m-1}^k]^T$ and

f^k = f (x,t_k,u,g) at k time step this mean the term of function add to the stander heatequation, A is the matrix of coefficients, and is the sum of a lower triangular matrix and super-diagonal matrix. The matrix entries A_{ij} for i=1,2,…,m-1 ; and j=1,2,…,m-1,defined by :

$$A = \begin{cases} 0 & if\ \ j \ge i+2 \\ 1+g_1 & if\ \ j = i \\ rg_{i-j+1} & otherwise \end{cases}$$

While $A_{0,0}$=1, $A_{0,j}$=0 for j=1,2,…,m $A_{m,m}$=1 and $A_{m,j}$=0 for j=0,1,2,…,m-1 with notes(a,b,c and d) at(2-1) ,and by the Greschgorin theorem the eigenvalue of matrix A lie in the union of the circles centered at A_{ii} with radius $R_i \sum_{j\ne i} A_{ij}$ we have A_{ii} =1+ r g_1 = 1 – r β and for R_i we have: $R_i = \sum_{\substack{j=0\\ j\ne i}}^{m} A_{ij} = \sum_{j=0}^{i+1} A_{ij} = r\sum_{\substack{j=0\\ j\ne i}}^{m} g_{ij} \le r\beta = 1 - A_{ii}$

Therefore $A_{ii}+R_i \le 1$. We have $A_{ii}-R_i = 1 - r\beta - R_i \ge 1 - 2r\beta$. So that we have for spectral radius of the matrix A to be at most one, it suffices to have $(1-2r\beta) \ge -1$.which yields the following condition of r,

$$r = \frac{\tau^\alpha}{h^\beta} \le \frac{1}{\beta} .$$

Under this condition on r the spectral radius of matrix A is bounded by one ,with spectral radius so bounded, the numerical error do not grow , and the explicit method defined above is conditionally stable. Moreover the explicit method defined above is consistent with order $O(\Delta t^n) + O(\Delta h^m)$' where n,m are integer numbers with $(n-1\le \alpha \le n)$ and $(m-1\le \beta \le m)$.This mean **explicit method** consistent and conditionally stable then it is converging, the one of special case is;

if α=1 and β=2 the condition become r ≤ 1/2, this condition of classical parabolic of PDE.

5-2 Stability and Convergence of implicit finite difference approximate equation (33):

5-2-1 Stability:

the following lemma will be proved for the system of equations, which are using to approximate solution of eq(1) by using implicit way, the coefficients b_k and g_j , where (k=0,1,2,…); (j=1,2,…) satisfy the following:

- $b_j > b_{j+1}$ for all j=1,2,…
- $b_0=1$; $b_j > 0$ for all j =0,1,2,…
- $g_1 = -\beta$; $g_j \ge 0$ for all $j \ne 1$; $\sum_{j=0}^{\infty} g_j = 0$
- for any positive integer n; $\sum_{j=0}^{n} g_j < 0$

Suppose that $\tilde{U}_i^k$; i=0, 1… m; k=0,1,…n is approximate solutions of equation (33). Define error as: $\tilde{\varepsilon}_i^k = \tilde{u}_i^k - u_i^k$ for all i; k, the error satisfies system equations then:

$$(1+\beta r - c)\,\varepsilon_i^1 - r\sum_{\substack{j=0\\ j\ne 1}}^{i+1} g_j \varepsilon_{i-j+1}^1 = (2+3c-b_1)\,\varepsilon_i^0$$

(40)

$$(1+\beta r - c)\,\varepsilon_i^{k+1} - r\sum_{\substack{j=0\\ j\ne 1}}^{i+1} g_j \varepsilon_{i-j+1}^{k+1} = (1+2c-b_1)\,\varepsilon_i^k + $$
$$+ (c+b_k)\,\varepsilon_i^0 + 2c\sum_{j=1}^{k-1}\varepsilon_i^j + \sum_{j=1}^{k-1}(b_j - b_{j+1})\,\varepsilon_i^{k-1}$$

(41)

Equations (33 and 34) written by using matrix form as:

$$\left\{\begin{array}{l} AE^1 = (2+3c-b_1)\,E^0 \\ AE^{k+1} = (1+2c-b_1)\,E^k + (c+b_k)\,E^0 + \\ 2c\sum_{j=1}^{k-i} E^j + \sum_{j=1}^{k-i}(b_j - b_{j+1})\,E^{k-1} \end{array}\right\}$$

(42)

Where $E^k = [\varepsilon_1^k, \varepsilon_2^k, \ldots, \varepsilon_{m-1}^k]^T$;

Now we use mathematical induction to prove

$\|E^k\|_\infty \le \|E^0\|_\infty$ for all k=1, 2,…,so that the theorem will be done then fractional implicit difference method defined in equation (33) is unconditionally stable.

Now when k=1 not that, and $g_j > 0$, $j\ne 1$,then from equation (40 and 41)

$$(1+\beta r - c)\,\varepsilon_i^1 - r\sum_{\substack{j=0\\ j\ne 1}}^{i+1} g_j \varepsilon_{i-j+1}^1 = (2+3c-b_1)\,\varepsilon_i^0$$

$$M_1\varepsilon_i^1 - M_2\sum_{\substack{j=0\\ j\ne 1}}^{i+1} g_j \varepsilon_{i-j+1}^1 = (2+3c-b_1)\,\varepsilon_i^0$$

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Where $M_1=\frac{(1+\beta r-c)}{(2+3c-b_1)}$; $M_2=\frac{r}{(2+3c-b_1)}$;

And since (2-3c - b_1) > 0.

Let $\|E^1\|_\infty=\|\varepsilon_i^1\|_\infty=\underset{0<i<m-1}{MAX}|\varepsilon_i^1|$

$$\|E^1\|_\infty=\|\varepsilon_i^1\|_\infty\le M_1|\varepsilon_i^1|-M_2\sum_{\substack{j=0\\j\ne1}}^{i+1}g_j|\varepsilon_i^1|$$

$$\le M_1|\varepsilon_i^1|-M_2\sum_{\substack{j=0\\j\ne1}}^{i+1}g_j|\varepsilon_{i-j+1}^1|$$

$$\le\left|M_1\varepsilon_i^1-M_2\sum_{\substack{j=0\\j\ne1}}^{i+1}g_j\varepsilon_{i-j+1}^1\right|=\|\varepsilon_i^1\|\le\|E^0\|_\infty$$

So that we have $\|E^1\|_\infty\le\|E^0\|_\infty$, (true at k=1).

assume that it is true for k= j, this mean:
$\|E^j\|_\infty\le\|E^0\|_\infty$ for j=1,2,…,k, now

for k+1 we have, $\|E^{k+1}\|_\infty\le\|\varepsilon_i^{k+1}\|_\infty=\underset{0<i<m-1}{MAX}|\varepsilon_i^{k+1}|$

$$\|E^{k+1}\|_\infty=\|\varepsilon_i^{k+1}\|_\infty\le(1+\beta r-c)|\varepsilon_i^{k+1}|-r\sum_{\substack{j=0\\j\ne1}}^{i+1}g_j|\varepsilon_i^{k+1}|$$

$$\le(1+\beta r-c)|\varepsilon_i^{k+1}|-r\sum_{\substack{j=0\\j\ne1}}^{i+1}g_j|\varepsilon_{i-j+1}^{k+1}|$$

$$\le\left|(1+\beta r-c)\,\varepsilon_i^{k+1}-r\sum_{\substack{j=0\\j\ne1}}^{i+1}g_j\,\varepsilon_i^{k+1}\right|$$

$$=\left|\begin{array}{l}(1+2c-b_1)\,\varepsilon_i^k\,(c+b_k)\,\varepsilon_i^0+2c\sum_{j=1}^{k-1}\varepsilon_i^j\\+\sum_{j=1}^{k-1}(b_j-b_{j+1})\,\varepsilon_i^{k-1}\end{array}\right|$$

$$\le(1+2c-b_1)\|E^k\|_\infty(c+b_k)\|E^0\|_\infty+2c\sum_{j=1}^{k-1}\|E^j\|_\infty+\sum_{j=1}^{k-1}(b_j-b_{j+1})\|E^{k-1}\|_\infty$$

$$\le(1+2c-b_1)\|E^0\|_\infty(c+b_k)\|E^0\|_\infty+2c\sum_{j=1}^{k-1}\|E^0\|_\infty+\sum_{j=1}^{k-1}(b_j-b_{j+1})\|E^0\|_\infty$$

$$=\|E^0\|_\infty\ so\ \ that\|E^{k+1}\|_\infty\le\|E^0\|_\infty$$

5-2-2Convergence:

Let U_i^k be the numerical solution of equation (33) at mesh-points (x_i, t_k), where i=1,2,…,m; k=1,2,…,n, now, define error as:

$e_j^k= u(x_i, t_k) -U_i^k$ for all i and k.

since$e^k =(e^k_1,e^k_2,…,e^k_{m-1})^T$,

substitution e_j^k and e^0 into equation (41) we have:

$$(1+\beta r-c)\,e_i^1-r\sum_{\substack{j=0\\j\ne1}}^{i+1}g_j\,e_{i-j+1}^1=(2+3c-b_1)\,e_i^0$$

$$M_1e_i^1-M_2\sum_{\substack{j=0\\j\ne1}}^{i+1}g_j\,e_{i-j+1}^1=e_i^0=R_i^1$$

$$(1+\beta r-c)\,e_i^{k+1}-r\sum_{\substack{j=0\\j\ne1}}^{i+1}g_j\,e_{i-j+1}^{k+1}=(1+2c-b_1)\,e_i^k$$

$$(c+b_k)\,e_i^0+2c\sum_{j=1}^{k-1}e_i^j+\sum_{j=1}^{k-1}(b_j-b_{j+1})\,e_i^{k-1}+R_i^{k+1}$$

We have (i=1, 2, ... , m-1; k=1, 2,…, n) and
$|R_1^k|\le C(\tau^{2+\alpha}+\tau^\alpha h)$

Then, using of the mathematical induction to give the convergence analysis as follows:

for k=1, $\|e^1\|_\infty=\|e_i^1\|_\infty=\underset{0<i<m-1}{MAX}|e_i^1|$ Then

$$\|e^1\|_\infty\le M_1|e_i^1|-M_2\sum_{\substack{j=0\\j\ne1}}^{i+1}g_j|e_i^1|$$

$$\le M_1|e_i^1|-M_2\sum_{\substack{j=0\\j\ne1}}^{i+1}g_j|e_{i-j+1}^1|$$

$$\le\left|M_1e_i^1-M_2\sum_{\substack{j=0\\j\ne1}}^{i+1}g_j\,e_{i-j+1}^1\right|=|e_i^o+R_i^k|$$

Since e^0=0 and $|R_1^k|\le C(\tau^{2+\alpha}+\tau^\alpha h)$

So $\|e^1\|_\infty\le C(\tau^{2+\alpha}+\tau^\alpha h)$

Suppose that $\|e^j\|_\infty\le C\,b_{j-1}^{-1}(\tau^{2+\alpha}+\tau^\alpha h)$

For j= 1,2,…,k, we prove that true for k+1

Let $\|e_i^{k+1}\|_\infty=\underset{0<i<m-1}{MAX}|e_i^{k+1}|$

Not $b_j^{-1}\le b_k^{-1};\ j=0,1,\ldots,k$

$$|e_i^{k+1}|_\infty\le(1+\beta r-c)|e_i^{k+1}|-r\sum_{\substack{j=0\\j\ne1}}^{i+1}g_j|e_i^{k+1}|$$

$$\le(1+\beta r-c)|e_i^{k+1}|-r\sum_{\substack{j=0\\j\ne1}}^{i+1}g_j|e_{i-j+1}^{k+1}|$$

$$\le\left|(1+\beta r-c)\,e_i^{k+1}-r\sum_{\substack{j=0\\j\ne1}}^{i+1}g_j\,e_i^{k+1}\right|$$

$$\left| (1+2c-b_1)e_i^k (c+b_k)e_i^0 + 2c\sum_{j=1}^{k-1} e_i^j + \sum_{j=1}^{k-1}(b_j - b_{j+1})e_i^{k-1} + R_i^{k+1} \right|$$

$$\le (1+2c-b_1)\left\|e^k\right\|_\infty + (c+b_k)\left\|e^0\right\|_\infty + + 2c\sum_{j=1}^{k-1}\left\|e^j\right\|_\infty + \sum_{j=1}^{k-1}(b_j - b_{j+1})\left\|e^{k-1}\right\|_\infty + \left|R_i^{k+1}\right|$$

Since $\left\|e^j\right\|_\infty \le C b_{j-1}^{-1}(\tau^{2+\alpha} + \tau^\alpha h)$; and $\left\|e^j\right\|_\infty = 0$

$$\le [(1+2c-b_1)b_{j-1}^{-1} + 2c\sum_{j=1}^{k-1} b_{k-j-1}^{-1} + \sum_{j=1}^{k-1}(b_j - b_{j+1})b_{k-j-1}^{-1}] \; C(\tau^{1+\alpha} + \tau^\alpha h) + \left|R_1^{k+1}\right|$$

Using $b_j^{-1} \le b_k^{-1}$ for j=0,1,…,k and $\left|R_1^k\right| \le C(\tau^{1+\alpha} + \tau^\alpha h)$

$$\le [(1+2c-b_1)b_k^{-1} + 2c\sum_{j=1}^{k-1} b_k^{-1} + \sum_{j=1}^{k-1}(b_j - b_{j+1})b_k^{-1}] \; C(\tau^{1+\alpha} + \tau^\alpha h) + C(\tau^{1+\alpha} + \tau^\alpha h)$$

$$\left\{ b_k^{-1}[(1+2c-b_1)b_k^{-1} + 2c\sum_{j=1}^{k-1} b_k^{-1} + \sum_{j=1}^{k-1}(b_j - b_{j+1})b_k^{-1}] + 1 \right\} C(\tau^{1+\alpha} + \tau^\alpha h)$$

$$b_k^{-1} C(\tau^{1+\alpha} + \tau^\alpha h)\left[(1+2c-b_1) + (2c(k-1)\sum_{j=1}^{k-1}(b_j - b_{j+1}) + b^k\right]$$

so that $\left\|e^{k+1}\right\|_\infty \le C b_k^{-1}(\tau^{1+\alpha} + \tau^1 h)\; for k = 0,1,\ldots$

Hence there is constant C such that:

$\left\|e^{k+1}\right\|_\infty \le C b_k^{-1}(\tau^{1+\alpha} + \tau^1 h)$ for k=0,1,2,…

If $k\tau \le T$ is finite the convergence is given by the following theorem:

Theorem3: let U_i^k be approximate value of $u(x_i, t_k)$ computed by using equation (33), then there is a positive constant C such that:

$$\left|U_i^k - u(x_i, t_k)\right| \le C(\tau + h); i = 1,2,\ldots,m-1; k = 1,2,\ldots,n$$

6- Numerical Examples:

Three examples with known exact solutions are considered. The examples are chosen such that the behavior of the solution has different characterizations with space and time ranging from polynomial, sinusoidal and exponentially decay.

Example 1: consider equation (1), with

$$q(x,t) = (x^2 - x^3)(\sin(t - \frac{\alpha}{4})) - (\sin(t)+1)(\frac{\Gamma(3)x^{2-\beta}}{\Gamma(3-\beta)} - \frac{\Gamma(4)x^{3-\beta}}{\Gamma(4-\beta)}) - \Gamma(2.6275)\,(x^2 - x^3) \quad (43)$$

the boundary conditions u(0,t) = u(1,t) = 0, t > 0 ; and the initial condition $(x,0)=(x^2-x^3)$, $x \in [0,1]$, where the exact solution is $u(x, t)=(x^2-x^3)(\sin(t)+1)$

Table 1 shows different choice of n, m, α and β, for two methods.

cha	α	β	n	m	τ1	τ2	err1	err2
1	.8	1.3	10^4	10	.0001	.0007	.018	.013
2	.5	1.5	50	50	.2	.06	.01	.0038
3	.8	1.3	50	20	.2	.05	.01	.0038

Table 1

Figure1 illustrates the exact solution and the numerical solutions obtained by using explicit method table1 shone the choice of n, m to achieve condition of stability, the large step of time gives small maximum error, with fixed α, β.

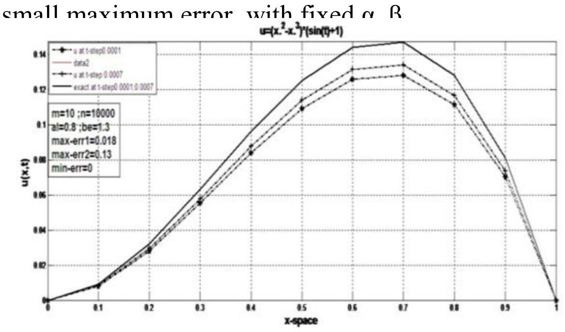

Figure1. numerical and anlaytic graph of soluthins using explicit method of examble 1

Figure (2) illustrates the exact and the numerical solution by using implicit method, for α = and β = at two time-steps with different choice of (h, τ).

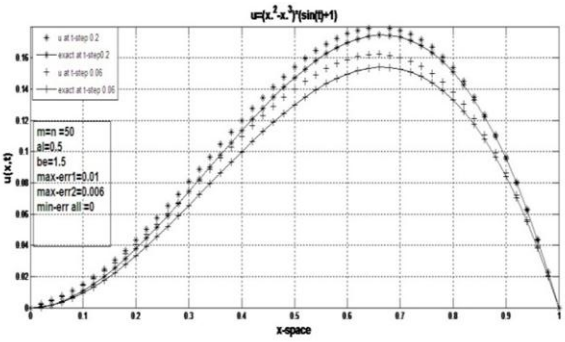

Figure2. numerical and anlaytic graph of soluthins using implicit method of examble 1

Figure (3)illustrates the exact and the numerical solution by using implicit method, for α = and β = at two time-steps with different choice of (h, τ),

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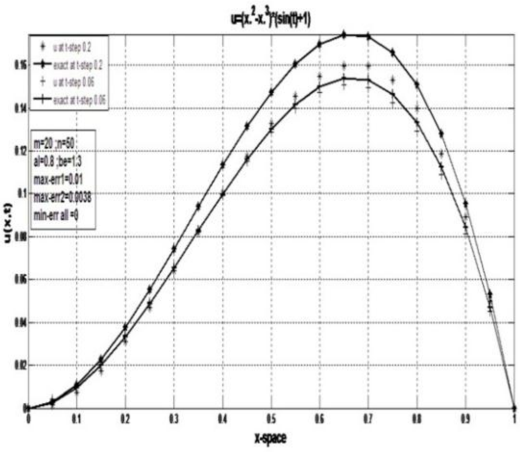

Figure3. numerical and anlaytic graph of soluthi using implicit method of examble 1.

Example 2:consider equation (1), with

$$q(x,t)=(e^{-t})[(x^4-x^3)-(\frac{\Gamma(4)\,x^{4-\beta}}{\Gamma(3-\beta)}-\frac{\Gamma(5)\,x^{3-\beta}}{\Gamma(4-\beta)})] -(\Gamma(3)-\Gamma(2.54))\,(x^4-x^3)$$

(44)

the boundary conditions u(0,t) = u(1,t)=0, t>0 ; and the initial condition $u(x,0) = (x^4-x^3)$, $x \in [0,1]$, whose exact solution has the form $u(x,t) = (x^4-x^3)(\exp(-t))$.

Table 2 shone different choice of n, m, α and β, for two methods.

cha	α	β	n	m	τ1	τ2	err1	err2
4	.5	1.5	16e4	20	St_2	St_9	.026	.0098
5	.5	1.5	2500	50	St_2	St_9	.007	. 005
6	.5	1.5	40	100	.01	.02	.027	.004

Table 2

Figure4 shows the exact and approximate solutions using explicit method; goes to exact solution with high time step, different in error with different choice of τ at table 2 shone that.

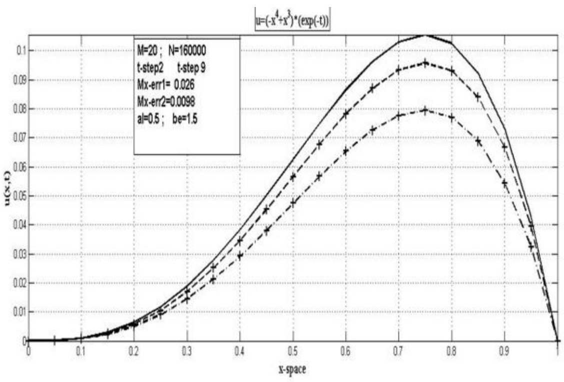

Figure4. numerical and anlaytic graph of soluthins using explicit method of examble 2

Figure5 and 6 show exact and approximate solutions by using implicit method, with different values of α and β. Both are choosing to show how the approximate solution go to exact solution with large values of n and m.

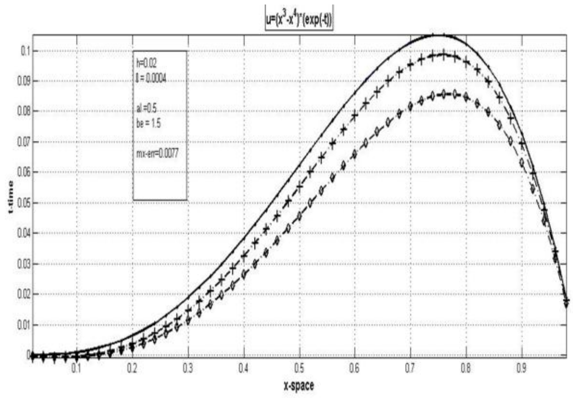

Figure5. numerical and anlaytic graph of soluthins using implicit method of examble 2

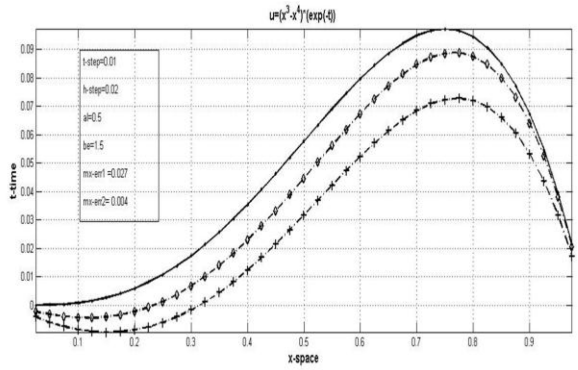

Figure6. numerical and anlaytic graph of soluthins using implicit method of examble 2

Example 3: consider equation (1), where

$$q(x,t)=(e^{-t})[\sin(\pi x)-(\pi^{\beta}\sin(\pi x+\frac{\pi\beta}{4})] -(\Gamma(3)-\Gamma(2.54))\sin(\pi x) \qquad (45)$$

With boundary and initial conditions: u(0,t) = u(1,t) =0; $t \in [0,1]$;u (x,0) =sin(πx), $x \in [0,1]$; with the exact solution u (x,t) = sin(πx) (exp(- t)),

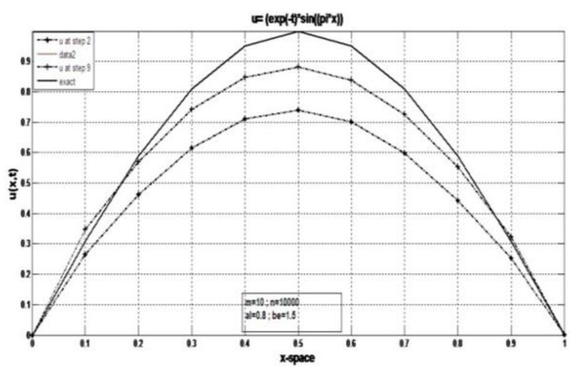

Figure7. numerical and anlaytic graph of soluthins using explicit method of examble 3

Figure7 shows how the approximate solution goes to the exact solution with choose the higher time step, with fixed ($\alpha = 0.8$, $\beta = 1.5$) and choose ($n = 10^4$; $m = 10$) to achieve the condition of stability.

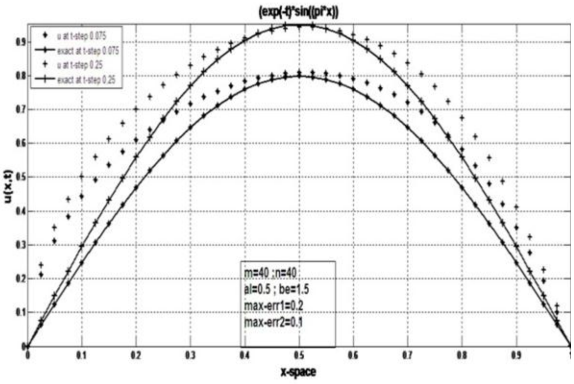

Figure8. numerical and anlaytic graph of soluthins using implicit method of examble 3

Figure 8 shows how the maximum error become small with high time step ($\tau = 0.075$; $\tau = 0.25$), with fixed ($\alpha = 0.5$, $\beta = 1.5$) and choose ($n = m = 40$).

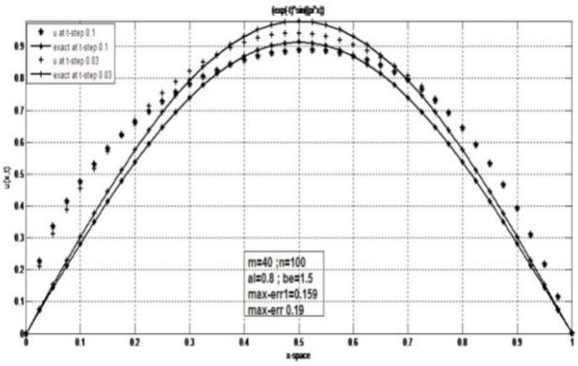

Figure9. numerical and anlaytic graph of soluthins using implicit method of examble 3

Figure 9 shows how the maximum error become small with high time step ($\tau = 0.075$; $\tau = 0.25$), and with large choice of ($n = 100$; $m = 40$), fixed ($\alpha = 0.5$, $\beta = 1.5$).

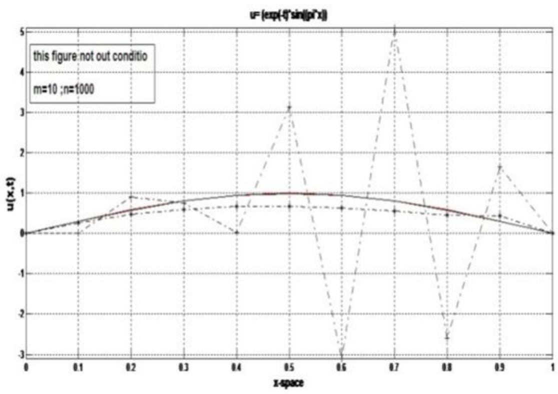

Figure10. where (m,n) are not satisfy condation

Figure 10 shows what happen to approximate solution with chosen values of n, m that didn't achieve the condition of stability.

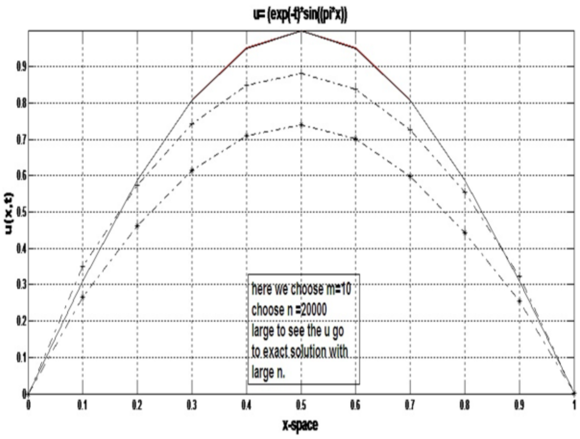

Figure11. where (m,n) are satisfy condation

Figure 11 shows that good approximation with chosen large n ,m with fixed n and m achieved condition of stability and fixed α and β.

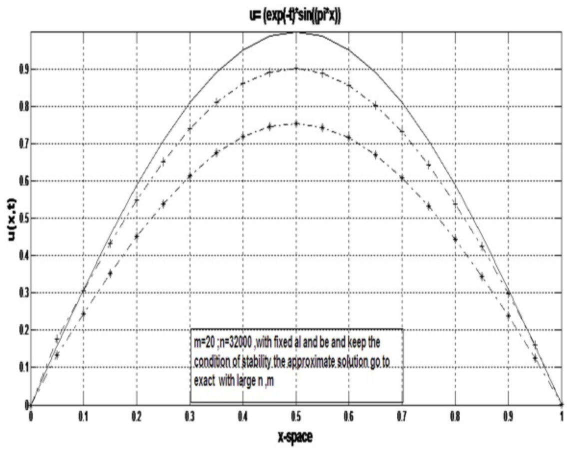

Figure12. shown the same example with choice n,m large

Figure 12 shows the choice of $n = 20$, $m = 32000$ the condition of stability is done and with fixed α and β good approximation with chosen large n ,m.

Conclusion

in this work implicit method gives approximate solution better than explicit with the same time and space split periods i.e same choice of (m,n). see Figure (1-9). Morever implicit method is unconditionlly stability and it's faster than explicit method because it isn't need high value of m or n to give small error. The explicit method has stability with this condition $r = \frac{\tau^{\alpha}}{h^{\beta}} \leq \frac{1}{\beta}$,this mean if we choose m integer number (i.e choose h= 1/m) then we must choose n (i.e τ=1/n) tosatisfy this quality, see Figure10 where (m,n are not satisfy condation) ,Figure11 shows the same examble but with n,m to satisfy condation ,Figure 12 shows the same example with choice n,m larger than Figure 10,11.The adding of any terms, like the integral term, will don't give any changing in stability and converg ,because, (since we use the method of trapezoidal to approximate integral term and it has error smaller than the order error of exiplicit or implicit methods, with this not: exiplicit method don't change in condition of stibilety).

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تعميم طريقة الفروق المنتهية لحل معادلات تفاضلية جزئية معينة ذات رتب كسورية نوع القطع المكافيء

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المستخلص :

تم دراسة معادلات تفاضلية جزئية ذات رتب كسورية في الزمن والمكان معا.(S-TFDE)
تم استخدام الطريقة الضمنية والصريحة مع طريقة شبه المنحرف لايجاد صيغة خاصة لحل هذا النوع من المعادلات. تم مناقشة التقارب والاستقراية لهذه الطريقة وايجاد شرط التقارب. كذلك تم دراسة وبيان تأثير اضافة الحد التكاملي على المعادلة التفاضلية. تم حل ثلاث امثل وايجاد الرسومات للحلول العددية والحل الحقيقي وبيان تفاصيل النتائج من خلال هذه الرسومات. برامج ايجاد الرسومات وبرامج ايجاد النتائج تمت بالاستعانة ببرنامج الماتلاب.

الكلمات المفتاحية:

PDE ذات رتب كسورية ، معادلات تفاضلية-تكاملية ذات رتب كسو، معادلات التفاضلية نوع القطع المكافيء ذات الرتب الكسورية.

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Certain Types of Complex Lie Group Action

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Abstract

The main aim in this paper is to look for a novel action with new properties on $Complex\ \ Lie\ Group$ from the $Lemma\ of\ Schure$, the Literature are concerned with studying the action of $Lie\ algebra$ of two representations , one is usual and the other is the dual, while our interest in this work is focused on some actions on complex Lie group[10] . Let G be a matrix complex Lie group , and π is representation of G. In this study we will present and analytic the concepts of action of complex Lie group on $Hom-space$. We recall the definition of tensor product of two representations of Lie group and construct the definition of action of Lie group on $Hom-space$, then by using the equivalent relation $Hom(w_2, w_1) \cong w_2^* \otimes w_1$ between Hom and $Tensor\ product$, we get a new action : $Action - complex\ Lie\ Group\ on\ tensor\ product$. The two actions are forming smooth representation of G [8], [9]. This we have new action which called $triple\ action\ \ of\ Complex\ Lie\ Group\ G$ denoted by $TAC - complex\ Lie\ group$ which acting on $Hom(Hom((w_5 \oplus w_4), w_3^*), Hom(w_2 \oplus w_1, w^*))$.
This TAC is smooth representation of G. The theoretical Justifications are developed and prove supported by some concluding remarks and illustrations.

Key words : Hom – Space , Tensor Product , Action of Lie Group , Complex Lie Group .

Mathematics subject classification: 64S40.

1- Introduction

A complex Lie group is a finite dimensional analytic manifold G together with a group structure on G, Such that the multiplication $GxG \to G$ and attaching of an inverse $g \to g^{-1} : G \to G$ is analytic map [4],[6].

A matrix Lie group is any subgroup G of $GL(n,\mathbb{C})$ with the following property [7] . If A_m converges to some Matrix A, then $A \in G$ or A is not invertible [5]. *The Schur's lemma* introduced the concepts of Lie algebra on the space of Linear maps from W_2 *into* W_1 ,which denoted by $Hom(W_2,W_1)$ [1],[3]. Also introduced the concepts of action on $Hom-space$ of two representations of *Lie algebr* [1].

Also the main work here is to give a representation of complex Lie group by intertwine these actions (representations) and to give representation by intertwine duel of these actions (representations) and Then generalizing them.

2- The TcoA of complex Lie Groups on Hom - Space

In [2- P327], *the Schur's lemma* introduced the concepts of action of *Lie algebra* on *Hom space* of Two representations of Lie algebra.

Lemma (2.1) [2]:

Suppose that π_1 *and* π_2 are two representations of *Lie algebra* g action on finite dimensional space W_1 *and* W_2 respectively . Define an Co-action of g on $Hom_k(W_2,W_1)$, $\pi: g \to gl(Hom_k(W_2,W_1))$ for all $x \in g$, $F \in Hom_k(W_2,W_1), \pi_1(x)F - F\pi_2(x)$ $\pi(x)(v) = \pi_1(x)F(v)$ *and* $Hom\ (W_2,W_1) = W_2^* \otimes W_1$ as equivalence of representations.

Lemma (2.2):

Put $Hom(Hom(W_5 \oplus W_4), W_3^*), (Hom(W_2 \oplus W_1, W^*)))$ the $K-vector-space$ of all Linear maps $(Hom(W_5 \oplus W_4), W_3^*)$ *onto* $(Hom(W_2 \oplus W_1, W^*)$.

Define $\pi: G \to GL(Hom(Hom(W_5 \oplus W_4), W_3^*), (Hom(W_2 \oplus W_1, W^*)))$,

by $\pi(a) = \pi^*(a) \circ F_1 \circ (\pi_2(a) \oplus \pi_1(a))^{-1} \circ F_2\pi_3^*(a) \circ F_3 \circ (\pi_5(a) \oplus \pi_4(a))^{-1}$,

for all $a \in G$, $F_1 \in Hom(W_2^* \oplus W_1, W)$

$F_2 \in Hom(W_5 \oplus W_4, W_3^*)$

$F_3 \in Hom\big(Hom(W_5 \oplus W_4, W_3^*), Hom(W_2 \oplus W_1, W^*)\big)$.

$$\pi(a)_v = \pi(a) \circ F_1 \circ (\pi_2(a) \oplus \pi_1(a))^{-1} \circ F_2 \circ (\pi_3(a) \circ F_3 \circ (\pi_5(a)^{-1} \oplus \pi_4(a)^{-1})_{(v)}$$

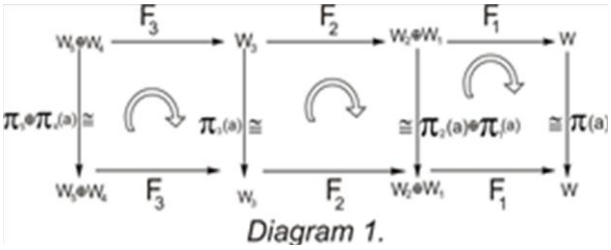

Diagram 1.

For all $a \in G$, $v \in (W_5 \oplus W_4)$

$$\pi(a)_v = \pi(a) \circ F_1 \circ (\pi_2(a)^{-1} \oplus \pi_1(a)^{-1}) \circ F_2 \circ (\pi_3(a) \circ F_3 \circ (\pi_5(a)^{-1} \oplus \pi_4(a)^{-1})_{(v)}$$

For all $a \in G$, $v \in (W_5 \oplus W_4)$.

Where the arrow that makes the diagram 1 commutative π is *homomorphism* of groups G *into* $GL((Hom(W_5 \oplus W_4), W_3^*), (Hom(W_2 \oplus W_1, W^*)))$.

Let $\pi_i: G \to GL(W_i)$, and $\pi_i^*: G \to GL(W_i^*)$, $for\ i = 1,2,3,4,5$.

The TAS of complex Lie group G on $Hom_k(Hom((W_5 \oplus W_4), W_3^*), Hom(W_2 \oplus W_1, W^*))$ is given by a representation π such that $\pi(a) = \pi(a) \circ F_1 \circ (\pi_2(a)^{-1} \oplus \pi_1(a))^{-1}) \circ F_2 \circ (\pi_3(a) \circ F_3 \circ \pi_5(a)^{-1} \oplus \pi_4(a)^{-1})$,

For all $a \in G$.

Then the TAS of complex Lie group G on $Hom_k(Hom((W_5 \oplus W_4), W_3^*), Hom(W_2 \oplus W_1, W^*))$ is also given by representation π^* such that

$$\pi^*(a) = \pi^*(a) \circ F_1 \circ (\pi_2^*(a)^{-1} \oplus \pi_1^*(a))^{-1}) \circ F_2 \circ (\pi_3^*(a) \circ F_3 \circ \pi_5^*(a)^{-1} \oplus \pi_4^*(a)^{-1}).$$

Proof of Lemma (2.2):

Let $TCoA$ of complex Lie group G on $Hom(Hom(W_5 \oplus W_4), W_3^*), Hom(W_2 \oplus W_1, W))$ is

induced by the representation

π : G →

GL(Hom$\big(Hom\ (W_5 \oplus W_4, W_3^*), Hom(W_2 \oplus W_1, W^*)\big)$

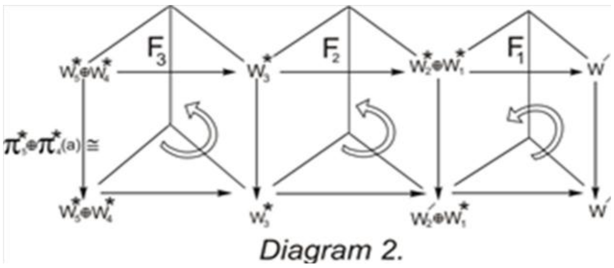

Diagram 2.

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such that $\pi(a) = \acute{\pi}(a) \circ F_1 \circ (\pi_2(a)^{-1} \oplus \pi_1(a))^{-1}) \circ F_2 \circ (\pi_3(a) \circ F_3 \circ \pi_5(a)^{-1} \oplus \pi_4(a)^{-1})$
For all $a \in G$. Thus π^* is a representation from G to $Hom-space$

This arrow makes the diagram 2 commutative .

Remark (2.4):
Since $\mathrm{Hom}\big(Hom\,(W_5 \oplus W_{4,}W_3^*), Hom\big(W_2 \oplus W_1, W^{*/}\big)\big) \cong ((W_5 \oplus W_{4,}W_3^*)^* \otimes ((W_2 \oplus W_1)^* \otimes W^{*/}) \cong ((W_5 \oplus W_4) \otimes W_3) \otimes ((W_2^* \oplus W_1^*) \otimes W^{*/})$
So we construct an action of G on the $product$,
Let $\pi(G) \to GL(((W_5 \oplus W_4) \otimes W_3) \otimes ((W_2^* \oplus W_1^*) \otimes W^{*/}))$, then π forms a representation of G acting on $vector\ space$ $(((W_5 \oplus W_4) \otimes W_3) \otimes ((W_2^* \oplus W_1^*) \otimes W^{*/}))$.

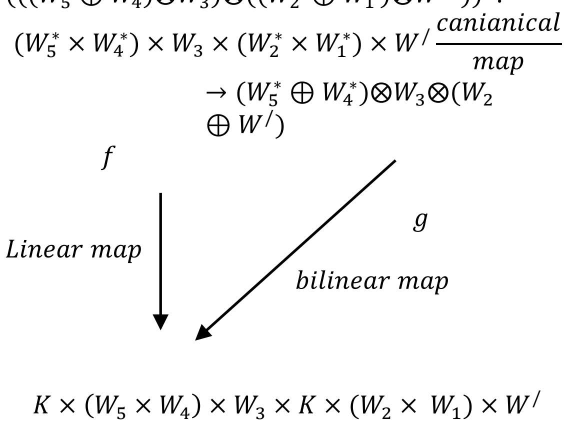

Diagram 3.

$$f\Big((W_5^* \times W_4^*) \times W_3 \times (W_2^* \times W_1^*) \times W^/\Big) = w^/ \circ (w_2^* \oplus w_1^*) \circ w_3 \circ (w_5^* \times w_4^*)$$

$$g\Big((W_5^* \oplus W_4^*) \otimes W_3 \otimes (W_2^* \oplus W_1^*) \otimes W^/\Big) = w^/ \circ (w_2^* \oplus w_1^*) \circ w_3 \circ (w_5^* \times w_4^*)$$

For all $w_5 \in W_5, w_5 \in W_5, w_4 \in W_4, w_3 \in W_3, w_2 \in W_2, w_1 \in W_1, w^/ \in W^/$.

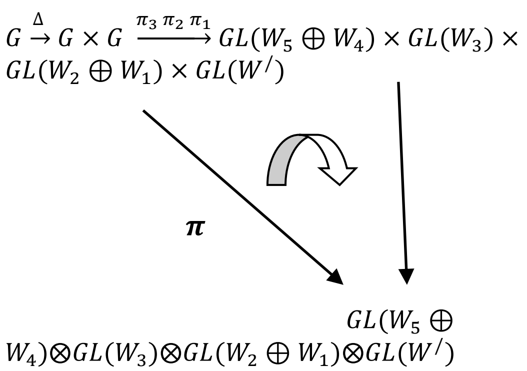

***Diagram* 4.**

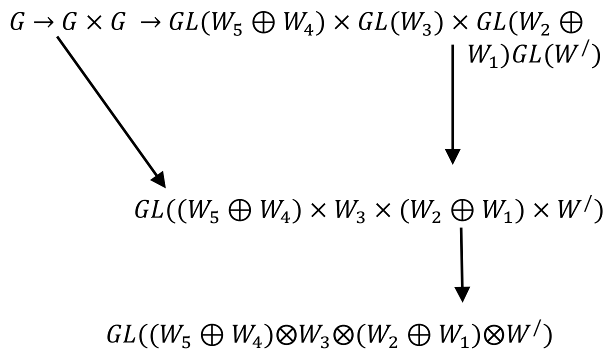

***Diagram* 5.**

That π is a representation of G acting on $GL((W_5 \oplus W_4) \otimes W_3 \otimes (W_2 \oplus W_1) \otimes W^/)$ where $\pi_i, i = 1,2,3,4,5$ are five representations of G acting on W_i , $i = 1,2,3,4,5$ respectively ,thus :

$\pi(ab) = \pi^/\,(ab) \circ W^/ \otimes (W_2 \oplus W_1) \otimes W_3 \otimes (W_5 \oplus W_4)$
$\pi^/\,(ab) \otimes \pi_1^/(ab) \circ (W^/)_{(v)} \circ \pi_2(ab)^{-1}\,(W_2 \oplus W_1)_{(v)} \circ \pi_3(ab)\,W_3(v) \circ\ \pi_4(ab)^{-1}\,(W_5 \oplus W_4)_{(v)} = \pi_4(ab)^{-1} \oplus \pi_4(ab)) \otimes W_3(ab)^{-1} \otimes \pi_2(ab) \otimes \pi_2(ab) \otimes \pi_1(ab)^{-1}\ \otimes \pi^/\,(ab)$
$\pi(a) \circ \pi(b) = \pi(b)(\pi(a)\,) = \pi(b)(\,\pi^/(a) \circ (W^/)_{(v)} \circ \pi_1(a)(W_2 \oplus W_1)_{(v)} \circ \pi_2(a) W_3(v) \circ \pi_3(a)(W_5 \oplus W_4)_{(v)} = \pi^/(b)(\,\pi^/(a) \circ (W^/)_{(v)} \circ \pi_1(b)\pi_1(a)(W_2 \oplus W_1)_{(v)} \circ \pi_2(b)\pi_2(a) W_3(v) \circ \pi_3(b)\pi_3(a)(W_5 \oplus W_4)_{(v)} = \pi^/(ab) \circ W^/ \otimes (W_2 \oplus W_1)\ \otimes W_3 \otimes (W_5 \oplus W_4)$.

$$\pi(ab) = \pi(a) \circ \pi(b)$$

π is a group homomorphism of G on $GL((W_5 \oplus W_4) \otimes W_3 \otimes (W_2 \oplus W_1) \otimes W^/)$.

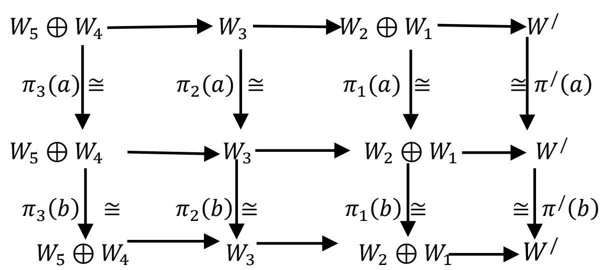

***Diagram* 6.**

3- The $TCoA$ of Complex Lie Groups on Tensor Product

We have been introduced the triple Co-action of complex Lie groups by the tensor product of the five representations, which are $TCoA$-complex Lie groups on tensor product

($W_5^* \otimes (W_4 \otimes ((W_3^* \otimes W_2) \oplus (W_3^* \otimes W_1)))$ and constructed this definition. Depending on what has been mentioned above, Π is called Triple Co-Action of complex Lie groups on tensor product denoted by ($TCoA$-complex Lie groups).

Example (3.1):

Let $\Pi_1:\mathbb{R} \longrightarrow GL(2,\mathbb{C})$ such that $\Pi_1(a) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}_{2\times 2}$, for all $a \in \mathbb{R}$, $\Pi_2:\mathbb{R} \longrightarrow GL(2,\mathbb{C})$ such that $\Pi_2(a) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}_{2\times 2}$, for all $a \in \mathbb{R}$, $\Pi_3:\mathbb{R} \longrightarrow GL(2,\mathbb{C})$ such that $\Pi_3(a) = \begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix}_{2\times 2}$, for all $a \in \mathbb{R}$, $\Pi_4:\mathbb{R} \longrightarrow GL(2,\mathbb{C})$ such that $\Pi_4(a) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}_{2\times 2}$, for all $a \in \mathbb{R}$ and $\Pi_5:\mathbb{R} \longrightarrow GL(3,\mathbb{C})$ such that $\Pi_5(a) = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}_{3\times 3}$, for all $a \in \mathbb{R}$.

The representation Π of $GL(W_5^* \otimes (W_4 \otimes ((W_3^* \otimes W_2) \oplus (W_3^* \otimes W_1))))$ is:

$\Pi:G \longrightarrow GL(W_5^* \otimes (W_4 \otimes ((W_3^* \otimes W_2) \oplus (W_3^* \otimes W_1)))) \cong GL(M(8\times 3),\mathbb{C})$, such that

$\Pi(a) = (((\Pi_1(a) \otimes \Pi_3^*(a)^{-1} \oplus \Pi_2(a) \otimes \Pi_3^*(a)^{-1}) \otimes \Pi_4(a)) \otimes \Pi_5^*(a)^{-1})$, where Π^* is dual representation

$$= \left(\left[\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & -2 \\ 0 & 2 \end{pmatrix} \oplus \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 0 & 2 \end{pmatrix}\right] \otimes \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\right) \otimes \begin{pmatrix} 0 & 0 & 1 \\ -\frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 1 & \frac{1}{2} \end{pmatrix}$$

$$= \left(\left[\begin{pmatrix} 1 & 0 & -2 & 0 \\ 1 & 1 & -2 & -2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 2 \end{pmatrix}_{4\times 4} \oplus \begin{pmatrix} 1 & 2 & -2 & -4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 2 \end{pmatrix}_{4\times 4}\right] \otimes \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}_{2\times 2}\right) \otimes \begin{pmatrix} 0 & 0 & 1 \\ -\frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 1 & \frac{1}{2} \end{pmatrix}_{3\times 3}$$

$$= \left[\begin{pmatrix} 2 & 2 & -4 & -4 \\ 1 & 2 & -2 & -4 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 2 & 4 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}_{2\times 2}\right] \otimes \begin{pmatrix} 0 & 0 & 1 \\ -\frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 1 & \frac{1}{2} \end{pmatrix}_{3\times 3}$$

$$= \begin{pmatrix} 2 & 2 & -4 & -4 & 0 & 0 & 0 & 0 \\ 1 & 2 & -2 & -4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 4 & 0 & 0 & 0 & 0 \\ 2 & 2 & -4 & -4 & 2 & 2 & -4 & -4 \\ 1 & 2 & -2 & -4 & 1 & 2 & -2 & -4 \\ 0 & 0 & 4 & 4 & 0 & 0 & 4 & 4 \\ 0 & 0 & 2 & 4 & 0 & 0 & 2 & 4 \end{pmatrix}_{8\times 8} \otimes \begin{pmatrix} 0 & 0 & 1 \\ -\frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 1 & \frac{1}{2} \end{pmatrix}_{3\times 3}$$

$$= \begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&2&-4&-4&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&2&-2&-4&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&4&4&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&4&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&2&-4&-4&2&2&-4&-4 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&2&-2&-4&1&2&-2&-4 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&4&4&0&0&4&4 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&4&0&0&2&4 \\
-1&-1&2&2&0&0&0&0&0&0&0&0&0&0&0&0&-1&-1&2&2&0&0&0&0 \\
-\frac{1}{2}&-1&1&2&0&0&0&0&0&0&0&0&0&0&0&0&-\frac{1}{2}&-1&1&2&0&0&0&0 \\
0&0&-2&-2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-2&-2&0&0&0&0 \\
0&0&-1&-2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&-2&0&0&0&0 \\
-1&-1&2&2&-1&-1&2&2&0&0&0&0&0&0&0&0&-1&-1&2&2&-1&-1&2&2 \\
-\frac{1}{2}&-1&1&2&-\frac{1}{2}&-1&1&2&0&0&0&0&0&0&0&0&-\frac{1}{2}&-1&1&2&-\frac{1}{2}&-1&1&2 \\
0&0&-2&-2&0&0&-2&-2&0&0&0&0&0&0&0&0&0&0&-2&-2&0&0&-2&-2 \\
0&0&-1&-2&0&0&-1&-2&0&0&0&0&0&0&0&0&0&0&-1&-2&0&0&-1&-2 \\
-1&-1&2&2&0&0&0&0&2&2&-4&-4&0&0&0&0&1&1&-2&-2&0&0&0&0 \\
-\frac{1}{2}&-1&1&2&0&0&0&0&1&2&-2&-4&0&0&0&0&\frac{1}{2}&1&-1&-2&0&0&0&0 \\
0&0&-2&-2&0&0&0&0&0&0&4&4&0&0&0&0&0&0&2&2&0&0&0&0 \\
0&0&-1&-2&0&0&0&0&0&0&2&4&0&0&0&0&0&0&1&2&0&0&0&0 \\
-1&-1&2&2&-1&-1&2&2&2&2&-4&-4&2&2&-4&-4&1&1&-2&-2&1&1&-2&-2 \\
-\frac{1}{2}&-1&1&2&-\frac{1}{2}&-1&1&2&1&2&-2&-4&1&2&-2&0&-3&3&0&-\frac{3}{2}&\frac{3}{2}&0&\frac{3}{2}&-\frac{3}{2} \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&6&0&0&6&0&0&-3&0&0&3 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&-3&3&0&-3&3&0&\frac{3}{2}&-\frac{3}{2}&0&-\frac{3}{2}&\frac{3}{2}
\end{pmatrix}_{24\times 24}$$

Proposition (3.2):

Let $\Pi_i:G \longrightarrow GL(Wi)$, $\Pi_i^*:G \longrightarrow GL(W_i^*)$ for $i = 1,2,3,4,5$ and the $TCoA$-complex Lie groups of G on $(W_5^* \otimes (W_4 \otimes ((W_3^* \otimes W_2) \oplus (W_3^* \otimes W_1))))$ is given by a representation Π such that

$\Pi(a) = [(\Pi_1(a) \circ W_1 \circ W_3^* \circ \Pi_3^*(a^{-1})) \oplus (\Pi_2(a) \circ W_2 \circ W_3^* \circ \Pi_3^*(a)^{-1}) \circ \Pi_4(a) \circ W_4 \circ W_5^* \circ \Pi_5^*(a)^{-1}$,

for all $a \in G$.

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Then the *TCoA*-complex Lie group of G on (W_5^* ⊗(W_4⊗((W_3^*⊗W_2)⊕(W_3^*⊗W_1)))) is also given by a representation Π*, such that:

$$\Pi^*(a)=\Pi_5^*(a)^{-1}\circ W_5\circ W_3^*\circ(\Pi_4^*(a)\circ(\Pi_3^*(a)^{-1}\circ W_3\circ W_2^*\circ\Pi_2^*(a)\oplus\Pi_3^*(a)^{-1}\circ W_3\circ W_1^*\circ\Pi_1^*(a)))$$

,
for all $a \in$ G.

Proof: Let *TCoA*-complex Lie group G on (W_5^* ⊗(W_4⊗((W_3^*⊗W_2)⊕ (W_3^*⊗W_1)))) is induced by the representation Π:G ⟶ GL((W_5^*⊗(W_4⊗ ((W_3^*⊗W_2)⊕(W_3^*⊗W_1)))) such that

$\Pi(a)$ = $(((\Pi_1(a)\circ W_1\circ W_3^*\circ\Pi_3(a^{-1}))\oplus(\Pi_2(a)\circ W_2\circ W_3^*\circ\Pi_3(a)^{-1})\circ\Pi_4(a)\circ W_4\circ W_5^*\circ\Pi_5(a)^{-1})$,

for all $a\in$G, $W_3'\times W_1'\in W_3\times W_1$, $\Pi_3\in(W_3,(W_2\times W_1))$, $\Pi_4\in W_5\times W_4$.

To show that Π*:G ⟶ GL(W_5^*⊗(W_4⊗((W_3^*⊗W_2)⊕ (W_3^*⊗W_1))))* is a representation, such that

$\Pi^*(a)$ = $(((\Pi_5^*(a)^{-1}\circ W_5\circ W_4^*\circ(\Pi_4^*(a)\circ(\Pi_3^*(a)^{-1}\circ W_3\circ W_2^*\circ\Pi_2^*(a)\oplus\Pi_3^*(a)^{-1}\circ W_3\circ W_1^*\circ\Pi_1^*(a)))$

is a representation for all $a\in$G and $\Pi_4^*\in(W_5^*\otimes W_4)^*$, $\Pi_3^*\in(W_3,(W_2\otimes W_1))^*$, $\Pi_2^*\times\Pi_1^*\in(W_2\otimes W_1))^*$ since

$$\Pi^*(a)=\Pi_5(a)^{-1}\circ W_5\circ W_4^*\circ(\Pi_4(a)\circ((\Pi_3(a)^{-1}\circ W_3\circ W_2^*\circ\Pi_2^*(a))\oplus(\Pi_3(a)^{-1}\circ W_3\circ W_1^*\circ\Pi_1^*(a))$$

For all $a\in$ G, $\Pi_4^*:W_4^*\longrightarrow W_5^*$ and $\Pi^*(ab)=(\Pi(ab))^*=(\Pi(b)\circ\Pi(a))^*=\Pi^*(a)\circ\Pi^*(b)$. Thus Π* is a representation from *G* (Π* is a group *homoorphism of G*)

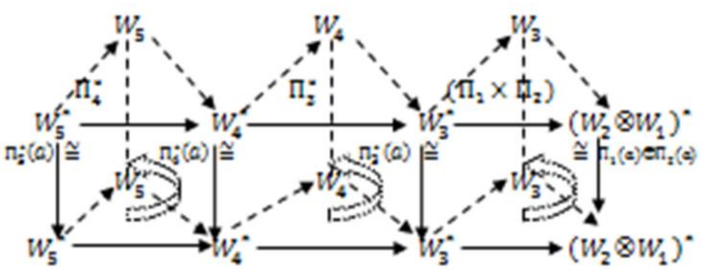

Diagram 7.

This arrow makes the diagram 7 commutative.

Proposition (3.3):

Let *Wi* for i = 1, 2, 3, 4, 5 are finite vector spaces, W_i^* is the dual of vectors W_i, for i = 1, 2, 3, 4, 5 then the following assertions are equivalent:

(1) [(W_5^*⊗(W_4⊗((W_3^*⊗W_2)⊕(W_3^*⊗W_1)))]*.

(2) (((W_3^*⊗W_2)*⊕ (W_3^*⊗W_1)*)⊗W_4^*)*⊗ W_5^{**}).

(3) (W_2^*⊗W_3)⊕(W_1^*⊗W_3)⊗ W_4^*)⊗W_5).

(4) ((($W_2^*\otimes W_1^*$)⊗W_3)⊗ W_4^*)⊗W_5).

(5) (($W_2^*\otimes W_1^*$)⊗(W_3^*,K)⊗ W_4^*)⊗W_5).

(6) (($W_2^*\otimes W_1^*$)⊗W_3) ⊗ W_4^*)⊗(W_5^{**},K)).

(7) ((W_1⊗ W_2)*⊗ W_4^*)⊗(W_3,K)⊗ W_4^*))⊗W_5).

(8) [(W_5⊗(W_4⊗(W_3⊗W_2)⊕(W_3⊗W_1))]***

$$=\begin{cases}(W_5\otimes(W_4\otimes(W_3\otimes W_2)\oplus(W_3\otimes W_1)) & \text{if } n \text{ is an even number}\\ (W_3\otimes W_2)^*\oplus(W_3\otimes W_1)^*\otimes W_4^*)\otimes W_5 & \text{if } n \text{ is an odd number}\end{cases}$$

Proof:

(1) ≅ (2) To show [(W_5^*⊗(W_4⊗((W_3^*⊗W_2)⊕(W_3^*⊗W_1))]* ≅ (((W_3^*⊗W_2)*⊕(W_3^*⊗W_1)*)⊗ W_4^*)*⊗ W_5^{**}).

Let $\Pi_4\in(W_5^*\otimes W_4)$, $\Pi_3\in(W_4\otimes W_3^*)$, $\Pi_2\times\Pi_1\in(W_3^*\otimes(W_2\otimes W_1))$, $\Pi_4^*\in(W_4^*\otimes W_5)$, $\Pi_3^*\in(W_3,W_4^*),\Pi_1^*\times\Pi_2^*=(\Pi_2\times\Pi_1)^*\in((W_1^*\otimes W_2^*)\otimes W_3)$ and there exists an intertwining map

ψ: (W_5^*⊗(W_4⊗((W_3^*⊗W_2)⊕(W_3^*⊗W_1)))) ⟶ (((W_3^*⊗W_2)*⊕(W_3^*⊗W_1)*)⊗ W_4^*)*⊗ W_5^{**},

such that

$\psi(\Pi^*(a))(v)=\Pi^*(a)\psi(v)$, for all v ∈ $W_1^*\times W_2^*$ and ψ is an invertible map.

(1) ≅ (3) To show (W_5^*⊗(W_4⊗((W_3^*⊗W_2)⊕

(2) (W_3^*⊗W_1)))* ≅

(($W_2^*\otimes W_3^*$)⊕($W_1^*\otimes W_3^*$)⊗ W_4^*)⊗W_5), since (W_3^*,W_2)* ≅ (W_2^*,W_3^{**}), (W_3^*,W_1)* ≅ (W_1^*,W_3^{**}), $W_5^{**}\cong W_5$ and $W_3^{**}\cong W_3$.

By the same methods, we have the other parts. ■

Example (3.4):

Let Π_i, i = 1,2,3,4, $\Pi_i:S^1 \longrightarrow SO(2) \subset GL(2,\mathbb{C})$ and $\Pi_5:S^1 \longrightarrow O(3) \subset GL(3,\mathbb{C})$, where G = S^1, (n = 2, m = 3) and Wi, i = 1,2,3,4 are the ℂ-vector spaces of dimensional 2 and W_5 is the ℂ-vector space of dimensional 3 such that,

$$\Pi_1(e^{i\theta})=\Pi_2(e^{i\theta})=\Pi_3(e^{i\theta})=\Pi_4(e^{i\theta})=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$$

, $0\le\theta\le2\pi$, $i^2=-1$, $\Pi_5(e^{i\theta})=\begin{pmatrix}1&0&0\\0&\cos\theta&-\sin\theta\\0&\sin\theta&\cos\theta\end{pmatrix}$, $0 \le \theta \le 2\pi$. The $TCoA$-complex Lie group G on ($W_5^*\otimes(W_4\otimes((W_3^*\otimes W_2)\oplus(W_3^*\otimes W_1)))$ is a representation:

Π: $S^1 \longrightarrow GL(W_5^*\otimes(W_4\otimes((W_3^*\otimes W_2)\oplus(W_3^*\otimes W_1)))$ such that

$$\Pi^*(a)=(\Pi_5(a)^{-1}\circ W_5\circ W_4^*\circ(\Pi_4^*(a)\circ((\Pi_3(a)^{-1}\circ W_3\circ W_2^*\circ\Pi_2^*(a)\oplus(\Pi_3(a))^{-1}\circ W_1\circ W_1^*\circ\Pi_1^*(a))$$

$$\Pi^*(e^{i\theta})=\Pi_5(e^{-i\theta})\otimes(\Pi_4^*(e^{i\theta})\otimes((\Pi_3(e^{-i\theta})\otimes\Pi_2^*(e^{i\theta}))\oplus(\Pi_3(e^{-i\theta})\otimes\Pi_1^*(e^{i\theta}))$$

$$=\begin{pmatrix}1&0&0\\0&\cos\theta&-\sin\theta\\0&\sin\theta&\cos\theta\end{pmatrix}\otimes\begin{pmatrix}\cos\theta&-\sin\theta\\ \sin\theta&\cos\theta\end{pmatrix}\otimes\left(\left(\begin{pmatrix}\cos\theta&\sin\theta\\-\sin\theta&\cos\theta\end{pmatrix}\otimes\begin{pmatrix}\cos\theta&-\sin\theta\\ \sin\theta&\cos\theta\end{pmatrix}\right)\oplus\right.$$

$$\left.\left(\begin{pmatrix}\cos\theta&\sin\theta\\-\sin\theta&\cos\theta\end{pmatrix}\otimes\begin{pmatrix}\cos\theta&-\sin\theta\\ \sin\theta&\cos\theta\end{pmatrix}\right)\right)$$

$$=\begin{pmatrix}\cos\theta&0&0&-\sin\theta&0&0\\0&\cos^2\theta&-\sin\theta\cos\theta&0&-\sin\theta\cos\theta&\sin^2\theta\\0&\sin\theta\cos\theta&\cos^2\theta&0&-\sin^2\theta&-\sin\theta\cos\theta\\ \sin\theta&0&0&\cos\theta&0&0\\0&\sin\theta\cos\theta&-\sin^2\theta&0&\cos^2\theta&-\sin\theta\cos\theta\\0&\sin^2\theta&\sin\theta\cos\theta&0&\sin\theta\cos\theta&\cos^2\theta\end{pmatrix}_{6\times6}\otimes$$

$$\left[\begin{pmatrix}\cos^2\theta&\sin\theta\cos\theta&-\sin\theta\cos\theta&-\sin^2\theta\\-\sin\theta\cos\theta&\cos^2\theta&\sin^2\theta&-\sin\theta\cos\theta\\ \sin\theta\cos\theta&\sin^2\theta&\cos^2\theta&\sin\theta\cos\theta\\-\sin^2\theta&\sin\theta\cos\theta&-\sin\theta\cos\theta&\cos^2\theta\end{pmatrix}_{4\times4}\oplus\right.$$

$$\left.\begin{pmatrix}\cos^2\theta&\sin\theta\cos\theta&-\sin\theta\cos\theta&-\sin^2\theta\\-\sin\theta\cos\theta&\cos^2\theta&\sin^2\theta&-\sin\theta\cos\theta\\ \sin\theta\cos\theta&\sin^2\theta&\cos^2\theta&\sin\theta\cos\theta\\-\sin^2\theta&\sin\theta\cos\theta&-\sin\theta\cos\theta&\cos^2\theta\end{pmatrix}_{4\times4}\right]$$

$$=\begin{pmatrix}\cos\theta&0&0&-\sin\theta&0&0\\0&\cos^2\theta&-\sin\theta\cos\theta&0&-\sin\theta\cos\theta&\sin^2\theta\\0&\sin\theta\cos\theta&\cos^2\theta&0&-\sin^2\theta&-\sin\theta\cos\theta\\ \sin\theta&0&0&\cos\theta&0&0\\0&\sin\theta\cos\theta&-\sin^2\theta&0&\cos^2\theta&-\sin\theta\cos\theta\\0&\sin^2\theta&\sin\theta\cos\theta&0&\sin\theta\cos\theta&\cos^2\theta\end{pmatrix}_{6\times6}\otimes$$

$$\begin{pmatrix}2\cos^2\theta&2\sin\theta\cos\theta&-2\sin\theta\cos\theta&-2\sin^2\theta\\-2\sin\theta\cos\theta&2\cos^2\theta&2\sin^2\theta&-2\sin\theta\cos\theta\\2\sin\theta\cos\theta&2\sin^2\theta&2\cos^2\theta&2\sin\theta\cos\theta\\-2\sin^2\theta&2\sin\theta\cos\theta&-2\sin\theta\cos\theta&2\cos^2\theta\end{pmatrix}_{4\times4}$$

$$\left(\begin{array}{cccccccccccc}
2\cos^3\theta&0&0&-2\cos^2\theta\sin\theta&0&0&2\sin\theta\cos^2\theta&0&0&-2\sin^2\theta\cos\theta&0&0\\
0&2\cos^4\theta&-2\sin\theta\cos^3\theta&0&-2\sin\theta\cos^3\theta&2\sin^2\theta\cos^2\theta&0&2\sin\theta\cos^3\theta&-2\sin^2\theta\cos^2\theta&0&-2\sin^2\theta\cos^2\theta&2\sin^3\theta\cos\theta\\
0&2\sin\theta\cos^3\theta&2\cos^4\theta&0&-2\sin^2\theta\cos^2\theta&-2\sin\theta\cos^3\theta&0&2\sin^2\theta\cos^2\theta&2\sin\theta\cos^3\theta&0&-2\sin^3\theta\cos\theta&-2\sin^2\theta\cos^2\theta\\
2\sin\theta\cos^2\theta&0&0&2\cos^3\theta&0&0&2\sin^2\theta\cos\theta&0&0&2\sin\theta\cos^2\theta&0&0\\
0&2\sin\theta\cos^2\theta&-2\sin^2\theta\cos^2\theta&0&2\cos^4\theta&-2\sin\theta\cos^3\theta&0&2\sin^2\theta\cos^2\theta&-2\sin^3\theta\cos\theta&0&2\sin\theta\cos^3\theta&-2\sin^2\theta\cos^2\theta\\
0&2\sin^2\theta\cos^2\theta&2\sin\theta\cos^3\theta&0&2\sin\theta\cos^3\theta&2\cos^4\theta&0&2\sin^3\theta\cos\theta&2\sin^2\theta\cos^2\theta&0&2\sin^2\theta\cos^2\theta&2\sin\theta\cos^3\theta\\
-2\sin\theta\cos^2\theta&0&0&2\sin^2\theta\cos\theta&0&0&2\cos^3\theta&0&0&-2\sin\theta\cos^2\theta&0&0\\
0&-2\sin\theta\cos^3\theta&2\sin^2\theta\cos^2\theta&0&2\sin^2\theta\cos^2\theta&-2\sin^3\theta\cos\theta&0&2\cos^4\theta&-2\sin\theta\cos^3\theta&0&-2\sin\theta\cos^3\theta&2\sin^2\theta\cos^2\theta\\
0&-2\sin^2\theta\cos^2\theta&-2\sin\theta\cos^3\theta&0&-2\sin^3\theta\cos\theta&2\sin^2\theta\cos^2\theta&0&2\sin\theta\cos^3\theta&2\cos^4\theta&0&-2\sin^2\theta\cos^2\theta&-2\sin\theta\cos^3\theta\\
-2\sin\theta\cos^2\theta&0&0&-2\sin\theta\cos^2\theta&0&0&2\sin\theta\cos^2\theta&0&0&2\cos^3\theta&0&0\\
0&-2\sin^2\theta\cos^2\theta&2\sin^3\theta\cos\theta&0&-2\sin\theta\cos^3\theta&2\sin^2\theta\cos^2\theta&0&2\sin\theta\cos^3\theta&-2\sin^2\theta\cos^2\theta&0&2\cos^4\theta&-2\sin\theta\cos^3\theta\\
0&-2\sin^3\theta\cos\theta&-2\sin^2\theta\cos^2\theta&0&-2\sin^2\theta\cos^2\theta&-2\sin\theta\cos^3\theta&0&2\sin^2\theta\cos^2\theta&2\sin\theta\cos^3\theta&0&2\sin\theta\cos^3\theta&2\cos^4\theta\\
2\sin\theta\cos^2\theta&0&0&-2\sin^2\theta\cos\theta&0&0&2\sin^2\theta\cos\theta&0&0&-2\sin^3\theta&0&0\\
0&-2\sin\theta\cos^3\theta&-2\sin^2\theta\cos^2\theta&0&-2\sin^2\theta\cos^2\theta&2\sin^3\theta\cos\theta&0&2\sin^2\theta\cos^2\theta&-2\sin^3\theta\cos\theta&0&-2\sin^3\theta\cos\theta&2\sin^4\theta\\
0&2\sin^2\theta\cos^2\theta&2\sin\theta\cos^3\theta&0&-2\sin^3\theta\cos\theta&-2\sin^2\theta\cos^2\theta&0&2\sin^3\theta\cos\theta&2\sin^2\theta\cos^2\theta&0&-2\sin^4\theta&-2\sin^3\theta\cos\theta\\
2\sin^2\theta\cos\theta&0&0&2\sin\theta\cos^2\theta&0&0&2\sin^3\theta&0&0&2\sin^2\theta\cos\theta&0&0\\
0&2\sin^2\theta\cos^2\theta&-2\sin^3\theta\cos\theta&0&2\sin\theta\cos^3\theta&-2\sin^2\theta\cos^2\theta&0&2\sin^3\theta\cos\theta&-2\sin^4\theta&0&2\sin^2\theta\cos^2\theta&-2\sin^3\theta\cos\theta\\
0&2\sin^3\theta\cos\theta&2\sin^2\theta\cos^2\theta&0&2\sin^2\theta\cos^2\theta&2\sin\theta\cos^3\theta&0&2\sin^4\theta&2\sin^3\theta\cos\theta&0&-2\sin^3\theta\cos\theta&2\sin^2\theta\cos^2\theta\\
2\sin^2\theta\cos\theta&0&0&2\sin^3\theta&0&0&2\sin\theta\cos^2\theta&0&0&-2\sin^2\theta\cos\theta&0&0\\
0&-2\sin^2\theta\cos^2\theta&2\sin^3\theta\cos\theta&0&2\sin^3\theta\cos\theta&-2\sin^4\theta&0&2\sin\theta\cos^3\theta&-2\sin^2\theta\cos^2\theta&0&-2\sin^2\theta\cos^2\theta&2\sin^3\theta\cos\theta\\
0&-2\sin^3\theta\cos\theta&-2\sin^2\theta\cos^2\theta&0&2\sin^4\theta&2\sin^3\theta\cos\theta&0&2\sin^2\theta\cos^2\theta&2\sin\theta\cos^3\theta&0&-2\sin^3\theta\cos\theta&-2\sin^2\theta\cos^2\theta\\
-2\sin^3\theta&0&0&-2\sin^2\theta\cos\theta&0&0&2\sin^2\theta\cos\theta&0&0&2\sin\theta\cos^2\theta&0&0\\
0&-2\sin^3\theta\cos\theta&2\sin^4\theta&0&-2\sin^2\theta\cos^2\theta&2\sin^3\theta\cos\theta&0&2\sin^2\theta\cos^2\theta&-2\sin^2\theta\cos\theta&0&2\sin\theta\cos^3\theta&-2\sin^2\theta\cos^2\theta\\
0&-2\sin^4\theta&-2\sin^3\theta\cos\theta&0&-2\sin^3\theta\cos\theta&-2\sin^2\theta\cos^2\theta&0&2\sin^3\theta\cos\theta&2\sin^2\theta\cos^2\theta&0&2\sin^2\theta\cos^2\theta&2\sin\theta\cos^3\theta
\end{array}\right.$$

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$$
\left.
\begin{matrix}
-2\sin\theta\cos^2\theta & 0 & 0 & 2\sin^2\theta\cos^2\theta & 0 & 0 & -2\sin^2\theta\cos\theta & 0 & 0 & 2\sin^3\theta & 0 & 0 \\
0 & -2\sin\theta\cos^3\theta & 2\sin^2\theta\cos^2\theta & 0 & 2\sin^2\theta\cos^2\theta & -2\sin^3\theta\cos\theta & 0 & -2\sin^2\theta\cos^2\theta & 2\sin^3\theta\cos\theta & 0 & 2\sin^3\theta\cos\theta & -2\sin^4\theta \\
0 & -2\sin^2\theta\cos^2\theta & -2\sin\theta\cos^3\theta & 0 & -2\sin^3\theta\cos\theta & 2\sin^2\theta\cos^2\theta & 0 & -2\sin^3\theta\cos\theta & -2\sin^2\theta\cos^2\theta & 0 & 2\sin^4\theta & 2\sin^3\theta\cos\theta \\
-2\sin^2\theta\cos\theta & 0 & 0 & -2\sin\theta\cos^2\theta & 0 & 0 & -2\sin^3\theta & 0 & 0 & -2\sin^2\theta\cos\theta & 0 & 0 \\
0 & -2\sin^2\theta\cos^2\theta & 2\sin^3\theta\cos\theta & 0 & -2\sin\theta\cos^3\theta & 2\sin^2\theta\cos^2\theta & 0 & -2\sin^3\theta\cos\theta & 2\sin^4\theta & 0 & -2\sin^2\theta\cos^2\theta & 2\sin^3\theta\cos\theta \\
0 & -2\sin^3\theta\cos\theta & -2\sin^2\theta\cos^2\theta & 0 & -2\sin^2\theta\cos^2\theta & -2\sin\theta\cos^3\theta & 0 & -2\sin^4\theta & -2\sin^3\theta\cos\theta & 0 & -2\sin^3\theta\cos\theta & -2\sin^2\theta\cos^2\theta \\
2\sin^2\theta\cos\theta & 0 & 0 & -2\sin^3\theta & 0 & 0 & -2\sin\theta\cos^2\theta & 0 & 0 & 2\sin^2\theta\cos\theta & 0 & 0 \\
0 & 2\sin^2\theta\cos^2\theta & -2\sin^3\theta\cos\theta & 0 & -2\sin^3\theta\cos\theta & 2\sin^4\theta & 0 & -2\sin\theta\cos^3\theta & 2\sin^2\theta\cos^2\theta & 0 & -2\sin^2\theta\cos^2\theta & -2\sin^3\theta\cos\theta \\
0 & 2\sin^3\theta\cos\theta & 2\sin^2\theta\cos^2\theta & 0 & -2\sin^4\theta & -2\sin^3\theta\cos\theta & 0 & -2\sin^2\theta\cos^2\theta & -2\sin\theta\cos^3\theta & 0 & -2\sin^3\theta\cos\theta & 2\sin^2\theta\cos^2\theta \\
2\sin^3\theta & 0 & 0 & 2\sin^2\theta\cos\theta & 0 & 0 & -2\sin^2\theta\cos\theta & 0 & 0 & -2\sin\theta\cos^2\theta & 0 & 0 \\
0 & 2\sin^3\theta\cos\theta & -2\sin^4\theta & 0 & 2\sin^2\theta\cos^2\theta & -2\sin^3\theta\cos\theta & 0 & -2\sin^2\theta\cos^2\theta & 2\sin^3\theta\cos\theta & 0 & -2\sin\theta\cos^3\theta & 2\sin^2\theta\cos^2\theta \\
0 & 2\sin^4\theta & 2\sin^3\theta\cos\theta & 0 & 2\sin^3\theta\cos\theta & 2\sin^2\theta\cos^2\theta & 0 & -2\sin^3\theta\cos\theta & -2\sin^2\theta\cos^2\theta & 0 & -2\sin^2\theta\cos^2\theta & -2\sin\theta\cos^3\theta \\
2\cos^3\theta & 0 & 0 & -2\sin\theta\cos^2\theta & 0 & 0 & -2\sin\theta\cos^2\theta & 0 & 0 & -2\sin^2\theta\cos\theta & 0 & 0 \\
0 & 2\cos^4\theta & -2\sin\theta\cos^3\theta & 0 & -2\sin\theta\cos^2\theta & 2\sin^2\theta\cos^2\theta & 0 & -2\sin\theta\cos^3\theta & -2\sin^2\theta\cos^2\theta & 0 & -2\sin^2\theta\cos^2\theta & -2\sin^3\theta\cos\theta \\
0 & 2\sin\theta\cos^3\theta & 2\cos^4\theta & 0 & -2\sin^2\theta\cos^2\theta & -2\sin\theta\cos^3\theta & 0 & 2\sin^2\theta\cos^2\theta & -2\sin\theta\cos^3\theta & 0 & -2\sin^3\theta\cos\theta & -2\sin^2\theta\cos^2\theta \\
2\sin\theta\cos^2\theta & 0 & 0 & 2\cos^3\theta & 0 & 0 & 2\sin^2\theta\cos\theta & 0 & 0 & 2\sin\theta\cos^3\theta & 0 & 0 \\
0 & 2\sin\theta\cos^3\theta & -2\sin^2\theta\cos^2\theta & 0 & 2\cos^4\theta & -2\sin\theta\cos^3\theta & 0 & 2\sin^2\theta\cos^2\theta & -2\sin^3\theta\cos\theta & 0 & 2\sin\theta\cos^3\theta & -2\sin^2\theta\cos^2\theta \\
0 & 2\sin^2\theta\cos^2\theta & 2\sin\theta\cos^3\theta & 0 & 2\sin\theta\cos^3\theta & 2\cos^4\theta & 0 & 2\sin^3\theta\cos\theta & -2\sin^2\theta\cos^2\theta & 0 & 2\sin^2\theta\cos^2\theta & 2\sin^2\theta\cos^2\theta \\
-2\sin\theta\cos^2\theta & 0 & 0 & 2\sin^2\theta\cos\theta & 0 & 0 & 2\cos^3\theta & 0 & 0 & -2\sin\theta\cos^2\theta & 0 & 0 \\
0 & -2c\sin\theta os^3\theta & 2\sin^2\theta\cos^2\theta & 0 & 2\sin^2\theta\cos^2\theta & -2\sin^3\theta\cos\theta & 0 & 2\cos^4\theta & -2\sin\theta\cos^3\theta & 0 & -2\sin\theta\cos^3\theta & 2\sin^2\theta\cos^2\theta \\
0 & -2\sin^2\theta\cos^2\theta & -2\sin\theta\cos^3\theta & 0 & 2\sin^3\theta\cos\theta & 2\sin^2\theta\cos^2\theta & 0 & 2\sin\theta\cos^3\theta & 2\cos^4\theta & 0 & -2\sin^2\theta\cos^2\theta & -2\sin\theta\cos^3\theta \\
-2\sin^2\theta\cos\theta & 0 & 0 & -2\sin\theta\cos^2\theta & 0 & 0 & 2\sin\theta\cos^2\theta & 0 & 0 & 2\cos^3\theta & 0 & 0 \\
0 & -2\sin^2\theta\cos^2\theta & 2\sin^3\theta\cos\theta & 0 & -2\sin\theta\cos^3\theta & 2\sin^2\theta\cos^2\theta & 0 & 2\sin\theta\cos^3\theta & -2\sin^2\theta\cos^2\theta & 0 & 2\cos^4\theta & -2\sin\theta\cos^3\theta \\
0 & -2\sin^3\theta\cos\theta & -2\sin^2\theta\cos^2\theta & 0 & -2\sin^2\theta\cos^2\theta & -2\sin\theta\cos^3\theta & 0 & 2\sin^2\theta\cos^2\theta & 2\sin\theta\cos^2\theta & 0 & 2\sin\theta\cos^3\theta & 2\cos^4\theta
\end{matrix}
\right]_{24\times 24}
$$

Proposition (3. 5):

Let Π_i, i = 1,2,3,4,5 be representations of G acting on K-finite dimensional vector spaces W_i, i = 1,2,3,4,5 respectively, then the $TCoA$-reductive Lie group of G on $\underset{K}{\mathrm{Hom}}(W_5,\mathrm{Hom}(W_4^*,Hom((W_3,W_2)\oplus Hom(W_3,W_1)))$ is equivalent to the representation $\Pi_5^*\otimes(\Pi_4\otimes((\Pi_3^*\otimes\Pi_2)\oplus(\Pi_3^*\otimes\Pi_1))))$ of G on $GL(W_5^*\otimes(W_4\otimes((W_3^*,W_2)\oplus(W_3^*,W_1))))$.

Proof: To show that:

$\psi:(W_5^*\otimes(W_4\otimes((W_3^*\otimes W_2)\oplus(W_3^*\otimes W_1))))\longrightarrow \underset{K}{\mathrm{Hom}}(W_5,\mathrm{Hom}(W_4^*,Hom((W_3,W_2)\oplus Hom(W_3,W_1)))$ is bilinear map, defined by $\psi(W_5^*,w_1) = F$ for all $W_5^* \in W_5$ and $w_1 \in W_1$,

where $F: W_5 \longrightarrow W_1$ is a linear map defined by $F(v) = w_5^*(v)w_1$, for all w_5^*, $w_5^{*'} \in W_5^*$, $v \in W_5$, $\alpha, \beta \in K$, $w_1 \in W_1$

$\psi(\alpha w_5^* + \beta w_5^{*'},w_1) = (\alpha w_5^* + \beta w_5^{*'}(v))w_1$

$= \alpha w_5^*(v)w_1 + \beta w_5^{*'}(v)w_1$

$= \alpha\psi(w_5^*,w_1) + \beta\psi(w_5^{*'},w_1)$

Other for all w_1, $w_1' \in W_1$ and $w_5^* \in W_5^*$

$\psi(,\alpha w_1 + \beta w_1') = (w_5^*(v)(\alpha w_1 + \beta w_1')$

$= w_5^*(v)(\alpha w_1) + w_5^{*'}(v)(\beta w_1')$

$= \alpha w_5^*(v)w_1 + \beta w_5^{*'}(v)w_1'$

$\psi(W_5^*,\alpha w_1 + \beta w_1') = \alpha\psi(W_5^*,w_1) + \beta\psi(W_5^*,w_1')$.

So $\psi: W_5^*\times(W_4\times((W_3^*\times W_2)\oplus(W_3^*\times W_1))))\longrightarrow \underset{K}{\mathrm{Hom}}(W_5,\mathrm{Hom}(W_4^*,Hom((W_3,W_2)\oplus Hom(W_3,W_1)))$ is a bilinear map, thus by using the tensor product and universal property of this tensor product, we get a unique linear map ϕ.

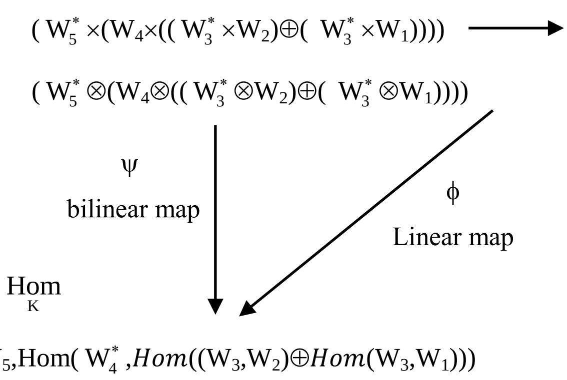

Diagram 8.

So by universal property of tensor product $W_5^*\times(W_4\times((W_3^*\times W_2)\oplus(W_3^*\times W_1))))$ there exists a unique linear map $\phi: W_5^*\otimes(W_4\otimes((W_3^*\otimes W_2)\oplus(W_3^*\otimes W_1))))\longrightarrow \underset{K}{\mathrm{Hom}}(W_5,\mathrm{Hom}(W_4^*,Hom((W_3,W_2)\oplus Hom(W_3,W_1)))$. This makes the above diagram commutative:

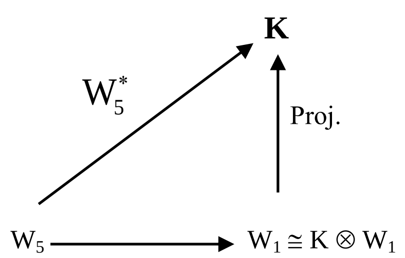

Diagram 9.

Consider the composition of linear maps where $W_5^*(v)$ is defined as follows:

$F(v) = w_1$, ∃! $k \in K$, such that $w_1 \longrightarrow (k,w_1)$ since all maps are linear and k is unique, put $W_5^*(v) = k$ related to w_1.

Define $\zeta: \underset{K}{\text{Hom}}(W_5,\text{Hom}(W_4^*,Hom((W_3,W_2)\oplus Hom(W_3,W_1)))$ $\longrightarrow W_5^* \otimes(W_4 \otimes((W_3^* \otimes W_2)\oplus (W_3^* \otimes W_1)))$ by $\zeta(F') = W_5^*(v)w_1$.

Define $W_5^* : W_5 \longrightarrow K$ by $W_5^*(v) = k$, where k is given by $\zeta(F'(v)) = (k, F'(v))$

We can show that W_5^* is linear put $F'(v) = w_1$, for all $F' \in \underset{K}{\text{Hom}}(W_5,\text{Hom}(W_4^*,Hom((W_3, W_2)\oplus Hom(W_3,W_1)))$, $w_1 \in W_1$ and $w_5^* \in W_5$ and is related to W_1.

$F'(\alpha v_1 + \beta v_2) = \alpha F'(v_1) + \beta F'(v_2)$

$= \alpha k_1 + \beta k_2$

$= \alpha W_5^*(v) + \beta W_5^*(v)$, for all $W_5^* \in W_5$

Where: $w_5^*(v_1) = k_1 \Rightarrow w_5^*(\alpha v_1) = \alpha k_1$,

$w_5^*(\alpha v_1 + \beta v_2) = (\alpha k_1 + \beta k_2)$

$w_5^*(v_2) = k_2 \Rightarrow w_5^*(\beta v_2) = \beta k_2$,

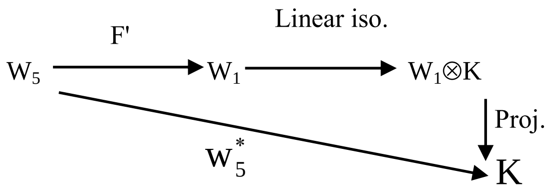

Diagram 10.

Clear F' is a linear and $\zeta^{-1} = \phi$, thus ζ is linear map. Related between the $TCoA$ of reductive Lie groups of G on $\underset{K}{\text{Hom}}(W_5,\text{Hom}(W_4^*,Hom((W_3, W_2)\oplus Hom(W_3,W_1)))$ and $TCoA$ of reductive Lie groups of G on $W_5^* \otimes(W_4 \otimes((W_3^* \otimes W_2)\oplus (W_3^* \otimes W_1)))$ up to the representation given above:

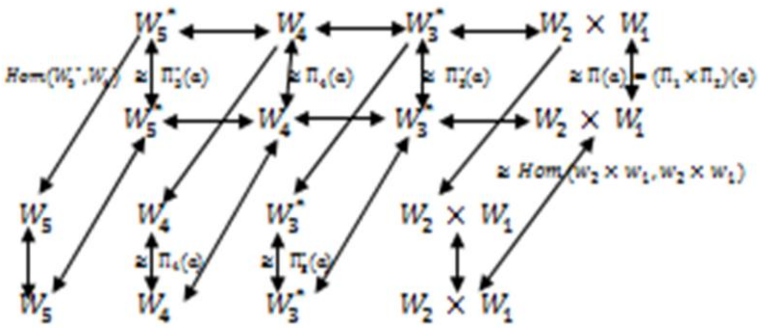

Diagram 11.

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أنواع معينة لفعل زمر لي المركبة

احمد خلف راضي تغريد حر مجيد
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المستخلص :

الهدف الرئيسي من البحث هو الحصول على فعل بصفات جديدة في زمر لي المركبة من بديهية $Schure$ التي درست وركزت على فعل جبر لي لتمثيليين احدهما عادي والأخر ثنائي ، والشئ المهم والممتع في العمل هو التركيز على بعض الأفعال لزمرة لي المركبة. في هذة الدراسة قمنا بتحليل مفاهيم من فعل زمر لي المركبة على فضاءات $\boldsymbol{Hom}$ وتعريف الضرب التنسوري لتمثيلات اثنتين في زمر لي وركزنا على فعل زمر لي على فضاءات $\boldsymbol{Hom}$، باستخدام التكافؤ

$$Hom(w_2, w_1) \cong w_2^* \otimes w_1$$

بين فضاءات $\boldsymbol{Hom}$ والضرب التنسوري للحصول على فعل زمر لي المركبة على الضرب التنسوري.
الفعل الثاني هو بصيغة تمثيلات ملساء للمجموعة $\boldsymbol{G}$. هذا الفعل هو فعل ثلاثي لزمر لي المركبة $\boldsymbol{G}$ ويرمز لها (فعل زمر لي المركبة $\boldsymbol{TAC}$)على

$$Hom(Hom((w_5 \oplus w_4), w_3^*), Hom(w_2 \oplus w_1, w^*)).$$

وهذا TAC هوتمثيلات ملساء في G ، أن النظريات المقدمة في هذا البحث انشات وبرهنت وجهزت ببعض النتائج كملاحظات و رسوم.

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Anti – fuzzy AT – ideals of AT – algebras

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Abstract.

In this paper , we introduce the notion of anti – fuzzy AT – ideals in AT – algebra, several appropriate examples are provided and theirsome properties are investigated. The image and the inverse imageof anti – fuzzy AT – ideals in AT – algebra are defined and how theimage and the inverse image of anti – fuzzy AT – ideals in AT – algebra become anti – fuzzy AT – ideals are studied. Moreover, the Cartesian product of anti – fuzzy AT – ideals are given .

Keywords : AT – ideal, anti – fuzzy AT – ideals , image and pre-image of anti – fuzzy AT – ideals .

Mathematics subject classification: 06F35, 03G25, 08A72.

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1 Introduction

BCK − algebras form an important class of logical algebras introducedby K. Iseki [4] and was extensively investigated by several researchers. Theclass of all BCK − algebras is quasi variety. J. Meng and Y. B. Jun posed an interesting problem (solved in[7]) whether the class of all BCK − algebras is a variety. In connection with this problem , Komori introduced in [6]a notion of BCC − algebras. W.A. Dudek (cf.[2],[5]) redefined the notion of BCC − algebras by using a dual form of the ordinary definition inthe sense of Y. Komori and studied ideals and congruences of BCC-algebras. In ([10],[11]), C. Prabpayak and U. Leerawat introduced a new algebraic structure, which is called KU − algebra. They gave the concept of homomorphisms of KU − algebras and investigated some related properties. L.A. Zadeh [13] introduced the notion of fuzzy sets. At present this concept has been applied to many mathematical branches, such as group, functional analysis , probability theory , topology, and soon. In 1991 , O.G. Xi [12] applied this concept to BCK − algebras, and heintroduced the notion of fuzzy sub-algebras (ideals) of the BCK − algebras with respect to minimum, and since then Jun et al studiedfuzzy ideals (cf.[1],[5],[12]), and moreover several fuzzystructures in BCC-algebras are considered (cf.[2],[6]). S. Mostafa , M. Abd-Elnaby, F. Abdel-Halim and A.T. Hameed (in [7]) introduced the notion of fuzzy KUS − ideals of KUS − algebras and they investigated severalbasic properties which are related to fuzzy KUS − ideals. they describedhow to deal with the homomorphism image and inverse image of fuzzy KUS − ideals. And in [8], the anti − fuzzy KUS − ideals of KUS − algebras is introduced. Several theorems are stated and proved. In [3], Areej Tawfeeq Hameed introduced and studied new algebraic structure, called AT − algebra and investigate some of its properties. She introduced the notion of fuzzy AT − ideal of AT − algebra, several theorems, properties are stated and proved.

In this paper, we introduce the notion of anti − fuzzy AT − ideals of AT − algebras and then we study the homomorphism image and inverse image of anti − fuzzy AT − ideals.We also prove that the Cartesian product of anti − fuzzy AT − ideals are anti − fuzzy AT − ideals .

2. Preliminaries

In this section we give some basic definitions and preliminaries lemmas of AT − ideals and fuzzy AT − ideals of AT − algebra.

Definition 2.1[3]. An **AT-algebra** is a nonempty set X with a constant (0) and a binary operation ($*$) satisfying the following axioms: for all x, y, z∈X,

(i) (x*y)*((y*z)*(x*z))=0,

(ii) 0* x =x,

(iii) x* 0 =0.

In X we can define a binary relation (≤) by :x ≤ y if and only if , y * x = 0 .

Remark 2.2[3]. (X ;*, 0) is an AT − algebra if and only if, it satisfies that: for allx, y, z∈X,

(i') : (y *z) *(x *z) ≤ x *y ,

(ii') : x ≤ y if and only if, y * x = 0.

Example 2.3 [3]. Let X = {0, 1, 2, 3, 4} in which (*) is defined by the following table:

*	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	3	4
2	0	1	0	3	3
3	0	0	2	0	2
4	0	0	0	0	0

It is easy to show that (X ;*, 0) is an AT − algebra.

Example 2.4[3]. Let X = {0, 1, 2, 3,4} be a set with the following table:

*	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	3	4
2	0	1	0	3	3
3	0	0	2	0	2
4	0	0	0	0	0

Then (X ;*, 0)is an AT − algebra.

Proposition 2.5 [3]. In any AT − algebra (X ;*, 0), the following properties holds: for all x, y, z ∈X;

a) z * z = 0,
b) z*(x *z) = 0 ,
c) y * ((y* z) * z) = 0,
d) x * y = 0 implies that x * 0 = y * 0 ,
e) 0*x=0*y implies that x=y .

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Proposition 2.6[3]. In any AT – algebra (X ;*, 0), the following properties holds: for all x, y, z ∈X;

a) $x \le y$ implies that $y * z \le x * z$,
b) $x \le y$ implies that $z * x \le z * y$
c) $z * x \le z * y$ implies that $x \le y$(left cancellation law).

Proposition 2.7[3]. In any AT – algebra (X ;*, 0), the following properties holds: for all x, y,z ∈X;

a) $x = 0 *(0*x)$,
b) $x* y \le z$ imply $z * y \le x$.

Definition 2.8[3]. A nonempty subset S of an AT – algebra X is called **an AT – subalgebra of AT – algebra X** if x*y∈S , whenever x, y∈S.

Definition 2.9[3]. A nonempty subset I of an AT – algebra X is called **an AT-ideal of AT-algebra X** if it satisfies the following conditions: for all x, y, z ∈ X .

AT_1) 0 ∈ I ;
AT_2) x * (y *z) ∈ I and y ∈ I imply x*z∈ I .

Proposition 2.10[3]. Every AT – ideal of AT – algebra X is an AT – subalgebra.

Definition 2.11[3]. Let X be an AT – algebra.

A fuzzy set μ in X is called a fuzzy AT – subalgebra of X if it satisfies the following conditions: for all x, y ∈ X,

$\mu(x*y) \ge \min\{\mu(x),\mu(y)\}$.

Definition 2.12[3]. Let X be an AT – algebra.

A fuzzy set μ in X is called a fuzzy AT – ideal of X if it satisfies the following conditions: for all x, y and z ∈ X,

(AT_1) $\mu(0) \ge \mu(x)$.

(AT_2) $\mu(x*z) \ge \min\{\mu(x*(y*z)),\mu(y)\}$.

Proposition 2.13[3]. Every fuzzy AT – ideal of AT – algebra X is fuzzy AT – subalgebra.

3. Anti-fuzzy AT-ideals of AT-algebras

In this section, we will introduce a new notion called an anti – fuzzy AT – ideal of AT – algebra and study several basic properties of it.

Definition 3.1[13]. Let X be a nonempty set, a fuzzy set μ in X is a function $\mu : X \to [0, 1]$.

Definition 3.2. Let X be an AT – algebra. A fuzzy set μ in X is called an anti- fuzzy AT – ideal of X if it satisfies the following conditions: for all x, y and z ∈ X,

(AAT_1) $\mu(0) \le \mu(x)$.

(AAT_2) $\mu(x*z) \le \max\{\mu(x*(y*z)),\mu(y)\}$.

Example 3.3. Let X = {0, 1, 2, 3} be a set with the following table:

*	0	1	2	3
0	0	1	2	3
1	0	0	2	3
2	0	0	0	3
3	0	0	0	0

Then (X ;*, 0) is an AT – algebra. It is easy to show that I_1 ={0, 1}and I_2={0, 3} are AT-ideals of X .

Define a fuzzy set μ : X ⟶ [0, 1] by $\mu(0) = t_1, \mu(1) = \mu(2) = \mu(3) = t_2$, where $t_1, t_2 \in [0, 1]$ with $t_1 < t_2$.

Routine calculation gives that μ is an anti- fuzzy AT – ideal of AT – algebras X.

Lemma 3.4. Let μ be an anti- fuzzy AT – ideal of AT – algebra X and if $x \le y$, then $\mu(y) \le \mu(x)$, for all x, y∈ X.

Proof: Assume that $x \le y$, then $y * x = 0$, and $\mu(0*y) = \mu(y) \le \max\{\mu(0*(x*y)), \mu(x)\} = \max\{\mu(0), \mu(x)\} = \mu(x)$.

Hence $\mu(y) \le \mu(x)$. ⌂

Proposition 3.5. Let μ be an anti- fuzzy AT – ideal of AT – algebra X. If the inequality $y*x\leq z$ hold in X, then $\mu(x) \leq \max\{\mu(y),\mu(z)\}$.

Proof: Assume that the inequality $y*x\leq z$ hold in X, by lemma(3.4),

$\mu(z) \leq \mu(y*x)$ --- (1).

By(AAT_2),$\mu(z*x) \leq \max\{\mu(z*(y*x)), \mu(y)\}$.Put z=0,then

$\mu(0*x) =\mu(x) \leq \max\{\mu(0*(y*x)), \mu(y)\}= \max\{\mu(y*x),\mu(y)\}$--- (2) .

From (1) and (2), we get $\mu(x) \leq \max\{\mu(y), \mu(z)\}$, for all x, y, z $\in$X. ⌂

Theorem 3.6. Let μ be an anti-fuzzy set in X then μ is an anti – fuzzy AT – ideal of X if and only if, it satisfies:

For all $\alpha\in[0, 1]$, $U(\mu,\alpha) \neq\varnothing$ implies $U(\mu,\alpha)$ is an AT – ideal of X----(A),

where $U(\mu,\alpha) = \{x\in X|\mu(x) \leq\alpha\}$.

Proof: Assume that μ is an anti – fuzzy AT – ideal of X, let $\alpha\in[0,1]$ be such that $U(\mu,\alpha) \neq\varnothing$, and let x, y $\in$ X be such that $x \in U(\mu,\alpha)$, then $\mu(x) \leq\alpha$and so by (AAT_1),$\mu(0) \leq\mu(x)\leq\alpha$. Thus $0 \in U(\mu,\alpha)$.

Now let $(z*(y*x)),y\in U(\mu,\alpha)$. It follows from ($AAT_2$) that

$\mu(z*x) \leq \max\{\mu(z*(y*x)),\mu(y)\}= \alpha$, so that $(z*x)\in U(\mu,\alpha)$. Hence $U(\mu,\alpha)$ is an AT – ideal of X.

Conversely, suppose that μ satisfies (A), assume that (AAT_1) is false, then there exist $x \in X$ such that $\mu(0) >\mu(x)$.If we take $t = \frac{1}{2}(\mu(x)+\mu(0))$, then $\mu(0) > t$ and $0 \leq\mu(x) < t \leq 1$, thus$x \in U(\mu,t)$ and $U(\mu,t) \neq\varnothing$.As $U(\mu,t)$ is an AT – ideal of X, we have $0 \in U(\mu,t)$, and so $\mu(0) \leq t$.This is a contradiction.

Hence $\mu(0) \leq\mu(x)$ for all $x \in X$. Now, assume (AAT_2) is not true thenthere exist x, y, z $\in$Xsuch that

$\mu(z*x) > \max\{\mu(z*(y*x)),\mu(y)\}$,

taking $\beta_0 = \frac{1}{2}[\mu(z*x) + \max\{\mu(z*(y*x)),\mu(y)\}]$,

we have $\beta_0\in[0, 1]$ and

$\max\{\mu(z*(y*x)),\mu(y)\}, <\beta_0<\mu(z*x)$, it follows that

$\max\{\mu(z*(y*x)),\mu(y)\}, \in U(\mu,\beta_0)$and $z*y\notin U(\mu,\beta_0)$, this is a contradiction and therefore μ is an anti – fuzzy AT – ideal of X. ⌂

4. Characterization of anti-fuzzy AT-ideals by their level AT-ideals

Theorem 4.1. A fuzzy subset μ of an AT – algebra X is an anti – fuzzy AT – ideal of X if and only if, for every $t\in[0,1]$, μ_t is an AT – ideal of X, where

$\mu_t = \{x\in X|\mu(x)\leq t\}$.

Proof: Assume that μ is an anti – fuzzy AT – ideal of X, by (AAT_1), we have

$\mu(0) \leq\mu(x)$ for all $x \in X$, therefore $\mu(0) \leq\mu(x) \leq t$, for $x \in \mu_t$ and so $0 \in \mu_t$.

Let $(z*(y*x)) \in \mu_t$ and $(y)\in \mu_t$, then $\mu(z*(y*x))\leq t$ and $\mu(y) \leq t$,since μ is an anti – fuzzy AT – ideal it follows that $\mu(z*x) \leq\{\mu(z*(y*x)),\mu(y)\}\leq t$ and that

$(z*x) \in \mu_t$ .Hence μ_t is an AT − ideal of X.

Conversely, we only need to show that (AAT_1) and (AAT_2) are true. If (AAT_1) is false, then there exist $x \in X$ such that $\mu(0) > \mu(x)$. If we take $t = \frac{1}{2}(\mu(x) + \mu(0))$, then $\mu(0) > t$ and $0 \le \mu(x) < t \le 1$ thus $x \in \mu_t$ and $\mu_t \neq \emptyset$. As μ_t is an AT − ideal of X, we have $0 \in \mu_t$ and so $\mu(0) \le t$. This is a contradiction.

Now, assume (AAT_2) is not true, then there exist x, y and $z \in X$ such that,

$\mu(z*x) > \max\{\mu(z*(y*x)), \mu(y)\}$.

Putting $t = \frac{1}{2}[\mu(z*x) + \max\{\mu(z*(y*x)), \mu(y)\}]$,

then $\mu(z*x) > t$ and $0 \le \max\{\mu(z*(y*x)), \mu(y)\}, < t \le 1$, hence $\mu(z*(y*x)) < t$ and $\mu(y) < t$, which imply that $(z*y) \in \mu_t$ and $(y*x) \in \mu_t$, since μ_t is an anti − fuzzy AT − ideal, it follows that $(z*x) \in \mu_t$ and that $\mu(z*x) \le t$, this is also a contradiction. Hence μ is an anti − fuzzy AT − ideal of X . ⌂

Corollary 4.2. If a fuzzy subset μ of AT − algebra X is an anti − fuzzy AT − ideal, then for every $t \in Im(\mu)$, μ_t is an AT − ideal of X.

Definition 4.3. Let μ be an anti − fuzzy AT − ideal of AT − algebra X, then the AT − ideal μ_t, $t \in [0,1]$ are called level AT − ideals of μ.

Corollary 4.4. Let I be an AT − ideal of an AT − algebra X, then for any fixed number t in an open interval (0,1), there exist an anti − fuzzy AT − ideal μ of X such that $\mu_t = I$.

Proof: Define $\mu : X \to [0:1]$ by $\mu(x) =$

$$\begin{cases} 0, & \text{if } x \in I; \\ t, & \text{if } x \notin I. \end{cases}$$

Where t is a fixed number in (0,1). Clearly, $\mu(0) \le \mu(x)$ and we have one two level sets $\mu_0 = I$, $\mu_t = X$, which are AT − ideals of X, then from Theorem (4.1) μ is an anti − fuzzy AT − ideal of X. ⌂

5. Image and Pre-image of anti-fuzzy AT-ideals

Definition 5.1. $f : (X; *, 0) \to (Y; *`, 0`)$ be a mapping from a nonempty set X to a nonempty set Y. If β is a fuzzy subset of X, then the fuzzy subset μ of Y defined by: $f(\mu)(y) = \beta(y) =$

$$\begin{cases} \inf_{x \in f^{-1}(y)} \mu(x) & \text{if } f^{-1}(y) = \{x \in X, f(x) = y\} \neq \phi \\ 0 & \text{otherwise} \end{cases}$$

is said to be the image of μ under f .

Similarly if μ is a fuzzy subset of Y, then the fuzzy subset $\mu = (\beta \circ f)$ in X (i.e., the fuzzy subset defined by $\mu(x) = \beta(f(x))$, for all $x \in X$) is called the pre-image of β under f .

Theorem 5.2. An into homomorphic pre-image of anti − fuzzy AT − ideal is also an anti − fuzzy AT − ideal.

Proof: Let $f:(X;*,0) \to (Y;*`,0`)$be an onto homomorphism of AT – algebras, β is an anti – fuzzy AT – ideal of Y and μ the pre-image of β under f, then

β(f (x)) = μ(x), for all x ∈ X. Let x ∈ X, then μ(0) = β(f (0)) <β(f (x)) = μ(x).

Now let x, y, z ∈ X, then μ (z∗x) = β(f (z∗x)) = β(f (z)∗' f (x))

≤ max{β(f (z)∗' f (y)),β(f (y)∗' f (x))}

= max{β(f (z∗y)),β(f (y∗x))}

= max{μ (z∗y),μ (y∗x)},and the proof is completed. ⌂

Definition 5.3. An anti fuzzy subset μ of X has inf property if for any subset T of X, there exist t_0∈ T such that μ (t) = $\inf_{t\in T} \mu(t)$.

Theorem 5.4. Let $f:(X;*,0) \to (Y;*`,0`)$be a homomorphism between AT – algebras X and Y respectively . For every anti – fuzzy AT – ideal μ in X, f (μ) is an anti – fuzzy AT – ideal of Y.

Proof: By definition β(y') = f (μ)(y')=

$\inf_{x\in f^{-1}(y')} \mu(x)$, for all y'∈Y and ∅ = 0.

We have to prove that β(z'∗x') ≤ max{β(z'∗(y'*x')),β(y')}, for all x', y', z'∈ Y.

Let f : X → Y be an onto homomorphism of AT – algebras, μ is an anti – fuzzy AT – ideal of X with inf property and β the image of μ under f , since μ is anti – fuzzy AT – ideal of X, we haveμ (0) ≤μ (x) for all x ∈ X.

Note that 0 ∈ f^{-1}(0'), where 0, 0'are the zero of X and Y, respectively.

Thus β(0') = $\inf_{t\in f^{-1}(x')} \mu(t)$ =β(x'), for all x ∈ X, which implies that

β(0') ≤ $\inf_{t\in f^{-1}(x')} \mu(t)$ = β(x'), for any x'∈ Y.For any x', y', z'∈ Y, let x_0∈ f^{-1} (x'), y_0∈ f^{-1} (y'), z_0∈ f^{-1} (z') be such that

μ (z_0∗ (y_0*x_0)) = $\inf_{t\in f^{-1}(z'*(y'*x'))} \mu(t)$,μ (y_0) = $\inf_{t\in f^{-1}(y')} \mu(t)$ and

μ (z_0∗ x_0)= $\inf_{t\in f^{-1}(z'*x')} \mu(t)$. Then

β (z'∗x') = $\inf_{t\in f^{-1}(z'*x')} \mu(t)$ = μ (z_0∗ x_0)

≤ max{μ (z_0∗ (y_0*x_0)) ,μ (y_0)}

= max[$\inf_{t\in f^{-1}(z'*(y'*x'))} \mu(t)$, $\inf_{t\in f^{-1}(y')} \mu(t)$]

= max{β(z'∗(y'*x')),β(y')}.

Hence β is an anti – fuzzy AT – ideal of Y. ⌂

6. Cartesian product of anti-fuzzy AT-ideals

Definition 6.1 ([1],[9]). A fuzzy relation R on any set S is a fuzzy subset

R: $S\times S \to [0,1]$.

Definition 6.2 ([1]). If R is a fuzzy relation on sets S and β is a fuzzy subset of S , then Ris a fuzzy relation on β if R(x, y) ≥ max {β(x), β(y)}, for all x, y∈S .

Definition 6.3([1]). Let μ and β be fuzzy subsets of a set S . The Cartesian product of μ and β is defined by($\mu\times\beta$)(x,y) = max {μ(x), β(y)}, , for all x, y∈S .

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Lemma 6.4([1]). Let S be a set and μ and β be fuzzy subsets of S . Then,

(1) $\mu\times\beta$ is a fuzzy relation on S ,

(2) $(\mu\times\beta)_t = \mu_t\times\beta_t$, for all t ∈[0,1].

Definition 6.5([1]). Let S be a set and β be fuzzy subset of S . The strongest fuzzy relation on S , that is, a fuzzy relation on β is R_β given by

R_β (x,y) = max {β(x) , β(y)},for all x, y∈S .

Lemma 6.6([1]). For a given fuzzy subset β of a set S, let R_β be the strongest fuzzy relation on S. Then for t ∈[0,1], we have $(R_\beta)_t = \beta_t\times\beta_t$.

Proposition 6.7. For a given fuzzy subset β of an AT – algebra X , let R_β be the strongest fuzzy relation on X . If β is an anti – fuzzy AT – ideal of $X\times X$, then

R_β (x,x) ≥ R_β (0,0) , for all x∈X .

Proof: Since R_β is a strongest fuzzy relation of $X\times X$, it follows from that,

R_β (x,x) = max{β(x), β(x)}≥ max{β(0), β(0)}= R_β (0,0),which implies that

R_β (x,x) ≥ R_β (0,0).⌂

Proposition 6.8. For a given fuzzy subset β of an AT – algebra X , let R_β be the strongest fuzzy relation on X . If R_β is an anti – fuzzy AT – ideal of $X\times X$, then

β(0) ≤ β(x) , for all x∈X .

Proof: Since R_β is an anti – fuzzy AT – ideal of $X\times X$, it follows from (AAT_1),

R_β (x,x) ≥ R_β (0,0),where (0,0) is the zero element of $X\times X$. But this means that , max{β(x), β(x)} ≥ max{β(0), β(0)}which implies that β(0) ≤ β(x) . ⌂

Remark 6.9([9]). Let X and Y be AT – algebras, we define (*) on $X\times Y$ by: for all(x,y),(u,v) ∈ $X\times Y$, (x,y) ∗ (u,v) = (x ∗ u,y ∗ v). Then clearly ($X\times Y$; ∗ .(0,0)) is anAT-algebra.

Theorem 6.10. Let μ and β be an anti – fuzzy AT – ideal of AT – algebra X . Then $\mu\times\beta$ is an anti – fuzzy AT – ideal of $X\times X$.

Proof: Note first that for every (x,y)∈ $X\times X$,

$(\mu\times\beta)$(0,0) = max {μ(0) , β(0)} ≤ max {μ(x) , β(y)} = $(\mu\times\beta)$(x,y) .

Now let (x_1,x_2) , (y_1,y_2) , (z_1,z_2) ∈ $X\times X$. Then

$(\mu\times\beta)(x_1 * z_1, x_2 * z_2)$ = max { $\mu(x_1 * z_1)$,$\beta(x_2 * z_2)$}

≤ max { max{$\mu(x_1 * (y_1 * z_1))$,$\mu(y_1)$}, max{$\beta(x_2 * (y_2 * z_2))$,$\beta(y_2)$}}

= max { max{$\mu((x_1 * (y_1 * z_1)))$, $\beta(x_2 * (y_2 * z_2))$}, max{ $\mu(y_1)$,$\beta(y_2)$}}

= max { $(\mu\times\beta)((x_1 * (y_1 * z_1)) ,(x_2 * (y_2 * z_2)))$, $(\mu\times\beta)(y_1,y_2)$}

Hence $(\mu\times\beta)$ is an anti – fuzzy AT – ideal of $X\times X$.⌂

Theorem 6.11. Let μ and β be anti-fuzzy subsets of AT – algebra X such that $\mu\times\beta$ is an anti – fuzzy AT – ideal of $X\times X$. Then for all x∈X,

(i) either μ (0) ≤ μ (x) or β (0) ≤ β (x) .

(ii) $\mu(0) \le \mu(x)$ for all $x \in X$, then either $\beta(0) \le \beta(x)$ or $\beta(0) \le \mu(x)$.

(iii)

If $\beta(0) \le \beta(x)$ for all $x \in X$, then either $\mu(0) \le \mu(x)$ or $\mu(0) \le \beta(x)$.

(iv) Either μ or β is an anti – fuzzy AT – ideal of X .

Proof.

(i) Suppose that $\mu(0)>\mu(x)$ and $\beta(0)>\beta(y)$ for some x , y $\in X$. Then

$(\mu\times\beta)(x,y) = \max\{\mu(x), \beta(y)\} < \max\{\mu(0), \beta(0)\} = (\mu\times\beta)(0,0)$. This is a contradiction and we obtain (i).

(ii) Assume that there exist x , y $\in X$ such that $\beta(0) > \mu(x)$ and $\beta(0) > \beta(y)$. Then

$(\mu\times\beta)(0,0) = \max\{\mu(0),\beta(0)\} = \beta(0)$ it follows that

$(\mu\times\beta)(x,y) = \max\{\mu(x), \beta(y)\} < \beta(0) = (\mu\times\beta)(0,0)$ which is a contradiction. Hence (ii) holds.

(iii) Is by similar method to part (ii).

(iv) Suppose $\beta(0) \le \beta(x)$ by (i), then form (iii) either $\mu(0) \le \mu(x)$ or $\mu(0) \le \beta(x)$ for all $x \in X$.

If $\mu(0) \le \beta(x)$, for any $x \in X$, then $(\mu\times\beta)(0,x) = \max\{\mu(0), \beta(x)\} = \beta(x)$. Let (x_1,x_2) , (y_1,y_2) , $(z_1,z_2) \in X\times X$, since $(\mu\times\beta)$ is an anti-fuzzy AT-ideal of $X\times X$, we have

$(\mu\times\beta)(x_1 * z_1, x_2 * z_2) \le \max\{(\mu\times\beta)((x_1 * (y_1 * z_1)),(x_2 * (y_2 * z_2))), (\mu\times\beta)(y_1,y_2)\}$---- (A)

If we take $x_1 = y_1 = z_1 = 0$, then

$\beta(x_2 * z_2) = (\mu\times\beta)(0,x_2 * z_2)$

$\le \max\{(\mu\times\beta)(0,(x_2 * (y_2 * z_2))), (\mu\times\beta)(0,y_2)\}$

$= \max\{\max\{\mu(0), \beta((x_2 * (y_2 * z_2)))\}, \max\{\mu(0), \beta(y_2)\}\}$

$= \max\{\beta((x_2 * (y_2 * z_2))), \beta(y_2)\}$

This prove that β is an anti – fuzzy AT – ideal of X .

Now we consider the case $\mu(0) \le \mu(x)$ for all $x \in X$.

Suppose that $\mu(0) > \mu(y)$ for some $y \in X$. then $\beta(0) \le \beta(y) < \mu(0)$.

Since $\mu(0) \le \mu(x)$ for all $x \in X$, it follows that $\beta(0) < \mu(x)$ for any $x \in X$.

Hence $(\mu\times\beta)(x,0) = \max\{\mu(x), \beta(0)\} = \mu(x)$

taking $x_2 = y_2 = z_2 = 0$ in (A), then

$\mu(x_1 * z_1) = (\mu\times\beta)(x_1 * z_1, 0)$

$\le \max\{(\mu\times\beta)((x_1 * (y_1 * z_1)),0), (\mu\times\beta)(y_1,0)\}$

$= \max\{\max\{\mu(x_1 * (y_1 * z_1)), \beta(0)\}, \max\{\mu(y_1), \beta(0)\}\}$

$= \max\{\mu(x_1 * (y_1 * z_1)), \mu(y_1)\}$

Which proves that μ is an anti – fuzzy AT – ideal of X . Hence either μ or β is an anti – fuzzy AT – ideal of X . ⌂

Theorem 6.12. Let β be a fuzzy subset of an AT – algebra X and let R_β be the strongest fuzzy relation on X , then β is an anti – fuzzy AT – ideal of X if and only if R_β is an anti – fuzzy AT – ideal of $X\times X$.

Proof: Assume that β is an anti – fuzzy AT – ideal of X . By proposition (6.7) ,we get, $R_\beta(0,0) \le R_\beta(x,y)$, for any $(x,y) \in X\times X$.

Let (x_1,x_2) , (y_1,y_2) , $(z_1,z_2) \in X\times X$, we have from (AAT_2) :

$R_\beta(z_1 * x_1, z_2 * x_2) = \max\{\beta(z_1 * x_1), \beta(z_2 * x_2)\}$

$\leq \max\{\max\{\beta(z_1 * (y_1 * x_1)), \beta(y_1)\}, \max\{\beta(z_2 * (y_2 * x_2)), \beta(y_2)\}\}$

$= \max\{\max\{\beta(z_1 * (y_1 * x_1)), \beta(z_2 * (y_2 * x_2))\}, \max\{\beta(y_1), \beta(y_2)\}\}$

$= \max\{R_\beta((z_2 * (y_2 * x_2)), (z_2 * (y_2 * x_2))), R_\beta(y_1, y_2)\}$

Hence R_β is an anti – fuzzy AT – ideal of $X \times X$.

Conversely, suppose that R_β is an anti – fuzzy AT – ideal of $X \times X$, by proposition (6.8)$\beta(0) \leq \beta(x)$ for all $x \in X$, which prove (AAT_1) .

Now, let $(x_1,x_2), (y_1,y_2), (z_1,z_2) \in X \times X$. Then,

$\max\{\beta(z_1 * x_1), \beta(z_2 * x_2)\} = R_\beta(z_1 * x_1, z_2 * x_2)$

$\leq \max\{R_\beta((z_1,z_2) * ((y_1, y_2) * (x_1, x_2)), R_\beta(y_1,y_2)\}$

$= \max\{R_\beta((z_1 * (y_1 * x_1)), (z_2 * (y_2 * x_2))), R_\beta(y_1,y_2)\}$

$= \max\{\max\{\beta((z_1 * (y_1 * x_1)), \beta(z_2 * (y_2 * x_2))\}, \max\{\beta(y_1), \beta(y_2)\}\}$

In particular if we take $x_2 = y_2 = z_2 = 0$, then $\beta(z_1 * x_1) \leq \max\{\beta(z_1 * (y_1 * x_1)), \beta(y_1)\}$ .This proves (AAT_2) and β is an anti – fuzzy AT – ideal of X . ⌂

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المثاليات الضبابية المضاد (AT) في الجبريات (AT)

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المستخلص :
في هذا البحث ، نقدم مفهوم المثاليات الضبابية المضاد "AT " في AT-الجبر ، وفيه يتم تقديم العديد من الأمثلة المناسبة ويتم التحقيق في خصائصها. يتم تعريف الصورة والصورة المعكوسة للمثالي الضبابي المضاد "AT " في الجبر -AT وكيف يتم دراسة خواص الصورة والصورة العكسيّة للمثاليات الضبابية المضادة AT في -ATالجبر. علاوة على ذلك ، يتم إعطاء الضرب الديكارتي للمثالي الضبابي المضاد AT.

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Trivial Extension of Armendariz Rings and Related Concepts

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Abstract.

This paper investigate the possibility of inheriting the properties of the ring R to the trivial extension ring $T(R,R)$ and present the relationship between the trivial extension $T(R,R)$ and many types of rings. The concepts of 2-primal, reversible, central reversible, semicommutative, nil–semicommutative, $\pi-$Armendariz and central $\pi-$Armendariz rings are studied with trivial extension $T(R,R)$ and some characterizations of π-Armendariz and central $\pi-$Armendariz rings are given.

Keywords. Trivial extension**,** Armendariz**,** α-Armendariz, nil-Armendariz, central $\pi-$ Armendariz .

Mathematics subject classification: 11T55,13F20,11CXX.

1.Introduction.

Throughout this paper, all rings are commutative with identity unless otherwise stated. For a ring R, we denote by $C(R)$ the center of a ring R. A ring R which has no non zero nilpotent elements is called reduced. In (1997) the concept of Armendariz rings is introduced by Rage and Chhawchharia [1]. An Armendariz ring (ARM ring, for short) R is a ring that satisfies if $f(x)=\sum_{i=0}^{m} s_i x^i$ and $g(x)=\sum_{j=0}^{m} t_j x^j \in R[x]$ satisfy $f(x)g(x)=0$ implies that $s_i t_j=0$ for each i, j. Rege and Chhawchharia in [1] showed that every reduced ring is ARM. Consequently, this class of rings are related to nilpotent elements. Anderson and Camillo [2] proved that if $n\geq 2$, then $R[x]/x^n$ is an ARM ring if and only if R is reduced. A ring R is said to reversible if $ab=0$ then $ba=0$, for all $,b\in R$. According to shin [3], semicommutative rings introduced as a generalization of commutative rings. A ring R that satisfies $st=0$ implies $sRt=0$ for each $s,t\in R$ is called semicommutative if. Clearly every reduced ring is reversible and every reversible ring is semicommutative but the converse is not true in general [4]. The set of all nilpotent elements in R, the set of all nilpotent polynomials in $R[x]$ and the intersection of all prime ideal are denoted by $N(R)$ ($N(R[x])$ and $P(R)$, respectively). Due to Birkenmeier et al. [5], a ring R is said to be 2-primal if $N(R)=P(R)$. Every semicommutative ring is 2-primal , Semicommutative rings also studied under the name zero insertive by Habeb [6]. Many of authors have been written on ARM property [7] and [8]. A ring R is said to be π-Armendariz (π-ARM, for short) if for any two polynomials $f(x)=\sum_{i=0}^{m} s_i x^i$ and $g(x)=\sum_{j=0}^{m} t_j x^j \in R[x]$ such that $f(x)g(x)\in N(R[x])$, then $s_i t_j\in N(R)$ for each i,j [9].

Each ARM (2-primal) ring is π- ARM, but the converse may not true [9]. Recall that a ring R is said to be weak Armendariz (WARM, for short) if whenever $f(x)=\sum_{i=0}^{m} s_i x^i$ and $g(x)=\sum_{j=0}^{m} t_j x^j \in R[x]$ satisfy $f(x)g(x)=0$ then $s_i t_j \in N(R)$ for each i,j [10]. It is easy to see that every ARM ring is WARM, but the convers is not true in general.

This further encourages to the study of nilpotent elements, which is a generalization of ARM rings which have been studied in [10]. The ARM feature of a ring was generalized to one of skew polynomial as in [11] and [12]. Let $\alpha: R \to R$ be an endomorphism for a ring R. The ring that produced by giving the polynomial ring over R with the multiplication $xr=\alpha(r)x$, for all $r \in R$ is called skew polynomial ring $R[x;\alpha]$ of R. Some properties of skew polynomial rings have been studied in [13] and [14]. As a generalization of the notion of ARM rings, the concepts of α-Armendariz(α −ARM, for short) rings and α-skew Armendariz (α −SARM, for short) are introduced in [11] and [12] respectively. A ring R is called α-SARM (respectively, α-ARM) if for any two polynomials $f(x)=\sum_{i=0}^{m} s_i x^i$, $g(x)=\sum_{j=0}^{n} t_j x^j \in R[x;\alpha]$ such that $f(x)g(x)=0$ then $s_i \alpha^i(t_j)=0$ (respectively, $s_i t_j=0$) for each i, j [11] (Respectively [12]). Agayev et. al. [15] introduced the concept of central Armendariz rings (CARM, for short) which is a generalization of the concept of ARM rings. The notion of CARM rings lies exactly between the notions of abelian rings and ARM rings. A ring R is said to be CARM if there exists any two polynomials $f(x)=\sum_{i=0}^{m} s_i x^i$ and $g(x)=\sum_{j=0}^{m} t_j x^j \in R[x]$ such that $f(x)g(x)=0$, then $s_i t_j \in C(R)$ for each i,j. As a generalization of CARM rings, Abdali in [16] introduced and studied the sense of central π-ARM rings(C π-ARM, for short). A ring R is called C π-ARM if for all $f(x)=\sum_{i=0}^{m} s_i x^i$, $g(x)=\sum_{j=0}^{n} t_j x^j \in R[x]$ such that $f(x)g(x) \in N(R[x])$ then $s_i t_j \in C(R)$ for each i,j. Note that every CARM ring is Cπ-ARM.

The main idea is to set M may be ($M=R$) in a commutative ring T=(R,M)=$R \oplus M$ (if R is commutative) so that the structure of M as an R-module is essentially the same as that of M as an T-module, that is, as an ideal of T [17], and [18]. The advantage of the trivial extension is:

(a) Transfer results relating to modules to the ideal case including the R-module$R$, (b) Extending results from rings to modules, (c) It is easier to find counterexamples of rings especially those with zero divisors.

Generally, this paper studied and explained the relationship between some types of rings and some related concepts with trivial extension ring $T(R,R)$.

Moreover, we extended the terms of domain, central reversible, 2-primal, symmetric, reversible, semicommutative, nil– semicommutative, π −ARM, central reduced, central reversible and Cπ −ARM to the trivial extension $T(R,R)$. We showed that if $R[x]$ is domain, then $T(R,R)$ is α-ARM. Also if R is a central reversible ring, then $T=T\,(R\,,R)$ is nil-ARM. In addition, R is 2-primal ring iff $T=T\,(R\,,R)$ is ARM ring according to specific conditions and characterizations of π-ARM and Cπ −ARM rings are given.

There is a considerable number of authors showed interest in studying trivial extension rings related or not to the family of ARM rings. Kim and Lee in [7] studied trivial extension rings of ARM rings. In 2006, Liu and Zhao show that trivial extension of WARM ring is also WARM [10]. The property of α-ARM has been studied in [12]. Also the trivial extension of central reduced rings has been discussed in [19]. The π -regular rings fact of trivial extension has recently been explained in more details by Abduldaim in [20].

It is worth to mention that many suggestions could be put to the study trivial extensions of some new generalizations like α-skew π-McCoy rings [21]. Moreover, the trivial extension rings can be employed in the field of cryptography as in [22] and [23].

The structure of this paper is as follows. Section 2 is devoted to recall previous known definitions and information about the trivial extension. In section 3, the paper presents the relationship between the trivial extension rings and some kind of rings. Several examples are given to clarify the ideas used within the section.

2. Preliminaries

In this section we survey known results concerning $T=T(R,M)=R\oplus M$. The theme throughout is how properties of T=$T(R,M)$=$R\oplus M$ are related to those of R and M([17], [18]).

Let M be an (R,R)-bimodule. Recall that the trivial extension of R by M (also called the idealization of M over R) is given to be the set $T=T(R,M)$ of all pairs (r,m) where $r \in R$, $m \in M$, that is:

$T=T(R,M)$=$R \oplus M$=$\{(r,m)|r \in R, m \in M\}$

With addition defined component wise as

$$(r_1,m_1)+(r_2,m_2)=(r_1+r_2,m_1+m_2)$$

And multiplication defined according to the rule

$$(r_1,m_1)(r_2,m_2)=(r_1r_2,r_1m_2+m_1r_2)$$

For all $r_1,r_2 \in R$ and $m_1,m_2 \in M$. Clearly $T=T(R,M)$ forms a ring (even an R-algebra) and it is commutative if and only if R is commutative. Note that $T(R,0) \cong R$ via $r \to (r,0)$, then R can be embedded into $T(R,M)$, this means M is identified

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with $T(0,M) = \{(0,m)|m \in M\}$ becomes a nonzero nilpotent ideal of $T(R,M)$ of index 2, which explains the term idealization. If N is a submodule of M, then $T(0,N)$, is an ideal of $T(R,M)$, and that $(T(R,M)/T(0,M) \cong R$. The ring $T = T(R,M)$ has identity element (1,0) and any idempotent element of the trivial extension ring $T(R,R)$ is of the form $(e,0)$ where $e^2 = e \in R$. In fact, there is another realization of the trivial extension.

Let $T = T(R,M) \cong \left\{\begin{pmatrix} r & m \\ 0 & r \end{pmatrix} | \ r \in R, m \in M\right\}$, then T is a subring of the ring of 2×2 matrices over with the usual matrix operations and T is a commutative ring with identity. If $M = R$, then $T = T(R,R) \cong R[x]/< x^2 >$where $R[x]$ denote the ring of all polynomials over R and $< x^2 >$ is the ideal generated by$< x^2 >$. Recall that a ring R is said to be semiprime such that $sRs = 0$ then $s = 0$ for $s \in R$, [16].

Proposition 2.1 [24, Proposition 2.18] Let R be a semiprime ring R. Then the following are equivalent:

(1) R is reduced

(2) R is symmetric

(3) R is reversible

(4) R is semicommutative

(5) R is nil-semicommutative

(6) R is 2-primal

Corollary 2.2 [24, Corollary 2.19] Let R be a Von Neumann regular ring R. Then the following are equivalent:

(1) R is reduced
(2) R is symmetric
(3) R is reversible
(4) R is semicommutative
(5) R is nil-semicommutative
(6) R is 2-primal

Theorem 2.3 [2, Theorem 5]: Let R be a ring and $n \geq 2$ a natural number. Then $R[x]/(x^n)$ is ARM if and only if R is reduced.

3. Main Results

It is known that if R is a domain, then $R \oplus R$ is ARM [1]. In addition, R is an α–SARM ring for any endomorphism α of a domain ring R [11, Proposition10]. Hong et al. [12, Example 1.9] illustrate that a domain may not be an α–ARM ring for an arbitrary endomorphism α.

Theorem 3.1 Let $R[x]$ be a domain. Then $T = T(R,R)$ is α-ARM.

Proof Suppose that $f(x) = \sum_{i=0}^{m} A_i\, x^i$, $g(x) = \sum_{j=0}^{n} B_j\, x^j$ in $T = T(R,R)[x,\alpha]$ with $f(x)g(x)$=0 where $A_i = (a_i\,, u_i\,)$, $B_j = (\,b_j\,, v_j\,)$ for all $0 \le i \le m$, $0 \le j \le n$ and $f_0(x) = \sum_{i=0}^{m} a_i\, x^i$, $f_1(x) = \sum_{i=0}^{m} u_i\, x^i$, $g_0(x) = \sum_{j=0}^{n} b_j\, x^j$ and $g_1(x) = \sum_{j=0}^{n} v_j\, x^j$ are elements in $R[x]$. In the other words $f(x) = (f_0(x), f_1(x))$ and $g(x) = (g_0(x), g_1(x))$. So, we have

$f(x)g(x)$=(0,0)=($f_0(x)\, g_0(x)$, $f_0(x)\, g_1(x)$+ $f_1(x)\, g_0(x)$) … (1)

And this implies that $f_0(x)\, g_0(x)$=0 and $f_0(x)\, g_1(x)$+ $f_1(x)\, g_0(x)$=0.

Since $R[x]$ is domain, then

(a) Either $f_0(x)$ =0 which means that a_i=0 for all $0 \le i \le m$ and by Eq. (1) we have $f_0(x)\, g_1(x)$+ $f_1(x)\, g_0(x)$=0=$f_1(x)\, g_0(x)$= 0 in $R[\,x,\alpha]$.

Again since $R[x]$ is domain, then either $f_1(x)$ =0 or $g_0(x)$=0, thus $A_i\, B_j$=0 , $0 \le i \le m$, $0 \le j \le n$.

(b) Or $g_0(x)$= 0 which means that b_j=0 for $0 \le j \le n$ and by Eq. (1) we have $f_0(x)\, g_1(x)$+ $f_1(x)\, g_0(x)$=0=$f_0(x)\, g_1(x)$= 0 in $R[\,x,\alpha]$.

Since $R[x]$ is domain, so we get either $f_0(x) = 0$ or $g_1(x) = 0$. The two cases (a) and (b) yields $\begin{pmatrix} a_i b_j & a_i v_j + u_i b_j \\ 0 & a_i b_j \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, thus $A_i\, B_i = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $0 \le i \le m$, $0 \le j \le n$.

Hence $T = T\,(\,R,R\,)$ is α–ARM ring.

Using Theorem 3.1 to get the following corollary.

Corollary 3.2 Let $R[x]$ be a domain ring. Then T= $T(R,R)$ is α –SARM ring.

Proof Since $R[x]$ is domain, then T= $T(R,R)$ is α –ARM by Theorem 3.1 and by [12, Theorem 1.8] T= $T(R,R)$ is α–SARM ring.

Kose et al. in [25] introduced the definition of central reversible rings. A ring R is said to be central reversible if for each $s,t \in R$ such that $st = 0$ then $ts \in C(R)$. Clearly, every reversible ring is central reversible, but the converse need not be true reversible. The concept of nil-ARM (n-ARM, for short) rings is introduced by Ramon Antoine in 2008 [26]. A ring R is called n-ARM if $f(x), g(x) \in R[x]$ achieves

$f(x)g(x) \in N(R)[x]$ then $a_i b_j \in N(R)$ for all i, j.

Our next result is to determine conditions in case the trivial extension of a ring is n-ARM.

Theorem 3.3 Let R be a central reversible ring, then T= $T(R,R)$is n-ARM.

Proof Since R is central reversible, so by [25, Theorem 2.19] R is 2- primal, and by [9, Proposition 1.3], $\frac{R}{P(R)}[x] \cong \frac{R[x]}{P(R)[x]}$ is reduced which implies that R is ARM, hence R is n- ARM by [26, Proposition 2.7] and by [26, Proposition 4.1] we get $T= T(R,R)$is n- ARM .

Using Theorem 3.3 to get the following corollaries:

Corollary 3.4 Let R be a central reversible ring, then $T= T(R,R)$ is WARM.

Proof Since R is central reversible, thus by Theorem 3.4 $T= T(R,R)$ is n- ARM.

Also since each n – ARM ring is WARB, so we get $T= T(R,R)$is WARM.

Corollary 3.5 Let R be a central reversible ring, then $T= T(R,R)$ is π-ARM.

Proof Since R is a central reversible ring, by Corollary 3.4, $T= T(R,R)$is WARM and using [16, Proposition 2.2.12], implies that $T= T(R,R)$ is π-ARM.

Recall that if every nilpotent element of R is central, then a ring R is called central reduced [19]. The following proposition explain the relationship between 2-primal and ARM rings on the one hand with trivial extension on the other.

Proposition 3.6 Let R be a semiprime ring, then R is 2-primal ring iff $T= T(R,R)$ is ARM ring.

Proof Firstly, assume that R is 2 – primal ring. By Proposition 2.1 and since R is semiprime, then R is reduced ring and so by [1, Proposition 2.5] we get $T= T(R,R)$is ARM ring.

Conversely, suppose that $T= T(R,R)$ is ARM ring. By Theorem 2.3, we have R is reduced ring which implies that R is central reduced and so by using [19, Theorem 2.15] implies that R is a 2- primal ring.

As proof of Proposition 3.6, the next corollary can be obtained

Corollary 3.7 Let R be a von Neumann regular ring, then R is 2-primal iff $T= T(R,R)$is ARM ring.

Proposition 3.8 Let R be a semiprime ring, then R is symmetric iff $T= T(R,R)$ is ARM ring.

Proof We have R is symmetric ring, since R is semiprime by Proposition 2.1, R is reduced ring and by [1, Proposition 2.5], $T= T(R,R)$ is ARM.

Conversely, suppose that $T= T(R,R)$ is ARM ring, by Theorem 2.3, we have R is reduced, thus by Proposition 2.1 we get R is symmetric.

In a manner comparable to the proof of Proposition 3.8, the next corollaries can be obtained.

Corollary 3.9 Let R be a von Neumann regular ring, then R is symmetric iff $T= T(R,R)$ is ARM ring.

Corollary 3.10 Let R be a semiprime ring, then R is reversible iff $T= T(R,R)$ is ARM ring**.**

Corollary 3.11 Let R be a von Neumann regular ring R , R is reversible iff $T= T(R,R)$ is ARM ring.

Corollary 3.12 Let R be a semiprime ring, then R is semicommutative iff$T= T(R,R)$ is ARM ring.

Corollary 3.13 Let R be a von Neumann regular ring, then R is semicommutative iff $T= T(R,R)$ is ARM ring.

Corollary 3.14 Let R be a semiprime ring, then R is nil – semi commutative iff $T(R,R)$ is ARM ring.

Corollary 3.15 Let R be a Von Neumann regular ring, then R is nil – semicommutative iff $T(R,R)$ is ARM ring.

The next proposition illustrates the relationship between reduced α-ARM and WARM ring with trivial extension.

Proposition 3.16 Let R be a reduced α- ARM, then $T= T(R,R)$ is WARM ring.

Proof Suppose that R is reduced α-ARM, we have to prove that $T= T(R,R)$ is WARM. Let $f(x)= \sum_{i=0}^{m}(a_i,c_i)\, x^i$ and $g(x)= \sum_{j=0}^{n}(b_j,d_j)\, x^j \in T= T(R,R)$ [x] such that $f(x)\, g(x)$=0 Where $f(x)= \big(f_0(x)\, , f_1(x)\big)$ and $g(x)= \big(g_0(x)\, , g_1(x)\big)\, , f_0(x)= \sum_{i=0}^{m} a_i\, x^i \in R$[x , α] ,

$f_1(x)= \sum_{i=0}^{m} c_i\, x^i \in R$[x ,α] , $g_0(x)= \sum_{j=0}^{n} b_j\, x^j \in R$[x , α] and $g_1(x)= \sum_{j=0}^{n} d_j\, x^j \in R[x\, ,\alpha]$.

$f(x)\, g(x)= \big(f_0(x)\, g_0(x)\, , f_0(x)\, g_1(x) + f_1(x)g_0(x)\big) =$ (0 ,0) …(2),

i.e. $f_0(x)g_0(x)$=0 …(3)

And $f_0(x)g_1(x)+ f_1(x)g_0(x)$=0 … (4)

In R[x,α], since R is reduced $\alpha -$ ARM ring, then by [12, Proposition 1.7] R is $-rigid$, and by [11, Proposition 3] yields $R\,[x\, ,\alpha]$ is reduced, so from Eq .(2) and by [27, Proposition 1.6], we get $g_0(x)f_0(x)$=0 . Multiply Eq. (4) from the right hand by $f_0(x)$ to get

$f_0(x)g_1(x)f_0(x)+ f_1(x)g_0(x)f_0(x)$=0, so we get $f_1(x)g_0(x)f_0(x)$=0 and since R is reduced, then by [27, Proposition 1.6] $T= T(R,R)$ is reversible, so we have $f_0(x)f_0(x)g_1(x)$=0 in R[x,α] which implies that$(f_0(x)g_1(x))^2$=0. Since R [x,α] is reduced, then $f_0g_1(x)$=0. …(5)

Substitute Eq. (5) in Eq. (4) to get

$f_1(x)g_0(x)$=0. …(6)

Since R is reduced α −ARM and from Eq. (3) , Eq. (5) and Eq. (6), then

$a_ib_j = 0$ (resp. $a_id_j = 0$ and $c_ib_j = 0$) for all i and j this means $(a_i\, , c_i)\,(b_j, d_j) = 0$ for all i and j . Therefor $(a_i\, , c_i)\,(b_j\, , d_j) \in N\,(R)$ so $T(R,R)$ is WARM.

Recall that a ring R is called α-skew π-ARM (α-Sπ-ARM, for short) ring if two polynomials $f(x) = \sum_{i=0}^{m} a_ix^i\, ,\ g(x) = \sum_{j=0}^{n} b_jx^j \in R[x;\alpha]$ such that $f(x)g(x) \in N(R[x;\alpha])$, then $a_i\alpha^i(b_i) \in N(R)$ for each $0 \le i \le m,\ 0 \le j \le n$, [16].

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We must remember that if α is an endomorphism of a ring R, then map $\bar{\alpha}: R[x] \to R[x]$ given by $\bar{\alpha}(\sum_{i=0}^{m} a_i x^i) = \sum_{i=0}^{n} \alpha(a_i) x^i$ is an endomorphism of R[x], and it is extended of α.

Proposition 3.17 Let $T = T(R,R)$ be $\bar{\alpha} -$ ARM ring. Then $T = T(R,R)$ is $\bar{\alpha}$ −S $\pi -$ ARM.

Proof Let $f(x) = \sum_{i=0}^{m}(a_i, r_i)\, x^i$ and $g(x) = \sum_{j=0}^{n}(b_j, v_j)\, x^j \in T = T(R,R)[x, \bar{\alpha}]$, where $f(x) = (f_0(x), f_1(x))$ and $g(x) = (g_0(x), g_1(x))$, $f_0(x) = \sum_{i=0}^{m} a_i\, x^i$, $f_1(x) = \sum_{i=0}^{m} r_i\, x^i$, $g_0(x) = \sum_{j=0}^{n} b_j\, x^j$ and $g_1(x) = \sum_{j=0}^{n} x^j$ in $R[x, \alpha]$.

Since T is $\bar{\alpha} -$ ARM, then $f(x)\, g(x) = 0$ in $T = T(R,R)[x, \bar{\alpha}]$, and this means $f(x)\, g(x) \in N(T) = N(T(R,R)[x, \bar{\alpha}])$, so $f(x)\, g(x) = (f_0(x)\, g_0(x), f_0(x)\, g_1(x) + f_1(x)\, g_0(x)) = 0$ which give

$f_0(x)\, g_0(x) = 0$ in $R[x, \alpha]$ …. (7)

And $f_0(x)\, g_1(x) + f_1(x)\, g_0(x) = 0$ in $R[x, \alpha]$ … (8)

Note that R is α −rigid by [12, Proposition 2.4], and $R[x, \alpha]$ is reduced by [11, Proposition 3].

From Eq. (7), and [27, Proposition 1.6], then $g_0(x)\, f_0(x) = 0$ … (9)

Multiply Eq. (8) on the right-hand side by $f_0(x)$ to get

$f_0(x)\, g_1(x)\, f_0(x) + f_1(x)\, g_0(x)\, f_0(x) = 0$

By Eq. (9), the last equation become $f_0(x)\, g_1(x)\, f_0(x) = 0$ and by [27, Proposition 1.6], $f_0(x)\, f_0(x)\, g_1(x) = f_0^2(x)\, g_1(x) = 0$.

Since $R[x, \alpha]$ is reduced, then $f_0(x) g_1(x) = 0$. … (10)

By substituting Eq. (10) in Eq. (8), we get $f_1(x)\, g_0(x) = 0$. …(11)

Now, from Eq. (3), (4) and Eq. (5) and since R is $\alpha -$ ARM, then $a_i b_j = 0$, $a_i v_j$ and $r_i b_j = 0$ for each $i = 1,2, \dots m$, $j = 1,2, \dots n$.

By [12, Theorem 1.8], we get $a_i \alpha^i(b_j) = 0$, $a_i \alpha^i(v_j) = 0$ and $r_i \alpha^i(b_j) = 0$ for each $i = 1,2, \dots m$, $j = 1,2, \dots n$. Therefore it follows that

$(a_i, r_i)\, \bar{\alpha}^i\, (b_j, v_j) = (0,0) \in N(T(R,R))$, hence $T(R,R)$ is $\bar{\alpha}$ −S π −ARM ring.

As an example of the concept of π −ARM rings in the sense of trivial extension, we have the following:

Theorem 3.18 Let R be π −ARM ring. Then the trivial extension ring $T = T(R,R) = R \oplus R$ is π-ARM.

Proof Let $f(x) = \begin{pmatrix} a_0 & u_0 \\ 0 & a_0 \end{pmatrix} + \begin{pmatrix} a_1 & u_1 \\ 0 & a_1 \end{pmatrix} x + \cdots + \begin{pmatrix} a_n & u_n \\ 0 & a_n \end{pmatrix} x^n = \begin{pmatrix} f(x) & l(x) \\ 0 & f(x) \end{pmatrix}$,

$$g(x) = \begin{pmatrix} b_0 & v_0 \\ 0 & b_0 \end{pmatrix} + \begin{pmatrix} b_1 & v_1 \\ 0 & b_1 \end{pmatrix} x + \cdots + \begin{pmatrix} b_n & v_n \\ 0 & b_n \end{pmatrix} x^n = \begin{pmatrix} g(x) & h(x) \\ 0 & g(x) \end{pmatrix} \in T[x]$$

Where $f(x) = a_0 + a_1 x + \cdots + a_n x^n$, $l(x) = u_0 + u_1 x + \cdots + u_n x^n$, $g(x) = b_0 + b_1 x + \cdots + b_m x^m$, $h(x) = v_0 + v_1 x + \cdots + v_m x^m$ are in $R[x]$. To prove that $T = T(R,R)$ is π-ARM, assume that $f(x)g(x) \in N(T[x])$, which means that $f(x)g(x) = \begin{pmatrix} f_1(x)g_1(x) & f_1(x)g_2(x) + f_2(x)g_1(x) \\ 0 & f_1(x)g_1(x) \end{pmatrix} \in N(T[x])$, then

$\begin{pmatrix} f_1(x)g_1(x) & f_1(x)g_2(x) + f_2(x)g_1(x) \\ 0 & f_1(x)g_1(x) \end{pmatrix}^n = \begin{pmatrix} f_1(x)g_1(x)^n & * \\ 0 & f_1(x)g_1(x)^n \end{pmatrix} = 0$, for some positive integer n.

So, we get $(f_1(x)g_1(x))^n = 0$, hence $f_1(x)g_1(x) \in N(R[x])$. Since R is π- ARM, then $a_i b_j \in N(R)$, $0 \le i \le n$, $0 \le j \le m$, i.e, hence there exists some positive integer p_{ij} such that $(a_i b_j)^{pij} = 0$.Take $p = max\{p_{ij}\}, 0 \le i \le n, 0 \le j \le m$. Then $\begin{pmatrix} a_i b_j & a_i v_j + u_i b_j \\ 0 & a_i b_j \end{pmatrix}^{pn} = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}^n = 0$. Thus $\begin{pmatrix} a_i b_j & a_i v_j + u_i b_j \\ 0 & a_i b_j \end{pmatrix} \in N(R \oplus R)$, $\begin{pmatrix} a_i b_j & a_i v_j + u_i b_j \\ 0 & a_i b_j \end{pmatrix}^p = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$, and $\begin{pmatrix} a_i b_j & a_i v_j + u_i b_j \\ 0 & a_i b_j \end{pmatrix}^{p+1} = 0$. Hence, $T = T(R,R) = R \oplus R$ is π-ARM.

Lemma 3.19 Let R be a central reduced ring. Then $T(R,R)$ is C π −ARM.

Proof Since R is central reduced, then by [19, Theorem 2.34], $T(R,R)$ is CARM and by using [16, Proposition 3.5.1] this implies that R is a $T(R,R)$ is C$\pi -$ ARM.

As a special case of Lemma 3.19, the following propositions can be obtained.

Lemma 3.20 Let R be a semiprime nil-semicommutative ring. Then $T(R,R)$is C π −ARM.

Proof By Proposition 2.1, R is reduced. Since every reduced is central reduced and by Lemma 3.19, then $T(R,R)$is Cπ −ARM.

Lemma 3.21 Let R be a domain ring. Then $T(R,R)$ is Cπ −ARM.

Proof Since R is domain, then by [19, Proposition 2.7] and Lemma 3.19 we have R is Cπ −ARM.

The following propositions explain the relationship between Cπ −ARM with trivial extension on the one hand and some kinds of rings on the other.

Proposition 3.22 Let R be a central reversible semiprime ring. Then $T(R,R)$ is Cπ −ARM.

Proof Since R is central reversible, then by [16, Theorem 1.4.20 (1)] and by Proposition 2.1, R is reduced and by Lemma 3.19 the proof is complete.

New fact of commutative ring and $C\pi$ −ARM ring with trivial extension can be obtained.

Proposition 3.23 Let R be a ring. Then R is commutative iff $T(R,R)$ is $C\pi$ −ARM.

Proof Assume that R is commutative, then by [19, Proposition 2.29], $T(R,R)$is central reduced. Let $f,g \in T(R,R)$ such that $f=(f_1,f_2)$, $g=(g_1,g_2)$ and $fg \in N(R[x])$. Since $T(R,R)$ is central reduced, then by [19, Theorem 2.31], R is nARM. That is, $fg \in N(R[x]) \subseteq N(R)[x]$ which implies that $a_ib_j \in N(R)$. Again, since R is central reduced, then $a_ib_j \in C(R)$. Hence $T(R,R)$ is $C\pi$ −ARM.

Conversely, suppose that $T(R,R)$ is $C\pi$ −ARM. To prove that R is commutative, let $a,b \in R$ and $f(x)=\begin{pmatrix}0 & a\\ 0 & 0\end{pmatrix}, g(x)=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \in T(R,R)[x]$. Since $T(R,R)$ is $C\pi$ −ARM and $f(x)g(x) = \begin{pmatrix}0 & a\\ 0 & 0\end{pmatrix} \in N(T(R,R)[x])$, then $\begin{pmatrix}0 & a\\ 0 & 0\end{pmatrix} \in C(T(R,R))$ and this implies that $\begin{pmatrix}0 & ab\\ 0 & 0\end{pmatrix} = \begin{pmatrix}0 & a\\ 0 & 0\end{pmatrix}\begin{pmatrix}b & 0\\ 0 & b\end{pmatrix}=\begin{pmatrix}b & 0\\ 0 & b\end{pmatrix}\begin{pmatrix}0 & a\\ 0 & 0\end{pmatrix}=\begin{pmatrix}0 & ba\\ 0 & 0\end{pmatrix}$, that is $ab=ba$, for all $a,b \in R$. Hence is commutative.

The next theorem explains new fact of $C\pi$ −ARM ring with trivial extension.

Theorem 3.24 Let R be a $C\pi$ −ARM that satisfies the property: if $f(x)g(x)$ is a central nilpotent element and $f(x)$ is a central element of $R[x]$, then $g(x) \in N(R[x])$. Then $T(R,R)$is $C\pi$ −ARM.

Proof Let $f(x) = \begin{pmatrix}a_0 & a_0'\\ 0 & a_0\end{pmatrix} + \begin{pmatrix}a_1 & a_1'\\ 0 & a_1\end{pmatrix}x + \cdots + \begin{pmatrix}a_n & a_n'\\ 0 & a_n\end{pmatrix}x^n = \begin{pmatrix}f_1(x) & f_2(x)\\ 0 & f_1(x)\end{pmatrix}$,

$g(x) = \begin{pmatrix}b_0 & b_0'\\ 0 & b_0\end{pmatrix} + \begin{pmatrix}b_1 & b_1'\\ 0 & b_1\end{pmatrix}x + \cdots + \begin{pmatrix}b_m & b_m'\\ 0 & b_m\end{pmatrix}x^m$

$= \begin{pmatrix}g_1(x) & g_2(x)\\ 0 & g_1(x)\end{pmatrix} \in T(R,R)[x]$ such that

$f(x)g(x) = \begin{pmatrix}f_1(x)g_1(x) & f_1(x)g_2(x) + f_2(x)g_1(x)\\ 0 & f_1(x)g_1(x)\end{pmatrix} \in N(T(R,R)[x])$ ...(12)

By some computations on Eq. (1), we can conclude that $f_1(x)g_1(x) \in C(R[x])$ and this can be reduced Eq. (12) to

$$\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix} = \big(f(x)g(x)\big)^n$$

$$= \begin{pmatrix}(f_1(x)g_1(x))^n & \sum_{i+j=n-1} \big(f_1(x)g_1(x)\big)^i (f_1(x)g_2(x) + f_2(x)g_1(x))\big(f_1(x)g_1(x)\big)^j\\ 0 & (f_1(x)g_1(x))^n\end{pmatrix}$$

$$= \begin{pmatrix}(f_1(x)g_1(x))^n & n\big(f_1(x)g_1(x)\big)^{n-1}(f_1(x)g_2(x) + f_2(x)g_1(x))\\ 0 & (f_1(x)g_1(x))^n\end{pmatrix}$$ ... (13)

Eq. (13) implies that $(f_1(x)g_1(x))^n = 0$ and since R is $C\pi$ −ARM, then$a_ib_j \in C(R)$. Also, $n\big(f_1(x)g_1(x)\big)^{n-1}\big(f_1(x)g_2(x) + f_2(x)g_1(x)\big) = 0$ and using [19, Lemma 2.33] implies that $nf_1(x)g_1(x)\big(f_1(x)g_2(x) + f_2(x)g_1(x)\big)$ is central nilpotent element of $R[x]$.

Hence, by hypothesis we get the required result.

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التوسيع التافهة لحلقات ارميندرايز ومفاهيم ذات صلة

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المستخلص :

هذا البحث يحقق في إمكانية انتقال خصائص الحلقة R إلى حلقة التوسيع التافهة $T(R,R)$ وعرض العلاقة بين حلقة التوسيع التافهة$T(R,R)$ والعديد من أنواع الحلقات. ان مفاهيم برايمل من النمط ٢، رفيرسبل، رفيرسبل المركزية، شبه الابدالية، شبه الابدالية الصفرية، أرمندريزمن النمط π و أرمندريزالمركزية من النمط π قد تم دراستها مع حلقة التوسيع التافهة $T(R,R)$ وبعض المكافئات لحلقات أرمندريز من النمط π وحلقات أرمندريز المركزية من النمط π.

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Analysis of Heat and Mass Transfer in a Tapered Asymmetric Channel During Peristaltic Transport of (Pseudoplastic Nanofluid) with Variable Viscosity Under the Effect of (MHD)

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Abstract.

In this paper, a study and an analysis of a heat and mass transfer during peristaltic flow for a pseudoplastic fluid in asymmetric tapered channel, and a variable viscosity dependent of a fluid temperature with exist of slip conditions through porous medium and the influence of this conditions on the velocity and pressure, where the wavelength of the peristaltic flow is a long and the Reynold number is very small. The solution of equations for the momentum and energy have been on the basis of a perturbation technique for a found the stream function, velocity, pressure gradient and temperature and also have been discussed the trapping phenomenon by the graphs.

Key words. Peristaltic transport,Reynolds number, Hartmann number, pseudoplastic, Porous Medium

Mathematics subject classification: 80Axx, 76Dxx, 76Txx.

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1.Introduction:

The process of peristalsis has become a subject of great importance for the researchers in view of its wide-ranging biomedical, physiological applications. You can see it in the food movement in the Intestine tract,the urine passage from a kidney to the bladder, Blood transfusion in the capillary blood vessels, also machines used in heart and lung operations and the movement of sperm in the male reproductive system and the transfer of the egg in the womb in women. Latham [1] and the physiology of the gastrointestinal tract, esophagus, stomach, intestine, and ureter associated with the phenomenon of peristalsis was discussed in the book "Biomathematics"[2].Also on the industrial field where has encouraged the complex nature of fluids of which, oils, chemicals, petroleum and other fluids, has encouraged extensive studies and research into the properties of these fluids. where many researchers presented basic equations for various non-Newtonian liquids [3-7].

Heat transfer in peristalsis is beneficial in the applications such as blood pumps in heart operations, Kidney dialysis operations, and Magnetohydrodynamics (MHD) is a topic important to many researchers in the problems they treated conductive fluids e.g., blood, blood pumping appliances, magnetic resonance imaging (MRI) for brain diagnosis. MHD has many implementations in geology (in the study of earthquakes and the subsoil of the earth) [8-15].

The word nanofluid is referring to a fluid containing nanometer-sized particles. Choi [16], the Nanofluids have applications in numerous medical, biochemical and engineeering including neuro-electronic interfaces, nanoporous materials (carbon, nanofibers), cancer diagnosis, drug delivery systems and many others.

The pseudoplastic fluids consider is a category of shear fluffy materials. In this materials, the viscosity decreases by enhancing shear rate. it is clear that non-Newtonian materials are involved in many qualities and ingredients and processes including food mixing, food movement in the intestine, blood flow in arteries and capillaries, the flow of metal fluids and alloys.

Most of the researchers on the peristalsis channels studies consider fluid viscosity is constant ,But some of them showed great importance to the situations which can attention variable viscosity of the fluid.And from these [17-24].

In a study recent also, Misra et al.[25] the influence of heat and mass transfer in asymmetric channels during peristaltic transport of an MHD fluid having temperature-dependent properties and Sinha et al.[26] Peristaltic transport of MHD flow and heat transfer in an asymmetric channel: effects of variable viscosity, velocity-slip and temperature jump and Hayat et al. [27] Influence of convective conditions in radiative peristaltic flow of pseudoplastic nanofluid in a tapered asymmetric channel.

In this paper, we will study the heat and mass transfer in a tapered asymmetric channel under the effect of a magnetohydrodynamic during peristaltic transport of pseudoplastic nanofluid with slip conditions in peristaltic flow for a variables viscosity for this fluid, where the wavelength of the peristaltic flow is long and the Reynolds number is small. The equations for the momentum and energy have been linearized on the basis of these considerations. Expressions for the stream function, velocity, pressure gradient and temperature have been obtained. Pumping characteristics of the peristaltic flow and the trapping phenomenon have been discussed, and we obtained numerical results of different physical parameters and a graphs by using the software MATHEMATICA. Accordingly, we will analyzed these data based on these figures.

2. Mathematical Formulation:

In the present study, we consider the flow of an incompressible magnetohydrodynamic (MHD) pseudoplastic nanofluid in a two-dimensional tapered asymmetric channel through a porous medium (see Fig.(1)) and the flux is induced by sinusoidal wave traveling propagating with constant velocity c along the channel walls and the effect this on a heat and mass transfer with a velocity of peristaltic waves. its walls are defined as:

$$\tilde{Y} = \tilde{H}_1\left(\tilde{X}, t'\right) = -d_1 - m'\tilde{X} - b_1 \,\mathrm{Sin}\left[\frac{2\pi}{\lambda}\left(\tilde{X} - ct'\right) + \phi\right] \qquad (1)$$

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$$\tilde{Y} = \tilde{H}_2\left(\tilde{X}, t'\right) = d_2 + m'\tilde{X} + b_2 \,\text{Sin}\left[\frac{2\pi}{\lambda}\left(\tilde{X} - ct'\right)\right] \qquad (2)$$

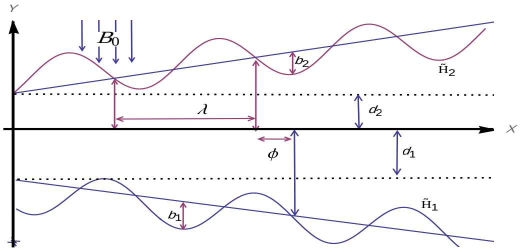

Fig.(1): Geometry of the tapered asymmetric channel.

Where $(d_1),(d_2)$ is the channel width , (b_1) and (b_2) are the amplitudes of right and left walls respectively, (c) is the phase speed of the wave, m'(□ 1) is the non-uniform parameter, (λ) is the wave length, (t') the time, the phase difference (ϕ) varies in the range $(0 \le \phi \le \pi)$ where $(\phi = 0)$ corresponds to symmetric channel, with waves out of phase i.e. both walls move towards outward or inward simultaneously and further b_1, b_2, d_1, d_2, ϕ satisfy the following condition at the inlet of a divergent channel.

$$b_1^2 + b_2^2 + 2b_1 b_2 \,\text{Cos}\,\phi \le \left(d_1 + d_2\right)^2 \qquad (3)$$

Here we assume the fluid to be electrically conducting in the presence of a magnetic field $B = \left(0, B_0, 0\right)$. To calculate the Lorentz force we will apply a magnetic field just in $\tilde{Y} - direction$ and then we study the effect of it on the fluid flow.

3. Constitutive Equations:

The expression of an extra stress tensor in the pseudoplastic fluid is [٢٨]

$$\tilde{S} + \lambda_1 \frac{D\tilde{S}}{Dt'} + \frac{1}{2}\left(\lambda_1 - \mu_1\right)\left(A_1\tilde{S} + \tilde{S}A_1\right) = \mu A_1 \qquad (4)$$

In which λ_1 and μ_1 are the relaxation times.

Also $A_1 = \left[\nabla\bar{V} + \left(\nabla\bar{V}\right)^T\right]$, A_1 is the first Rivlin-Ericksen tensor with the velocity gradient, and

$$d\tilde{S}/dt' = \partial\tilde{S}/\partial t' + \bar{V}.\nabla\tilde{S} \qquad (5)$$

And

$$\frac{D\tilde{S}}{Dt'} = d\tilde{S}/dt' - \left(\nabla\bar{V}\right)\tilde{S} - \tilde{S}\left(\nabla\bar{V}\right)^T \qquad (6)$$

where $\bar{V} = [\tilde{U}, \tilde{V}, 0]$ is the velocity field, $\frac{D}{Dt'}$ is the upper-convected derivative d/dt' is the material time derivative and

$$\left.\begin{aligned} \tilde{S} &= \begin{pmatrix} \tilde{S}_{\tilde{X}\tilde{X}} & \tilde{S}_{\tilde{X}\tilde{Y}} \\ \tilde{S}_{\tilde{Y}\tilde{X}} & \tilde{S}_{\tilde{Y}\tilde{Y}} \end{pmatrix}, \\ A_1 &= \begin{pmatrix} 2\dfrac{\partial\tilde{U}}{\partial\tilde{X}} & \left[\dfrac{\partial\tilde{U}}{\partial\tilde{Y}} + \dfrac{\partial\tilde{V}}{\partial\tilde{X}}\right] \\ \dfrac{\partial\tilde{V}}{\partial\tilde{X}} + \dfrac{\partial\tilde{U}}{\partial\tilde{Y}} & 2\dfrac{\partial\tilde{V}}{\partial\tilde{Y}} \end{pmatrix} \end{aligned}\right\} \qquad (7)$$

The stress components $\tilde{S}_{\tilde{X}\tilde{X}}, \tilde{S}_{\tilde{X}\tilde{Y}}$ and $\tilde{S}_{\tilde{Y}\tilde{Y}}$ can be obtained through the following relation :

$$\tilde{S}_{\tilde{X}\tilde{X}} + \lambda_1\left[\left(\frac{\partial}{\partial t'} + \tilde{U}\frac{\partial}{\partial\tilde{X}} + \tilde{V}\frac{\partial}{\partial\tilde{Y}}\right)\tilde{S}_{\tilde{X}\tilde{X}} - 2\tilde{S}_{\tilde{X}\tilde{X}}\frac{\partial\tilde{U}}{\partial\tilde{X}} - 2\tilde{S}_{\tilde{X}\tilde{Y}}\frac{\partial\tilde{U}}{\partial\tilde{Y}}\right] + \frac{1}{2}\left(\lambda_1 - \mu_1\right)\left[4\tilde{S}_{\tilde{X}\tilde{X}}\frac{\partial\tilde{U}}{\partial\tilde{X}} + 2\tilde{S}_{\tilde{X}\tilde{Y}}\left(\frac{\partial\tilde{U}}{\partial\tilde{Y}} + \frac{\partial\tilde{V}}{\partial\tilde{X}}\right)\right] = 2\mu_0\frac{\partial\tilde{U}}{\partial\tilde{X}} \qquad (8)$$

$$\tilde{S}_{\tilde{X}\tilde{Y}} + \lambda_1\left[\left(\frac{\partial}{\partial t'} + \tilde{U}\frac{\partial}{\partial\tilde{X}} + \tilde{V}\frac{\partial}{\partial\tilde{Y}}\right)\tilde{S}_{\tilde{X}\tilde{Y}} - \tilde{S}_{\tilde{X}\tilde{X}}\frac{\partial\tilde{V}}{\partial\tilde{X}} - \tilde{S}_{\tilde{Y}\tilde{Y}}\frac{\partial\tilde{U}}{\partial\tilde{Y}}\right] + \frac{1}{2}\left(\lambda_1 - \mu_1\right)\left(\tilde{S}_{\tilde{X}\tilde{X}} + \tilde{S}_{\tilde{Y}\tilde{Y}}\right)\left(\frac{\partial\tilde{U}}{\partial\tilde{Y}} + \frac{\partial\tilde{V}}{\partial\tilde{X}}\right) = \mu_0\left(\frac{\partial\tilde{U}}{\partial\tilde{Y}} + \frac{\partial\tilde{V}}{\partial\tilde{X}}\right) \qquad (9)$$

$$\tilde{S}_{\tilde{Y}\tilde{Y}} + \lambda_1\left[\left(\frac{\partial}{\partial t'} + \tilde{U}\frac{\partial}{\partial\tilde{X}} + \tilde{V}\frac{\partial}{\partial\tilde{Y}}\right)\tilde{S}_{\tilde{Y}\tilde{Y}} - 2\tilde{S}_{\tilde{Y}\tilde{X}}\frac{\partial\tilde{V}}{\partial\tilde{X}} - 2\tilde{S}_{\tilde{Y}\tilde{Y}}\frac{\partial\tilde{U}}{\partial\tilde{Y}}\right] + \frac{1}{2}\left(\lambda_1 - \mu_1\right)\left[2\tilde{S}_{\tilde{X}\tilde{Y}}\left(\frac{\partial\tilde{U}}{\partial\tilde{Y}} + \frac{\partial\tilde{V}}{\partial\tilde{X}}\right) + 4\tilde{S}_{\tilde{Y}\tilde{Y}}\frac{\partial\tilde{V}}{\partial\tilde{Y}}\right] = 2\mu_0\frac{\partial\tilde{V}}{\partial\tilde{Y}} \qquad (10)$$

4. Governing Equations:

Based on the consideration made above, the governing equations that describe the flow in the present study as follows:

$$\frac{\partial \tilde{U}}{\partial \tilde{X}}+\frac{\partial \tilde{V}}{\partial \tilde{Y}}=0 \qquad (11)$$

$$\rho_f\left(\frac{\partial \tilde{U}}{\partial t'}+\tilde{U}\frac{\partial \tilde{U}}{\partial \tilde{X}}+\tilde{V}\frac{\partial \tilde{U}}{\partial \tilde{Y}}\right)=-\frac{\partial \tilde{P}}{\partial \tilde{X}}+\frac{\partial}{\partial \tilde{X}}\left(\tilde{S}_{\tilde{X}\tilde{X}}\right)+\frac{\partial}{\partial \tilde{Y}}\left(\tilde{S}_{\tilde{X}\tilde{Y}}\right)-\sigma' B_0^2\tilde{U}-\frac{\tilde{\mu}(\tilde{T})}{K_0}\tilde{U} \qquad (12)$$

$$\rho_f\left(\frac{\partial \tilde{V}}{\partial t'}+\tilde{U}\frac{\partial \tilde{V}}{\partial \tilde{X}}+\tilde{V}\frac{\partial \tilde{V}}{\partial \tilde{Y}}\right)=-\frac{\partial \tilde{P}}{\partial \tilde{Y}}+\frac{\partial}{\partial \tilde{Y}}\left(\tilde{S}_{\tilde{Y}\tilde{Y}}\right)+\frac{\partial}{\partial \tilde{X}}\left(\tilde{S}_{\tilde{Y}\tilde{X}}\right)-\frac{\tilde{\mu}(\tilde{T})}{K_0}\tilde{V} \qquad (13)$$

$$(\rho c')_f\left[\frac{\partial \tilde{T}}{\partial t'}+\tilde{U}\frac{\partial \tilde{T}}{\partial \tilde{X}}+\tilde{V}\frac{\partial \tilde{T}}{\partial \tilde{Y}}\right]=\kappa\left[\frac{\partial^2 \tilde{T}}{\partial \tilde{X}^2}+\frac{\partial^2 \tilde{T}}{\partial \tilde{Y}^2}\right]+(\rho c')_p\frac{D_T}{T_m}\left[\left(\frac{\partial \tilde{T}}{\partial \tilde{X}}\right)^2+\left(\frac{\partial \tilde{T}}{\partial \tilde{Y}}\right)^2\right] \qquad (14)$$

In above equations, (t') denotes the dimensional time, $(\tilde{P})$ the pressure, $(\tilde{S}_{\tilde{X}\tilde{X}}, \tilde{S}_{\tilde{X}\tilde{Y}}, \tilde{S}_{\tilde{Y}\tilde{X}}, \tilde{S}_{\tilde{Y}\tilde{Y}})$ the components of stress tensor, (σ') the fluid electrical conductivity,

$(\tilde{U})$ and $(\tilde{V})$ are the velocity components in the axial and transverse directions respectively,

(κ) the thermal conductivity of fluid, (B_0) is the magnetic parameter, (ρ_f) the density of fluid, (ρ_p) the density of nano-particles, $\tilde{\mu}(\tilde{T})$ the variable viscosity,$(\tilde{T})$ the temperature of fluid, (T_m) the fluid mean temperature, (D_T) the thermophoretic diffusion coefficient, (μ_0) a constant viscosity and (K_0) the permeability parameter.

The appropriate boundary conditions comp-rising wall slip and convective boundary cond-itions are given as follows:

$$\tilde{u}=-c,\quad \tilde{T}=T_0 \quad at \quad \tilde{Y}=\tilde{H}_1 \qquad (15)$$

$$\tilde{u}=-c,\quad \tilde{T}=T_1 \quad at \quad \tilde{Y}=\tilde{H}_2 \qquad (16)$$

Now we treat the wave frame having coordinates (X, Y) moving in the X-direction with wave velocity (c). The velocities, pressure, time and coordinates in two frames are a related by:

$$\left.\begin{aligned}&\tilde{x}=\tilde{X}-ct',\tilde{y}=\tilde{Y},\tilde{u}(\tilde{x},\tilde{y})=\tilde{U}(\tilde{X},\tilde{Y},t')-c,\\&\tilde{v}(\tilde{x},\tilde{y})=\tilde{V}(\tilde{X},\tilde{Y},t'),\tilde{p}(\tilde{x},\tilde{y})=\tilde{P}(\tilde{X},\tilde{Y},t')\end{aligned}\right\} \qquad (17)$$

Where $\tilde{u},\tilde{v}$ are the velocity components in the wave frame $(\tilde{x},\tilde{y})$ and $\tilde{p},\tilde{P}$ are the pressures in the wave and laboratory setting respectively.

Now we will define the non-dimensional quantities and stream function through the equations below:

$$\left.\begin{aligned}&x=\frac{\tilde{x}}{\lambda},y=\frac{\tilde{y}}{d_1},t=\frac{ct'}{\lambda},u=\frac{\tilde{u}}{c},v=\frac{\tilde{v}}{c},\delta=\frac{d_1}{\lambda},\theta=\frac{T-T_0}{T_1-T_0},\\&\mu(\theta)=\frac{\tilde{\mu}(T)}{\mu_0},h_1(x)=\frac{\tilde{H}_1(\tilde{x})}{d_1},h_2(x)=\frac{\tilde{H}_2(\tilde{x})}{d_1},\\&p=\frac{d_1^2\tilde{p}}{\lambda\mu_0 c},S_{ij}=\frac{d_1}{c\mu_0}\tilde{S}_{\tilde{i}\tilde{j}},m=\frac{m'\lambda}{d_1},K=\frac{K_0}{d_1^2},\\&a=\frac{b_1}{d_1},b=\frac{b_2}{d_1},R_e=\frac{\rho_f c d_1}{\mu},P_r=\frac{\mu_0 c'_f}{\kappa},\\&Nt=\frac{\dot{\tau}D_T(T-T_0)}{T_m\nu},M=\sqrt{\frac{\sigma'}{\mu}}d_1B_0,\\&u=\frac{\partial\psi}{\partial y},v=-\delta\frac{\partial\psi}{\partial x}\end{aligned}\right\} \qquad (18)$$

Where (ψ) is the stream function, (x) is the non-dimensional axial coordinate, (y) is the non-dimensional transverse coordinate, (t) is the dimensionless time, (u) and (v) are non-dimensional axial and transverse velocity components respectively, (p) is the dimensionless pressure, (a) and (b) are amplitudes of upper and lower walls, (δ) is the wave number, (m) is the non-uniform parameter, (K) is the Darcy number, (Re) is the Reynolds number, (ν) is the nanofluid kinematic viscosity, $\dot{\tau}=(\rho c')_p/(\rho c')_f$ is the ratio of the effective heat capacity of nanoparticle material and heat capacity of the fluid, (θ) is the dimensionless temperature, (P_r) is the Prandtl number, (M) is the Hartmann number, (ρ_p) the density of nano-particles, (Nt) is the thermophoresis parameter.

Since the flow is a steady and using the shifts in Eq. (17) and by introducing non-dimensional quantities [Eq.(18)] into constitutive relations (11)-(14), and conduct some algebraic processes we get the following equations:

$$\delta\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad (19)$$

$$\left.\begin{aligned}&Re\left[\delta(u+1)\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right]=-\frac{\partial p}{\partial x}\\&+\delta\frac{\partial}{\partial x}(S_{xx})+\frac{\partial}{\partial y}(S_{xy})\\&-\left[M^2+\frac{1}{K}\mu(\theta)\right](u+1)\end{aligned}\right\} \qquad (20)$$

$$\left.\begin{aligned}&Re\left[\delta(u+1)\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right]=-\frac{1}{\delta}\frac{\partial p}{\partial y}\\&+\delta\frac{\partial}{\partial x}(S_{xy})+\frac{\partial}{\partial y}(S_{yy})\\&-\frac{d_1^2}{K_0}\mu(\theta)v\end{aligned}\right\} \qquad (21)$$

$$\left.\begin{aligned}&Re.Pr\left[d\,(u+1)\frac{\partial q}{\partial x}+v\frac{\partial q}{\partial y}\right]=\\&\left[d^2\frac{\partial^2 q}{\partial x^2}+\frac{\partial^2 q}{\partial y^2}\right]+Nt.Pr\left[d^2\left(\frac{\partial q}{\partial x}\right)^2+\left(\frac{\partial q}{\partial y}\right)^2\right]\end{aligned}\right\} \qquad (22)$$

$$\begin{aligned}&S_{xx}+\lambda_1\left\{\delta\left[\frac{\partial}{\partial t}+\left(\frac{\partial\psi}{\partial y}+1\right)\frac{\partial}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\right]S_{xx}-2\delta S_{xx}\frac{\partial^2\psi}{\partial x\partial y}-2S_{xy}\frac{\partial^2\psi}{\partial y^2}\right\}\\&+\frac{1}{2}(\lambda_1-\mu_1)\left[4\delta S_{xx}\frac{\partial^2\psi}{\partial x\partial y}+2S_{xy}\left(\frac{\partial^2\psi}{\partial y^2}-\delta^2\frac{\partial^2\psi}{\partial x^2}\right)\right]=2\delta\frac{\partial^2\psi}{\partial x\partial y}\end{aligned} \qquad (23)$$

$$\begin{aligned}&S_{xy}+\lambda_1\left\{\delta\left[\frac{\partial}{\partial t}+\left(\frac{\partial\psi}{\partial y}+1\right)\frac{\partial}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\right]S_{xy}+\delta^2S_{xx}\frac{\partial^2\psi}{\partial x^2}-S_{yy}\frac{\partial^2\psi}{\partial y^2}\right\}\\&+\frac{1}{2}(\lambda_1-\mu_1)(S_{xx}+S_{yy})\left(\frac{\partial^2\psi}{\partial y^2}-\delta^2\frac{\partial^2\psi}{\partial x^2}\right)=\frac{\partial^2\psi}{\partial y^2}-\delta^2\frac{\partial^2\psi}{\partial x^2}\end{aligned} \qquad (24)$$

$$\begin{aligned}&S_{yy}+\lambda_1\left\{\delta\left[\frac{\partial}{\partial t}+\left(\frac{\partial\psi}{\partial y}+1\right)\frac{\partial}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\right]S_{yy}+2\delta^2S_{xy}\frac{\partial^2\psi}{\partial x^2}+2\delta S_{yy}\frac{\partial^2\psi}{\partial x\partial y}\right\}\\&+\frac{1}{2}(\lambda_1-\mu_1)\left[-4\delta S_{yy}\frac{\partial^2\psi}{\partial x\partial y}+2S_{xy}\left(\frac{\partial^2\psi}{\partial y^2}-\delta^2\frac{\partial^2\psi}{\partial x^2}\right)\right]=-2\delta\frac{\partial^2\psi}{\partial x\partial y}\end{aligned} \qquad (25)$$

The low Reynolds number and long wavelength approximation are widely in the solution of issues concerning the peristaltic flows. The long wavelength approximation is based on a supposition that wavelength of the peris-taltic wave is considerably major as compared with the half width of the channel.

And since the stream functions (ψ)is connected with the velocity components by the relations $u=\frac{\partial\psi}{\partial y}$, $v=-\frac{\partial\psi}{\partial x}$ and $\delta \ll 1$ then:

$$\frac{\partial p}{\partial x}=\frac{\partial}{\partial y}\left(S_{xy}\right)-\left[M^2+\frac{1}{K}\mu(\theta)\right]\left(\frac{\partial\psi}{\partial y}+1\right) \quad (26)$$

$$\frac{\partial p}{\partial y}=0 \quad (27)$$

$$\frac{\partial^2\theta}{\partial y^2}+Nt.Pr\left(\frac{\partial\theta}{\partial y}\right)^2=0 \quad (28)$$

With

$$S_{xx}=\left(\lambda_1+\mu_1\right)S_{xy}\frac{\partial^2\psi}{\partial y^2} \quad (29)$$

$$S_{xy}+\frac{1}{2}\left(\lambda_1-\mu_1\right)\left(S_{xx}+S_{yy}\right)\frac{\partial^2\psi}{\partial y^2}-\lambda_1 S_{yy}\frac{\partial^2\psi}{\partial y^2}=\frac{\partial^2\psi}{\partial y^2} \quad (30)$$

$$S_{yy}=-\left(\lambda_1-\mu_1\right)S_{xy}\frac{\partial^2\psi}{\partial y^2} \quad (31)$$

By simplifying Eqs.(29)-(31), we get

$$S_{xy}=\frac{\frac{\partial^2\psi}{\partial y^2}}{1-\xi\left(\frac{\partial^2\psi}{\partial y^2}\right)^2} \quad (32)$$

Here $\xi=\left(\mu_1^2-\lambda_1^2\right)$ is the pseudoplastic fluid parameter.

For the simplicity of analysis, most research on fluid mechanics takes fluid viscosity as a constant quantity.But in many processes, the viscosity is a function of heat, and in the present study, we will take this into account by treating viscosity as an exponential function of temperature. Let us take, $\mu(\theta)=e^{-\alpha\theta}$, where (α) is the Reynolds model viscosity parameter, which is a constant. For ($\alpha \ll 1$) neglecting the border which contains the powers of (α) more than two , we write

$$\mu(\theta)=1-\alpha\theta \quad \text{for} \quad \alpha \ll 1 \quad (33)$$

And by offsetting it in Eq. (26), we get

$$\frac{\partial p}{\partial x}=\frac{\partial^3\psi}{\partial y^3}+3\xi\left(\frac{\partial^2\psi}{\partial y^2}\right)^2\frac{\partial^3\psi}{\partial y^3}-\left(N_1^2-\frac{\alpha}{K}\theta\right)\left(\frac{\partial\psi}{\partial y}+1\right) \quad (34)$$

Where $N_1^2=M^2+\frac{1}{K}$

The Eq.(27) shows that (p) is not a function of (y) of the non-dimensional axial coordinate Y, from this and also of Eq.(34), we get:

$$\left.\begin{array}{l}\frac{\partial^4\psi}{\partial y^4}+3\xi\left(\frac{\partial^2\psi}{\partial y^2}\right)^2\frac{\partial^4\psi}{\partial y^4}\\+6\xi\frac{\partial^2\psi}{\partial y^2}\left(\frac{\partial^3\psi}{\partial y^3}\right)^2-\left(N_1^2-\frac{\alpha}{K}\theta\right)\frac{\partial^2\psi}{\partial y^2}\\+\frac{\alpha}{K}\frac{\partial\theta}{\partial y}\left(\frac{\partial\psi}{\partial y}+1\right)=0\end{array}\right\} \quad (35)$$

Similarly, the Eqs. (15) and (16), by using Eqs. (17), (18), then the boundary conditions for the dimensionless stream function and the temperature in the wave frame are:

$$\left.\begin{array}{l}\psi=\frac{F}{2},\frac{\partial\psi}{\partial y}=-1,\ \theta=0\ \ at\ \ y=h_1,\\ \psi=-\frac{F}{2},\frac{\partial\psi}{\partial y}=-1,\theta=1\ at\ \ y=h_2\end{array}\right\} \quad (36)$$

where (F) is the dimensionless mean flows in the wave frame.where

$$F(x,t)=Q+aSin\left[2\pi x+\phi\right]+bSin\left[2\pi x\right] \quad (37)$$

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Where(Q) is the dimensionless mean flows in the wave frame. Here

$$\left.\begin{aligned} F &= \int_{h_1(x)}^{h_2(x)} \frac{\partial \psi}{\partial y}\, dy \\ &= \psi(h_2) - \psi(h_1) \;, Q \equiv \frac{\tilde{q}}{cd} \end{aligned}\right\} \qquad (38)$$

$$\tilde{q} = \int_{\tilde{H}_1(\tilde{X},t)}^{\tilde{H}_2(\tilde{X},t)} \tilde{u}(\tilde{x},\tilde{y})\, d\tilde{y} \qquad (39)$$

5. Solution technique:

The Eqs. (28) and (35) are cannot find an exact solution to it. And in order to solve this problem, we resort to perturbation method where is applied to find series solving for the small parameters (S_{xy} ,ψ, F, p) about fluid parameter (ξ) and (θ) about Prandtl number (Pr). as shown in the following equations.

$$\left.\begin{aligned} &\psi=\psi_0+\xi\psi_1+\xi^2\psi_2+\dots , \\ &p=p_0+\xi p_1+\xi^2 p_2+\dots , \\ &F=F_0+\xi F_1+\xi^2 F_2+\dots , \\ &\theta=\theta_0+Pr.\theta_1+Pr^2.\theta_2+\dots . \end{aligned}\right\} \qquad (40)$$

furthermore, the series solution is used only up to first order.

6. Perturbed Systems:

To find the parameters values we say

(I) The solution (by perturbation technique) for the temperature in Eq.(28) which satisfies the boundary conditions (40), becomes:

(i) Zero order: $$\frac{\partial^2 \theta_0}{\partial y^2} = 0 \qquad (41)$$

(ii) First order: $$\frac{\partial^2 \theta_1}{\partial y^2} + Nt\left(\frac{\partial \theta_0}{\partial y}\right)^2 = 0 \qquad (42)$$

With the dimensionless boundary conditions

$$\left.\begin{aligned} &\theta_0 = 0\,, \theta_1 = 0 \quad at \quad y = h_1 \quad where \\ &h_1 = -1 - m(x+t) - aSin[2\pi x + \phi] \\ &\theta_0 = 1\,, \theta_1 = 0 \quad at \quad y = h_2 \quad where \\ &h_2 = d + m(x+t) + bSin[2\pi x] \end{aligned}\right\} \qquad (43)$$

Then

$$(Eq.41) \Rightarrow \quad \theta_0 = C_1 + yC_2 \quad \text{where}$$

$$C_1 = \frac{h_1}{h_1 - h_2}, C_2 = \frac{1}{h_1 - h_2}$$

$$(Eq.42) \Rightarrow \quad \theta_1 = -\frac{Nt.y^2}{2(h_1 - h_2)^2} + C_3 + y\, C_4$$

Where

$$C_3 = -\frac{h_1 h_2 Nt}{2(h_1 - h_2)^2}\,,\; C_4 = -\frac{-h_1 Nt - h_2 Nt}{2(h_1 - h_2)^2}$$

Since $\theta = \theta_0 + Pr.\theta_1$ (by Eq.(44)) we get:

$$\left.\begin{aligned} \theta = &\frac{h_1}{h_1 - h_2} - \frac{y}{h_1 - h_2} + \\ Pr &\left(-\frac{h_1 h_2\, Nt}{2(h_1 - h_2)^2} - \frac{(-h_1 Nt - h_2 Nt)\, y}{2(h_1 - h_2)^2} - \frac{Nt\, y^2}{2(h_1 - h_2)^2} \right) \end{aligned}\right\} \qquad (44)$$

(II) The solution (by perturbation technique) for the momentum equation (35) which satisfies the boundary conditions (40), we get:

(i) zreo order: $$\frac{\partial^4 \psi_0}{\partial y^4} - \left(N_1^2 - \frac{\alpha}{K}\theta\right)\frac{\partial^2 \psi_0}{\partial y^2} + \frac{\alpha}{K}\frac{\partial \theta}{\partial y}\left(\frac{\partial \psi_0}{\partial y} + 1\right) = 0 \qquad (45)$$

With the dimensionless boundary conditions

$$\left.\begin{aligned} &\psi_0 = -\frac{F_0}{2}, \frac{\partial \psi_0}{\partial y} = -1 \; at \quad y = h_1 \\ &\psi_0 = \frac{F_0}{2}, \frac{\partial \psi_0}{\partial y} = -1 \; at \quad y = h_2 \end{aligned}\right\} \qquad (46)$$

(ii) First order:

$$\frac{\partial^4 \psi_1}{\partial y^4} + 3\left(\frac{\partial^2 \psi_0}{\partial y^2}\right)^2 \frac{\partial^4 \psi_0}{\partial y^4} + 6\frac{\partial^2 \psi_0}{\partial y^2}\left(\frac{\partial^3 \psi_0}{\partial y^3}\right)^2 - \left(N_1^2 - \frac{\alpha}{K}\theta\right)\frac{\partial^2 \psi_1}{\partial y^2} + \frac{\alpha}{K}\frac{\partial \theta}{\partial y}\frac{\partial \psi_1}{\partial y} = 0 \qquad (47)$$

With the dimensionless boundary conditions

$$\left.\begin{aligned}\psi_1 &= -\frac{F_1}{2},\ \frac{\partial\psi_1}{\partial y} = 0 \ at \quad y = h_1\\ \psi_1 &= -\frac{F_1}{2},\ \frac{\partial\psi_1}{\partial y} = 0 \ at \quad y = h_2\end{aligned}\right\} \qquad (48)$$

$$(\text{Eq}.45) \Rightarrow \psi_0 = a3 + a4y + \frac{a1e^{-yN_1} + a2e^{yN_1}}{N_1^2} + \alpha(a7 + a8y$$

$$+\frac{1}{48(h_1 - h_2)^2 KN_1^6} e^{-yN_1}(-147(a1 + a2e^{2yN_1})Nt.Pr$$

$$+51\left(a1 - a2e^{2yN_1}\right)\left(h_1(-2 + Nt.Pr) + h_2(2 + Nt.Pr) - 2Nt.Pr.y\right)N_1$$

$$-30\left(a1 + a2e^{2yN_1}\right)(-h_1 + y)\left(2h_1 - h_2(2 + Nt.Pr) + Nt.Pr.y\right)N_1^2$$

$$-2\left(a1 - a2e^{2yN_1}\right)y\left(-12h_1^2 + y(-3h_2(2 + Nt.Pr) + 2Nt.Pr.y)\right.$$

$$\left.+3h_1\left(2h_2(2 + NtPr) + (2 - Nt.Pr)y\right)\right)N_1^3 + 48(a5 + a6e^{2yN_1})(3h_1 - 3h_2)^2 KN_1^4)) \qquad (49)$$

$$(\text{Eq}.47) \Rightarrow \psi_1 = b3 + b4y + \frac{1}{8N_1}(-2e^{-yN_1}\left(-6a1^2\, a2y - \frac{15a1^2\, a2 + 4b1}{N_1}\right)$$

$$+2e^{yN_1}\left(-6a1a2^2\, y + \frac{15a1a2^2 + 4b2}{N_1}\right) - \frac{a1^3\, e^{-3yN_1}}{N_1} - \frac{a2^3\, e^{3yN_1}}{N_1})$$

$$-1080b1e^{2yN_1} - 8316a1a2^2\, e^{4yN_1} - 1080b2e^{4yN_1} + 151a2^3\, e^{6yN_1})NtPr$$

$$-3(-72a1^2\, a2e^{2yN_1}(12h_1(-2 + NtPr) + 12h_2(2 + NtPr) - 37NtPry)$$

$$+72a1a2^2\, e^{4yN_1}\left(12h_1(-2 + NtPr) + 12h_2(2 + NtPr) - 37NtPry\right)$$

$$+35a1^3\left(h_1(-2 + NtPr) + h_2(2 + NtPr) - 2NtPry\right) - 7e^{2yN_1}(24b1 - 24b2e^{2yN_1}$$

$$+5a2^3\, e^{4yN_1})\left(h_1(-2 + NtPr) + h_2(2 + NtPr) - 2NtPry\right))N_1 + 18(7a1^3\,(-h_1 + y)$$

$$\left(2h_1 - h_2(2 + NtPr) + NtPry\right) - 6a1^2\, a2e^{2yN_1}(-68h_1^2 + 7y(-4h_2(2 + NtPr)$$

$$+5NtPry) + h_1\left(34h_2(2 + NtPr) - 28(-2 + NtPr)y\right)) - 6a1a2^2\, e^{4yN_1}(-68h_1^2$$

$$+7y\left(-4h_2(2 + NtPr) + 5NtPry\right) + h_1\left(34h_2(2 + NtPr) - 28(-2 + NtPr)y\right))$$

$$+e^{2yN_1}(8b1\left(10h_1^2 + 3y\left(h_2(2 + NtPr) - NtPry\right) - h_1\left(5h_2(2 + NtPr) - 3(-2 + NtPr)y\right)\right)$$

$$+e^{2yN_1}(7a2^3\, e^{2yN_1}(-h_1 + y)\left(2h_1 - h_2(2 + NtPr) + NtPry\right) + 8b2(10h_1^2$$

$$+3y\left(h_2(2 + NtPr) - NtPry\right) - h_1\left(5h_2(2 + NtPr) - 3(-2 + NtPr)y\right)))))N_1^2$$

$$+6y(3a1^3\left(-12h_1^2 + y\left(-3h_2(2 + NtPr) + 2NtPry\right) + 3h_1\left(2h_2(2 + NtPr) + (2 - NtPr)y\right)\right)$$

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$$-e^{2yN_1}\left(8b1-8b2e^{2yN_1}+3a2^3e^{4yN_1}\right)(-12h_1^2+y\left(-3h_2\left(2+NtPr\right)+2NtPry\right)$$
$$+3h_1\left(2h_2\left(2+NtPr\right)+\left(2-NtPr\right)y\right))-6a1^2a2e^{2yN_1}(-120h_1^2+y(-39h_2\left(2+NtPr\right)$$
$$+32NtPry)+3h_1\left(20h_2\left(2+NtPr\right)-13\left(-2+NtPr\right)y\right))+6a1a2^2e^{4yN_1}(-120h_1^2$$
$$+y\left(-39h_2\left(2+NtPr\right)+32NtPry\right)+3h1\left(20h_2\left(2+NtPr\right)-13\left(-2+NtPr\right)y\right)))N_1^3$$
$$-24(2e^{2yN_1}(9a2^2e^{2yN_1}\left(-10a5+a6e^{2yN_1}\right)\left(h_1-h_2\right)^2K-2(12\left(b5+b6e^{2yN_1}\right)\left(h_1-h_2\right)^2K$$
$$-b4e^{yN_1}y^2\left(h1\left(6-3NtPr\right)-3h_2\left(2+NtPr\right)+2NtPry\right)))+3a1a2e^{2yN_1}(-120a5\left(h_1-h_2\right)^2K$$
$$+e^{2yN_1}(-120a6\left(h_1-h_2\right)^2K+a2y^2(-12h_1^2+y\left(-3h_2\left(2+NtPr\right)+2NtPry\right)$$
$$+3h_1\left(2h_2\left(2+NtPr\right)+\left(2-NtPr\right)y\right))))+3a1^2(6a5\left(h_1-h_2\right)^2K+e^{2yN_1}(-60a6\left(h_1-h_2\right)^2K$$
$$+a2y^2\left(-12h_1^2+y\left(-3h_2\left(2+NtPr\right)+2NtPry\right)+3h_1\left(2h_2\left(2+NtPr\right)+\left(2-NtPr\right)y\right)\right)))N_1^4$$
$$-1728e^{2yN_1}\left(-a1^2a6+a2^2a5e^{2yN_1}-2a1a2\left(a5-a6e^{2yN_1}\right)\right)\left(h_1-h_2\right)^2KyN_1^5)) \qquad (50)$$

And from there it will be

$$\psi=\psi_0+\xi\psi_1 \qquad (51)$$

Where a1,a2,…,a8 and b1,…,b6 are constant . Note also, $u=\dfrac{\partial\psi}{\partial y}$

(III) The solution for the pressure equation (34) which satisfies the boundary conditions (40) and the Eq. (44):

(i) Zero order :

$$\frac{\partial p_0}{\partial y}=\frac{\partial^3\psi_0}{\partial y^3}-\left(N_1^2-\frac{\alpha}{K}\theta\right)\left(\frac{\partial\psi_0}{\partial y}+1\right) \qquad (52)$$

(ii) First order:

$$\frac{\partial p_1}{\partial y}=\frac{\partial^3\psi_1}{\partial y^3}+3\left(\frac{\partial^2\psi_0}{\partial y^2}\right)^2\frac{\partial^3\psi_0}{\partial y^3}-\left(N_1^2-\frac{\alpha}{K}\theta\right)\left(\frac{\partial\psi_1}{\partial y}+1\right) \qquad (53)$$

Now by Eqs.(52) ,(53) and (40) we get:

$$\left.\begin{aligned}\frac{\partial p}{\partial y}&=\frac{\partial^3\psi_0}{\partial y^3}-\left(N_1^2-\frac{\alpha}{K}\theta\right)\left(\frac{\partial\psi_0}{\partial y}+1\right)\\&+\xi\left[\begin{aligned}&\frac{\partial^3\psi_1}{\partial y^3}+3\left(\frac{\partial^2\psi_0}{\partial y^2}\right)^2\frac{\partial^3\psi_0}{\partial y^3}\\&-\left(N_1^2-\frac{\alpha}{K}\theta\right)\left(\frac{\partial\psi_1}{\partial y}+1\right)\end{aligned}\right]\end{aligned}\right\} \qquad (54)$$

7. Average pressure rise:

By The Eq.(54) the pressure rise (Δp) per wavelength and the walls shear stress can be obtained by the formula

$$\Delta p=\int_0^1\frac{\partial \mathrm{p}}{\partial \mathrm{x}}dx \qquad (55)$$

8. Numerical results and discussion :

In this section, the graphs are a description of values various parameters under the effect of an (MHD) during peristaltic transport for the flow of pseudoplastic nano-fluids in the tapered asymmetric channel through a porous medium with variable viscosity.

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The complicated behavior of the non-Newtonian fluids can be transacted more swiftly with the assist of numerical solutions, where solved numerically using perturbation technique for the nonlinear equations, therefore our numerical approach relies upon the linear equation solvers by Mathematica program to find the results numerically and graphically.

8.1. The Pressure gradient distribution :

In this paragraph, describe the effect of different parameters which have an impact on the pressure gradient (dp/dx) per wavelength. The influence of these parameters is observed for a Figs. (2)-(7), where shown that in the wider part of the channel (-0.7≤ x ≤-0.55) and (0≤ x ≤0.2), effect these parameters on pressure gradient are very small, which means that the flow can pass easily without imposing a large gradient pressure.Where, in the tight part of the channel (0.55≤ x ≤0), there must be a large pressure gradient in order to keep the same flow of fluid in the channel, especially for the narrowest place of approximately in (x = 0.25) and the values of (y) and (t) are fixed at y = 0.3 and t =2. This is illustrated by the Figs.(2)-(5) the increase of a value of parameters (M, ϕ, Q , ξ) is leading to the pressure gradient is increasing, but we observe an opposite in Figs.(6)&(7) where the pressure gradient is decreased when increasing the values of the parameters (m, K) .

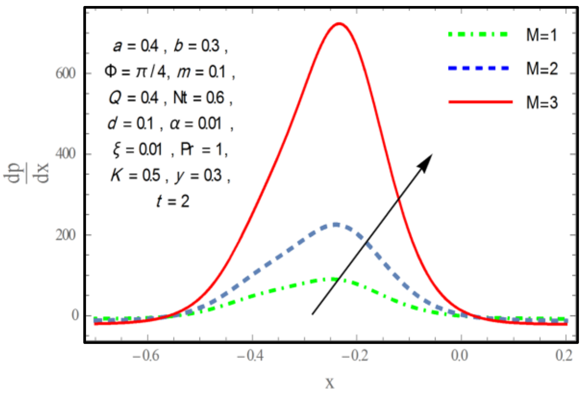

Fig.(2): Variation of $\frac{dp}{dx}$ with increasing of M

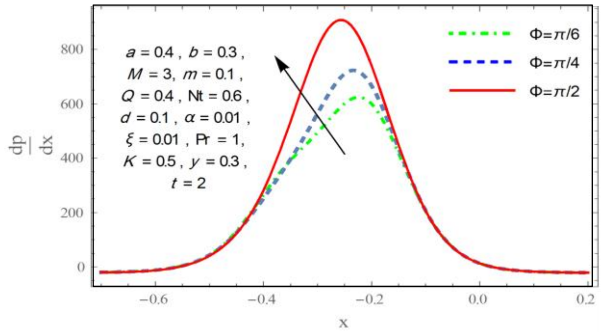

Fig.(3): Variation of $\frac{dp}{dx}$ with increasing of ϕ

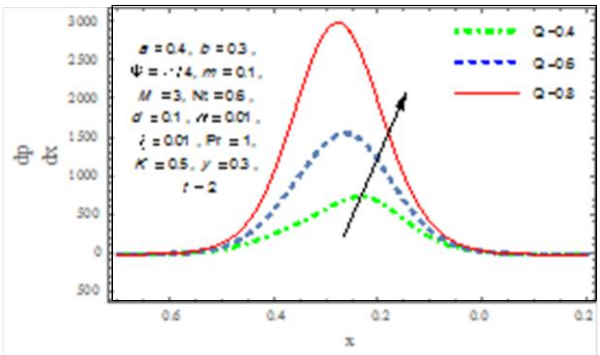

Fig.(4): Variation of $\frac{dp}{dx}$ with increasing of Q

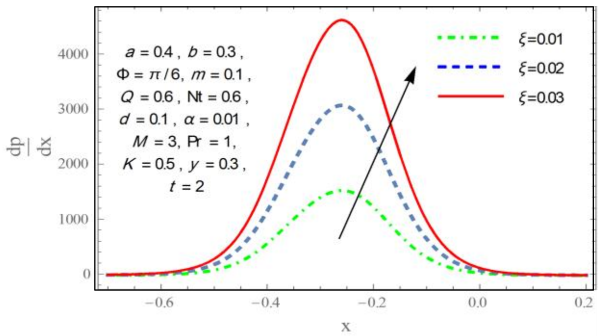

Fig.(5): Variation of $\frac{dp}{dx}$ with increasing of ξ

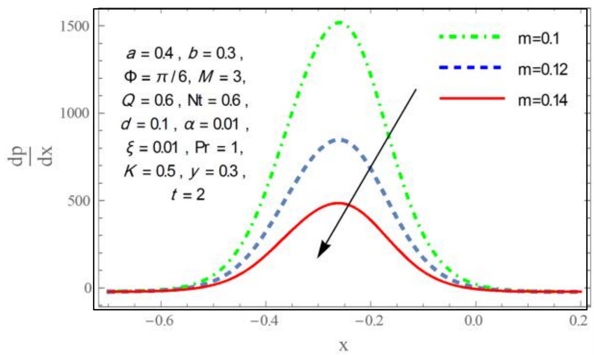

Fig.(6): Variation of $\frac{dp}{dx}$ with increasing of m

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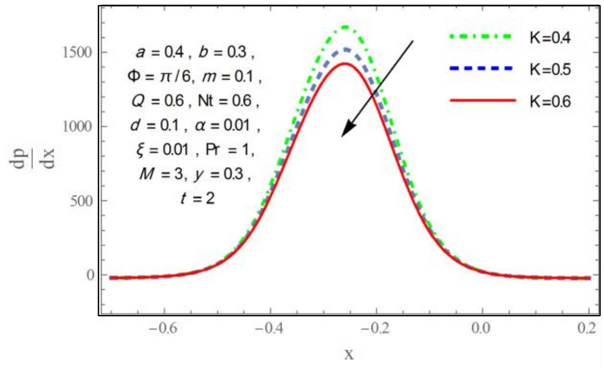

Fig.(7): Variation of $\frac{dp}{dx}$ with increasing of K

8.2. Pumping Characteristics :

The Figs.(8)-(13) show the relation between an average pressure rise (Δp) and the mean flow rate (Q) with a various physical parameters which are the Hartman number (M), the phase difference (ϕ), the pseudoplastic fluid parameter (ξ) ,the non-uniform parameter (m) , the Darcy number (K) and the Prandtl number (Pr). And effect these parameters on the average pressure rise (Δp). We observe in Fig.(8) the pumping rate decrease in the co-pumping region
(Q>0, Δp<0) while the opposite is happening in the retrograde pumping region (Q<0, Δp>0)
with an increase of parameter (M). Illustrated by the Fig.(9), the pumping rate increase in the co-
pumping region (Q>0, Δp<0) and decreasing in the retrograde pumping region (Q<0, Δp>0) with an increase in (m). Fig.(10) shows the effect of (ξ) on Δp, where the pumping rate decrease with an increase in (ξ) in the co-pumping region Δp < 0. Fig.(11) shows the impact of (ϕ) on average pressure rise (Δp), in a co-pumping region (Q>0, Δp<0) is increased up to point (0.93,-29.73) but the situation is reflected after that point, where there is a decrease in the pressure rate with enhancing of (ϕ). The Fig. (12), demonstrate the effect of (K) on (Δp). It is noted that (Δp) is increasing in the co-pumping region (Q>0, Δp<0) with values enhancing (K). Finally, the effect of the parameter (Pr) is very simple is negligible on Δp, this is illustrated by Fig. (13).

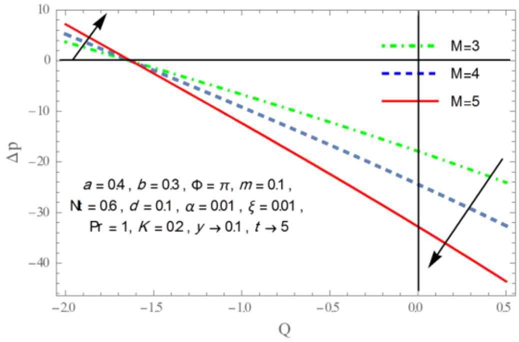

Fig.(8): Variation of Δp with increasing of M

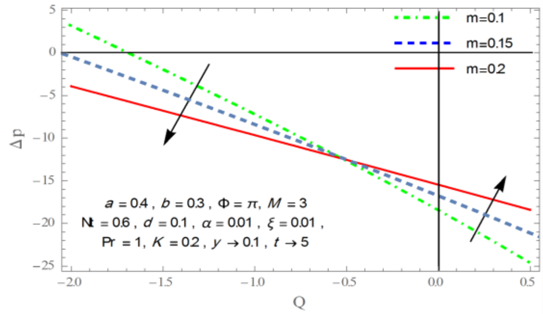

Fig.(9): Variation of Δp with increasing of m

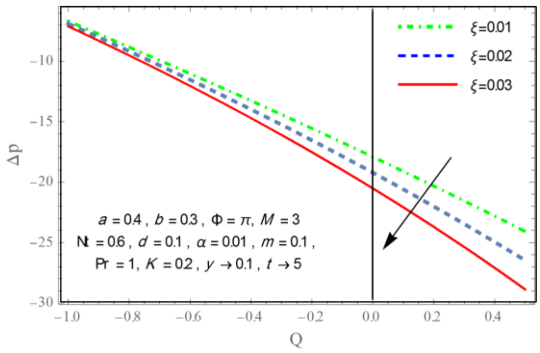

***Fig.(10):** Variation of Δp with increasing of ξ*

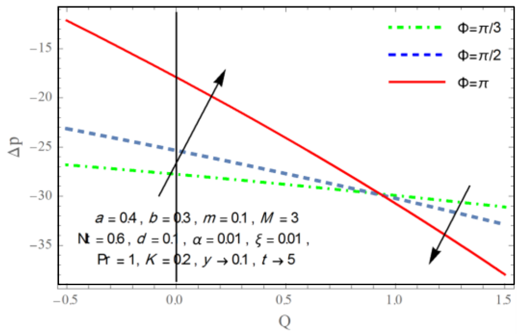

***Fig.(11):** Variation of Δp with increasing of ϕ*

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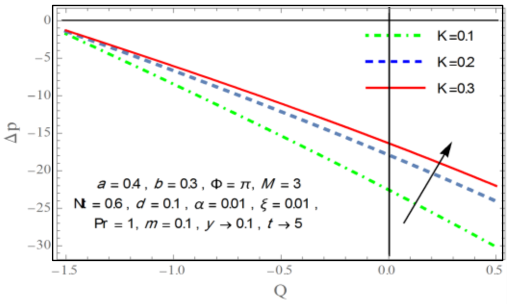

Fig.(12): *Variation of* Δp *with increasing of* K

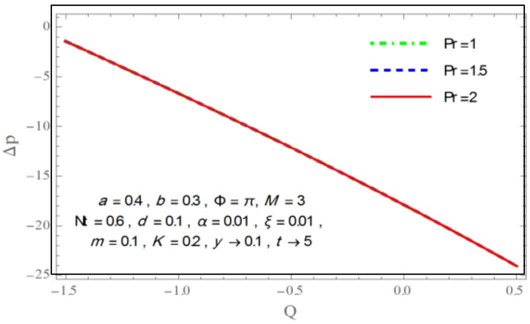

Fig.(13): *Variation of* Δp *with increasing of* Pr

8.3. Velocity Profile :

The Fig.(14)-(17) illustrate the velocity profile (u) versus y-axis, and to see the effects of a change in the values for different parameters at the fixed values of x=0.3 and t=2. The behavior of velocity profile is parabola as seen through figures.We observed from the Fig.(14) that the axial velocity (u) increases with an increase in the Hartmann number (M) at the core part of the channel, but it decreasing for near to walls where this result is expected because the fact that an effect of magnetic field generates a Lorentz force which is a resistant force.This force tends to oppose the fluid movement causing the flux to decelerate. In Fig.(15) we observe the opposite. The velocity decreases in the center and increases near the walls by increasing the values of the thermophoresis parameter (Nt). The Figs.(16)&(17) shows the effect of the Darcy number (K), the mean flow rate (Q) on the velocity profile (u) ,where the velocity decreases with an increase values for these parameters, in the center of the channel .

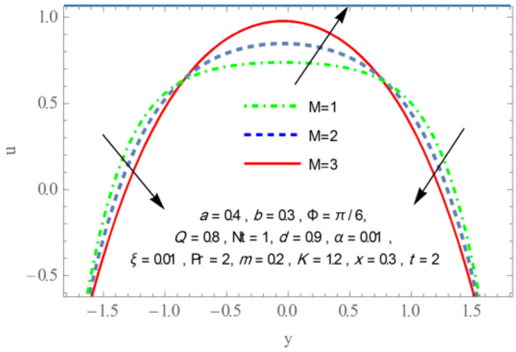

Fig.(14): *Variation of* (u) *with increasing of* M

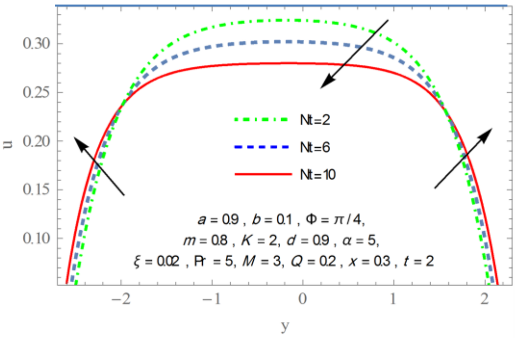

Fig.(15): *Variation of* (u) *with increasing of* Nt

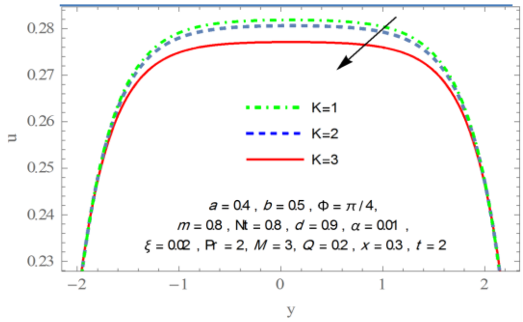

Fig.(16): *Variation of* (u) *with increasing of* K

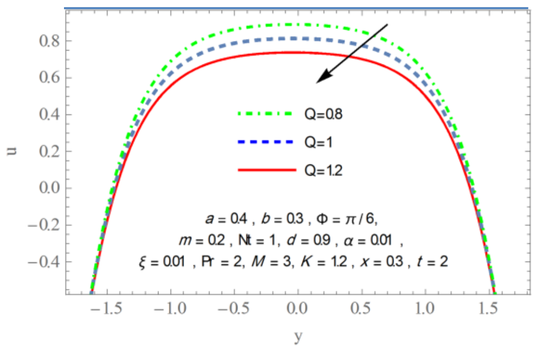

Fig.(17): *Variation of* (u) *with increasing of* Q

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8.4. Temperature distribution:

The Figs.(18)-(21) shows the variation in the distribution of temperature between the center of the channel and the layers near the walls, during the peristaltic motion of the fluid for the fixed values of x=1 and t=0.5.

The Figs.(18)&(19) illustrate the effect a phase difference (ϕ) and the non-uniform parameter (m) on the temperature of fluid (θ),

where (θ) is increasing near to walls with enhances to (ϕ & m), but the temperature is almost not affected by with height values of those parameters in center of the channel. While is the opposite with parameters (Nt) and (Pr) where the temperature of the fluid is decreasing near to walls of the channel and the effect of these parameters is gradually fading in the middle of the channel [see Figs.(20)&(21)].

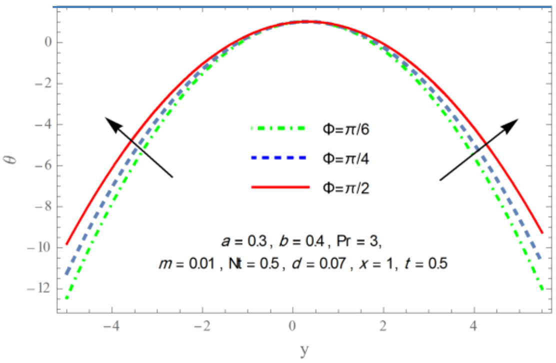

Fig.(18): Variation of (θ) with increasing of ϕ

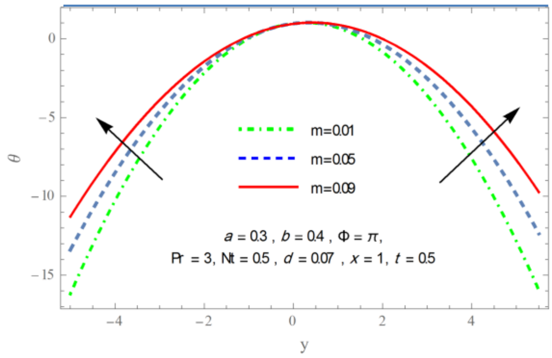

Fig.(19): Variation of (θ) with increasing of m

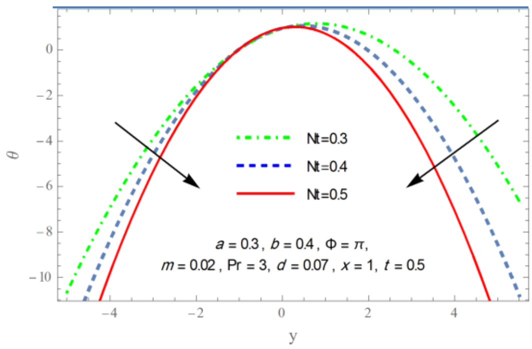

Fig.(20): Variation of (θ) with increasing of Nt

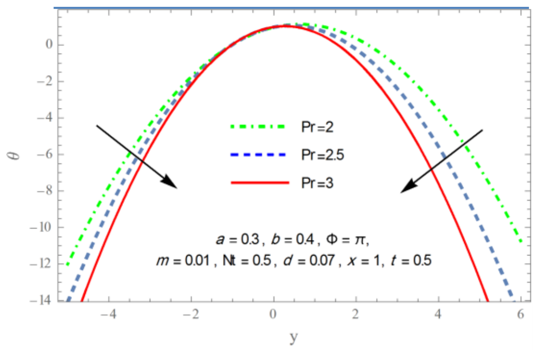

Fig.(21): Variation of (θ) with increasing of Pr

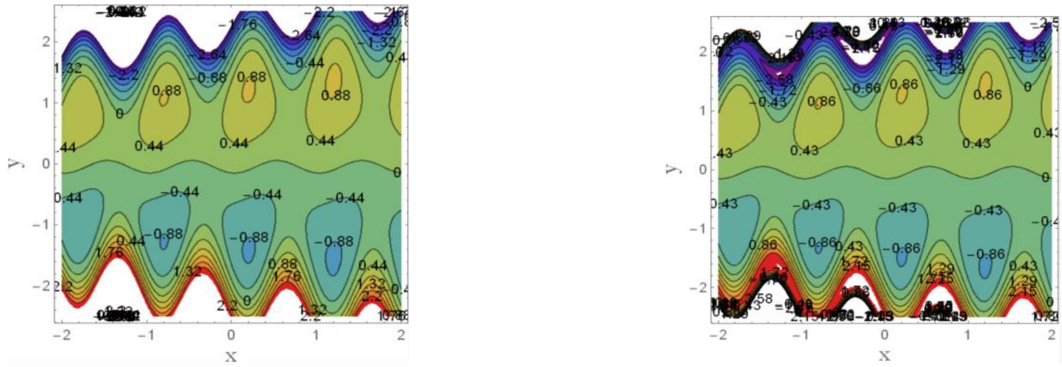

Fig.(22): Streamlines for $a = 0.4$, $b = 0.3$, $\phi = \pi / 6$, $m = 0.2$, $Q = 0.8$, $Nt = 1.1$, $d = 0.9$, $\alpha = 0.01$, $\xi = 0.002$, $Pr = 1$, $K = 1.2$, $t = 2$ **and for different M : (a) M=1, (b) M=2 .**

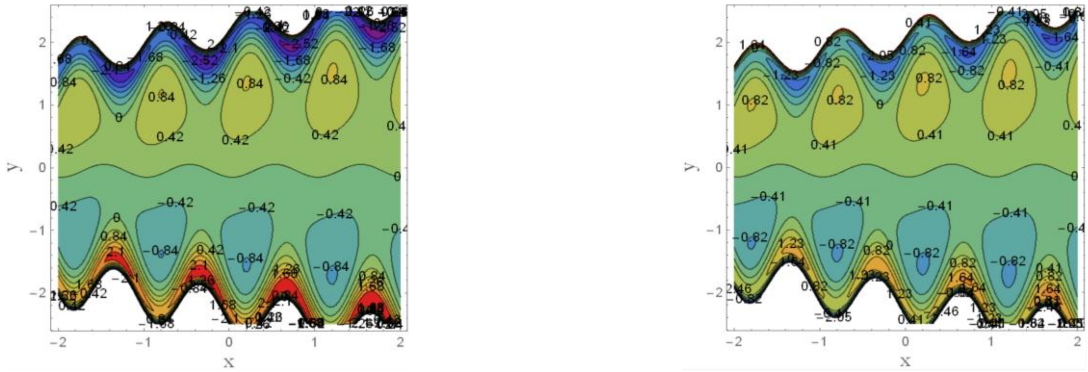

Fig.(23): Streamlines for $a = 0.4$, $b = 0.3$, $\phi = \pi / 6$, $m = 0.2$, $Q = 0.8$, $Nt = 1.1$, $d = 0.9$, $\alpha = 0.01$, $Pr = 1$, $M = 3$, $K = 1.2$, $t = 2$ **and for different** ξ **: (a)** $\xi = 0.001$ **, (b)** $\xi = 0.002$ **.**

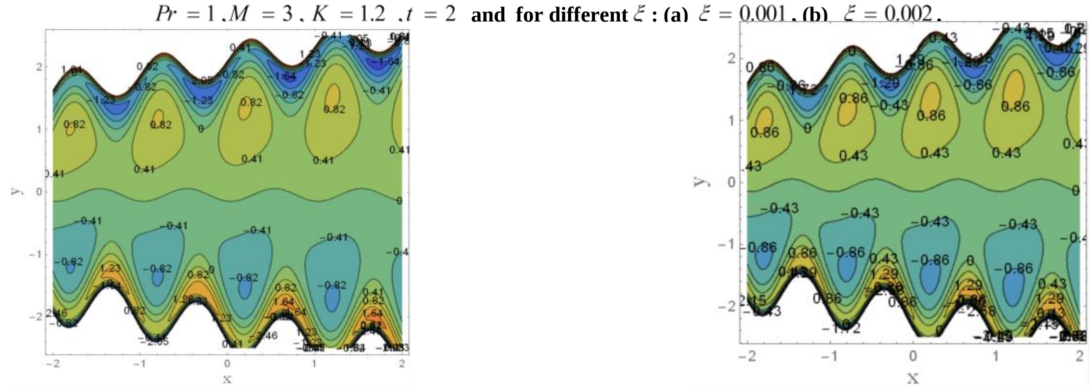

Fig.(24): Streamlines for $a = 0.4$, $b = 0.3$, $\phi = \pi / 6$, $m = 0.2$, $Nt = 1.1$, $d = 0.9$, $\alpha = 0.01$, $\xi = 0.002$, $Pr = 1$, $M = 3$, $K = 1.2$, $t = 2$ **and for different Q : (a) Q=0.8 , (b) Q=1 .**

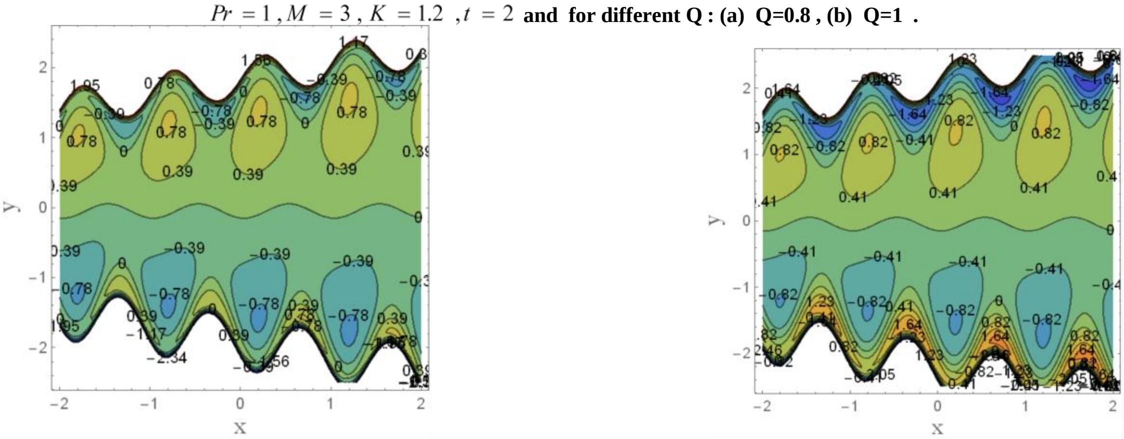

Fig.(25): Streamlines for $a = 0.4$, $b = 0.3$, $\phi = \pi / 6$, $m = 0.2$, $Q = 0.8$, $Nt = 1.1$, $d = 0.9$, $\alpha = 0.01$, $\xi = 0.002$, $M = 3$, $Pr = 1$, $t = 2$ and for different K : **(a)** K=0.1 , **(b)** K=2.2 .

12. Conclusions:

In this paper, we succeeded in presenting a mathematical model to study the effect of no-slip conditions with variable viscosity on peristaltic transport of a non-Newtonian pseudoplastic fluid inside an asymmetric channel. A regular perturbation method is employed to obtain the expression for the pressure gradient and pressure rise over a wavelength, velocity, temperature distribution, the heat transfer coefficient, the Nusselt number and the stream function.

We have discussed the effect peristaltic flow and the rheological parameters of the fluid.

- The increases of a value of parameters (M, ϕ, Q ,ξ) is leading to the pressure gradient is increased and decreased when increasing the values of the parameters (m, α , K).
- The pumping rate (Δp) is decre-asing with an increase in (M, ξ) , But the opposite happens with the parameter (K) in the co-pumping region.
- The profiles of axial velocity (u) take parabolic shape for it is curves.
- The axial velocity (u) increases with the increase in (M) at the core part of the channel but it decreases for near to walls, and the opposite happens with the parameters (Nt).
- The axial velocity (u) decreases with an increase in (K, Q).
- The temperature increases near to walls and almost not affected in center of the channel with enhances to (ϕ ,m) and the opposite with parameters (Nt, Pr).
- The size of the trapping bolus increases with increasing of the parameters (Q,K), while it has decrease with increases of parameters (M, ξ).

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تحليل نقل الحرارة والكتلة في قناة غير متناظرة مدببة اثناء النقل التمعجي لمائع (البسيدوبلاستك) مع لزوجة متغيرة تحت تأثير المجال مغناطيسي

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المستخلص :

في هذا البحث ، دراسة وتحليل انتقال الحرارة والكتلة أثناء التدفق التمعجي لمائع (pseudoplastic) في قناة مدببة غير متماثلة ، ولزوجة متغيرة تعتمد على درجة حرارة السائل مع وجود شروط الانزلاق من خلال وسط مسامي وتأثير هذه الظروف على السرعة والضغط ، حيث يكون الطول الموجي للتدفق التموجي طويلًا و عدد رينولد صغيرًا جدًا. ان حل معادلات الزخم والطاقة قد تم على اساس تقنية الاضطراب لايجاد دالة التدفق والسرعة وتدرج الضغط ودرجة الحرارة ، كما تمت مناقشة ظاهرة المحاصرة باستخدام الرسومات البيانية.

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Permuting Jordan Left Tri – Derivations On Prime and semiprime rings

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Abstract :

Let $\mathfrak{K}$ be a 2 and 3 – torsion free prime ring then if $\mathfrak{K}$ admits a non-zero Jordan left tri- derivation $B: \mathfrak{K} \times \mathfrak{K} \times \mathfrak{K} \to \mathfrak{K}$, then R is commutative ,also we give some properties of permuting left tri - derivations.

Keywords: prime ring , semiprime ring , left tri-derivation, Jordan left tri-derivations.

Mathematics Subject Classification: 16A12; 16A68; 12A72

1.Introduetion:

Throughout this paper we will use $\mathfrak{K}$ to represent an associative ring with center $Z(\mathfrak{K})$, $\mathfrak{K}$ is said to be n-torsion free if $na = 0$, $a \in \mathfrak{K}$ implies $a = 0$ [5].

A ring $\mathfrak{K}$ is called prime(semiprime) if $a\mathfrak{K}b = 0$ ($a\mathfrak{K}a = 0$) implies that $a = 0$ or $b = 0$($a = 0$) [3] .A mapping $D: \mathfrak{K} \times \mathfrak{K} \times \mathfrak{K} \longrightarrow \mathfrak{K}$ is said to be permuting if

$$D(x_1, x_2, x_3) = D(x_{\pi_1}, x_{\pi_2}, x_{\pi_3})$$

hold for all $x_1, x_2, x_3 \in \mathfrak{K}$ and every permute π_1, π_2, π_3 .

A mapping $d: \mathfrak{K} \to \mathfrak{K}$ defined by $d(x) = D(x, x, x)$ is called the trace of $D(.,.,.)$ where $D: \mathfrak{K} \times \mathfrak{K} \times \mathfrak{K} \to \mathfrak{K}$ is permuting tri- additive mapping [3] , a tri-additive mapping $D: \mathfrak{K} \times \mathfrak{K} \times \mathfrak{K} \to \mathfrak{K}$ is called tri – derivation if

$D(x_1x_2, y, z) = x_1D(x_2, y, z) + D(x_1, y, z)x_2$,
$D(x, y_1y_2, z) = y_1D(x, y_2, z) + D(x, y_1, z)y_2$ and
$D(x, y, z_1z_2) = z_1D(x, y, z_2) + D(x, y, z_1)z_2$ are hold for all $x, y, z, x_i, y_i, z_i \in \mathfrak{K}$ [3] .

The trace d of D satisfy the relation

$$d(x + y) = d(x) + d(y) + 3D(x, x, y) + 3D(x, y, y)$$

for all $x, y \in \mathfrak{K}$ [7].

A. K. Faraj in [1] and R. C. Shaheenin [6] define the permuting left tri- derivation as follows a permuting tri-additive mapping $D: \mathfrak{R}\times\mathfrak{R}\times\mathfrak{R} \to v$ is called permuting left tri – derivation if

$D(x_1x_2,y,z) = x_1D(x_2,y,z) + x_2D(x_1,y,z)$

$D(x,y_1y_2,z) = y_1D(x,y_2,z)$
$+y_2D(x,y_1,z)$

and

$D(x,y,z_1z_2) = z_1D(x,y,z_2)$
$+z_2D(x,y,z_1)$

are hold for all $x,y,z,x_i,y_i,z_i \in \mathfrak{R}$, $i = 1,2$ also D is called permuting Jordan left tri – derivation if $D(x^2,y,z) = 2xD(x,y,z)$, $D(x,y^2,z) = 2yD(x,y,z)$ and $D(x,y,z^2) = 2zD(x,y,z)$ are hold for all $x,y,z \in \mathfrak{R}$

In this paper , we gave some properties of permeating left tri-derivation, also we prove that if $\mathfrak{R}$ is a prime ring of characteristic not equal 2 and 3 and $\mathfrak{R}$ is admit anon-zero Jordan left-tri-derivation on $\mathfrak{R}$, Then $\mathfrak{R}$ is commutative.

2.Permuting left tri-derivations:-

In the following theorem we introduce some properties of permuting Jordan left tri –derivation on a ring

Theorem 2.1:

Let v a 2 - torsion free ring . If $B: \mathfrak{R}\times\mathfrak{R}\times\mathfrak{R} \to \mathfrak{R}$ is a Jordan left tri – derivation then for all $a,b,y,z \in \mathfrak{R}$ we have.

i) $B(ab+ba,y,z) = 2aB(b,y,z) + 2bB(a,y,z)$

ii) $B(aba,y,z) = a^2B(b,y,z) + 3abB(a,y,z) - baB(a,y,z)$

iii) $B(abc+cba,y,z) = (ac+ca)B(b,y,z) + 3abB(c,y,z) + 3cbB(a,y,z)$
$-baB(c,y,z) - bcB(a,y,z)$

iv) $[a,b]aB(a,y,z) = a[a,b]B(a,y,z)$

v) $[a,b]\{B(ab,y,z) - aB(b,y,z) - bB(a,y,z)\} = 0$

Proof:

i) Since $B(a^2,y,z) = 2aB(a,y,z)$

replace a by $a+b$

$B((a+b)^2,y,z) = B(a^2+ba+ab+b^2,y,z)$
$= B(a^2,y,z) + B(b^2,y,z) + B(ab+ba,y,z)$
$= 2aB(a,y,z) + 2bB(b,y,z) + B(ab+ba,y,z)$ … (1)

Now

$B((a+b)^2,y,z) = 2(a+b)B(a+b)y,z)$
$= 2aB(a+b),y,z) + 2bB(a+b,y,z)$
$= 2aB(ay,z) + 2aB(b,y,z) + 2bB(a,y,z) + 2bB(b,y,z)$ ...(2)

comparing (1) and(2)we get

$B(ab+ba,y,z) = 2aB(b,y,z) + 2bB(a,y,z)$

ii) From(i) we have for $a,b,y,z \in \mathfrak{R}$.

$B(a(ab+ba)+(ab+ba)a,y,z) = 2aB(ab+ba,y,z) + 2(ab+ba)B(a,y,z)$
$= 2a\{2aB(b,y,z) + 2bB(a,y,z)\} + 2abB(a,y,z) + 2baB(a,y,z)$
$= 4a^2B(b,yz) + 6abB(a,y,z) + 2baB(a,y,z)$ ….(3)

On the other hand , we have

$B(a(ab+ba)+(ab+ba)a,y,z) = B(a^2b+aba+aba+ba^2,y,z)$
$= B(a^2b+ba^2,y,z) + 2B(aba,y,z)$
$= 2a^2B(b,y,z) + 4baB(a,y,z) + 2B(aba,y,z)$ …(4)

Composing (3)and(4) we get

$2B(aba,y,z) = 2a^2B(b,y,z) + 6abB(a,y,z) - 2baB(a,y,z)$

So that

$2B(aba,y,z) = 2(a^2B(b,y,z) + 3abB(a,y,z) - baB(a,y,z))$

Since R is 2-torsion free we have

$B(aba,y,z) = a^2B(b,y,z) + 3abB(a,y,z)$
$- baB(a,y,z)$.

iii) Linearizing (ii) on a we get

$B\big((a+c)b(a+c),y,z\big)$
$= (a+c)^2B(b,y,z) + 3(a+c)bB(a+c,y,z) - b(a+c)B(a+c,y,z)$
$= a^2B(b,y,z) + acB(b,y,z) + caB(b,y,z)$
$+ c^2B(b,y,z)$
$+ 3abB(a+c,y,z)$
$+3cbB(a+c,y,z) - baB(a+c,y,z) - bcB(a+c,y,z)$
$= a^2B(b,y,z) + acB(b,y,z) + caB(b,y,z) + c^2B(b,y,z) + 3abBca,y,z)$

$+3abB(c,y,z) + 3cbB(a,y,z) + 3cbB(c,y,z) - baB(a,y,z)$
$-baB(c,y,z) - bcB(a,y,z) - bcB(c,y,z)$ …(5)

In other hand

$B\big((a+c)b(a+c),y,z\big) = B(aba+abc+cba+cbc,y,z) = B(aba,y,z) + B(abc+cba,y,z) + B(cbc,y,z) = a^2B(b,y,z) + 3abB(a,y,z) - baB(a,y,z) + B(abc+cba,y,z) + c^2B(b,y,c) + 3cbB(c,y,z) - bcB(c,y,z)$ …(6)

Comparing (5) and(6) we have.

$B(abc+cba,y,z) = (ac+ca)B(b,y,z) + 3abB(c,y,z) + 3cbB(a,y,z)$

$-baB(c,y,z) - bcB(a,y,z)$

(iv) Assume that $w = B(ab(ab) + (ab)ba, y, z)$
Then by (iii) we obtain

$$w = (a(ab) + (ab)a)B(b,y,z) + 3abB(ab,y,z) + 3(ab)bB(a,y,z) - baB(ab,y,z) - b(ab)B(a,y,z)$$

$$w = (a^2b + aba)B(b,y,z) + 3ab\,B(ab,y,z) + 3ab^2B(a,y,z) - ba\,B(ab,y,z) - babB(a,y,z) \quad \ldots(7)$$

On the other hand

$$w = B\big((ab)(ab) + (ab)ba, y, z\big) = B((ab)^2,y,z) + B(ab^2a,y,z)$$
$$= 2abB(ab,y,z) + a^2B(b^2,y,z) + 3ab^2\,B(a,y,z) - b^2aB(a,y,z)$$

So by definition of B

$$w = 2abB(ab,y,z) + 2a^2bB(b,y,z) + 3ab^2\,B(a,y,z) - b^2aB(a,y,z) \quad \ldots(8)$$

By comparing (7) and (8)

$$[a,b]B(ab,y,z) + abaB(b,y,z) - b(ab)B(a,y,z) - a^2bB(b,y,z) + b(ba)B(a,y,z) = 0$$

Then

$$[a,b]B(ab,y,z) - a[a,b]B(b,y,z) - b[a,b]B(a,y,z) = 0$$

so

$$[a,b]B(ab,y,z) = a[a,b]B(b,y,z) + b[a,b]B(a,y,z) \quad \ldots(9)$$

Replace b by $a+b$ in (9)

$$[a,a+b]B(a(a+b),y,z) = [a,b]B(a^2,y,z) + [a,b]B(ab,y,z)$$
$$= 2[a,b]aB(a,y,z) + a[a,b]B(b,y,z) + b[a,b]B(a,y,z)$$
$$= a[a,a+b]B(a+b,y,z) + (a+b)[a,a+b]B(a,y,z)$$
$$= a[a,b]B(a,y,z) + a[a,b]B(b,y,z) + a[a,b]B(a,y,z) + b[a,b]B(a,y,z)$$

Hence

$$2[a,b]aB(a,y,z) = 2a[a,b]B(a,y,z)$$

Since $\mathfrak{K}$ is 2-torsion free, then

$$[a,b]aB(a,y,z) = a[a,b]B(a,y,z) \quad \ldots(10)$$

(v) In (10) replace a by $+b$,
the left hand give

$$w = [a+b,b](a+b)B(a+b,y,z)$$
$$= [a,b]aB(a,y,z) + [a,b]a\,B(b,y,z) + [a,b]bB(a,y,z) + [a,b]bB(b,y,z) \ldots(11)$$

The right hand give

$$w = (a+b)[a+b,b]B(a+b,y,z)$$
$$= a[a,b]B(a,y,z) + a[a,b]B(b,y,z) + b[a,b]B(a,y,z) + b[a,b]B(b,y,z) \quad \ldots(12)$$

from(9)we have

$$[a,b]B(ab,y,z) = a[a,b]B(b,y,z) + b[a,b]B(a,y,z)$$

So that by using (10)

$$[a,b]B(ab,y,z) = [a,b]\big(aB(b,y,z) + bB(a,y,z)\big)$$
$$[a,b]\{B(ab,y,z) - aB(b,y,z) - bB(a,y,z)\} = 0$$

3.The Main Results:

Theorem 3.1:-

let $\mathfrak{K}$ be prime ring of char $\mathfrak{K} \neq 2,3$,then if R admits a non –zero Jordan left – tri derivation $B: \mathfrak{K} \times \mathfrak{K} \times \mathfrak{K} \to \mathfrak{K}$, then $\mathfrak{K}$ is commutative

Proof:

we divide proof to some steps.

Step1: If $B\ (a,y,z) \neq 0$ for some $a,y,z \in \mathfrak{K}$ then $(a[a,x] - [a,x]a)^2 = 0$ for all $x \in \mathfrak{K}$.

Let a be a fixed element in $\mathfrak{K}$ and $\emptyset: \mathfrak{K} \to \mathfrak{K}$ be a mapping defined by

$$\emptyset(x) = [a,x]$$

for all $x \in \mathfrak{K}$

Now [Theorem 2.1 ,iv] can be written in the form

$$\emptyset^2(x)B(a,y,z) = 0$$

for all $x \in \mathfrak{K}$ …….(13)

Since the mapping $\emptyset(x)$is a derivation, we have

$$\emptyset^2(x_1x_2) = \emptyset^2(x_1)x_2 + 2\emptyset(x_1)\emptyset(x_2) + x_1\emptyset^2(x_2)$$

And from (13) it follows that $\emptyset^2(x_1x_2)B(a,y,z) = 0$

hence

$$\big(\emptyset^2(x_1)x_2 + 2\emptyset(x_1)\emptyset(x_2) + x_1\emptyset^2(x_2)\big)B(a,y,z) = 0$$

So that

$$(\emptyset^2(x_1)x_2 + 2\emptyset(x_1)\emptyset(x_2))B(a,y,z) = 0$$

Hold for all $x_1, x_2 \in \mathfrak{K}$.

In the above relation replace $x_2 by\ \emptyset(x_2x_3)$and the relation (13) we get

$$(\emptyset^2(x_1)\emptyset(x_2)x_3 + \emptyset^2(x)x_2\emptyset(x_2))B(a,y,z) = 0 \ldots\ldots(14)$$

For all $x_1, x_2, x_3 \in \mathfrak{K}$.

In (14) substitute $\emptyset(x_3) for\ x_3, we\ get$

$$\emptyset^2(x_1)\emptyset(x_2)\emptyset(x_3)B(a,y,z) = 0 \quad \ldots(15)$$

Now in (14) replace x_2 by $\emptyset(x_2)$ and using(15) we have

$$\emptyset^2(x_1)\emptyset^2(x_2)x_3B(a,y,z) = 0 \quad \ldots(16)$$

holds for all $x_1, x_2, x_3 \in \mathfrak{K}$

since the relation (16) hold for all $x_3 \in \mathfrak{K}$, we are forced to conclude that B(a,y,z) to which implies that

$$\emptyset^2(x_1)\emptyset^2(x_2) = 0\ for\ all\ x_1, x_2 \in \mathfrak{K}$$

in particular $\left(\emptyset^2(x_1)\right)^2 = 0$ as requaried

Step 2: If $a^2 = 0$ then $B(a,y,z) = 0$ for all $y,z \in \mathfrak{K}$

Let $w = B(a(xay + yax)a, y, z)$

Then by using (ii) in Theorem 2.1 We get

$$w = a^2B(xay + yax, y, z) + 3a(xay + yax)B(a,y,z) - (xay + yax)aB(a,y.z) \quad \ldots(17)$$

Since $a^2 = 0$ we have
$B(a^2, y, z) = 0 = 2aB(a, y, z)$
But char $\mathfrak{K} \neq 2$, then $aB(a, y, z) = 0$
hance (17) becomes
$w = 3a(xay + yax)B(a, y, z)$
$= 3axayB(a, y, z) + 3ayaxB(a, y, z)$ ...(18)
From (ii) Theorem 2.1 we have
$B(axa, y, z) = 3axB(a, y, z)$
According to (iii) in Theorem 2.1, we find
$B(ax(axa) + (axa)xa, y, z)$
$= (a^2xa + axa^2)B(x, y, z)$
$+3axB(axa, y, z)$
$+3(axa)xB(a, y, z)$
$-x(axa)B(a, y, z)$
Since $a^2 = 0$, we have
$w = 9axaxB(a, y, z) + 3axaxB(a, y, z)$ ...(19)
Comparing(18)and (19)we get
$6axaxB(a, y, z) = 0$
But $\mathfrak{K}$ is of characteristic not equal 2 and 3 so that
$axaxB(a, y, z) = 0$ for all x, y and z are arbitrary elements of $\mathfrak{K}$.
so that
$B(a, y, z) = 0 \; or \; a = 0$
If $a = 0$ we have $B(a, y, z) = 0$
So that in any case we have
$B(a, y, z) = 0$ for all $y, z \in \mathfrak{K}$

Step3 : $\mathfrak{K}$ is commutative.
Take: $a, y, z \in \mathfrak{K}$ such that $B(a, y, z) \neq 0$.
From step1 and step2, it follows in this case that.
$B(a[a, x] - [a, x]a), y, z) = 0$ for all $x \in \mathfrak{K}$ ... (20)
By using (i) and (ii) in Theorem 2.1, and since
$a[a, x] - [a, x]a = a^2x - 2axa + xa^2$
So we obtain from (20)
$0 = B(a^2x + xa^2, y, z) - 2B(axa, y, z)$
$= 2a^2B(x, y, z) + 2xB(a^2, y, z) - 2a^2B(x, y, z)$
$- 6axB(a, y, z) + 2xaB(a, y, z)$
$= 4xaB(a, y, z) - 6axB(a, y, z) + 2xab(a, y, z)$
$= 6xaB(a, y, z) - 6axB(a, y, z)$
hence
$6[x, a]B(a, y, z) = 0$
since $char\ (\mathfrak{K}) \neq 2,3$, we get
$[x, a]B(a, y, z) = 0$ for all $x, y \in \mathfrak{K}$
For all $x, y \in \mathfrak{K}$ we have
$0 = [yx, a]B(a, y, z) = y[x, a]B(a, y, z) + [y, a]xB(a, y, z) = [y, a]xB(a, y, z)$
Since we have assumed that $B(a, y, z) \neq 0$
So that $[y, a] = 0$ for all $y \in \mathfrak{K}$
consequentially $a \in Z(\mathfrak{K})$
Thus we have proved that $\mathfrak{K}$ is the union of its proper subsets $Z(\mathfrak{K})$ and
$kerB = \{a \in R | B(a, y, z) = 0$ for all $y, z \in \mathfrak{K}\}$
It is clear that both subsets $Z(\mathfrak{K})$ and $kerB$ are additive subgroups of , but a group cannot be the union of it is two proper subgroups , so that either
$\ker B = \mathfrak{K}$ or $Z(\mathfrak{K}) = \mathfrak{K}$ and since $B \neq 0$ by hypothesis we have $ker\ B \neq \mathfrak{K}$
Consequentially $\mathfrak{K} = Z(\mathfrak{K})$ and hence $\mathfrak{K}$ is commutative

Theorem 3.2 : Let $\mathfrak{K}$ be a prime ring and $B: \mathfrak{K} \times \mathfrak{K} \times \mathfrak{K} \to \mathfrak{K}$ a left tri-derivation. If $B \neq 0$ then $\mathfrak{K}$ is commutative .

Proof:
Consider $B(aba, y, z)$ for all $a, b, y, z \in \mathfrak{K}$
Then
$B(a(ba), y, z) = aB(ba, y, z) + baB(a, y, z)$
$= a^2B(b, y, z) + abB(a, y, z)$
$+baB(a, y, z)$ ...(21)
On the other hand
$B((ab)a, y, z) = abB(a, y, z) + aB(ab, y, z)$
$= ab\ B(a, y, z) + a^2B(b, y, z) + abB(a, y, z)$
$= 2abB(a, y, z) + a^2B(b, y, z)$ ...(22)
Comparing (21) and (22) we have
$[a, b]B(a, y, z) = 0$ for all $a, b, y, z \in \mathfrak{K}$ ....(23)
Replace b by cb in (23) we get
$0 = [a, cb]B(a, y, z)$
$= (a(cb) - (cb)a)B(a, y, z)$
$= (ac - ca)b\ B(a, y, z) + c(ab - ba)B(a, y, z)$
$= [a, c]bB(a, y, z)$ for all $a, b, c, y, z \in \mathfrak{K}$
So that
$[a, c]bB(a, y, z) = 0$ for all $a, b, c, z, y \in \mathfrak{K}$ ...(24)
It follows that for each $a \in \mathfrak{K}$ we have either
$a \in Z(\mathfrak{K})$ or $B(a, y, z) = 0$
But , since $Z(\mathfrak{K})$ and
$kerB = \{a \in \mathfrak{K} \mid B(a, y, z) = 0 \quad for\ all\ y, z \in \mathfrak{K}\}$ are additive subgroups of $\mathfrak{K}$.
We have either $\mathfrak{K} = Z(\mathfrak{K})$ or $\mathfrak{K} = ker\ B$ but $B \neq 0$, so that $\mathfrak{K} = Z(\mathfrak{K})$ and hence $\mathfrak{K}$ is commutative.

Theorem 3.3 : Let $\mathfrak{K}$ be a semi-Prime ring and $B: \mathfrak{K} \times \mathfrak{K} \times \mathfrak{K} \to \mathfrak{K}$ be a left tri –derivations then B is tri-derivation that maps $\mathfrak{K}$ into it is center .

Proof :
Linearize (24)
$[a + d, c]bB(a + d, y, z) = 0$
Which gives that
$[a, c]bB(d, y, z) + [d, c]bB(a, y, z) = 0$
Which implies that
$[a, c]bB(d, y, z) = -[d, c]bB(a, y, z)$
Since $\mathfrak{K}$ is a ring , for all $a, b, c, d, y, z \in \mathfrak{K}$ we have
$[a, c]bB(d, y, z)x[a, c]bB(a, y, z)$
$= -[d, c]bB(d, y, z)x[a, c]bB(a, y, z) = 0$
Since $\mathfrak{K}$ is semiprime , this relation yields
$[a, c]bB(d, y, z) = 0$

In particular
$\{aB(d,y,z) - B(d,y,z)a\}b\{aB(d,y,z) - B(d,y,z)a\} = 0$
semiprimeness of $\mathfrak{K}$ implies that
$aB(d,y,z) - B(d,y,z)a = 0$
Consequently B is a tri-derivation and $B(d,y,z) \in Z(\mathfrak{K})$

Theorem3.4: Let $\mathfrak{K}$ be a prime ring of $char\mathfrak{K} \neq 2,3$ and $B: \mathfrak{K} \times \mathfrak{K} \times \mathfrak{K} \rightarrow \mathfrak{K}$ a Jordan left tri-derivation, suppose that $ax = 0, a, x \in \mathfrak{K}$ implies that $a = 0$ *or* $x = 0$ if $B \neq 0$ then $\mathfrak{K}$ is commutative .

Proof:

From (v) of Theorem 2.1 , we have for each pair $a, b \in \mathfrak{K}$

$$[a,b] = 0 \; or \; B\,(ab,y,z) = aB(b,y,z) + bB(a,y,z)$$

Given $a \in \mathfrak{K}$ and let
$G_a = \{b \in \mathfrak{K} | a[a,b] = 0\}$ and
$H_a = \{b \in \mathfrak{K} | B(ab,y,z) = aB(b,y,z) + bB(a,y,z)\}$
We see that $\mathfrak{K}$ is the union of it's additive subgroups $G_a and\; H_a$
Hence $= G_a \; or \; \mathfrak{K} = H_a$, on the other words , $\mathfrak{K}$ is the union of its subgroup's
$G = \{a \in \mathfrak{K} | G_a = \mathfrak{K}\} = Z(\mathfrak{K})$ and
$H = \{a \in \mathfrak{K} | H_a = \mathfrak{K}\}$
$= \{a \in \mathfrak{K} | B(ab,y,z) = aB(b,y,z) + bB(a,y,z) \, , for \; all \; a,b,y,z \in R\}$
Clearly $G \; and \; H$ are additive subgroups of $\mathfrak{K}$
Hence $G = \mathfrak{K}$ or $H = \mathfrak{K}$
If $G = \mathfrak{K}$ then $\mathfrak{K}$ is commutative.
If $H = \mathfrak{K}$ then B is a left tri-derivation and hence $\mathfrak{K}$ is commutative by Theorem 3.3
Thus in any way $\mathfrak{K}$ is commutative.

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اشتقاقات جوردان اليسارية الثلاثية التبادلية على الحلقات الاولية والحلقات شبه الاولية

مازن عمران كريم
جامعة القادسية / كلية التربية / قسم الرياضيات

الملخص :
لتكن Rحلقة اولية طليقة الالتواء من النمط 2 و 3 . اذا كانت R تسمح بوجود اشتقاق جوردان اليساري الثلاثي $B: R \times R \times R \to R$ فانها تكون ابدالية .
كذلك قدمنا بعض الخواص لمشتقات جوردان اليسارية الثلاثية التبادلية

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The new exponential identities

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ABSTRACT: We have obtained new exponential identities. By ten original propositions we have proved them.

Keywords: Identities, Pascal's triangle, Binomial coefficients.

Mathematics subject classification: 11D61.

1. Introduction

Pascal's triangle can be arranged in a triangular array of numbers, as follows:

$$\binom{0}{0}$$
$$\binom{1}{0} \quad \binom{1}{1}$$
$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$
$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$
$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$
$$\dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots$$
$$\binom{n}{0} \quad \binom{n}{1} \dots \binom{n}{k-1} \quad \binom{n}{k} \dots \binom{n}{n-1} \quad \binom{n}{n}$$
$$\binom{n+1}{0} \quad \binom{n+1}{1} \quad \dots \quad \binom{n+1}{k} \quad \dots \quad \binom{n+1}{n} \quad \binom{n+1}{n+1}$$

Where $n \geq k$.
It has the following properties.

- The first number and the last number in each row is 1.
- Every other number in the array can be obtained by adding the two numbers appearing directly above it. This property is equivalent to the following identity:

$$\binom{n}{n-1} + \binom{n}{k} = \binom{n+1}{k} \qquad (1.1)$$

- The numbers equidistant from the ends are equal. This property is equivalent to the following identity:

$$\binom{n}{k} = \binom{n}{n-k} \qquad (1.2)$$

Now since the numbers appearing in Pascal's triangle are the binomial coefficients, and here is some of identities satisfied by them.

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n \tag{1.3}$$

$$\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1} \tag{1.4}$$

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0 \tag{1.5}$$

See [3] for more details.
Can we obtain new identities?
By using the identities above. This paper has answered this question by ten original propositions.

2. Notation and Definitions

We denote the set of natural numbers $\mathbb{N} := \{1,2,3, \dots\}$. By $\mathbb{Z}$ we denote the set of integers numbers. By $\mathbb{C}$ we denote the complex numbers. The set of $\mathbb{C}^*$ is defined by $\mathbb{C}^* := \{ z \in \mathbb{C},\ z \neq 0\}$. The set of all nonzero polynomials over the set $\mathbb{C}$ with indeterminate z is denoted by $\mathbb{C}[z]$. Thus $\mathbb{C}[z] \coloneqq \{f(z): f(z)$ is a polynomial , $f(z) \neq 0\}$.

Definition 2.1. (See [1]). A number P is called a *composite prime*, and $P \in \mathbb{Z}$, if $P = p_1 \cdot p_2 \cdots p_i$. Where $p_1, p_2, \dots, p_i$ are distinct primes.

Definition 2.2. (See [2]). We call $\mathbb{A}$ a *set of basic numbers*.

$$\mathbb{A} := \{p, P \in \mathbb{Z} :\ p \text{ is a prime number}, \ P \text{ is a composite prime}\}.$$

Definition 2.3. (See [2]). A number a is called a *basic number* if $a \in \mathbb{A}$.

Definition 2.4 (See [2]). A polynomial $P(z)$ is called a *composite primary polynomial*, and $P(z) \in \mathbb{C}[z]$, if $P(z) = cp_1(z) \cdot p_2(z) \cdot p_i(z)$. Where $p_1(z), p_2(z), \dots , p_i(z)$ are irreducible distinct polynomials and $c \neq 0$ is a constant.

Definition 2.5. (See [2]). We call $\mathbb{A}[z]$ a *set of basic polynomials*.

$$\mathbb{A}[z] \coloneqq \{ p(z), P(z) \in \mathbb{C}[z]:\ p(z) \text{ is } a \text{ irreducible polynomial}, \ P(z) \text{ is a composite primary polynomial} \}.$$

Definition 2.6. (See [2]). A polynomial $A(z)$ is called a *basic polynomial* if $A(z) \in \mathbb{A}[z]$.

Definition 2.7. We will define two types powers triangle:

- A *powers triangle of number* can be obtained by z^r (or a) , where $z \in \mathbb{C}^*$ and $r \in \mathbb{N}$, as follows:

$$\begin{array}{ccccccccccc}
 & & & 1 & & z^{r\binom{0}{0}} & & 1 & & & \\
 & & 1 & & z^{r\binom{1}{0}} & & z^{r\binom{1}{1}} & & 1 & & \\
 & 1 & & z^{r\binom{2}{0}} & & z^{r\binom{2}{1}} & & z^{r\binom{2}{2}} & & 1 & \\
 \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
 1 & z^{r\binom{n}{0}} & \dots & z^{r\binom{n}{k-1}} & & z^{r\binom{n}{k}} & \dots & z^{r\binom{n}{n}} & 1 & & \\
 1 & z^{r\binom{n+1}{0}} & \dots & & z^{r\binom{n+1}{k}} & & \dots & & z^{r\binom{n+1}{n+1}} & & 1
\end{array}$$

Or

$$\begin{array}{ccccccccccc}
 & & & 1 & & a^{\binom{0}{0}} & & 1 & & & \\
 & & 1 & & a^{\binom{1}{0}} & & a^{\binom{1}{1}} & & 1 & & \\
 & 1 & & a^{\binom{2}{0}} & & a^{\binom{2}{1}} & & a^{\binom{2}{2}} & & 1 & \\
 \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
 1 & a^{\binom{n}{0}} & \dots & a^{\binom{n}{k-1}} & & a^{\binom{n}{k}} & \dots & a^{\binom{n}{n}} & 1 & & \\
 1 & a^{\binom{n+1}{0}} & \dots & & a^{\binom{n+1}{k}} & & \dots & & a^{\binom{n+1}{n+1}} & & 1
\end{array}$$

It has the following interesting properties:

 – The first number and the last number in each row is 1.
 – Every other number in the array can be obtained by multiplying the two numbers appearing directly above it.
 – The numbers equidistant from the ends are equal.

- A *powers triangle of polynomial* can be obtained by $f(z)^r$ $\big($or $A(z)\big)$. By using the symbol $f(z)$ $\big($or $A(z)\big)$ instead of z (or a), likewise, we define a powers triangle of polynomial.

3. The Results

We have proved the following Results:

Proposition 3.1.

$$z^{r\binom{n+1}{k}} = z^{r\binom{n}{n-1}+r\binom{n}{k}}. \tag{3.1}$$

Corollary. If $z^r = a$, then

$$a^{\binom{n+1}{k}} = a^{\binom{n}{n-1}+\binom{n}{k}}. \tag{3.2}$$

Proposition 3.2.

$$z^{r\binom{n}{k}} = z^{r\binom{n}{n-k}} . \tag{3.3}$$

Corollary. If $z^r = a$, then

$$a^{\binom{n}{k}} = a^{\binom{n}{n-k}} . \tag{3.4}$$

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Proposition 3.3.

$$\prod_{k=0}^{n} z^{r\binom{n}{k}} = z^{r2^n}. \quad (3.5)$$

Corollary. If $z^r = a$, then

$$\prod_{k=0}^{n} a^{\binom{n}{k}} = a^{2^n}. \quad (3.6)$$

Proposition 3.4.

$$\prod_{k=0}^{n} \left(z^{r\binom{n}{k}}\right)^k = z^{rn2^{n-1}}. \quad (3.7)$$

Corollary. If $z^r = a$, then

$$\prod_{k=0}^{n} \left(a^{\binom{n}{k}}\right)^k = a^{n2^{n-1}}. \quad (3.8)$$

Proposition 3.5.

$$\prod_{k=0}^{n} \left(z^{r\binom{n}{k}}\right)^{(-1)^k} = 1. \quad (3.9)$$

Corollary. If $z^r = a$, then

$$\prod_{k=0}^{n} \left(a^{\binom{n}{k}}\right)^{(-1)^k} = 1. \quad (3.10)$$

Proposition 3.6.

$$f(z)^{r\binom{n+1}{k}} = f(z)^{r\binom{n}{n-1}+r\binom{n}{k}}. \quad (3.11)$$

Corollary. If $f(z)^r = A(z)$, then

$$A(z)^{\binom{n+1}{k}} = A(z)^{\binom{n}{n-1}+\binom{n}{k}}. \quad (3.12)$$

Proposition 3.7.

$$f(z)^{r\binom{n}{k}} = f(z)^{r\binom{n}{n-k}}. \quad (3.13)$$

Corollary. If $f(z)^r = A(z)$, then

$$A(z)^{\binom{n}{k}} = A(z)^{\binom{n}{n-k}}. \quad (3.14)$$

Proposition 3.8.

$$\prod_{k=0}^{n} f(z)^{r\binom{n}{k}} = f(z)^{r2^n}. \quad (3.15)$$

Corollary. If $f(z)^r = A(z)$, then

$$\prod_{k=0}^{n} A(z)^{\binom{n}{k}} = A(z)^{2^n}. \quad (3.16)$$

Proposition 3.9.

$$\prod_{k=0}^{n} \left(f(z)^{r\binom{n}{k}}\right)^k = f(z)^{rn2^{n-1}}. \quad (3.17)$$

Corollary. If $f(z)^r = A(z)$, then

$$\prod_{k=0}^{n} \left(A(z)^{\binom{n}{k}}\right)^k = A(z)^{n2^{n-1}}. \quad (3.18)$$

Proposition 3.10.

$$\prod_{k=0}^{n} \left(f(z)^{r\binom{n}{k}}\right)^{(-1)^k} = 1. \quad (3.19)$$

Corollary. If $f(z)^r = A(z)$, then

$$\prod_{k=0}^{n} \left(A(z)^{\binom{n}{k}}\right)^{(-1)^k} = 1. \quad (3.20)$$

Now, here are some examples to show the results.

Example 3.1. If a powers triangle is

$$\begin{array}{ccccccccccc} & & & & 1 & 4 & 1 & & & & \\ & & & 1 & 4 & & 4 & 1 & & & \\ & & 1 & 4 & & 4^2 & & 4 & 1 & & \\ & 1 & 4 & & 4^3 & & 4^3 & & 4 & 1 & \\ 1 & 4 & 4^4 & & & 4^6 & & & 4^4 & 4 & 1 \end{array}$$

$$1 \quad 4 \quad 4^5 \quad 4^{10} \quad 4^{10} \quad 4^5 \quad 4 \quad 1$$

Compute

a)

$$\prod_{k=0}^{5} 4^{\binom{5}{k}}.$$

b)

$$\prod_{k=0}^{5} \left(4^{\binom{5}{k}}\right)^k.$$

c)

$$\prod_{k=0}^{5} \left(4^{\binom{5}{k}}\right)^{(-1)^k}.$$

Solution:

a)

$$\prod_{k=0}^{5} 2^{2\binom{5}{k}} = 2^{64}.$$

b)

$$\prod_{k=0}^{5} \left(2^{2\binom{5}{k}}\right)^k = 2^{160}.$$

c)

$$\prod_{k=0}^{5} \left(2^{2\binom{5}{k}}\right)^{(-1)^k} = 1.$$

Example 3.2. If a powers triangle is

$$\begin{array}{ccccccccc} & & & 1 & 3 & 1 & & & \\ & & 1 & 3 & & 3 & 1 & & \\ & 1 & 3 & & 3^2 & & 3 & 1 & \\ 1 & 3 & & 3^3 & & 3^3 & & 3 & 1 \end{array}$$

$$1 \quad 3 \quad 3^4 \quad 3^6 \quad 3^4 \quad 3 \quad 1$$

Compute

d)

$$\prod_{k=0}^{4} 3^{\binom{4}{k}}.$$

e)

$$\prod_{k=0}^{4} \left(3^{\binom{4}{k}}\right)^k.$$

f)

$$\prod_{k=0}^{4}\left(3^{\binom{4}{k}}\right)^{(-1)^k}.$$

Solution:

d)

$$\prod_{k=0}^{4}3^{\binom{4}{k}} = 3^{16}.$$

e)

$$\prod_{k=0}^{4}\left(3^{\binom{4}{k}}\right)^{k} = 3^{32}.$$

f)

$$\prod_{k=0}^{4}\left(3^{\binom{4}{k}}\right)^{(-1)^k} = 1.$$

Example 3.3. Compute

a)

$$\prod_{k=0}^{5}(4z^2+2z+1)^{\binom{5}{k}}.$$

b)

$$\prod_{k=0}^{5}\left((4z^2+2z+1)^{\binom{5}{k}}\right)^{k}.$$

c)

$$\prod_{k=0}^{5}\left((4z^2+2z+1)^{\binom{5}{k}}\right)^{(-1)^k}.$$

Solution:

a)

$$\prod_{k=0}^{5}(2z+1)^{2\binom{5}{k}} = (2z+1)^{64}.$$

b)

$$\prod_{k=0}^{5}\left((2z+1)^{2\binom{5}{k}}\right)^{k} = (2z+1)^{160}.$$

c)

$$\prod_{k=0}^{5}\left((2z+1)^{2\binom{5}{k}}\right)^{(-1)^k} = 1.$$

Example 3.4. Compute

a)

$$\prod_{k=0}^{4}(2z+1)^{\binom{4}{k}}.$$

b)

$$\prod_{k=0}^{4}\left((2z+1)^{\binom{4}{k}}\right)^{k}.$$

c)

$$\prod_{k=0}^{4}\left((2z+1)^{\binom{4}{k}}\right)^{(-1)^k}.$$

Solution:

a)

$$\prod_{k=0}^{4}(2z+1)^{\binom{4}{k}} = (2z+1)^{16}.$$

b)

$$\prod_{k=0}^{4}\left((2z+1)^{\binom{4}{k}}\right)^{k} = (2z+1)^{32}.$$

c)

$$\prod_{k=0}^{4}\left((2z+1)^{\binom{4}{k}}\right)^{(-1)^k} = 1.$$

4. Proof of the Results

Proof of Proposition 3.1. Since $z = z$, now (1.1) leads to

$$z^{r\binom{n+1}{k}} = z^{r\binom{n}{n-1}+r\binom{n}{k}}.$$

□

Proof of Proposition 3.2. Since $z = z$, now (1.2) leads to

$$z^{r\binom{n}{k}} = z^{r\binom{n}{n-k}}.$$

□

Proof of Proposition 3.3. By definition 2.7 , in row n

$$\prod_{k=0}^{n} z^{r\binom{n}{k}} = z^{r\binom{n}{0}}\cdot z^{r\binom{n}{1}}\cdots z^{r\binom{n}{n}}$$

$$= z^{r\left(\binom{n}{0}+\binom{n}{1}+\cdots+\binom{n}{n}\right)} \qquad \text{[By (1.3)]}$$

$$= z^{r2^n}.$$ □

Proof of Proposition 3.4. We expand the left-hand side of (3.7)

$$\prod_{k=0}^{n}\left(z^{r\binom{n}{k}}\right)^{k} = \left(z^{r\binom{n}{0}}\right)^{0}\left(z^{r\binom{n}{1}}\right)^{1}\left(z^{r\binom{n}{2}}\right)^{2}\cdots\left(z^{r\binom{n}{n}}\right)^{n}$$

$$= z^{r\left(0\binom{n}{0}+1\binom{n}{1}+2\binom{n}{2}+\cdots+n\binom{n}{n}\right)} \qquad \text{[By (1.4)]}$$

$$= z^{rn2^{n-1}}.$$ □

Proof of Proposition 3.5. We expand the left-hand side of (3.9)

$$\prod_{k=0}^{n}\left(z^{r\binom{n}{k}}\right)^{(-1)^k}$$
$$= \left(z^{r\binom{n}{0}}\right)^1\left(z^{r\binom{n}{1}}\right)^{-1}\left(z^{r\binom{n}{2}}\right)^1\cdots\left(z^{r\binom{n}{n}}\right)^{(-1)^n}$$
$$= z^{r\left(\binom{n}{0}-\binom{n}{1}+\binom{n}{2}+\cdots+(-1)^n\binom{n}{n}\right)} \qquad \text{[By (1.5)]}$$
$$= 1.$$ □

Proof. By using the symbol $f(z)$ instead of z, likewise, we prove propositions 3.6, 3.7, 3.8, 3.9 and 3.10.
□

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المتطابقات الأسية الجديدة

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المستخلص :

تحصلنا على متطابقات أسية جديدة اثبتناها بعشر مبرهنات أصلية.

كلمات مفتاحية: متطابقات، مثلث باسكال، معاملات ذات الحدين.

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On the NBC-Bases of product hypersolvable arrangements

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Abstract

This paper aims centered around the product of hypersolvable arrangements by using the hypersolvable partition analogue by proving that each of A and B are hypersolvable if $A\times B$ is a hypersolvable, also each of A and B are supersolvable if $A\times B$ is supersolvable. Moreover, this paper show how to prove that the dimension of the first non-vanishing higher homotopy groups of the complement $M(A\times B)$ is $p(A\times B)=\min\{p(A),p(B)\}$.

Keyword: Hypersolvable**,** Supersolvable**,** Hypersolvable arrangements

List of symbols: *rk(A)* = rank of *A*, *T(A)* = maximal element of central *A*, *M(A)*= complement of A.

Mathematics subject classification: 55Q20

1. Introduction:

Let $A=\{H_1,\ldots,H_n\}$ be a complex hyperplane r-arrangement, with complement $M(A)=C^r \setminus \bigcup_{i=1}^{n} H_i$. The cohomology ring for the complement $M(A)$, with arbitrary constant coefficients was given by Arnold [1] and Brieskorn [2].

For a given total order $\trianglelefteq$ on A, if $C \subseteq A$ is a minimal (with respect to inclusion) dependent set, we call C a circuit of A and $\overline{C}=C\setminus\{H\}$ a broken circuit of C, where H is the smallest hyperplane in C via $\trianglelefteq$ and by NBC base $B \subseteq A$ we mean that B contains no broken circuit.

The hypersolvable class of hyperplane arrangements were originally introduced by M. Jambu and S. Papadima [3,4], as a combinatorial generalization of the supersolvable class of hyperplane arrangements and they showed that all the major results on the topology of the complements together with their algebraic and combinatorial aspects, may be extended and refined in this new framework. The hypersolvable class of hyperplane arrangements contains the supersolvable ones, the generic ones and many others.
We used the hypersolvable partition, the hypersolvable ordering which are defined by Ali and Al-Ta'ai [5], and their study of the NBC bases of a hypersolvable arrangement to complete the study of the product of hypersolvable arrangements which is studied by Mahdi in [7] he suggested a conjecture , namely , if $(A\times B, V\oplus W)$ is a hypersolvable arrangement , then A and B are hypersolvable arrangements . This conjecture is proved under some condition, namely, all the exponents of $A\times B$ are equal to 1. In section three we prove this conjecture without any condition, also we prove that the dimension of the first non-vanishing higher homotopy groups for complement $M(A\times B)$ is $p(A\times B)=\min\{p(A),p(B)\}$.

2. A hypersolvable partition of an arrangement

A hypersolvable class of arrangements was originally introduced by Jambu and Papadima ([3], [4]) Ali and Al-Ta'ai redefine this concept by using a partition which is called a hypersolvable partition as follows:

(2.1)Definition: [5]

Let A be an essential central complex r-arrangement(i.e. $\bigcap_{i=1}^{n} H_i = T(A) = T \neq \phi$ and $\mathrm{rk}(A)=\mathrm{rk}(T(A))=\mathrm{co}\dim(\bigcap_{H\in A} H = r = \dim(C^r)$). A partition $\Pi=(\Pi_1,\ldots,\Pi_\ell)$ of A is said to be a *hypersolvable partition* of A with *length* $\ell(A)=\ell$ denoted by Hp, if $|\Pi_1|=1$, (i.e. Π_1 is a singleton), and for fixed $2\le j\le \ell$, the block Π_j satisfies the following properties:

(*j-closed property of* Π_j) For each $H_1,H_2\in \Pi_1\cup\cdots\cup\Pi_j$, there is no hyperplane $H\in \Pi_{j+1}\cup\cdots\cup\Pi_\ell$ such that $\mathrm{rk}(H_1,H_2,H)=2$.

(*j-complete property of* Π_j) For each $H_1,H_2 \in \Pi_j$, there is a hyperplane $H \in \Pi_1 \cup \cdots \cup \Pi_{j-1}$ such that $\mathrm{rk}(H_1,H_2,H)=2$. Note that, from the closed properties of the blocks $\Pi_2,\ldots,\Pi_{j-1}$, the hyperplane H is unique and it is denoted in this case by $H_{1,2}$.

(*j-solvable property of* Π_j) If $H_1,H_2,H_3 \in \Pi_j$, the hyperplanes $H_{1,2},H_{1,3},H_{2,3} \in \Pi_1 \cup \cdots \cup \Pi_{j-1}$ are equal or $\mathrm{rk}(H_{1,2},H_{1,3},H_{2,3})=2$. Observe that, if $\mathrm{rk}(H_1,H_2,H_3)=2$, then from the closed properties of the blocks $\Pi_2,\ldots,\Pi_{j-1}$, we have $H_{1,2}=H_{1,3}=H_{2,3}$.

The vector of integers $d=(d_1,\ldots,d_\ell)$, is called the *exponent vector* of Π, where $d_i=|\Pi_i|$, $i=1,\ldots,\ell$. The *rank* of Π_i is defined to be $\mathrm{rk}(\Pi_i)=\mathrm{rk}(\Pi_1 \cup \cdots \cup \Pi_i)=\mathrm{rk}(\bigcap_{H\in\Pi_1\cup\cdots\cup\Pi_i} H)$, for $1 \le i \le \ell$. We call the block Π_i, a *singular block* of Π if $\mathrm{rk}(\Pi_i)=\mathrm{rk}(\Pi_{i-1})$ and we call it *non-singular block* otherwise. Notice that, in general $\mathrm{rk}(\Pi_i) \le \mathrm{rk}(\Pi_{i-1})+1$.

(2.2) Proposition: [6]

Let A be an essential central complex $r-$arrangement. A is hypersolvable if, and only if, A has a Hp $\Pi=(\Pi_1,\ldots,\Pi_\ell)$.

(2.3) Definition:[5]

Let A be a hypersolvable $r-$arrangement with Hp $\Pi=(\Pi_1,\ldots,\Pi_\ell)$. For a fixed $1 \le j \le \ell$, the properties of the hypersolvable partition give rise to a natural partition Π_j as follows:

1- Let $\Pi_{j*1}=\{H_{i_1},\ldots,H_{i_k}\}$ such that $rk(H_{i_1},\ldots,H_{i_k})=2$ and

2- Let $\Pi_{j*2}=\Pi_j \setminus \Pi_{j*1}$.

Define the *hypersolvable ordering* of A that is denoted by $\trianglelefteq$ as follows:

1- $H \in \Pi_i$ and $H' \in \Pi_j$ such that $1 \le i < j \le \ell$, put $H \trianglelefteq H'$.

2- For a fixed $1 < j \le \ell$, give the hyperplanes of the block Π_{j*1} *of* Π_j an arbitrary total order with preserving the order of Π_i in Π for each $1 \le i \le j-1$ and preserving the order of Π_{j*2} as if $H_1,H_2,H_3 \in \Pi_j$ with $rk(H_1,H_2,H_3)=3$, put $H_{i_1} \trianglelefteq H_{i_2} \trianglelefteq H_{i_3}$ if, and only if, $H_{i_1,i_2} \trianglelefteq H_{i_2,i_3} \trianglelefteq H_{i_1,i_3}$ such that $\{H_{i_1},H_{i_2},H_{i_3}\}=\{H_1,H_2,H_3\}$. Observe that, since $rk(H_1,H_2,H_3)=3$ then there is at least one of $H_1,H_2,H_3 \in \Pi_{j*2}$.

(2.4) proposition:[3]

Let A be a hypersolvable arrangement. Then A is said to be *supersolvable* if, and only if, $\ell(A)=rk(A)$.

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3. The Product of Hypersolvable Arrangement

(3.1) Definition:

Let (A,V) and (B,W) be two hyperplane arrangements. Define the product $(A\times B, V\oplus W)$ by $A\times B=\{H\oplus W : H\in A\}\cup\{V\oplus K : K\in B\}$.

Note that, $|A\times B|=|A|+|B|$. If we denote the sets $\{H\oplus W : H\in A\}$ and $\{V\oplus K : K\in B\}$ by $A\oplus W$ and $V\oplus B$ respectively, then one can easily denote the hyperplane arrangement $A\times B$ by $A\times B=(A\oplus W)\cup(V\oplus B)$.

(3.2) Proposition: [7]

Let $(A\times B, V\oplus W)$ be the product of (A,V) and (B,W) such that, $rk(A)=r$ and $rk(B)=k$. Then we have the following:

1. If each one of A and B is a hypersolvable arrangement, then $(A\oplus W, V\oplus W)$, $(V\oplus B, V\oplus W)$ and $(A\times B, V\oplus W)$ are hypersolvable arrangements.
2. If each one of A and B is a supersolvable arrangement, then $(A\oplus W, V\oplus W)$, $(V\oplus B, V\oplus W)$ and $(A\times B, V\oplus W)$ are supersolvable arrangements .

(3.3) Remark:

Suppose (A,V) and (B,W) be hypersolvable arrangements with hypersolvable partitions say; $\Pi^A=(\Pi_1^A,\ldots,\Pi_{\ell_1}^A)$ and $\Pi^B=(\Pi_1^B,\ldots,\Pi_{\ell_2}^B)$ respectively. From [7], then $(A\times B, V\oplus W)$ is a hypersolvable arrangement with a hypersolvable composition series ;

$$\Pi_1^A\oplus W\subseteq(\Pi_1^A\cup\Pi_2^A)\oplus W\subseteq\ldots$$
$$\subseteq(\Pi_1^A\cup\ldots\cup\Pi_{\ell_1}^A)\oplus W=(A\oplus W)$$
$$\subseteq(A\oplus W)\cup V\oplus\Pi_1^B\subseteq$$
$$\ldots\subseteq(A\oplus W)\cup V\oplus(\Pi_1^B\cup\ldots\cup\Pi_{\ell_2}^B)$$
$$=(A\oplus W)\cup(V\oplus B)\ldots\ldots(3.1)$$

From [5], $A\times B$ has a hypersolvable partition $\Pi^{A\times B}=(\Pi_1^{A\times B},\ldots,\Pi_{\ell_1+\ell_2}^{A\times B})$ induced from the composition series (3.1), as follows:

- For $1\le k\le \ell_1$; $\Pi_k^{A\times B}=\Pi_k^A\oplus W$ and;
- For $\ell_1+1\le k\le \ell_1+\ell_2$; $\Pi_k^{A\times B}=V\oplus\Pi_{k-\ell_1}^B$.

Ali in [5] showed that such partition forms a hypersolvable partition.

(3.4)Remark:[7]

There are no collinear relations among the hyperplanes of $A\oplus W$ and $V\oplus B$. Thus, for each $H_1,H_2\in A$, there is no hyperplane $K\in B$ such that $rk\{H_1\oplus W, H_2\oplus W, V\oplus K\}=2$ and for each $K_1,K_2\in B$, there is no $H\in A$ such that $rk\{H\oplus W, V\oplus K_1, V\oplus K_2\}=2$.

(3.5) Lemma:

Every broken circuit C in $A \oplus W$ has the following property; there is no hyperplane K in B such that $C \bigcup \{V \oplus K\}$ forms a circuit in $A \times B$. As well as, for any broken circuit C' in $V \oplus B$, there is no hyperplane H in A such that $C' \bigcup \{H \oplus W\}$ forms a circuit in $A \times B$. Thus,

$$NBC(A \oplus W) \bigcap NBC(V \oplus B) = \phi.$$

Proof: directly result of proposition (2.4) and remark (3.3).

(3.6) Proposition :

Let $A \times B$ be a hypersolvable $r+k-$ arrangement. Then;

$$NBC(A \oplus W) \subseteq NBC(A \times B) \text{ and}$$
$$NBC(V \oplus B) \subseteq NBC(A \times B).$$

Proof: By contrary, for $1 \le k \le r$, let $S_k = \{H_{i_1} \oplus W, \ldots, H_{i_k} \oplus W\}$ be a k-section of $\prod^{A \times B}$, such that $S_k \in NBC(A \oplus W)$ and $S_k \notin NBC(A \times B)$. Then S_k be a broken circuit in $A \times B$. That is, there exists a hyperplane $H' \in A \times B$ such that $H' \trianglerighteq H_{i_j} \oplus W$, $1 \le j \le k$ and $\{H'\} \bigcup S_k$ form a circuit, i.e. $rk\{H' \bigcup S_k\} = k$. It is clear that, $H' \notin A \oplus W$, since $S_k \in NBC(A \oplus W)$. On the other hand, $H' \notin V \oplus B$ as shown in lemma (3.5) above. Therefore, S_k must be an NBC base of $A \times B$.

Similarly, it is easy to show that $NBC(V \oplus B) \subseteq NBC(A \times B)$.

(3.7) Theorem*:*

Let $A \times B$ be a hypersolvable $r+k-$ arrangement then;

$$NBC(A \times B) = \{C \in A \times B \mid C = C_1 \bigcup C_2$$
$$: C_1 \in NBC(A \oplus W) \text{ and } C_2 \in NBC(V \oplus B)\}$$

Proof: By contrary, suppose that $C \in NBC(A \times B)$, such that C cannot be written as a union of an NBC base of $A \oplus W$ and NBC base of $V \oplus B$, i.e. either;

$$C \bigcap (A \oplus W) \notin NBC(A \oplus W) \text{ or}$$
$$C \bigcap (V \oplus B) \notin NBC(V \oplus B).$$

If $C \bigcap (A \oplus W) \notin NBC(A \oplus W)$, then there exists a hyperplane $H' \in A \times B$ such that $H' \bigcup \{C \bigcap (A \oplus W)\}$ forms a circuit in $A \times B$. But this contradicts our assumption that $C \in NBC(A \times B)$. By the same way, we deduce that $C \bigcap (V \oplus B) \notin NBC(V \oplus B)$.

(3.8) Corollary :

Let $A \times B$ be a hypersolvable $r+k-$ arrangement then $p(A) = p(A \oplus W)$ and $p(B) = p(V \oplus B)$.

(3.9) Theorem :

Let $A \times B$ be a hypersolvable $r+k-$ arrangement then

$$p(A \times B) = \min\{p(A), p(B)\}.$$

Proof: In general, deduce that $p(A \times B) \le p(A)$ and $p(A \times B) \le p(B)$. So by contrary suppose

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that, $p(A\times B)<\min\{p(A),p(B)\}$. So suppose that, there exists a section $S\in S_{p(A\times B)+1}$ such that S is a $(p(A\times B)+1)$-broken circuit and from our construction of $\prod^{A\times B}$ then $S=S^{A\oplus W}\bigcup S^{V\oplus B}$ where $S^{A\oplus W}=S\bigcap A\oplus W$ and $S^{V\oplus B}=S\bigcap V\oplus B$. It is clear that $S^{A\oplus W}\in NBC(A\oplus W)$ and $S^{V\oplus B}\in NBC(V\oplus B)$ since $p(A\times B)+1<\min\{p(A)+1,p(B)+1\}$.

Now, let H be the minimal hyperplane of $A\times B$ such that $\{H\}\bigcup S$ forms a $(p(A\times B)+1)$-circuit. If $S^{A\oplus W}\neq\phi$, then H minimal than H' via the hypersolvable ordering $\trianglerighteq$ on the hyperplanes of $A\times B$, for each $H'\in S^{A\oplus W}$. Thus, $\{H\}\bigcup S^{A\oplus W}$ is a circuit and this contradicts the fact that $S^{A\oplus W}$ is an NBC base of $A\oplus W$. On the other hand, if $S^{A\oplus W}=\phi$ then $S=S^{V\oplus B}$. That is, the hyperplane H minimal than K via hypesovable ordering $\trianglerighteq$ for each $K\in S^{V\oplus B}$, thus $\{H\}\bigcup S^{V\oplus B}$ is a circuit which contradicts that $S^{V\oplus B}\in NBC(V\oplus B)$. This ends the proof.

(3.10)Theorem:

If $A\times B$ be a hypersolvable $r+k-$ arrangement, then each of $A\oplus W$ and $V\oplus B$ are hypersolvable.

Proof: Since $A\times B$ be a hypersolvable $r+k-$arrangement, hence $A\times B$ has an Hp, $\prod^{A\times B}=(\prod_1,\ldots,\prod_\ell)$. From lemma (3.4), the partition $\prod^{A\times B}$ splits into two partitions as follows:

- Let $\prod_i^A=\prod_{j_i}^{A\times B}\subseteq A\oplus W$, for $1\leq i\leq \ell_1, 1\leq j_1<j_2<\ldots<j_{\ell_1}\leq\ell$ and;
- $\prod_i^B=\prod_{j_i}^{A\times B}\subseteq V\oplus B$, for $1\leq i\leq \ell_2, 1\leq j_1<j_2<\ldots<j_{\ell_2}\leq\ell$; where $\ell_1+\ell_2=\ell$.

Deduce that $\prod^A=(\prod_1^A,\ldots,\prod_{\ell_1}^A)$ form a partition of $A\oplus W$. We need to show that $\prod^A$ is a hypersolvable partition as follows:

1. If $\prod_1^A$ contains two hyperplanes say $H_1\oplus W$ and $H_2\oplus W$, then there exists a hyperplane $H\in\prod_1^{A\times B}\bigcup\ldots\bigcup\prod_{j_{n-1}}^{A\times B}$ such that $rk\{H_1\oplus W,H_2\oplus W,H\}=2$, from the complete property of block $\prod_{j_1}^{A\times B}$. Therefore, $H\in A\oplus W$, see lemma(3.4). But this contradicts our assumption that $\prod_{j_1}^{A\times B}$ is the first block of $\prod^{A\times B}$ such that $\prod_{j_1}^{A\times B}\subseteq A\oplus W$. Thus, $\left|\prod_1^A\right|=1$.
2. For $2\leq k\leq \ell_1$; it is clear that the block $\prod_k^A$ satisfies the closed , complete and solvable properties since it is a block from an Hp. Thus $A\oplus W$ is hypersolvable since it is has an Hp. In the same way $V\oplus B$ is a hypersolvable.

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(3.11) Corollary :

The product $r+k$-arrangement $A\times B$ is hypersolvable if, and only if, each of A and B are hypersolvable.

Proof: It is known that, if A and B are hypersolvable, then $A\times B$ is hypersolvable (see [7]). Conversely, If $A\times B$ is a hypersolvable arrangement, the canonical projections $q_A : V\times W \to V$ defined as $q_A(H\oplus W)=H$ and $q_B : W\times V \to V$ defined by $q_B(V\oplus K)=K$ preserve the dependent and independent relations. Therefore, each one of A and B are hypersolvable arrangements.

(3.12) Corollary:

$A\times B$ is supersolvable if, and only if, each of A and B is supersolvable.

Proof: It is known that, if A and B are supersolvale, then $A\times B$ is supersolvable (see [7]). Conversely if $A\times B$ is supersolvable then $\ell(A\times B)=\ell=r+k$ where $r=rk(A)$ and $k=rk(B)$, since $\ell_1 \geq rk(A)=r$ and $\ell_2 \geq rk(B)=k$, then $\ell=\ell_1+\ell_2 \geq r+k$, but $\ell=r+k$ which means ℓ_1 and ℓ_2 cannot be greater than r and k respectively. Hence, each of A and B is supersolvable.

(3.13) Example:

Let A be central complex 6-arrangements, define as follows:

$$Q(A) = x_2x_3(x_1-x_3)(x_1+x_3)(x_2-x_1)(x_2+x_1)(x_5+3x_6)(x_5+2x_6)x_5(x_5-x_6)x_6(x_4+x_5+x_6)(x_5-x_4+x_6)$$

A is a hypersolvable arrangement in C^6 since we can find a hypersolvable Hp as follows:

$$\Pi^A=(\Pi_1,\Pi_2,\Pi_3,\Pi_4,\Pi_5,\Pi_6,\Pi_7,\Pi_8) =$$

$$(\{H_1\},\{H_2,H_3\},\{H_4\},\{H_5H_6\},\{H_7\},\{H_8\},\{H_9\},\{H_{10}\},\{H_{11},H_{12},H_{13}\})$$ where

$$H_1=\{(x_1,x_2,x_3,x_4,x_5,x_6): x_1+x_3=0\}$$

$$H_2=\{(x_1,x_2,x_3,x_4,x_5,x_6): x_1-x_3=0\}$$

$$H_3=\{(x_1,x_2,x_3,x_4,x_5,x_6): x_1+x_2=0\}$$

$$H_4=\{(x_1,x_2,x_3,x_4,x_5,x_6): x_2=0\}$$

$$H_5=\{(x_1,x_2,x_3,x_4,x_5,x_6): x_2-x_1=0\}$$

$$H_6=\{(x_1,x_2,x_3,x_4,x_5,x_6): x_3=0\}$$

$$H_7=\{(x_1,x_2,x_3,x_4,x_5,x_6): x_5+3x_6=0\}$$

$$H_8=\{(x_1,x_2,x_3,x_4,x_5,x_6): x_5+2x_6=0\}$$

$$H_9=\{(x_1,x_2,x_3,x_4,x_5,x_6): x_5=0\}$$

$$H_{10}=\{(x_1,x_2,x_3,x_4,x_5,x_6): x_5-x_6=0\}$$

$$H_{11}=\{(x_1,x_2,x_3,x_4,x_5,x_6): x_6=0\}$$

$$H_{12}=\{(x_1,x_2,x_3,x_4,x_5,x_6): x_5-x_4+x_6=0\}$$

$$H_{13}=\{(x_1,x_2,x_3,x_4,x_5,x_6): x_4+x_5+x_6=0\}$$

From new hypersolvable ordering we rewrite a defining polynomial as

$$Q(A) = (x_1+x_3)(x_1-x_3)(x_1+x_2)x_2(x_2-x_1)x_3(x_5+3x_6)(x_5+2x_6)x_5(x_5-x_6)x_6(x_5-x_4+x_6)(x_4+x_5+x_6).$$

Note that by applying our construction we can split A into two arrangements A_1 and A_2 where:

$Q(A_1) = (x_1+x_3)(x_1-x_3)(x_1+x_2)x_2$
$(x_2-x_1)x_3$

and

$Q(A_2) = (x_5+3x_6)(x_5+2x_6)x_5$
$(x_5-x_6)x_6(x_5-x_4+x_6)(x_4+x_5+x_6).$

Observe that both of A_1 and A_2 are hypersolvable 3-arrangements since they have Hp as follows:

$\Pi^{A_1} = (\Pi_1,\Pi_2,\Pi_3,\Pi_4) = (\{H_1\},$
$\{H_2,H_3\},\{H_4\},\{H_5,H_6\})$

$\Pi^{A_2} = (\Pi_1,\Pi_2,\Pi_3,\Pi_4) = (\{K_1\},$
$\{K_2\},\{K_3\},\{K_4\},\{K_5,K_6,K_7\})$

Where;

$$H_1 = \{(x_1,x_2,x_3): x_1+x_3=0\}$$
$$H_2 = \{((x_1,x_2,x_3)): x_1-x_3=0\}$$
$$H_3 = \{(x_1,x_2,x_3): x_1+x_2=0\}$$
$$H_4 = \{(x_1,x_2,x_3): x_2=0\}$$
$$H_5 = \{(x_1,x_2,x_3): x_2-x_1=0\}$$
$$H_6 = \{(x_1,x_2,x_3): x_3=0\}$$
$$K_1 = \{(x_1,x_2,x_3): x_2+3x_3=0\}$$
$$K_2 = \{(x_1,x_2,x_3): x_2+2x_3=0\}$$
$$K_3 = \{(x_1,x_2,x_3): x_2=0\}$$
$$K_4 = \{(x_1,x_2,x_3): x_2-x_3=0\}$$
$$K_5 = \{(x_1,x_2,x_3): x_3=0\}$$
$$K_6 = \{(x_1,x_2,x_3): x_2-x_1+x_3=0\}$$
$$K_7 = \{(x_1,x_2,x_3): x_1+x_2+x_3=0\}$$

(3.14) Example :

Let A be central complex 6-arrangements, define as follows:

$Q(A) = x_2x_1(x_1+x_2)x_3(x_2-x_3)x_4x_5x_6$
$(x_4-x_5)(x_4+x_5)(x_6-x_5)(x_6+x_5)$

A is a hypersolvable arrangement in C^6 since we can find a hypersolvable Hp as follows:

$\Pi^A = (\Pi_1,\Pi_2,\Pi_3,\Pi_4,\Pi_5,\Pi_6) =$
$(\{H_1\},\{H_2,H_3\},\{H_4,H_5\},\{H_6\},\{H_7,$
$H_8,H_9\},\{H_{10},H_{11},H_{12}\})$ where

$$H_1 = \{(x_1,x_2,x_3,x_4,x_5,x_6): x_2=0\}$$
$$H_2 = \{(x_1,x_2,x_3,x_4,x_5,x_6): x_1=0\}$$
$$H_3 = \{(x_1,x_2,x_3,x_4,x_5,x_6): x_1+x_2=0\}$$
$$H_4 = \{(x_1,x_2,x_3,x_4,x_5,x_6): x_3=0\}$$
$$H_5 = \{(x_1,x_2,x_3,x_4,x_5,x_6): x_2-x_3=0\}$$
$$H_6 = \{(x_1,x_2,x_3,x_4,x_5,x_6): x_5=0\}$$
$$H_7 = \{(x_1,x_2,x_3,x_4,x_5,x_6): x_4=0\}$$
$$H_8 = \{(x_1,x_2,x_3,x_4,x_5,x_6): x_6=0\}$$
$$H_9 = \{(x_1,x_2,x_3,x_4,x_5,x_6): x_4-x_5=0\}$$
$$H_{10} = \{(x_1,x_2,x_3,x_4,x_5,x_6): x_4+x_5=0\}$$
$$H_{11} = \{(x_1,x_2,x_3,x_4,x_5,x_6): x_6-x_5=0\}$$
$$H_{12} = \{(x_1,x_2,x_3,x_4,x_5,x_6): x_6+x_5=0\}$$

Note that A is supersolvable arrangement since $\ell(A) = rk(A) = 6$

From new hypersolavble ordering we rewrite the defining polynomial of A as follow:

$Q(A) = x_2x_1(x_1+x_2)x_3(x_2-x_3)x_4$
$x_5(x_4-x_5)(x_4+x_5)x_6(x_6-x_5)(x_6+x_5)$

Note that by applying our construction we can split A into two 3-arrangements A_1 and A_2 where:

$Q(A_1) = x_2x_1(x_1+x_2)x_3(x_2-x_3)$ and

$Q(A_2) = x_2x_1(x_1-x_2)(x_1+x_2)x_3$
$(x_3-x_2)(x_3+x_2)$

Observe that both of A_1 and A_2 are supersolvable arrangements since they have Hp as follows:

$\prod^{A_1} = (\prod_1, \prod_2, \prod_3) = (\{H_1\},\{H_2,H_3\},$
$\{H_4,H_5\})$

$\prod^{A_2} = (\prod_1, \prod_2, \prod_3, \prod_4) = (\{K_1\},\{K_2,$
$K_3,K_4\},\{K_5,K_6,K_7\})$.

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(حول قواعد-NBC لضرب الترتيبة القابلة للحل الفوقية)

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المستخلص :

الهدف من هذا البحث يتمركز حول دراسه ضرب الترتيبه القابله للحل فوقيا التي تم دراستها باستخدام مفهوم تجزئه الترتيبه القابله للحل فوقيا ففي هذا البحث تمكنا من برهان انه اذا كان $A\times B$ ترتيبه قابله للحل فوقيا فأن كل من A وB ترتيبه قابله للحل فوقيا. كذلك في هذا البحث تطرقنا الى برهان اذا كان $A\times B$ ترتيبه السوبر القابله للحل فأن كل من A وB تكون ترتيبه سوبر قابله للحل وايضا كيفية برهان ان بعد اول زمرة غير متلاشية الاعلى هوموتوبي لمتممة $M(A\times B)$ تكون $p(A\times B)=\min\{p(A),p(B)\}$.

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Some Properties of Random Fixed Points and Random Periodic Points for Random Dynamical Systems

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Abstract. Our aim in this paper is to give some properties for random fixed points for random dynamical systems where we give the characteristic of random fixed points in terms of the random trajectory emanating from random variable and give. Also, the concept of random periodic points for random dynamical systems is studied where the sufficiently and necessarily conditions which make the random variable random periodic point for random dynamical systems. Also the authors prove that the set of all (continuous) random periodic points under certain conditions is $\mathbb{P}$ −closed.

Key Words: Random dynamical system; random fixed point; random periodic point.

Mathematics subject classification:37HXX.

1-Introduction. In common one can not anticipate that one point $x \in X$ is fixed by (almost) all mappings $\varphi(t, \omega)$. Though, there is a suitable generalization of the idea of a fixed point. In this paper some new properties of random fixed point and random periodic point for RDS's are considered.

L. Arnold and I.D. Chueshov [1] (1998) presented the general notion of an order-preserving random dynamical system, gave several examples and studied the properties of their random equilibria and attractor.

Gunter Ochs and Valery. Oseledets [2] (1999) establish that topological fixed point theorems have no canonical generalization to the case of random dynamical systems. This is prepared by exhausting implements from algebraic ergodic theory. They provide a condition for the existence of invariant probability measures for group valued cocycles. With that, examples of continuous random dynamical systems on a compact interval without random invariant points, which are an suitable generalization of fixed points, are created.

H.E. Kunze D. La Torreb and E.R. Vrscay[3](2007) they absorbed in the direct and inverse problems for certain class of random fixed point equations.

Chuanxi Zhu and Chunfang Chen[4](2008),they prove an essential inequality and inspect some new computing problems of random fixed point index.

"Ismat Beg and Mujahid Abbas[5](2008) they prove the existence of random fixed points of a non-expansive random operator defined on an unbounded subset of a Banach space".

In this paper some new properties of random fixed points and periodic random points for random dynamical system are introduced and proved. Here the time space considered any locally compact space and the phase space is any metric space. Also some new concepts are introduced here such as Topological metric dynamical system, $\mathbb{P}-$uniform converge and $\mathbb{P}-$closed set.

Through this paper the following notation are used.

Notations 1.1

(i) $\mathbb{G}$ =locally compact group.

(ii) $\mathbb{X}$=metric space.

(iii) $(\Omega, \mathbb{F}, \mathbb{P})$ is a probability space.

(iv) $\mathbb{X}_{\mathcal{B}}^{\Omega}$ = the set of all measurable functions from Ω to $\mathbb{X}$.

Definition 1.2[6,7]: The ***metric dynamical system*** (MDS) is the 5-tuple $(\mathbb{G}, \Omega, \mathbb{F}, \mathbb{P}, \theta)$ where $(\Omega, \mathbb{F}, \mathbb{P})$ is a probability space and $\theta: G \times \Omega \to \Omega$ is $(\beta(\mathbb{G}) \otimes \mathbb{F}, \mathbb{F})$ −measurable, with

(i) $\theta(e, \omega) = Id_{\Omega}$, (the identity function on Ω)

(ii) $\theta(g * h, \omega) = \theta(g, \theta(h, \omega))$ and

(iii) $\mathbb{P}(\theta_g F) = \mathbb{P}(F)$, $\forall F \in \mathbb{F}\ \forall g \in G$.

Definition 1.3 The MDS $(\mathbb{G}, \Omega, \mathbb{F}, \mathbb{P}, \theta)$ is said to be topological metric dynamical system (TMDS) if Ω is topological space and $\theta: G \times \Omega \to \Omega$ is continuous.

Definition 1.4[6,7,8] The mapping $\varphi: \mathbb{G} \times \Omega \times \mathbb{X} \to \mathbb{X}$ is said to be measurable random dynamical system on the measurable space $(\mathbb{X}, \beta(\mathbb{X}))$ over an MDS $(\mathbb{G}, \Omega, \mathbb{F}, \mathbb{P}, \theta)$ with if it has the following properties:

(i) φ is $\beta(\mathbb{G})\otimes\mathbb{F}\otimes\beta(\mathbb{X}), \beta(\mathbb{X})$ − measurable.

(ii) The mappings $\varphi(t, \omega) \coloneqq \varphi(g, \omega, \cdot): \mathbb{X} \to \mathbb{X}$ form a cocycle over $\theta(\cdot)$, that is, $\forall\ g, h \in \mathbb{G}, \omega \in \Omega$ they satisfy

$\varphi(e,\omega) = id_X\ \forall\omega \in \Omega$, (1.1)

$\varphi(g * h,\omega) = \varphi(g,\theta_h\omega) \circ \varphi(h,\omega)$ (1.2)

The RDS $(\mathbb{G},\Omega,\mathbb{X},\theta,\varphi)$ shall denote by (θ,φ).

If the function $\varphi(\cdot,\omega,\cdot): \mathbb{T} \times \mathbb{X} \to \mathbb{X}$, $(t,x) \mapsto \varphi(t,\omega,x)$, is continuous for every $\omega \in \Omega$ then the measurable dynamical system is called continuous or topological RDS.

Definition 1.5 [6,7,8]: A measurable function $v \in X_B^\Omega$ is said to be a ***random fixed point*** (R.F.P) for the RDS (θ,φ) if ,$\forall g \in G$,

$\mathbb{P}\{\omega: \varphi(g,\omega)v(\omega) = v(\theta_g\omega)\} = 1$.

Here some examples on R.F.P are stated (see[9]).

Example 1.6[9] Consider the probability space $(\Omega,\mathcal{F},\mathbb{P})$, where $\Omega \coloneqq [0,1]$, $\mathcal{F}$ be the $\sigma-$algebra of Lebesgue measurable sets and $\mathbb{P}$ be the Lebesgue measurer on Ω. Define $\theta: \mathbb{Z}_2 \times [0,1] \to [0,1]$ by $\theta(0,\omega) = \omega$ and $\theta(1,\omega) = 1-\omega$. Also define $\varphi: \mathbb{Z}_2 \times [0,1] \times \mathbb{R} \to \mathbb{R}$ by $\varphi(0,\omega)x = x$ and $\varphi(1,\omega)x = (1-2\omega) + x$. Then (θ,φ) is RDS. Define $\zeta: [0,1] \to \mathbb{R}$ defined by$(\omega) \coloneqq \omega^2$, then $\zeta \in \mathbb{R}_B^{[0,1]}$. It is easy to see that ζ is a fixed point of (θ,φ).■

Example1.7[9]: Let $\theta: \mathbb{R} \times \Omega \to \Omega$ be any non-trivial MDS and let $\eta: \Omega \to \mathbb{R}$ be any injective random variable. Define a cocycle $\varphi: \mathbb{R} \times \Omega \times \mathbb{R} \to \mathbb{R}$ over θ by $\varphi(t,\omega)x \coloneqq xe^{-\eta(\omega)}e^{\eta(\theta(t)\omega)}$. Then (θ,φ) is RDS. This RDS has no random fixed point.■

Definition 1.8[7] Let $v \in \mathbb{X}_B^\Omega$ and γ_v, γ_v^+ and γ_v^- be the mappings form X in to 2^X defined as follows

(1) $\gamma_v(\omega) = \{\varphi(t,\theta_{-t}\omega)v(\theta_{-t}\omega): t \in \mathbb{R}\}$

(2) $\gamma_v^+(\omega) = \{\varphi(t,\theta_{-t}\omega)v(\theta_{-t}\omega): t \in \mathbb{R}^+\}$

(3) $\gamma_v^-(\omega) = \{\varphi(t,\theta_{-t}\omega)v(\theta_{-t}\omega): t \in \mathbb{R}^-\}$

For every $v \in X_B^\Omega$, the sets γ_v, γ_v^+, and γ_v^- are respectively called the trajectory, the forward semi-trajectory and backward semi-trajectory.

2.Main Results

In this section the concept of random fixed point is stated and some new properties of such concept are studied. Also the concept of random periodic point for random dynamical systems is introduced and some new properties are given.

Lemma 2.1 If $v \in \mathbb{X}_B^\Omega$ and $\mathbb{P}\{\omega: \varphi(t,\omega)v(\omega) = v(\theta_t\omega)\} = 1$ for some $t \in \mathbb{R}$, then $\mathbb{P}\{\omega: \varphi(nt,\omega)v(\omega) = v(\theta_{nt}\omega)\} = 1$ for all integer n.

Proof. If $\mathbb{P}\{\omega: \varphi(t,\omega)v(\omega) = v(\theta_t\omega)\} = 1$, for some $t \in \mathbb{R}$, then

$\mathbb{P}\{\omega: \varphi(t,\omega)^{-1} \circ \varphi(t,\omega)v(\omega) =$
$\varphi(t,\omega)^{-1}v(\theta_t\omega)\} = 1$.

Hence

$\mathbb{P}\{\omega: v(\omega) = \varphi(-t,\theta_t\omega)v(\theta_t\omega)\} = 1$. Therefore we shall prove by induction the result for positive integers lone. If $n = 1$, then

$\mathbb{P}\{\omega: \varphi(t,\omega)v(\omega) = v(\theta_t\omega)\} = 1$

for some $t \in \mathbb{R}$.

Now, suppose that the statement is true for n. i.e., $\mathbb{P}\{\omega: \varphi(nt,\omega)v(\omega) = v(\theta_{nt}\omega)\} = 1$,

for some $t \in \mathbb{R}$.

To show that this statement true for $n+1$. Set

$\widetilde{\Omega} \coloneqq \{\omega: \varphi(nt,\omega)v(\omega) = v(\theta_{nt}\omega)\}$.

For $\omega \in \widetilde{\Omega}$,

$$\begin{aligned}
\varphi((n+1)t,\omega)v(\omega) &= \varphi(nt+t,\omega)v(\omega) \\
&= \varphi(nt,\theta_t\omega) \circ \varphi(t,\omega)v(\omega) \\
&= \varphi(nt,\theta_t\omega)v(\theta_t\omega) \\
&= \varphi(nt,\omega')v(\omega'), \quad \omega' \coloneqq \theta_t\omega \in \widetilde{\Omega} \\
&= v(\theta_{nt}\omega') = v(\theta_{nt}\theta_t\omega) \\
&= v(\theta_{(n+1)t}\omega)
\end{aligned}$$

Thus

$\mathbb{P}\{\omega: \varphi((n+1)t,\omega)v(\omega) = v(\theta_{(n+1)t}\omega)\} = 1$

This prove the lemma. ■

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Theorem 2.2 Let $v \in \mathbb{X}_{\mathcal{B}}^{\Omega}$. Then the following are equivalent:

1. v is random fixed point,
2. $\gamma_v(\omega) = \{v(\omega)\}$,
3. $\gamma_v^+(\omega) = \{v(\omega)\}$,
4. $\gamma_v^-(\omega) = \{v(\omega)\}$,

Proof.(1) ⟺ (2): Suppose(1) holds, then

$$\gamma_v(\omega) = \{\varphi(t, \theta_{-t}\omega)v(\theta_{-t}\omega): t \in \mathbb{R}\}$$
$$= \{\varphi(t, \omega')v(\omega'): t \in \mathbb{R}\},$$

where $\omega' \coloneqq \theta_{-t}\omega$

$$= \{v(\theta_t\omega'): t \in \mathbb{R}\}$$
$$= \{v(\theta_t\theta_{-t}\omega): t \in \mathbb{R}\} = \{v(\omega)\}.$$

Conversely, suppose (2) holds, then $\gamma_v(\omega) = \{v(\omega)\}$. But $\gamma_v(\omega) = \{\varphi(t, \theta_{-t}\omega)v(\theta_{-t}\omega): t \in \mathbb{R}\}$, then $\{\varphi(t, \theta_{-t}\omega)v(\theta_{-t}\omega): t \in \mathbb{R}\} = \{v(\omega)\}$ That is $\forall t \in \mathbb{R}$ and $\forall \omega \in \Omega$, $\varphi(t, \theta_{-t}\omega)v(\theta_{-t}\omega) = v(\omega)$. Thus $\forall t \in \mathbb{R}$ $\mathbb{P}\{\omega: \varphi(t, \theta_{-t}\omega)v(\theta_{-t}\omega) = v(\omega)\} = 1$. Set $\omega' \coloneqq \theta_{-t}\omega$. Then for every $\in \mathbb{R}$, $\varphi(t, \omega')v(\omega') = v(\theta_t\omega')$ and $\mathbb{P}\{\omega': \varphi(t, \omega')v(\omega') = v(\theta_t\omega')\} = 1$.

Consequently v is an R.F.P.

(2) ⟺(3). Suppose (2) holds. Since $\gamma_v(\omega) \supset \gamma_v^+(\omega) \neq \emptyset$, we conclude that $\gamma_v^+(\omega) = \{v(\omega)\}$. Conversely, suppose (3) holds. Then $\gamma_v^+(\omega) = \{v(\omega)\}$. That is, $\{\varphi(t, \theta_{-t}\omega)v(\theta_{-t}\omega): t \in \mathbb{R}^+\} = \{v(\omega)\}$. Then $\{\varphi(s, \theta_{-s}\omega)v(\theta_{-s}\omega): s \in \mathbb{R}^-\} = \{v(\omega)\}$, where $s = -t$. Thus $\{\varphi(t, \theta_{-t}\omega)v(\theta_{-t}\omega): t \in \mathbb{R}\} = \{v(\omega)\}$.

(2) ⟺ (4). As in (2) ⟺ (3).

This end the proof. ∎

Theorem 2.3 Let (φ, θ) be an RDS with θ considered as a topological MDS. $v \in \mathbb{X}_{\mathcal{B}}^{\Omega}$ is continuous, then the following are equivalent:

1. v is random fixed point,
2. There is a sequence $\{t_n\}$, $t_n > 0$, $t_n \longrightarrow 0$ with $\mathbb{P}\{\omega: \varphi(t_n, \omega)v(\omega) = v(\theta_{t_n}\omega)\} = 1$.

Proof To prove (1) ⟺(2). Assume (1). Since v is random fixed point, then for all $\in \mathbb{R}$,

$$\mathbb{P}\{\omega: \varphi(t, \omega)v(\omega) = v(\theta_t\omega)\} = 1.$$

Thus we can say that there exists a real sequence $\{t_n\}$, $t_n > 0$, $t_n \longrightarrow 0$ with

$$\mathbb{P}\{\omega: \varphi(t_n, \omega)v(\omega) = v(\theta_{t_n}\omega)\} = 1.$$

Conversely, assume (2) holds, let $t \in \mathbb{R}$. If $t = kt_n$ for some integers k and n, then by Lemma 2.1 we have

$$\mathbb{P}\{\omega: \varphi(kt_n, \omega)v(\omega) = v(\theta_{kt_n}\omega)\} = 1.$$

If $t \neq kt_n$, then for every n, there exists k_n such that $k_n t_n < t < (k_n + 1)t_n$ and moreover for any n there exist an integer $m > n$ with

$$k_n t_n < k_m t_m < t < (k_m + 1)t_m < (k_n + 1)t_n .$$

Thus clearly the so constructed sequence $\{k_n t_n\}$ has the property that $k_n t_n \longrightarrow t$. Now since $\varphi(\cdot, \omega, v(\omega)): \mathbb{R} \longrightarrow \mathbb{X}$ is continuous for every $\omega \in \Omega$, then

$$\varphi(k_n t_n, \omega)v(\omega) \longrightarrow \varphi(t, \omega)v(\omega),$$

for every $\omega \in \Omega$. Since

$$\mathbb{P}\{\varphi(k_n t_n, \omega)v(\omega) = v(\theta_{k_n t_n}\omega)\} = 1$$

for every n then for every $\omega \in \Omega$,

$$v(\theta_{k_n t_n}\omega) \longrightarrow \varphi(t, \omega)v(\omega).$$

Again, since $k_n t_n \longrightarrow t$ and $\theta(\cdot, \omega): \mathbb{R} \longrightarrow \Omega$ is continuous for every $\omega \in \Omega$, then $\theta_{k_n t_n}\omega \longrightarrow \theta_t\omega$, and since v is continuous, then $v(\theta_{k_n t_n}\omega) \longrightarrow v(\theta_t\omega)$ for every $\omega \in \Omega$. Thus $\varphi(t, \omega)v(\omega) = v(\theta_t\omega)$ for every $\omega \in \Omega$. This means that

$$\mathbb{P}\{\omega: \varphi(t, \omega)v(\omega) = v(\theta_t\omega)\} = 1$$

for all $t \in \mathbb{R}$. ∎

Note 2.4. The implication (1) ⟹(2) is true when θ is any MDS.

Definition 2.5 The set $\mathbb{X}_{\mathcal{B}}^{\Omega}$ is said to be distinguishable if for every $u, v \in \mathbb{X}_{\mathcal{B}}^{\Omega}$, there exist two random open sets U and V in $\mathbb{X}$ with $\mathfrak{D} := \{\omega : u(\omega) \neq v(\omega)\} \subseteq \Omega$ such that $\mathfrak{D} \neq \emptyset$, and for every $\omega \in \mathfrak{D}$ we have $u(\omega) \in U(\omega)$, $v(\omega) \in V(\omega)$ and $U(\omega) \cap V(\omega) = \emptyset$.The set $\mathfrak{D}$ is said to be distinguish set.

Lemma 2.6 Let $\mathbb{X}_{\mathcal{B}}^{\Omega}$ is distinguishable with distinguish set $\mathfrak{D}$. If $v \in \mathbb{X}_{\mathcal{B}}^{\Omega}$ is not random fixed point, then there exist two random open sets U and V such that for every $\omega \in \mathfrak{D}$ with $v(\omega) \in U(\omega)$ and $\varphi(t,\omega)v(\omega) \in V(\omega)$ we have $V(\omega) = \varphi(t,\omega)U(\omega)$ and $V(\omega) \cap U(\omega) = \emptyset$.

Proof. Note that if W is random open set in $\mathbb{X}$, then for every $\omega \in \Omega$, $\varphi(t,\omega)W(\omega)$ is random open set in $\mathbb{X}$, since $\varphi(t,\omega): \mathbb{X} \longrightarrow \mathbb{X}$ is homeomorphism. Since $\mathbb{X}_{\mathcal{B}}^{\Omega}$ is distinguishable with distinguish set $\mathfrak{D}$, then there exist two random open sets W_1 and W_2 in X such that for every $\omega \in \mathfrak{D}$ we have $v(\omega) \in W_1(\omega)$ and $\varphi(t,\omega)v(\omega) \in W_2(\omega)$ and $W_1(\omega) \cap W_2(\omega) = \emptyset$ for every $\omega \in \mathfrak{D}$. Since $v(\omega) \in W_1(\omega)$ for every $\in \mathfrak{D}$, then $\varphi(t,\omega)v(\omega) \in \varphi(t,\omega)W_1(\omega)$ for every $\omega \in \mathfrak{D}$. Set

$$V(\omega) := \begin{cases} \varphi(t,\omega)W_1(\omega) \cap W_2(\omega), & \omega \in \mathfrak{D} \\ \emptyset, & \omega \in \mathfrak{D}^c \end{cases}.$$

Then $\varphi(t,\omega)v(\omega) \in V(\omega)$, for every $\omega \in \mathfrak{D}$. Set

$$U(\omega) := \begin{cases} \varphi(t,\omega)^{-1}V(\omega), & \omega \in \mathfrak{D} \\ \emptyset, & \omega \in \mathfrak{D}^c \end{cases}.$$

Then $v(\omega) \in U(\omega)$, for every $\omega \in \mathfrak{D}$. Clearly that $V(\omega) \subseteq W_2(\omega)$ and $U(\omega) \subseteq W_1(\omega)$ for every $\omega \in \mathfrak{D}$. But $W_1(\omega) \cap W_2(\omega) = \emptyset$ for every $\omega \in \mathfrak{D}$, this implies that $U(\omega) \cap V(\omega) = \emptyset$, for every $\omega \in \mathfrak{D}$. ∎

Theorem 2.7 Let $\mathbb{X}_{\mathcal{B}}^{\Omega}$ is distinguishable with distinguish set $\mathfrak{D}$. Then $v \in \mathbb{X}_{\mathcal{B}}^{\Omega}$ is random fixed point if and only if every random neighborhood of v, contains semi-trajectory for all $\omega \in \mathfrak{D}$.

Proof Suppose that $v \in \mathbb{X}_{\mathcal{B}}^{\Omega}$ is random fixed point, then $\gamma_v(\omega) = \{v(\omega)\}$ so that $\gamma_v(\omega)$ contained in every random neighborhood of v. Conversely, suppose that every random neighborhood of v contains semi-trajectory. Assume contrary that $v \in \mathbb{X}_{\mathcal{B}}^{\Omega}$ is not random fixed point, then there exists $t \in \mathbb{R}^+$, such that for every $\tilde{\Omega} \subseteq \Omega$ with $\mathbb{P}(\tilde{\Omega}) = 1$, $\varphi(t,\omega)v(\omega) \neq v(\theta_t\omega)$, for some $\omega \in \tilde{\Omega}$. By Lemma 2.6 there exist two random open sets U and V such that for every $\omega \in \mathfrak{D}$ with $v(\omega) \in U(\omega)$ and $\varphi(t,\omega)v(\omega) \in V(\omega)$ we have $V(\omega) = \varphi(t,\omega)U(\omega)$ and $V(\omega) \cap U(\omega) = \emptyset$. Since for each $u(\omega) \in U(\omega)$, $\omega \in \mathfrak{D}$ we have $\varphi(t,\omega)u(\omega) \in V(\omega)$, $\omega \in \mathfrak{D}$, then $\varphi(t,\omega)u(\omega) \notin U(\omega)$, $\omega \in \mathfrak{D}$. But this is a contradiction. ∎

Theorem 2.8 Let $\mathbb{X}_{\mathcal{B}}^{\Omega}$ is distinguishable with distinguish invariant set $\mathfrak{D}$ .If $u, v \in X_{\mathcal{B}}^{\Omega}$ and $d(\varphi(t,\theta_{-t}\omega)v(\theta_{-t}\omega), u(\omega)) \longrightarrow 0$, for every $\omega \in \Omega$ as $t \longrightarrow +\infty$ (or $t \longrightarrow -\infty$). Then u is random fixed point.

Proof. Let U be a random neighborhood of u. Since $d(\varphi(t,\theta_{-t}\omega)v(\theta_{-t}\omega), u(\omega)) \longrightarrow 0$, for every $\omega \in \Omega$ as $t \longrightarrow +\infty$, there exists $T \geq 0$ such that

$$\varphi(t,\theta_{-t}\omega)v(\theta_{-t}\omega) \in U(\omega),$$

for every $\omega \in \Omega$, for all $t \geq T$. Hence for all $\omega \in \mathfrak{D}$ we have

$\{\varphi(s,\omega)\varphi(t,\theta_{-t}\omega)v(\theta_{-t}\omega) : s \in \mathbb{R}^+\} \in U(\omega)$.

Then for all $\omega \in \mathfrak{D}$,

$\{\varphi(s+t,\theta_{-t}\omega)v(\theta_{-t}\omega) : s \in \mathbb{R}^+\} \in U(\omega)$. Or for all $\omega \in \mathfrak{D}$,

$$\{\varphi(r,\omega')v(\omega') : r \in \mathbb{R}^+\} \in U(\theta_t\omega'),$$

where $r = s + t$ and $\omega' \coloneqq \theta_{-t}\omega$. That is U contains semi-trajectory, consequently, by Theorem 2.7 u is R.F.P. ■

Proposition 2.9 Let $v \in \mathbb{X}_{\mathcal{B}}^{\Omega}$ be a random fixed point. If $u \in \mathbb{X}_{\mathcal{B}}^{\Omega}$ with $\mathbb{P}\{\omega: v(\omega) \neq u(\omega)\} = 1$, then

$\mathbb{P}\{\omega: \varphi(t,\omega)u(\omega) = v(\theta_t\omega)\} \neq 1.$

Proof. Suppose that $v \in \mathbb{X}_{\mathcal{B}}^{\Omega}$ is a random fixed point. Let $u \in \mathbb{X}_{\mathcal{B}}^{\Omega}$ with

$\mathbb{P}\{\omega: v(\omega) \neq u(\omega)\} = 1.$

Assume contrary that

$\mathbb{P}\{\omega: \varphi(t,\omega)u(\omega) = v(\theta_t\omega)\} = 1.$

Then $\mathbb{P}\{\omega: \varphi(t,\omega)u(\omega) = \varphi(t,\omega)v(\omega)\} = 1$

Since $\varphi(t,\omega)$ is bijective, then

$\mathbb{P}\{\omega: u(\omega) = v(\omega)\} = 1.$

So $\mathbb{P}\{\omega: u(\omega) \neq v(\omega)\} = 0.$

Which is a contradiction. ■

In [9] I.J.Kadhim introduce the concept of random periodic point for random dynamical system. Here we define this concept in another manner which more suitable with our work .

Definition 2.10 A random variable $v \in \mathbb{X}_{\mathcal{B}}^{\Omega}$ is said to be ***random periodic point*** of a RDS (θ, φ) if there exists $\tau \neq 0$ such that

$\mathbb{P}\{\omega: \varphi(\tau,\omega)v(\omega) = v(\theta_\tau\omega)\} = 1.$ The number τ is called the period of v.

Remark 2.11 In any RDS every random fixed point is random periodic point.

Proposition 2.12 A random variable $v \in \mathbb{X}_{\mathcal{B}}^{\Omega}$ is random periodic point if and only if there exists $\tau \neq 0$ such that for every $t \in \mathbb{R}$, $\mathbb{P}\{\omega: \varphi(t+\tau,\omega)v(\omega) = \varphi(t,\theta_\tau\omega)v(\theta_\tau\omega)\} = 1$

Proof. Suppose that $v \in \mathbb{X}_{\mathcal{B}}^{\Omega}$ is random periodic point. Then there exists $\tau \neq 0$ such that

$\mathbb{P}\{\omega: \varphi(\tau,\omega)v(\omega) = v(\theta_\tau\omega)\} = 1.$

If and only if

$\mathbb{P}\{\omega: \varphi(t,\theta_\tau\omega)\varphi(\tau,\omega)v(\omega) = \varphi(t,\theta_\tau\omega)v(\theta_\tau\omega)\} = 1$, $\forall t \in \mathbb{R}$

If and only if

$\mathbb{P}\{\omega: \varphi(t+\tau,\omega)v(\omega) = \varphi(t,\theta_\tau\omega)v(\theta_\tau\omega)\} = 1$, $\forall t \in \mathbb{R}$

This complete the proof. ■

Theorem 2.13 Let (θ, φ) be an RDS with θ be a stable TMDS and let $v \in \mathbb{X}_{\mathcal{B}}^{\Omega}$ be a random periodic point and continuous but not R.F.P. Then there exists $T > 0$ such that T is the smallest positive period of v.Further, if τ is any other positive period of v, then $\tau = nT$ for some integer n.

Proof. Consider the set

$S \coloneqq \{t > 0: t \text{ the period of } v\}.$

If $T \neq 0$ period of v, then

$\mathbb{P}\{\omega: \varphi(T,\omega)v(\omega) = v(\theta_T\omega)\} = 1$

Let $\omega' \coloneqq \theta_T\omega$, then $\omega = \theta_{-T}\omega'$. Since

$\mathbb{P}\{\omega: \varphi(T,\omega)v(\omega) = v(\theta_T\omega)\} \equiv \mathbb{P}\{\omega': \varphi(T,\theta_{-T}\omega')v(\theta_{-T}\omega') = v(\omega')\}$,

then

$\mathbb{P}\{\omega': \varphi(T,\theta_{-T}\omega')v(\theta_{-T}\omega') = v(\omega')\} = 1.$

Now, set

$\widetilde{\Omega} \coloneqq \{\omega': \varphi(T,\theta_{-T}\omega')v(\theta_{-T}\omega') = v(\omega')\}.$ To show that

$\varphi(-T,\omega')v(\omega') = v(\theta_{-T}\omega'), \forall \omega' \in \widetilde{\Omega}$:

$$\begin{aligned}\varphi(-T,\omega')v(\omega') &= \varphi(-T,\omega')\varphi(T,\theta_{-T}\omega')v(\theta_{-T}\omega') \\ &= \varphi(-T+T,\omega')v(\theta_{-T}\omega') = v(\theta_{-T}\omega').\end{aligned}$$

Hence

$\mathbb{P}\{\omega': \varphi(-T,\omega')v(\omega') = v(\theta_{-T}\omega')\} = 1.$ Since either T or $-T$ is positive, then the set S is nonempty. Now, set $\inf S = T$. We calem that $T > 0$. Indeed $T \geq 0$, and if $T = 0$, then there exists a sequence $\{t_n\}$ in S with $t_n \longrightarrow 0$. Since

$\mathbb{P}\{\omega: \varphi(t_n,\omega)v(\omega) = v(\theta_{t_n}\omega)\} = 1$

for each n, then by Theorem 2.3 v is random fixed point which contradicts our hypothesis. Thus $T > 0$. Since $\inf S = T$, then there is a sequence $\{t_n\}$ in S with $t_n \longrightarrow T$. Since $\varphi(\cdot,\omega): \mathbb{R} \longrightarrow \mathbb{X}$ is continuous for every $\omega \in \Omega$, then for every $\omega \in \Omega$, $\varphi(t_n,\omega) \longrightarrow \varphi(T,\omega)$.So for every $\omega \in \Omega$,

$$\varphi(t_n,\omega)v(\omega) \longrightarrow \varphi(T,\omega)v(\omega).$$

Since $\theta(\cdot,\omega): \mathbb{R} \longrightarrow \Omega$ is continous for every $\omega \in \Omega$, then $\theta_{t_n}\omega \longrightarrow \theta_T\omega$ for every $\omega \in \Omega$. Again, since $v \in \mathbb{X}_{\mathcal{B}}^{\Omega}$ is continuous, then $v(\theta_{t_n}\omega) \longrightarrow v(\theta_T\omega)$ for every $\omega \in \Omega$. But $\mathbb{P}\{\omega: \varphi(t_n,\omega)v(\omega) = v(\theta_{t_n}\omega)\} = 1$,

i.e., there exists a full measure subset $\widetilde{\Omega}$ of Ω such that $\varphi(t_n,\omega)v(\omega) = v(\theta_{t_n}\omega)$ for every $\omega \in \widetilde{\Omega}$. Hence $\forall \omega \in \widetilde{\Omega}$,

$$v(\theta_{t_n}\omega) \longrightarrow \varphi(T,\omega)v(\omega),.$$

On the other hand, $v(\theta_{t_n}\omega) \longrightarrow v(\theta_T\omega)$, for every $\omega \in \widetilde{\Omega}$. Since $\mathbb{X}$ is metric space, then from the uniqenss of the limit we have $\varphi(T,\omega)v(\omega) = v(\theta_T\omega)$ $\forall \omega \in \widetilde{\Omega}$.

That is, $\mathbb{P}\{\omega: \varphi(T,\omega)v(\omega) = v(\theta_T\omega)\} = 1$. It follows that $T \in S$. By definition of T it is also the smallest positive period of v . Finally, let $t \in \mathbb{R}$ be a positive periodic. If $t \neq nT$, for any integer, then there is an integer n with $nT < t < (n+1)T$. Then by Lemma 2.1, we have

$$\mathbb{P}\{\omega: \varphi(nT,\omega)v(\omega) = v(\theta_{nT}\omega)\} = 1.$$

Since the TMDS θ is satble, then

$$\mathbb{P}\{\omega: \theta_t\omega = \theta_{nT}\omega\} = 1.$$

Therefore by Lemma 2.1 we have

$\mathbb{P}\{\omega: \varphi(nT,\omega)v(\omega) = \varphi(t,\omega)v(\omega)\} = 1$. So

$\mathbb{P}\{\omega: v(\omega) = \varphi(-nT,\theta_{nT}\omega) \circ \varphi(t,\omega)v(\omega)\} = 1$.

Thus

$$\varphi(nT,\omega)v(\theta_{t-nT}\omega) = \varphi(t,\omega)v(\omega)$$

$$v(\theta_{t-nT}\omega) = \varphi(-nT,\theta_{t-nT}\omega)\varphi(t,\omega)v(\omega)$$

$$\varphi(\tau,\omega)v(\omega) = v(\theta_\tau\omega)$$

where $\tau := t - nT$. Then τ satisfy (2.1). Since $0 < \tau < T$, we get a contradiction to the fact that T was the smallest positive period of period of v. This complete the prove. ∎

In the following we need to show that the set of random periodic point for random dynamical system (under certain conditions) is $\mathbb{P}$ −closed. To this end the following notations are introduced.

Definition 2.14 Let $(\Omega,\mathbb{F},\mathbb{P})$ be a probability space with Ω considered as a topological space and $\mathbb{X}$ be any metric space. A sequence $\{v_n\}$ in $\mathbb{X}_{\mathcal{B}}^{\Omega}$ is said to be $\mathbb{P}$ −uniform converge in $\mathbb{X}_{\mathcal{B}}^{\Omega}$ if there exist $v \in \mathbb{X}_{\mathcal{B}}^{\Omega}$ and $\widetilde{\Omega} \subseteq \Omega$ such that $v_n(\omega)$ converge uniformly (shortly u.c.) to $v(\omega)$ for every $\omega \in \widetilde{\Omega}$. That is for every $\varepsilon > 0$, there is a positive integer n_0 such that

$d(v_n(\omega), v(\omega)) < \varepsilon$, for every $\omega \in \widetilde{\Omega}$ and for every $n > n_0$.

Definition 2.15 Let $(\Omega,\mathbb{F},\mathbb{P})$ be a probability space with Ω considered as a topological space and $\mathbb{X}$ be any topological space. A subset of $\mathbb{X}_{\mathcal{B}}^{\Omega}$ is said to be $\mathbb{P}$ −closed if any sequence $\{v_n\}$ in $\mathbb{X}_{\mathcal{B}}^{\Omega}$ is $\mathbb{P}$ −uniform converge in $\mathbb{X}_{\mathcal{B}}^{\Omega}$.

Lemma 2.16 Let (θ,φ) be an RDS with θ is be a TMDS. If $\{v_n\}$ be a sequence of continous random periodic point in $\mathbb{X}_{\mathcal{B}}^{\Omega}$ with positive periodic $T_n \longrightarrow 0$, and $\mathbb{P}$ −uniform converge in $\mathbb{X}_{\mathcal{B}}^{\Omega}$, then v is random fixed point.

Proof. For a given $t \in \mathbb{R}$, there are integers k_n such that

$$k_nT_n \le t < k_nT_n + T_n.$$

Since $T_n \longrightarrow 0$, we have $k_nT_n \longrightarrow t$. Since v_n is $\mathbb{P}$ −uniform converge to v, then $\mathbb{P}\{\omega: v_n(\omega) \text{ u.c. to } v(\omega)\} = 1$. Let

$\Omega_1 := \{\omega: v_n(\omega) \text{ u.c. to } v(\omega)\}$,

then $\mathbb{P}(\Omega_1) = 1$ But $\varphi(\cdot,\omega,\cdot): \mathbb{R} \times \mathbb{X} \longrightarrow \mathbb{X}$ is continuous for every ω, then for every $\omega \in \Omega_1$,

$\varphi(k_nT_n,\omega)v_n(\omega) \longrightarrow \varphi(t,\omega)v(\omega)$.

Since $\theta(\cdot,\omega): \mathbb{R} \longrightarrow \Omega$ is continuous for every ω, then $\theta_{k_nT_n}\omega \longrightarrow \theta_t\omega$ for every ω. Therefore $v_n(\theta_{k_nT_n}\omega) \longrightarrow v_n(\theta_t\omega)$ for every ω. Since $v_n(\theta_t\omega) \longrightarrow v(\theta_t\omega)$, for every $\in \Omega_1$, then $v_n(\theta_{k_nT_n}\omega) \longrightarrow v(\theta_t\omega)$ for every $\omega \in \Omega_1$. Since v_n is random periodic point for every n, then

$\mathbb{P}\{\omega: \varphi(k_nT_n,\omega)v_n(\omega) = v_n(\theta_{k_nT_n}\omega)\} = 1$

Set

$\Omega_2 \coloneqq \{\omega: \varphi(k_nT_n,\omega)v_n(\omega) = v_n(\theta_{k_nT_n}\omega)\}$. Then $\mathbb{P}(\Omega_2) = 1$. So

$\varphi(k_nT_n,\omega)v_n(\omega) = v_n(\theta_{k_nT_n}\omega)$, for every $\omega \in \Omega_2$.

$\varphi(k_nT_n,\omega)v_n(\omega) \longrightarrow \varphi(t,\omega)v(\omega)$,

for every $\omega \in \Omega_1$. Then,

$v_n(\theta_{k_nT_n}\omega) \longrightarrow \varphi(t,\omega)v(\omega)$, for every $\omega \in \Omega_2 \cap \Omega_1$

Since $v_n(\theta_{k_nT_n}\omega) \longrightarrow v(\theta_t\omega)$ for every $\omega \in \Omega_1$, it follwos that

$\varphi(t,\omega)v(\omega) = v(\theta_t\omega)$, for every $\omega \in \Omega_2 \cap \Omega_1$.

Since $\mathbb{P}(\Omega_2 \cap \Omega_1) = 1$,then v is random fixed point.

Theorem 2.17 Let (θ,φ) be an RDS with θ is be a TMDS .Given any $\alpha > 0$, the set of all v such that v is (continuous) random periodic point with positive period $T \leq \alpha$ is $\mathbb{P}$ −closed.

Proof. Let P be a set of all random periodic point with positive period $T \leq \alpha$. Suppose that $\{v_n\}$ be $\mathbb{P}$ −uniform converge sequence in P. Then for every n, v_n is random periodic point with periods $T_n \leq \alpha$ and v_n then

$\mathbb{P} \coloneqq \{\omega: v_n(\omega) \longrightarrow v(\omega)\} = 1$.

Set

$\Omega_1 \coloneqq \{\omega: v_n(\omega) \longrightarrow v(\omega)\}$,

then $\mathbb{P}(\Omega_1) = 1$. Since

$\mathbb{P}\{\omega: \varphi(T_n,\omega)v_n(\omega) = v_n(\theta_{T_n}\omega)\} = 1$, then $\mathbb{P}(\Omega_2) = 1$, where

$\Omega_2 \coloneqq \{\omega: \varphi(T_n,\omega)v_n(\omega) = v_n(\theta_{T_n}\omega)\}$. Also for every $\in \Omega_1$,

$\mathbb{P}\{\omega: v_n(\theta_{T_n}\omega) \longrightarrow v(\theta_{T_n}\omega)\} = 1$.

Since $0 \leq T_n \leq \alpha$, either $T_n \longrightarrow 0$ in which case v is random fixed point by Lemma 2.15 and hence random periodic, or there is a subsequence $T_{n_k} \longrightarrow \tau$, $0 < \tau \leq \alpha$, then by the continuity axiom for every $\omega \in \Omega_1$

$\varphi(T_{n_k},\omega)v_{n_k}(\omega) \longrightarrow \varphi(\tau,\omega)v(\omega)$.

and also for every $\omega \in \Omega_2$,

$$\varphi(T_{n_k},\omega)v_{n_k}(\omega) \longrightarrow v(\theta_{T_{n_k}}\omega)$$

Since θ is continuous, then $\theta_{T_{n_k}}\omega \longrightarrow \theta_\tau\omega$ for every ω. Also we have v is continuous for every $\omega \in \Omega_1$, then $v(\theta_{T_{n_k}}\omega) \longrightarrow v(\theta_\tau\omega)$, for every $\omega \in \Omega_1$.Therefore

$\varphi(T_{n_k},\omega)v_{n_k}(\omega)) \longrightarrow v(\theta_\tau\omega)$.

for every $\omega \in \Omega_2 \cap \Omega_1$. Consequently

$\varphi(\tau,\omega)v(\omega) = v(\theta_\tau\omega)$,

for every $\omega \in \Omega_2 \cap \Omega_1$. Since $\mathbb{P}(\Omega_2 \cap \Omega_1) = 1$.,then v is random periodic point.

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بعض خواص النقاط الصامدة العشوائية و النقاط الدورية العشوية للنظم الديناميكية العشوائية

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المستخلص :

الهدف الرئيسي من هذا البحث هو اعطاء بعض الخواص النقاط الصامدة العشوائية للنظم الديناميكية العشوائية حيث . قدمنا وصف للنقاط الصامدة العشوائية بدلالة المسارات المنبثقة من متغير عشوائي. كذلك قمنا بدراسة مفهوم النقاط الدورية العشوائية للنظم الديناميكية العشوائية حيث قدمنا الشرطين الكافي و الضروري الذين بموجبهما يكون متغير عشوائي نقطة دورية عشوائية كما وقد برهنا ان مجموعة كل النقاط الدورية (المستمرة) تحت شروط معينة، تكون $\mathbb{P}$—مغلقة.

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Exact Solutions for nonlinear partial differential equation by modified F-expansion method

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Abstract

In this Paper, we use modified F-expansion method to construct new exact traveling wave solutions of nonlinear partial differential equation, The (2+1) – dimensional zoomeron equation, The obtained solutions include soliton – like solutions, trigonometric function solutions, rational solutions and exponential solutions.

Mathematics subject classification: 35-XX .

1- Introduction

Nonlinear partial differential equations are used to describe many phenomena in physics and other domains. The investigation of the exact solutions of Nonlinear PDE plays an important role in the study of physical phenomena.

In applied mathematics, it has importance to obtain and search the exact solutions of these equations. Therefore, recently, a lot of efficient and accurate methods to find exact solutions for Nonlinear PDES submit by many authors includes the$(G'/G)-$expansion method [3], Homogeneous Balance Method [6], Darboux transformation method [8], the inverse Scattering method [1], the F-expansion method [9], the sine-cosine method [7], Jacobi elliptic function [5], homotopy perturbation method [2] and several powerful methods which have been employed to obtain exact solutions.

2 – Description of the Method

Consider a general nonlinear partial differential equation, with two variables x , t

$P(u, u_t, u_{xx}, u_{xt}, u_{tt}, \dots) = 0 \quad \dots(2-1)$

Where $u = u(x, t)$ is the solution of (2-1)

Step I: The transformations which are used are as follows:

$u(x_1, x_2, \dots, x_l, t) = u(\delta)$

$\delta = k_1(x_1 + k_2x_2 + \cdots + k_lx_l - \lambda t) \quad \dots(2-2)$

Where $k_1, k_2, \dots, k_l, \lambda$ are constants to be determined Inserting (2-2) into (2-1) change the NPDE to ODE for $u(\delta)$

$p(u, \acute{u}, \acute{\acute{u}}, \dots) = 0 \quad \dots(2-3)$

Step II: According to the modified F-expansion method,

$u(\delta) = a_0 + \sum_{i=1}^{N} a_iF^i(\delta) + \sum_{i=1}^{N} b_iF^{-i}(\delta) \quad \dots(2-4)$

Where a_0, a_i, b_i constants to be determined are $F(\delta)$ satisfies the following eq. :

$\acute{F}(\delta) = A + BF(\delta) + CF^2(\delta) \quad \dots(2-5)$

Where A, B and C are constants to be determined. Integer N can determined by considering the homogeneous balance between the governing nonlinear term(s) and highest order derivatives of $V(\delta)$ in (2-3).Given different values of A , B and C , the different Riccati function solution , $F(\delta)$ can be obtained from (2-5) (see Table 1).

Step III: Substitute (2-4) into (2-3) and using (2-5), and collect coefficients of $F^i(\delta)$ $(i = -N, \dots, N)$, then set each coefficient to zero. Equating each coefficient of $F^i(\delta)$ to zero yields a system of algebraic equations for $a_i(i = N, \dots, 1, 0)$, $b_i(i = 1, \dots, N)$, $k_i(i = 1, \dots, m)$ and λ.

Step IV: Solve the system of algebraic equation,a_i, b_i, k_i, λ can be expressed by A, B, and C (or the coefficients of ODE (2-3)). Substituting these results into (2-4), we can obtain the general form of travelling wave solutions to Eq. (2-3).

StepV: From Table 1, and the general form of travelling wave Solutions, we can obtained a series of Soliton – like solutions, trigonometric function Solutions and rational solutions of Eq.(2-1).

Table1. Relations between values of A, B, C and corresponding $F(\delta)$ in Riccati equation $\acute{F}(\delta) = A + BF(\delta) + CF^2(\delta)$

A	B	C	$F(\delta)$
0	1	-1	$\frac{1}{2} + \frac{1}{2}\tanh(\frac{1}{2}\delta)$
0	-1	1	$\frac{1}{2} - \frac{1}{2}\coth(\frac{1}{2}\delta)$
$\frac{1}{2}$	0	$\frac{-1}{2}$	$\coth(\delta) \pm \mathrm{Csch}(\delta)$
			$\tanh(\delta) \pm i\,\mathrm{sech}(\delta)$
1	0	-1	$\tanh(\delta), \coth(\delta)$
$\frac{1}{2}$	0	$\frac{1}{2}$	$\sec(\delta) + \tan(\delta)$
			$\mathrm{Csc}(\delta) - \cot(\delta)$
$\frac{-1}{2}$	0	$\frac{-1}{2}$	$\sec(\delta) - \tan(\delta)$
			$\mathrm{Csc}(\delta) + \cot(\delta)$
1(-1)	0	1(-1)	$\tanh\delta$ $(\cot\delta)$
0	0	$\neq 0$	$\frac{-1}{c\delta + \eta}$ (η is anarbitrary
A is arbit-rary cons-tant	0	0	$A\delta$
A is arbit-rary constant	$\neq 0$	0	$\frac{\exp(B\delta) - A}{B}$

3-Exact solutions to zoomeron equation

In this section we apply this method to Construct exact solution of the (2+1)– dimensional zoomeron equation (see [4])

$\left(\frac{u_{xy}}{u}\right)_{tt} - \left(\frac{u_{xy}}{u}\right)_{xx} + (2u^2)_{xt} = 0 \quad \dots(3-1)$

Where function u (x, y, t) is the amplitude of the relevant wave mode. This highly nonlinear equation play an important role in describing the evolution of a single Scalar field and is a convenient model to display the noval phenomena associated with boomers and trapons. If y=t, the (2+1) – dimensional zoomeron equation reduces to its (1+1) – dimensional form which can be regarded as a sub case of the boomeron equation.

By using the transformation:

$u\,(x,y,t) = u(\delta)$ and

$\delta = x + y - \lambda t$ ... (3 – 2)

Where λ is arbitrary constant.

Substituting Eq. (3-2) with Eq. (3-1), the change will be:

$\lambda^2 (\frac{\acute{u}\acute{}}{u}) - (\frac{\acute{u}\acute{}}{u}) - 2\lambda (u^2\acute{}) = 0$ ... (3 – 3)

Now integrating (3-3) with respect λ twice, we get:

$(\lambda^2 - 1)\acute{\acute{u}} - 2\lambda\, u^3 + ku = 0$ ... (3 – 4)

Where k is an integral constant.

Considering the homogeneous balance between $u_{\delta\delta}$ and u^3 in (3-3),

The Solution to ordinary Eq. (3-3) can be expressed be

$U(\delta\,) = a_0 + a_1 F(\delta) + b_1 F^{-1}(\delta)$ **...**(3 – 5)

Where a_0, a_1 and b_1 are constants to be determined. By Substituting (3-5) with (3-3) and using (2-5) the left – hand side of Eq.(3-5) can be converted into a finite series in $F^j(\delta)$ $(j = -3, \dots ,3)$.

Equating each coefficient of $F^j(\delta)$ to zero yields a system of algebraic equations for a_0, a_1, b_1 and k.

$F^{-3} : (\lambda^2 - 1)(2b_1A^2) - 2\lambda b_1^3 = 0$

$F^{-2} : (\lambda^2 - 1)(3b_1AB) - 2\lambda(\,3a_0 b_1^2) = 0$

$F^{-1} : (\lambda^2 - 1)(2b_1AC + b_1B^2) - 2\lambda(\,3{a_0}^2 b_1 + 3\,a_1 b_1^2) + k\,b_1 = 0$

$F^0 : (\lambda^2 - 1)[\,a_1AB + \,b_1BC] - 2\lambda(a_0^3) + ka_0 + 6a_0a_1b_1 = 0$

$F : (\lambda^2 - 1)[\,a_1B^2 + \,2a_1AC] - 2\lambda[\,3a_0^3a_1 + 3a_1^2 b_1\,] + ka_1 = 0$

$F^2 : (\lambda^2 - 1)(3a_1BC) - 2\lambda(\,3a_0{a_1}^2) = 0$

$F^3 : (\lambda^2 - 1)(2a_1C^2) - 2\lambda({a_1}^3) = 0$

... (3 – 6)

Solving the algebraic equations (3-6), we have the following solutions of a_0, a_1, b_1 and k

Case I: $a_0 = 0$,$a_1 = c\sqrt{\lambda - \frac{1}{\lambda}}$,$b_1 = A\sqrt{\lambda - \frac{1}{\lambda}}$

$k = -4AC + \lambda^2(\,4AC - B^2)$

CaseII: $a_0 = B\sqrt{\lambda - \frac{1}{\lambda}}$, $a_1 = C\sqrt{\lambda - \frac{1}{\lambda}}$

$b_1 = A\sqrt{\lambda - \frac{1}{\lambda}}$

$k = (1 - \lambda^2)(-4AC + \frac{CB^2}{A} - 6B^2)$

CaseIII: $a_0 = \frac{B\sqrt{\lambda^2 - 1}}{2\lambda}$,

$a_1 = c\,\sqrt{\lambda^2 - 1}$,$b_1 = 0$

$k = \frac{(\lambda^2 - 1)}{2\lambda}\,(-4\lambda^2 AC + B^2)$

The solition – like solutions to Zoomeron equation:

1)When A=0, B=1, C=-1, from Table 1,

$F(\delta\,) = \frac{1}{2} + \frac{1}{2}\tanh(\frac{1}{2}\delta)$

By Case I, the exact Solution to equation (3-1) is given by:

$u_1 = \frac{-1}{2}\sqrt{\lambda - \frac{1}{\lambda}}(1 + \tanh(\frac{1}{2}x + \frac{1}{2}y - \frac{1}{2}\lambda t))$

By Case II, the exact solutions to equation (3-1) are given by:

$u_2 = \frac{1}{2}\sqrt{\lambda - \frac{1}{\lambda}}(1 - \tanh(\frac{1}{2}x + \frac{1}{2}y - \frac{1}{2}\lambda t))$

By Case III, the exact solutions to equation (3-1) are given by:

$u_3 = \frac{\sqrt{\lambda^2 - 1}}{2}\,(\frac{1 - \lambda}{\lambda} - \tanh(\frac{1}{2}x + \frac{1}{2}y - \frac{1}{2}\,\lambda t))$

2) When A=0, B=-1, C=1, from Table 1, $F(\delta) = \frac{1}{2} - \frac{1}{2}\coth(\frac{1}{2}\delta)$

By Case I, the exact solutions to equation (3-1) is given by

$u_4 = \frac{1}{2}\sqrt{\lambda - \frac{1}{\lambda}}\,(1 - \coth(\frac{1}{2}x + \frac{1}{2}y - \frac{1}{2}\lambda t))$

By Case II, the exact solutions to equation (3-1) is given by

$u_5 = \frac{-1}{2}\sqrt{\lambda - \frac{1}{\lambda}}\,(1 - \coth(\frac{1}{2}x + \frac{1}{2}y - \frac{1}{2}\lambda t))$

By Case III, the exact solutions to equation (3-1) is given by

$u_6 = \frac{\sqrt{\lambda^2 - 1}}{2}\,(\frac{\lambda - 1}{\lambda} - \coth(\frac{1}{2}x + \frac{1}{2}y - \frac{1}{2}\lambda t)))$

3)When $A = \frac{1}{2}$, $B = 0$, $C = \frac{-1}{2}$, from Table 1,

$F(\delta\,) = \coth(\,\delta\,) \pm \mathrm{Csch}(\,\delta)$

$F(\delta\,) = \tanh(\,\delta\,) \pm i\,\mathrm{sech}(\delta)$

By Case I, the exact solutions to equation (3-1) is given by

$u_7 = \frac{-1}{2}\sqrt{\lambda - \frac{1}{\lambda}}(\coth(x + y - \lambda t) \pm Csch(x + y - \lambda t))$

$+ \frac{1}{2}\sqrt{\lambda - \frac{1}{\lambda}}(\coth(x + y - \lambda\,t) \pm Csch(x + y - \lambda t))^{-1}$,

$u_8 = \frac{-1}{2}\sqrt{\lambda - \frac{1}{\lambda}}(\tanh(x + y - \lambda t) \pm i\, sech(x + y - \lambda t))$
$+\frac{1}{2}\sqrt{\lambda - \frac{1}{\lambda}}(\tanh(x + y - \lambda t) \pm i\, sech(x + y - \lambda t))^{-1}$.

By Case II, the exact solutions to equation (3-1) is given by

$u_9 = \frac{-1}{2}\sqrt{\lambda^2 - 1}\,(\, Coth\,(x + y - \lambda t) \pm \mathrm{Csch}(x + y - \lambda t))$,

$u_{10} = \frac{-1}{2}\sqrt{\lambda^2 - 1}(\tanh(x + y - \lambda t) \pm i\, sech(x + y - \lambda t))$

4)When A=1, B=0, C=-1, from Table 1, $F(\delta) = \tanh(\delta)$

$F(\delta) = \coth(\delta)$

By Case I, the exact solutions to equation (3-1) is given by

$u_{11} = -\sqrt{\lambda - \frac{1}{\lambda}}(\tanh(x + y - \lambda t) + \sqrt{\lambda - \frac{1}{\lambda}}(tanh(x + y - \lambda t))^{-1}$

$u_{12} = -\sqrt{\lambda - \frac{1}{\lambda}}(\mathrm{Coth}(x + y - \lambda t) + \sqrt{\lambda - \frac{1}{\lambda}}(coth(x + y - \lambda t))^{-1}$

By Case II, the exact solutions to equation (3-1) is given by

$u_{13} = -\sqrt{\lambda - \frac{1}{\lambda}}[\tanh(x + y - \lambda t) - (tanh(x + y - \lambda t))^{-1}]$

$u_{14} = -\sqrt{\lambda - \frac{1}{\lambda}}[\mathrm{Coth}(x + y - \lambda t) - \left(Coth(x + y - \lambda t)\right)^{-1}]$

By Case III, the exact solutions to equation (3-1) is given by

$u_{15} = -\sqrt{\lambda^2 - 1}(\tanh(x + y - \lambda t)$

$u_{16} = -\sqrt{\lambda^2 - 1}(\mathrm{Coth}(x + y - \lambda t)$

The Trigonometric Function solutions to Zoomeron equation:

1)When $A = \frac{1}{2}, B = 0, C = \frac{1}{2}$, from Table 1,

$F(\delta) = \sec(\delta) + \tanh(\delta)$

$or\ \ F(\delta) = \mathrm{Csc}(\delta) - \cot(\delta)$.

By Case I, the exact solutions to equation (3-1) is given by

$u_{17} = \frac{1}{2}\sqrt{\lambda - \frac{1}{\lambda}}[(\,\mathrm{Sec}(x + y - \lambda t) + \tan(x + y - \lambda t)) + (\,\mathrm{Sec}(x + y - \lambda t) + \tan(x + y - \lambda t\,)^{-1}]$,

$u_{18} = \frac{1}{2}\sqrt{\lambda - \frac{1}{\lambda}}[(\,\mathrm{Csc}(x + y - \lambda t) - \mathrm{Cot}(x + y - \lambda t)) + (\,\mathrm{Csc}(x + y - \lambda t) - \mathrm{Cot}(x + y - \lambda t\,)^{-1}$.

By Case III, the exact solutions to equation (3-1) is given by

$u_{19} = \frac{1}{2}\sqrt{\lambda^2 - 1}(\mathrm{Sec}(x + y - \lambda t) + \tan(x + y - \lambda t))$,

$u_{20} = \frac{1}{2}\sqrt{\lambda^2 - 1}(\mathrm{Csc}(x + y - \lambda t) + \mathrm{Cot}(x + y - \lambda t))$.

2) When A=1, B=0, C=1, from Table1,

$F(\delta) = \tan(\delta)(\cot(\delta))$

By Case I, the exact solutions to equation (3-1) is given by

$u_{21} = \sqrt{\lambda - \frac{1}{\lambda}}\tan(x + y - \lambda)(\mathrm{Cot}(x + y - \lambda)) + \sqrt{\lambda - \frac{1}{\lambda}}[\tan(x + y - \lambda)(\mathrm{Cot}(x + y - \lambda)]^{-1}$

By Case III, the exact solutions to equation (3-1) is given by

$u_{22} = \sqrt{\lambda^2 - 1}\,\mathrm{Sec}(x + y - \lambda t)(\mathrm{Cot}(x + y - \lambda))$

The rational solution to zoomeron equation:

1) When A=B=0, $C \neq 0$, from Table 1, $F(\delta) = \frac{1}{c\delta + \eta}(\,\eta\ is\ an\ arbitrary\ constant)$

By Case I, an exact solutions to equation (3-1) can be written as:

$u_{23} = \frac{-C\sqrt{\lambda - \frac{1}{\lambda}}}{C\,(x - \lambda t) + \eta}$

By Case III, the exact solutions to equation (3-1) can be written as:

$u_{24} = \frac{-C\sqrt{\lambda^2 - 1}}{C\,(x - \lambda t) + \eta}$

2) When A is an arbitrary constant, B=C=0, from Table 1,$F(\delta) = A\delta$,

By Case II, the exact solutions to equation (3-1) can be written as:

$u_{25} = \frac{\sqrt{\lambda - \frac{1}{\lambda}}}{(x - \lambda t)}$.

The exponential Solution zoomeron equation:

When A is an arbitrary constant, $B \neq 0, C = 0$, from Table 1, $F(\delta) = \frac{\exp(B\,\delta) - A}{B}$.

By Case I, the exact solutions to equation (3-1) can be obtained that:

$u_{26} = A\sqrt{\lambda - \frac{1}{\lambda}}(\frac{\exp(B(x - \lambda t) - A)}{B})^{-1}$.

By Case II, the exact solutions to equation (3-1) can be obtained that:

$u_{27} = B\sqrt{\lambda - \frac{1}{\lambda}} + A\sqrt{\lambda - \frac{1}{\lambda}}(\frac{\exp(B\,(x - \lambda t)) - A}{B})^{-1}$.

4- Conclusion

The powerful modified F-expansion method was employed of the (2+1) – dimensional zoomeron equation. This method is an efficient way to solve nonlinear PDES as it is used to solve differential equations which can be integrated or non-integrated. By using this method , we have been able to calculate many new exact solutions for nonlinear partial differential equation include soliton-like solutions, trigonometric function solutions, rational solutions and exponential solutions.

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الحلول المضبوطة لمعادلة تفاضلية جزئية لاخطية باستخدام طريقة F-expansion المعدلة

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المستخلص:
في هذه الورقة،نستخدم طريقةF-expansion المعدلة لإيجاد حلول مضبوطةجديدة لمعادلة تفاضليةجزئيةلاخطية،معادلةzomeronب(٢ + ١) من الأبعاد،وتشمل الحلول التي تم الحصول عليهاحلول-soliton like،حلول الدالةالمثلثية،حلول نسبية و الحلول الأسية.

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On Minimal and Maximal T_1 –space Via B^*c-open set

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Abstract

In this paper, we introduced study about properties of the spaces minimal T_1 – Space and maximal T_1 – space by using the set open (respace β - open, B^*c– open) sets and we concluded some propositions, remarks and relations between spaces m B^*c – T_1 space and M B^*c – T_1 space and study the relation between m – space and M – space to space T_1 – space. Where we find every m B^*c – T_1 space is M B^*c – T_1 space, but the converse is not true in general. Also we introduced hereditary properties and topological properties.

Key Words: minimal B^*c – T_1 space maximal B^*c – T_1 space minimal β – T_1 space maximal β – T_1 space

Mathematics subject classification: 54XX

1) Introduction:

The topological idea from study this type of the space came to determine the relation between minimal T_1 – space and maximal T_1 – space In [5] Abd El – Monsef M. E., El – Deeb S. N. Mahmoud R. A., the Category β – open set was defined which considered to study the B^*c – open set. In [1] and [2] F. Nakaoka and N. oda. , has been defined minimal and maximal open set. In [4] M. C. Gemignani, has been defined T_1 – space In[3]S.S Benchalli defined the function category strongly m – open function and m – irresolute function which is study open function continuity respectively. In [6] and [7] proved proposition about T_1–space by the B^*c – open set.

2) Basic Definitions and Remarks

Definition (2.1): [5]

Let X be a topological space, then a sub set A of X is said to be β - open set if A $\subseteq cl(int(cl(A)))$ is β - closed set if A^C is β - open.

Definition (2.2):

Let X be a topological space and $A \subseteq X$.
Then a β - open set A is said a B^*c - open set if ∀ x ∈ A ∃ F_x closed set ∋ $x \in F_x \subseteq A$. A is a B^*c – closed set if A^c is a B^*c – open.

Definition (2.3):

The family union of all B^*c - open set of a topological space X contained in A is said B^*c - interior of A is, denoted by A^{oB^*c}. i.e

$A^{oB^*c} = \cup \{ : G \subseteq A$ and G B^*c - open in X }.

Definition (2.4): [1]

Let X be a topological space A proper non empty open set U of X is said to be a minimal open set if any open set which is contained in V is ∅ or U.

Definition (2.5): [2]

Let X be a topological space A proper nonempty open set U of X is called to be a maximal open set if any open set which is contained in U is X or U.

Definition (2.6):

Let X be a topological space A proper non empty β - open set U of X is said to be
i) A minimal β - open set if any β - open set which is contained in U is ∅ or U.
ii) A maximal β - open set if any β - open set which contains in U is X or U.

Definition (2.7):

Let X be a topological space A proper non empty B^*c - open set U of X is said to be
i) A minimal B^*c - open set if any B^*c - open set which contains in U is ∅ or U.
ii) A maximal B^*c - open set if any B^*c - open set which is contained in U is X or U.

Definition (2.8): [4]

A topological space X is called T_1 – space iff for each x ≠ y in X, there exists open sets U and V such that x ∈ U, y ∉ U and y ∈ V. x ∉ V.

Definition (2.9):

A topological space X is called m - T_1 space (respace M - T_1) space iff for each x ≠ y in X, ∃ m – open (respace M – open) sets U and V such that x ∈ U, y ∉ U and y ∈ V. x ∉ V.

Definition (2.10):

A topological space X is called βT_1 – space iff ∀ x ≠ y in X there exists β - open sets U and V such that x ∈ U, y ∉ U and y ∈ V. x ∉ V.

Definition (2.11):

A topological space X is called mβ - T_1 space (respace Mβ T_1 - space) iff for each x ≠ y in X there exists mβ – open (respace Mβ – open) sets U and V such that x ∈ U, y ∉ U and y ∈ V. x ∉ V.

Definition (2.12):

A topological space X is called BcT_1 – space iff for each x ≠ y in X ,∃B*c - open sets U and V such that x ∈ U, y ∉ U and y ∈ V. x ∉ V.

Definition (2.13):

A topological space X is called mB*c - T_1 space (respace MB*c - T_1) space iff for each x ≠ y in X there exists mB*c – open (respace MB*c – open) sets U and V so that x ∈ U, y ∉ U and y ∈ V. x ∉ V.

Definition (2.14):

Let X,Y be a topological spaces and let F: X → Y be a function Then:
i) F is called strongly m – open [3], if ∀ m – open set U in X, then F(U) is m - open set in Y.
ii) F is called strongly M – open ,if ∀ M – open set U in X, then F(U) M - open set in Y.
iii) F is called strongly mβ – open (respace strongly Mβ – open),if for all mβ – open (respace Mβ – open) set U in X, then F(U) is mβ – open (respace Mβ – open) set in Y.[8].

iv) F is called strongly mB^*c–open (respace strongly $M B^*c$ – open), if ∀m B^*c – open (respace MB*c – open) set U in X, then F(U) is mB*c – open (respace MB*c – open) set in Y.

Definition (2.15):

Let X,Y be a topological spaces and let F: X → Y be a function Then:

i) F is called m – irresolute function [3], if ∀ m – open U in Y, then $F^{-1}(U)$ is m – open in X.

ii) F is called M –irresolute function, if ∀M –open U in Y, then $F^{-1}(U)$ is M – open in X.

iii) F is called mβ – irresolute (respace Mβ – irresolute) function. If ∀ mβ –open (respace Mβ – open) U in Y, then $F^{-1}(U)$ mβ – open (respace Mβ – open) set in X.[8].

iv) F is called mB^*c – irresolute (respace MB^*c – irresolute) function, if ∀ mB^*c –open (respace MB^*c – open) U in Y, then $F^{-1}(U)$ mB^*c – open (respace MB^*c – open) set in X.

Notion: We will use the symbol m to minimal sets and the symbol M to maximal sets.

The family a fall β – open set is denoted by βo(x) and the family of all B^*c – open is denoted by $B^*co(x)$.

Theorem (2.1):

Let X be a topological space and A ⊆ X. Then:

i) Every open set is β - open.

ii) Every B^*c – open set is β - open.

Proof:

i) Let A be open set, then A =int(A). Since A ⊆ cl(A), then A = int(A) ⊆ int(cl(A)), there for A ⊆cl(int(cl(A))), hence A β - open set in X.

ii) By definition (2.2)

The converse of above Theorem is not true in general.

Example (2.1):

Let X = {a, b, c}, t = {∅, X, {a}, {b}, {a, b} }.

βo (X) = {∅, X, {a}, {b}, {a, b}, {a, c},{b, c} }.

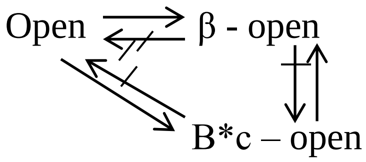
B*co (X) = {∅, X, {b,c}, {a, c} }.

Then let A = {b, c}, B = {b}. Note that

i) A is β - open, but not open.

ii) B is β - open, but not B*c – open.

The following diagram shows the relation among types of open sets



Remark (2.1):

Let X be a topological space . Then:

i) Every m – open (respace M – open) is open.

ii) Every mβ – open (respace Mβ – open) is β - open.

iii) Every mB*c – open (respace MB*c – open) is B*c - open.

The converse of above Theorem is not true in general.

Example (2.2):

In example (2.1) ,we notice:

i) A = {a,b} open, but not m – open also B = {a} open, but not M – open.

ii) A = {a,b}β - open, but not m β – open also B = {a} β - open, but not M – open.

iii) A = ∅ B*c - open, but not m B*c – open also B = X B*c - open, but not M – open.

Corollary (2.1):

i) Every m –open (respace M–open) is β - open.

ii) Every mB*c – open (respace MB*c – open) is β - open.

Remark (2.2):

Let X be a topological space Then

i) Every T_1 – space is β - T_1 space

ii) Every B*c – T_1 space is β - T_1 space

The converse of above Theorem is not true in general.

Example (2.3):

In example (2.1), we see that X is β - T_1 space But

i) X is not T_1 – space, since ∀ a, c ∈ X ∋ a ≠ c, but ∄ U,V open in X∋ a ∈ U, c ∉ U and c ∈ V, a ∉ V.

ii) X is not B*c – T_1 space ,since ∀ a, c ∈ X ∋ a ≠ c, but ∄ U, V B*c – open ∋ a ∈ U, c ∉ U and c ∈ V, a ∉ V.

Theorem (2.2):

Let X be a topological space and A ⊆ X. Then x ∈ A^{oB^*c} iff ∃ B*c – open in X ∋ x ∈ G ⊆ A.

Proof:

Let $x \in A^{oB^*c}$

Since $A^{oB^*C} = \cup$ { G : G ⊆ A, G is B*c - open set in X }.

Then x ∈ ∪ { G : G ⊆ A, G is B*c - open set in X }, and hence ∃ G B*c – open in X ∋ x ∈ G ⊆ A.

Conversely

Let x ∈ G ⊆ A and G is B*c – open ∋ x ∈ G ⊆ A. Then

x ∈ ∪ { G : G ⊆ A, G is B*c - open set in X },therefore $x \in A^{oB^*C}$.

Definition (2.16):

Let X be a topological space and A ⊆ X, x ∈ X. Then

i) The point x is called limit point of A [7] iff ∀ U open set ∋ x ∈ U, then (U ∩ A) - {x} ≠ ∅.

ii) The point x is called β-limit point of A iff ∀ U β -open set ∋x ∈U,then (U∩ A) - {x} ≠ ∅.

iii) The point x is called B*c - limit point of A iff ∀ U B*c - open set ∋ x ∈ U, then (U ∩ A) - {x} ≠ ∅.

Remark (2.2):

i) The set of all limit point of A is denoted by Á.

ii) The set of all β - limit point of A is denoted that $Á^{\beta}$.

iii) The set of all B*c - limit point of A is denoted that $Á^{B^*c}$.

Lemma (2.1):

Let X be a topological space and A ⊆ X, x ∈ X. Then

i) A is closed set iff Á ⊆ A. [7].

ii) A is β - closed set iff $Á^{\beta} \subseteq A$. [5].

Theorem (2.3):

Let X be a topological space and A ⊆ X. Then A is B*c – closed set iff $Á^{B^*c} \subseteq A$.

Proof:

Let A be B*c – closed and b ∉ A, then b ∈ A^C is B*c – open set, hence ∃ B*c – open set A^C ∋ $A^C \cap A = \emptyset$. Hence $x \notin Á^{B^*c}$, therefore $Á^{B^*c} \subseteq A$.

Conversely

Let $Á^{B^*c} \subseteq A$ and b ∉ A then $b \notin Á^{B^*c}$, hence ∃ B*c – open set G ∋ b ∈ G ∩ A = ∅, hence b ∈ G ⊆ A^C. Therefore A^C is B*c – open set in X by Theorem(2.2), hence A is B*c– closed .

3) m – T_1 (M – T_1) space by using the set (open, β - open, B*c – open).

Lemma (3.1) [6]

Let X be a topological space Then X is T_1 – space iff {x} is closed set in X ∀ x ∈ X.

Theorem (3.1):

Let X be a topological space Then X is β–T_1 space iff {x} is β– closed set in X ∀ x ∈ X.

Proof:

Let X be β– T_1 space and let y ∉ {x}, the x ≠ y. Since X β– T_1 space, then ∃ U, V β - open in X ∋ x ∈ U, y ∉ U and y ∈ V, x ∉ V. Then V β - open in X and y ∈ V, then (V ∩ {x}) – {Y} = ∅, then y not β - limit point of {x}, then $y \notin [\{x\}]́^{\beta}$, then $[\{x\}]́^{\beta} \subseteq \{x\}$. Hence {x} is β - closed by lemma (2.1) (ii).

Conversely

Let x, y ∈ X ∋ x ≠ y. Let {x} β - closed in X, then $\{x\}^C$ β - open in X. Let U = $\{x\}^C$, V = $\{y\}^C$ are β - open in X ∋ x ∈ V, y ∉ V and y ∈ X, x ∉ U, hence X β–T_1 space

Theorem (3.2):

Let X be a topological space Then X B*c –T_1 space iff {x} is B*c – closed set in X ∀ x ∈ X.

Proof:

Similarly of Theorem(3.2).

Theorem (3.3):

Let X be a topological space Then

i) Every m - T_1 space is T_1 space

ii) Every m β - T_1 space is β–T_1 space

ii) Every mB*c - T_1 space is B*c –T_1 space

Proof:

i) Let X be m - T_1 space and let x, y ∈ X ∋ x ≠ y. Since X m - T_1 space, then ∃ U, V m - open in X ∋ x ∈ U, y ∉ U and y ∈ V, x ∉ V, then by Remark (2.1) (i) and definition (2.8), we get the resulte.

ii) Let X be mβ- T_1 space and let x, y ∈ X ∋ x ≠ y. Since X is mβ- T_1 space, then ∃ U, V mβ- open sets in X ∋ x ∈ U, y ∉ U and y ∈ V, x ∉ V. then by Remark (2.1) (ii) and definition (2.10), we get the resulte.

iii) Let X be mB*c - T_1 space and let x, y ∈ X ∋ x ≠ y. Since X is mB*c - T_1 space, then ∃ U, V mB*c - open sets in X ∋ x ∈ U, y ∉ U and y ∈ V, x ∉ V. then by Remark (2.1) (iii) and definition (2.12), we get the resulte .The converse of above Theorem is not true in general.

Example (3.1)

Let X = IR with a usual top. Then X is T_1 – space, but not m – T_1 space

Proof:

Let x ∈ IR and let (x, ∞), (-∞, x) ∈ T. Since (x, ∞) ∪ (-∞, x) ∈ T, then IR – {(x, ∞) ∪ (-∞, x)} closed set in IR.

But IR – {(x, ∞) ∪ (-∞, x)} = {x}, then ∃ {x} closed set in IR ∀ x ∈ IR. Hence X is T_1 – space by lemma (3.1).

T. P X not m – T_1 space

Let x, y ∈ X ∋ x ≠ y, but ∄ U, V m – open in X ∋ x ∈ U, y ∉ U and y ∈ V, x ∉ V.

Example (3.2):

In example (3.1). Note that X is βT_1 – space by remark (2.2) (i), but not mβ - T_1 space Since ∀ x, y ∈ X , x ≠ y, but ∄ U, V mβ – open in X ∋ x ∈ U, y ∉ U and y ∈ V, x ∉ V.

Example (3.3)

Let X = IR with a usual topology. Then X is B*cT_1 – space, but not mB*c – T_1 space

Proof:

Let x, y ∈ Y, x ≠ y and let |x – y| = ε and let U = $(x - \frac{\varepsilon}{4}, x + \frac{\varepsilon}{4})$, V = $(y - \frac{\varepsilon}{4}, y + \frac{\varepsilon}{4})$, then U, V ∈ T ∋ x ∈ U, Y ∉ U and Y ∈ V, X ∉ V.

Choose U = (x, Y], V = [x, Y), then U, V are β - open set, then

∀ a ∈ U ∃ {a} closed set ∋ a ∈ {a} ⊆ U.

∀ b ∈ V ∃ {b} closed set ∋ a ∈ {b} ⊆ V.

Then U, V B*c – open sets in X ∋ y ∈ U, x ∉ U and x ∈ V, Y ∉ V.

Then X is B*c – T_1 space

T. P. X not mB*c – T_1 space Let x, y ∈ X ∋ x ≠ Y, but ∄ mB*c – open set U, V in X ∋ y ∈ U, x ∉ U and x ∈ V, Y ∉ V.

Corollary (3.1):

Let X be a topological space .Then:

i) If X m–T_1 space , then {x} closed set in X∀ x∈ X.

ii) If X mβ – T_1 space, then {x} β - closed set in X ∀ x ∈ X.

iii) If X mB*c – T_1 space, then {x} B*c - closed set in X ∀ x ∈ X.

Proof:

i) Follows from theorem (3.3) (i) and lemma (3.1).

ii) Follows from theorem (3.3) (ii) and Theorem(3.2).

iii) Follows from theorem (3.3) (iii) and Theorem(3.2).

Theorem (3.4):

Let X be a topological space Then:

i) If X is m – T_1 space, then X is β - T_1 space

ii) If X is mB*c – T_1 space, then X is β - T_1 space

Proof:

i) Follows from Theorem(3.3) (i) and remark (2.2) (i).

ii) Follows from Theorem(3.3) (iii) and remark (2.2) (ii).

Lemma (3.3)

Let X be a topological space and a ∈ X. Then:

i) [1]. If {a} open (respace closed), then {a} m – open (respace m – closed) set. So $[\{a\}]^C$ M – closed (respace M – open).

ii) If {a} β - open (respace β - closed), then {a} mβ – open (respace mβ– closed). So $[\{a\}]^C$ Mβ – closed (respace Mβ – open).

iii) If {a} B*c - open (respace B*c - closed), then {a} mB*c – open (respace mB*c – closed). So $[\{a\}]^C$ MB*c – closed (respace MB*c – open).

Theorem (3.5):

Let X be a topological space .Then X is M – T_1 space iff {x} closed set in X ∀ x ∈ X.

Proof:

Let X be M– T_1 space and let Y ∉ {x}, the x ≠ y. Since X M– T_1 space, then ∃ U, V M - open in X ∋ x ∈ U, y ∉ U and y ∈ V, x ∉ V, then by Lemma (3.1) we get {x} closed set in X.

Conversely

Let x, y ∈ X and x ≠ y. Let {x} be closed set in X, then by Lemma (3.3) (i), we get {x} m – closed, then $[\{x\}]^C$ M – open in X. Let U = $\{x\}^C$, V = $\{Y\}^C$ are M - open in X, then ∃ U, V M - open in X ∋ x ∈ V, y ∉ V and y ∈ U, x ∉ U, hence X is M – T_1 space

Theorem (3.6):

Let X be a topological space Then

i) X Mβ–T_1 space iff{x}β–closed set in X∀x∈ X.

ii) X MB*c – T_1 space iff {x} B*c – closed set in X ∀ x ∈ X.

Proof:

Similarly Theorem(3.5).

Corollary (3.2):

Let X be a topological space Then

i) X M – T_1 space iff X is T_1 – space

ii) X Mβ – T_1 space iff X is β–T_1 space

iii) X MB*c – T_1 space iff X is B*c –T_1 space

Proof:

i) Follows from Theorem(3.5) and Lemma (3.1).

ii) Follows from Theorem(3.6) (i) and Theorem(3.1).

iii) Follows from Theorem(3.6) (ii) and Theorem(3.2).

Theorem (3.7):

Let X be a topological space .Then

i) Every m – T_1 space is M –T_1 space

ii) Every mβ – T_1 space is Mβ–T_1 space

iii) Every mB*c – T_1 space is MB*c –T_1 space

Proof:

i) Follows from Theorem (3.3)(i)and Corollary.(3.13) (i).

ii)Follows from Theorem (3.3)(ii)and Corollary.(3.2) (ii).

iii) Follows from Theorem(3.3) (iii) and Corollary .(3.2) (iii).

The converse of above Theorem is not true in general.

Example (3.4):

i) In example (3.1), we note that X is T_1 – space, then X is M - T_1 space, by Coro. (3.13) (i), but not m - T_1 space Since ∀ x, Y ∈ X , x ≠ Y, but ∄ U, V m – open in X ∋ x ∈ U, Y ∉ U and Y ∈ V, X ∉ V.

ii) In example (3.2), note that X is βT_1 – space, then X is Mβ - T_1 space by (3.13) (ii), but not mβ - T_1 space Since ∀ x, Y ∈ X , x ≠ Y, but ∄ U, V m – open in X ∋ x ∈ U, Y ∉ U and Y ∈ V, X ∉ V.

iii) In example (3.3), note that X is B*c – T_1 space, then X is MB*c - T_1 space by corollary (3.2) (iii), but not mB*c - T_1 space Since ∀ x, Y ∈ X , x ≠ Y, but ∄ U, V mB*c – open in X ∋ x ∈ U, Y ∉ U and Y ∈ V, X ∉ V.

Theorem (3.8):

Let X be a topological space . Then

i) If X M – T_1 space, then X is β– T_1 space

ii) If X M – T_1 space, then X is Mβ– T_1 space

iii) If X MB*c – T_1 space, then X is β– T_1 space

iv) If X MB*c – T_1 space, then X is Mβ– T_1 space

Proof:

It is clear.

The converse of above theorem is not true in general.

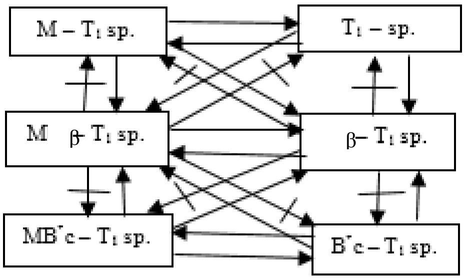
Example (3.5):

In example (3.1), note that X is βT_1 – space, and Mβ - T_1 space But

i) and (ii) Not M - T_1 space Since a, c ∈ X ∋ a ≠ c, but ∄ U, V M – open in X ∋ a ∈ U, c ∉ U and c ∈ V, a ∉ V.

iii) and (iv) Not MB*c - T_1 space Since a, c ∈ X ∋ a ≠ c, but ∄ U, V MB*c – open in X ∋ a ∈ U, c ∉ U and c ∈ V, a ∉ V.

The following diagram shows the relation among types of M - T_1 space



4) Hereditary properties:

Lemma (4.1): [7]

Let X be a topological space then

i) If V open in Y and Y open in X, then V open in X.

ii) If V closed in Y and Y closed in X, then V closed in X.

Lemma (4.2): [6]

Let X be a topological space Then G open set in X if and only if cl(G∩cl(A))=cl(G∩A) ∀ A ⊆ X.

Theorem (4.1):

Let X be a topological space and Y open in X. If A β - open in Y, then A β- open in X.

Proof:

Let A be β - open in Y

Let x ∈ A, then there exists U β - open in Y such that x ∈ U ⊆ A. Then

$$U \subseteq \overline{U}^{Y\,oY} = [\overline{\overline{U}\cap Y}^{Y}]^{oY} .$$

$$\subseteq [(\overline{\overline{U}\cap Y})^{oY} \cap Y].$$

$$\subseteq [(\overline{\overline{U}\cap Y})^{oY} \cap Y].$$

$$= [(\overline{\overline{U}\cap Y})^{oY} \cap Y^{o}].$$ by lemma (4.2).

$= \overline{\overline{(\overline{U} \cap Y)}^{\,o}}$

$= \overline{(U \cap Y)^{o}}$ by lemma (4.2).

$\subseteq \overline{U^{o}}$.

Then U is β - open. Since x ∈ U ⊆ X and U is β - open in X. Therefore A β - open in X.

Theorem (4.2):

Let X be a topological space and Y clopen in X. If A B*c - open in Y, then A B*c - open in X.

Proof:

Let A be B*c - open in Y, then A β - open in Y. Since Y clopen in X, then Y open and closed in X, then A β - open in X by Theorem(4.1). Let x ∈ A, then ∃ F closed set in Y such that x ∈ F_y ⊆ A, then F closed set in X by lemma (4.1) (ii). Then x ∈ F_x ⊆ A, hence A B*c – open in X.

Theorem (4.3):

Let X be a topological space and Y open in X. Then:

i) If U m–open in X, then U ∩ Y is m– open in Y.

ii) If U M–open in X, then U∩Y is M–open in Y.

Proof:

i) Let V open in Y ∋ V ⊆ U ∩ Y. T. P V = ∅ or V = U ∩ Y. Then V open in X by lemma (4.1). Since V ⊆ U and U m – open in X, then V = ∅ or V = U. Since V = V ∩ Y = U ∩ Y, then U ∩ Y m – open in Y.

ii) Let V open in Y ∋ U ∩ Y ⊆ V. T. P V = X or V = U ∩ Y. Then V open in X by lemma (4.1). Since U ⊆ V and U M – open in X, then V = X or V = U. Since V = V ∩ Y = U ∩ Y, then U ∩ Y M – open in Y.

Theorem (4.4):

Let X be a topological space and Y clopen in X. Then:

i) If U $mB^{*}c$–open in X, then U∩Y is mB*c– open in Y.

ii) If U MB*c– open in X, then U∩Y is MB*c – open in Y.

Proof : Similarly Theorem(4.3).

Theorem (4.5):

i) Every open sub space of m – T_1 space is m – T_1 space

ii) Every open sub space of M – T_1 space is M – T_1 space

Proof:

i) Let X be m – T_1 space and A is open sub space of X T. P A m – T_1 space

Let a_1, a_2 ∈ A ∋ $a_1 \neq a_2$. Since A ⊆ X and a_1, a_2 ∈ A, then a_1, a_2 ∈ X ∋ $a_1 \neq a_2$. Since m – T_1 space, then ∃ U, V m - open in X ∋ a_1 ∈ U, a_2 ∉ U and a_2 ∈ V, a_1 ∉ V. Since U, V m - open in X and A open sub space of X, then. Let U^{*} = U ∪ A, V^{*} = V ∪ A, then U^{*}, V^{*} are m – open in A by Theorem(4.3) (i). Since a_1 ∈ U, a_2 ∉ U and a_1, a_2 ∈ A, then a_1 ∈ A ∩ U = U^{*} and a_2 ∉ A ∩ U = U^{*}, then a_1 ∈ U^{*}, a_2 ∉ U^{*}. Since a_2 ∈ V, a_1 ∉ V and a_1, a_2 ∈ A, then a_2 ∈ A ∩ V = V^{*} and a_1 ∉ A ∩ V = V^{*}, then a_2 ∈ V^{*}, a_1 ∉ V^{*}. Hence A is m – T_1 space

ii) Similarly part (i)

Remark (4.1):

i) If A m – T_1 space sub space of X, the X not necessary m – T_1 space

ii) If A M – T_1 space sub space of X, the X not necessary M – T_1 space

Example (4.1)

Let X = {1,2,3}, t = {∅, X, {1}, {2}, {1,2} }.

Let A = {1,2}, t_A = {∅, X, {1}, {2}}. Note that

i) A m – T_1 space, but X not m – T_1 space Since 2, 3 ∈ X ∋ 2 ≠ 3, but ∄ U, V m – open in X ∋ 2 ∈ U, 3 ∉ U and 3 ∈ V, 2 ∉ V.

ii) A M – T_1 space, but X not M – T_1 space Since 2, 3 ∈ X ∋ 2 ≠ 3, but ∄ U, V M – open in X ∋ 2 ∈ U,3 ∉ U and 3∈ V,2 ∉ V.

Theorem (4.6):

i) Every open sub space of mβ–T_1 space is mβ–T_1 space

ii) Every open sub space of Mβ–T_1 space is mB*c–T_1 space

Proof:

Similarly Theorem(4.5).

Theorem (4.7):

i) Every clopen sub space of mB*c – T_1 space is mB*c – T_1 space

ii) Every clopen sub space of MB*c – T_1 space is MB*c – T_1 space

Proof:

i) Let X be a mB*c – T_1 space and A is clopen sub space of X T. P A mB*c – T_1 space

Let a_1, a_2 ∈ A ∋ $a_1 \neq a_2$. Since A ⊆ X and a_1, a_2 ∈ A, then a_1, a_2 ∈ X ∋ $a_1 \neq a_2$. Since X mB*c – T_1 space, then ∃ U, V mB*c - open in X ∋ a_1 ∈ U, a_2 ∉ U and a_2 ∈ V, a_1 ∉ V. Since U, V mB*c - open in X and A clopen sub space of X, then. Let $U^* = U \cup A$, $V^* = V \cup A$, then U^*, V^* are mB*c – open in A by Theorem(4.3) (i).

Since $a_1 \in U, a_2 \notin U$ and $a_1, a_2 \in A$, then $a_1 \in A \cap U = U^*$ and $a_2 \notin A \cap U = U^*$, then $a_1 \in U^*$, $a_2 \notin U^*$.

Since $a_2 \in V$, $a_1 \notin V$ and $a_1, a_2 \in A$, then $a_2 \in A \cap V = V^*$ and $a_1 \notin A \cap V = V^*$, then $a_2 \in V^*$, $a_1 \notin V^*$, hence A is mB*c – T_1 space

Remark (4.2):

i) If A mB*c – T_1 space sub space of X, the X not necessary mB*c – T_1 space

ii) If A MB*c – T_1 space sub space of X, the X not necessary MB*c – T_1 space

Example (4.2):

Let X = {a, b, c, d, e}

t = { ∅ , X, {a} , {b}, {e}, {a, b}, {a, e}, {b, e} {a, b, e}, {a, c, d}, {a,b,c,d}, {a, c, d, e} }.

B*co (X) = {∅, X, {b},{e},{b, e}, {a, c, d},{a, b, c, d},{a, c, d, e} }.

Let A = {b, e}, t_A = {∅, A, {b}, {e}}. B*co (A) = t_A Note that:

i) A mB*c – T_1 space, but X not mB*c – T_1 space

Since a, d ∈ X ∋ a ≠ d, but ∄ U, V mB*c – open in X ∋ a ∈ U, d ∉ U and d ∈ V, a ∉ V.

ii) A MB*c – T_1 space, but X not MB*c – T_1 space

Since a, d ∈ X ∋ a ≠ d, but ∄ U, V MB*c – open in X ∋ a ∈ U, d ∉ U and d ∈ V, a ∉ V.

5) Topological properties

Lemma (5.1): [3]

Let F: X → Y be abjection, strongly m – open, m – irresolute function. Then X is m – T_1 space iff Y is m – T_1 space

Theorem (5.1):

Let F: X → Y be abjection, strongly mβ – open. If X mβ – T_1 space, then Y is mβ – T_1 space

Proof:

Let X be mβ – T_1 space T. P Y mβ – T_1 space Let $y_1, y_2 \in Y \ni y_1 \neq y_2$. Since F on to, then $\exists\, x_1, x_2 \in X \ni F(x_1) = y_1, F(x_2) = y_2$. If $x_1 = x_2$, then $F(x_1) = F(x_2)$, then $y_1 = y_2$ which is contradiction, then $x_1 \neq x_2$. Since X mβ – T_1 space, then ∃ U, V mβ- open set in X $\ni x_1 \in U, x_2 \notin U$ and $x_2 \in V, x_1 \notin V$. Since F strongly mβ - open, then F (U) mβ-open in Y, then $y_1 = F(x_1) \in F(U)$, $y_2 = F(x_2) \notin F(U)$ and $y_2 = F(x_2) \in F(V)$, $y_1 = F(x_1) \notin F(V)$. Hence Y mβ–T_1 space

Theorem (5.2):

Let F:X→Y be abjection,mB*c–irresolute function. If Y is mB*c–T_1 space ,then X is mB*c–T_1 space

Proof:

Let X be mB*c – T_1 space T. P X mB*c – T_1 space Let $x_1, x_2 \in X \ni x_1 \neq x_2$ and let $F(x_1) = y_1$, $F(x_2) = y_2$, then $y_1, y_2 \in Y$.

If $y_1 = y_2$, then $F(x_1) = F(x_2)$, then $x_1 = x_2$ which is contradiction, then $y_1 \neq y_2$. Since F bijection, then $x_1 = F^{-1}(y_1)$, $x_2 = F^{-1}(y_2)$. Since Y mB*c – T_1 space, then ∃ U, V mB*c - open set in Y $\ni y_1 \in U, y_2 \notin U$ and $y_2 \in V$, $y_1 \notin V$. Since F mB*c – irresolute function, then $F^{-1}(U)$ mB*c - open in X, then $x_1 = F^{-1}(y_1) \in F^{-1}(U)$, $x_2 = F^{-1}(y_2) \notin F^{-1}(U)$ and $x_2 = F^{-1}(y_2) \in F^{-1}(V)$, $x_1 = F^{-1}(y_1) \notin F^{-1}(V)$. Hence Y mB*c – T_1 space

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حول فضاءات T_1 العضمى والصغرى بأستخدام مجموعات B^*c المفتوحه

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المستخلص :

في هذا البحث قدمنا دراسه حول خصائص الفضاء التبولوجي Maximal T_1-space and Minimal T_1- Space بأستخدام المجموعه B^*c-Open set وأستنتجنا بعض المبرهنات والملاحظات والعلاقات بالنسبه للفضاء MB^*c-T_1-Space و mB^*c-T_1-Space ودرسنا العلاقه بين الفضاء الأعظم MB^*c-T_1-Space والفضاء الأصغر mB^*c-T_1-Space حيث وجدنا أن كل فضاء mB^*c-T_1-Space يؤدي الى MB^*c-T_1-Space لكن العكس لايتحقق بصوره عامه . كذالك قدمنا دراسه حول الصفات الوراثيه والصفات التبولوجيه لهذا الفضاء .

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A Class of Small Injective Modules

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Abstract:

Let R be a ring. In this paper, a right R-module M is defined to be AS-injective if $\text{Ext}^1(R/K\ ,M) = 0$, for any annihilator-small right ideal K of R. We characterize rings over which every right module is AS-injective. Conditions under which the class of AS-injective right R-modules (ASI_R) is closed under quotient (resp. pure submodules, direct sums) are given. Finally, we study the definability of the class ASI_R.

Keywords: Injective module, Definable class, a-small right ideal, Pure submodule.

Mathematics subject classification: 13C11,16D50,16D10.

1. Introduction

Throughout R is an associative ring with identity and all modules are unitary R-modules. If not otherwise specified, by a module (resp. homomorphism) we will mean a right R-module (resp. right R-homomorphism). We use R-Mod (resp. Mod-R) to denote to the class of left (resp. right) R-modules. If $Y \subseteq R$, then $r(Y) = \{r \in R \mid Yr = 0\}$ (resp. $l(Y) = \{r \in R \mid rY = 0\}$)stands for the right (resp. left) annihilator of Y in R. We will use M^* to denote the character module $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ of a module M. Let $\mathcal{G}$ (resp. $\mathcal{F}$) be a class of right (resp. left) R-modules. A pair $(\mathcal{F}, \mathcal{G})$ is called almost dual pair if the class $\mathcal{G}$ is closed under direct products and summands, and for any left R-module M, $M \in \mathcal{F}$ if and only if $M^* \in \mathcal{G}$ [11, p. 66]. An exact sequence $0 \to A \overset{\alpha}{\to} B \overset{\beta}{\to} C \to 0$ of right R-modules is said to be pure if the sequence $0 \to \mathrm{Hom}_R(N, A) \to \mathrm{Hom}_R(N, B) \to \mathrm{Hom}_R(N, C) \to 0$ is exact, for every finitely presented right R-module N and we called that $\alpha(A)$ is a pure submodule of B [18]. A right R-module M is called FP-injective if every monomorphism $\alpha: M \to N$ is pure. A right R-module M is called pure injective if M is injective with respect to all pure short exact sequences [18]. Recall that a subclass $\mathcal{G}$ of Mod-R is called definable if it is closed under pure submodules, direct limits and direct products [14]. A right ideal X of a ring R is called small in R if $X + Y \neq R$, for any proper right ideal Y of R [8]. A right R-module M is called small injective if $\mathrm{Ext}^1(R/K, M) = 0$, for any small right ideal K of R. A right ideal I of R is called annihilator-small (a-small) and denoted by $I \subseteq^a R_R$ if for any right ideal K of R with $I + K = R$, then $l(K) = 0$ [13].

The sum of all the annihilator-small right ideals of a ring R is called the right AS-ideal of a ring R and denoted by A_r [13].

We refer the reader to [1, 7, 8, 14, 18], for general background materials.

In section 2 of this paper, we introduce the class of AS-injective modules. This class of modules lies between injective modules and small injective modules. We first characterize rings over which every module is AS-injective. Over a commutative ring R, we prove the equivalence of the following statements: (1) $A_r = 0$. (2) Every module is AS-injective. (3) Every principal a-small right ideal of R is AS-injective. (4) Every simple module is AS-injective and $A_r \subseteq^a R_R$. Conditions under which the class of AS-injective right R-modules (ASI_R) is closed under quotient are given. For instance, we prove that the following statements are equivalent: (1) The class ASI_R is closed under quotient. (2) If $K \subseteq^a R$, then K is projective. (3) ASI_R contains all sums of any two AS-injective submodules of any module. Also, we show that the class ASI_R is closed under pure submodules if and only if all a-small right ideals in R are finitely generated if and only if all FP-injective modules are AS-injective. Finally, we give conditions such that any direct sum of modules in the class ASI_R is also belong to ASI_R. For instance, we prove that if $A_r \subseteq^a R_R$, then the following are equivalent. (1) A_r is a noetherian module. (2) The class ASI_R is closed under direct sums.

Section 3 studies the definability of the class ASI_R. It is shown that the following assertions are equivalent: (1) ASI_R is definable. (2) The class ASI_R is closed under pure submodules and pure homomorphic images.

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(3) Every a-small right ideal in R is finitely presented. (4) A module $M \in ASI_R$ if and only if $M^{**} \in ASI_R$. Finally, we prove that if the class ASI_R is a definable, then the following are equivalent. (1) The class of flat left R-modules and the class $\{M \in R\text{-Mod} \,|\, M^* = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \in ASI_R\}$ are coincide. (2) Each module in ASI_R is FP-injective. (3) Each pure-injective module in ASI_R is injective.

2. *AS*-Injective modules

Definition 2.1. A module M is said to be annihilator-small injective (shortly, AS-injective), if $\mathrm{Ext}^1(R/K, M) = 0$, for any annihilator-small right ideal K of R; equivalently, if K is any annihilator-small right ideal in R, then any R-homomorphism $f: K \longrightarrow M$ extends to R_R. A ring R is said to be right AS-injective if R_R is AS-injective.

We will use ASI_R to denote to the class of AS-injective right R-modules.

Examples 2.2.

(1) It is clear that AS-injectivity implies small injectivity, but $\mathbb{Z}$ is a small injective $\mathbb{Z}$-module [17] and clearly, it is not AS-injective. Thus the class of small injective modules contains properly the class of AS-injective modules.

(2) All injective modules are AS-injective and generally the converse is not true, for example, let $\{F_i\}_{i\in I}$ be a family of fields and let $R = \prod_{i\in\Lambda} F_i$ be the ring product of F_i, for all $i \in \Lambda$, where addition and multiplication are define componentwise and let $K = \bigoplus_{i\in\Lambda} F_i$. If Λ is infinite, then K_R is not itself injective by [8, p. 140], but K_R is AS-injective, since $A_r = 0$. Therefore, AS-injective module is a proper generalization of injective modules.

Hence $Inj_R \subsetneq ASI_R \subsetneq SI_R$, where Inj_R (resp. SI_R) is the class of injective (resp. small injective) right R-modules.

Remarks 2.3.

(1) The two classes Inj_R and ASI_R are coinciding, when R is an integral domain, since all proper right ideals are a-small in any integral domain.

(2) All finitely generated $\mathbb{Z}$-modules are not AS-injective and this follows from (1) and the fact that every non-trivial finitely generated $\mathbb{Z}$-module is not injective [7, p.31]. Also, we have from [17, Theorem 2.8] that any $\mathbb{Z}$-module is small injective.

(3) From (1) and [9, p.410], we have that any ring R is a field if and only if it is an AS-injective integral domain.

Proposition 2.4. The class of AS-injective modules (ASI_R) is closed under direct summands, direct products and isomorphic copies.

Proof. Clear. □

Theorem 2.5. Consider the following conditions for a ring R.

(1) $A_r = 0$.

(2) All modules are AS-injective.

(3) All principal a-small right ideals of R are AS-injective.

(4) All principal a-small right ideals of R are direct summand in R_R.

(5) All simple modules are AS-injective and $A_r \subseteq^a R_R$.

Then (1) and (5) are equivalent and (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4). Moreover, if R is commutative, then (4) implies (1).

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Proof. (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) and (1) $\Longrightarrow$ (5) are obvious.

(3) $\Longrightarrow$ (4). Let $aR \subseteq^a R$, where $a \in R$. By hypothesis, aR is AS-injective and so there is a homomorphism $\alpha: R \to aR$ such that $\alpha i = I_{aR}$, where $I_{aR}: aR \to aR$ is the identity homomorphism and $i: aR \to R$ is the inclusion mapping. Thus aR is a direct summand in R_R.

(4) $\Longrightarrow$ (1). Let R be a commutative ring. Assume that $A_r \neq 0$, thus there is $(0 \neq) a \in A_r$. By hypothesis, $A_r \subseteq^a R_R$ and hence Lemma 1 in [13] implies $aR \subseteq^a R_R$. By hypothesis, aR is a direct summand in R_R and hence there exist a right ideal K with $aR \oplus K = R_R$. Since $aR \subseteq^a R_R$, $r(K) = 0$. Since $aR + K = R$, we have $r(aR \cap K) = r(aR) + r(K)$ and hence $r(aR) = R$. Thus $aR = 0$, a contradiction. Therefore, $A_r = 0$.

(5) $\Longrightarrow$ (1)Assume that $A_r \neq 0$, thus there is $(0 \neq) a \in A_r$. If $A_r + r(a) \neq R$, then $A_r + r(a) \subseteq C$, for some maximal right ideal C of R. Thus R/C is a simple right R-module. By hypothesis, R/C is an AS-injective module. Define $\alpha: aR \to R/C$ by $\alpha(ar) = r + C$. Clearly, α is a well-defined right R-homomorphism. By AS-injectivity, there exist $b \in R$ with $1 + C = ba + C$ and hence $1 - ba \in C$. Since $a \in A_r$ and A_r is a two sided ideal (by [13, Theorem 9 (1)]), we have $ba \in C$. Thus $C = R$, a contradiction. Therefore, $A_r + r(a) = R$. Since $A_r \subseteq^a R_R$ (by hypothesis), $l(r(a)) = 0$, so that $r(l(r(a))) = R$. By [1, Proposition 2.15, p.37], $r(a) = R$ and hence $a = 0$, a contradiction. Thus $A_r = 0$. □

Recall that a ring R is called regular if for any $x \in R$, there is an element $y \in R$ such that $x = xyx$ [8]

Corollary 2.6. If R is a commutative regular ring, then every module is AS-injective and $A_r = 0$.

Proof. By [8, Theorem 10.4.9, p. 262] and Theorem 2.5. □

It is not true in general that if $K \subseteq^a R_R$, then K is a projective right R-module, for example, if $R = Z_4$ and $K = 2R$, then $K \subseteq^a R_R$ but it is not projective right R-module.

Theorem 2.7. For a ring R, the following are equivalent.

(1) If $K \subseteq^a R_R$, then K is projective.

(2) The class ASI_R is closed under quotient.

(3) ASI_R contains all quotients of injective modules.

(4) ASI_R contains all sums of any two AS-injective submodules of any module.

(5) ASI_R contains all sums of any two injective submodules of any module.

Proof. (2) $\Longrightarrow$ (3) and (4) $\Longrightarrow$ (5) are obvious.

(1) $\Longrightarrow$ (2) Let $\alpha: N \longrightarrow M$ be any epimorphism, where N is an AS-injective module and M is any module. Let $\lambda: K \longrightarrow M$ be any homomorphism, where $K \subseteq^a R_R$. By hypothesis, K is projective and hence there is a homomorphism $\beta: K \longrightarrow N$ such that $\alpha\beta = \lambda$. By AS-injectivity of N, there is a homomorphism $\gamma: R \to N$ with $\gamma i = \beta$, where $i: K \to R$ is the inclusion mapping. Put $\varphi = \alpha\gamma: R \longrightarrow M$, so that $\varphi i = \alpha\gamma i = \alpha\beta = \lambda$ and hence M is an AS-injective module.

(3) $\Longrightarrow$ (1) Let $K \subseteq^a R_R$. Let $\alpha: E \to N$ be an epimorphism (where E is an injective module) and $\beta: K \to N$ a homomorphism. By hypothesis, $N \in ASI_R$ and hence there is a homomorphism $\lambda: R \longrightarrow N$ with $\lambda i = \beta$, where $i: K \to R$ is the inclusion mapping. By projectivity of R_R, there is a

homomorphism $\gamma: R \to E$ such that $\alpha\gamma = \lambda$. Let $\tilde{\alpha}: K \longrightarrow E$ be the restriction of γ over K. Clearly, $\alpha\tilde{\alpha} = \beta$ and hence from Proposition 5.2.10 in [2, p.148] we get that K is projective.

(2) $\Longrightarrow$ (4) Let M_1 and M_2 be two AS-injective submodules of module M. By Proposition 2.4, $M_1 \oplus M_2 \in ASI_R$. Since $M_1 + M_2$ is a homomorphic image of $M_1 \oplus M_2$, we have $M_1 + M_2 \in ASI_R$, by hypothesis.

(5) $\Longrightarrow$ (3). By similar argument as in the proof of Theorem 2.14 ((6) $\Longrightarrow$ (3)) in [12]. □

Proposition 2.8. For a ring R, consider the following conditions.

(1) Every module is AS-injective.

(2) R_R is AS-injective and the class ASI_R is closed under quotient.

(3) For any $x \in R$, if $xR \subseteq^a R_R$, then there is $y \in R$ such that $x = xyx$.

Then (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) and if R is commutative, then (3) implies (1).

Proof. (1) $\Longrightarrow$ (2). Clear.

(2) $\Longrightarrow$ (3). Let $x \in R$ such that $xR \subseteq^a R_R$. Since ASI_R is closed under quotient (by hypothesis), xR is projective, by Theorem 2.7. Define $\alpha: R \to xR$ by $\alpha(r) = xr$, for all $r \in R$. Clearly, α is an epimorphism, so that there is a homomorphism $f: xR \to R$ with $\alpha f(a) = a$, for all $a \in xR$. Since R_R is AS-injective (by hypothesis), there is a homomorphism $g: R \to R$ such that $gi = f$, where $i: xR \to R$ is the inclusion mapping. Thus $x = \alpha\big(f(x)\big) = \alpha\big(g(x)\big) = xyx$, where $y = g(1) \in R$.

(3) $\Longrightarrow$ (1). Suppose that R is a commutative ring. Let $xR \subseteq^a R_R$, where $x \in R$. By hypothesis, there is $y \in R$ with $x = xyx$. Let $e = xy$. Clearly, e is an idempotent of R and $xR = eR$, so that xR is a direct summand of R_R. Therefore, the result follows by Theorem 2.5. □

Proposition 2.9. For a ring R, the following are equivalent.

(1) All a-small right ideals in R are finitely generated.

(2) The class ASI_R is closed under pure submodules.

(3) All FP-injective modules are AS-injective.

Proof. (1) $\Longrightarrow$ (2). Let $M \in ASI_R$ and K a pure submodule of M. Let $I \subseteq^a R_R$, thus the hypothesis implies that I is finitely generated and so R/I is a finitely presented. Hence the sequence $\mathrm{Hom}_R(R/I, M) \to \mathrm{Hom}_R(R/I, M/K) \longrightarrow 0$ is exact. By [6, Theorem XII.4.4 (4), p. 491], the exact sequence $\mathrm{Hom}_R(R/I, M) \longrightarrow \mathrm{Hom}_R(R/I, M/K) \longrightarrow \mathrm{Ext}^1(R/I, K) \longrightarrow \mathrm{Ext}^1(R/I, M)$ and so $\mathrm{Ext}^1(R/I, K) = 0$. Thus, $K \in ASI_R$ and hence the class ASI_R is closed under pure submodules.

(2) $\Longrightarrow$ (3). If M is any FP-injective module, then M is a pure submodule of an AS-injective module. By hypothesis, $M \in ASI_R$.

(3) $\Longrightarrow$ (1). Let $I \subseteq^a R_R$ and $\alpha: I \to M$ a homomorphism, where M is an FP-injective module. By hypothesis, M is AS-injective and hence α extends to R_R. By [4], I is finitely generated. □

Corollary 2.10. If each a-small right ideal in a ring R is finitely generated, then the class ASI_R is closed under direct sums.

Proof. Let $\{M_i \mid i \in I\}$ be a subclass of ASI_R. By Proposition 2.4, $\prod_{i\in I} M_i \in ASI_R$. By [14, Proposition 2.1.10, p. 57], $\bigoplus_{i\in I} M_i$ is a pure submodule in $\prod_{i\in I} M_i$ and hence $\bigoplus_{i\in I} M_i \in ASI_R$, by Proposition 2.9. □

Theorem 2.11. For a ring R, consider the following conditions.

(1) A_r is a noetherian module.

(2) The class ASI_R is closed under direct sums.

(3) $M^{\mathbb{N}}$ is AS-injective, for any AS-injective module M_R.

(4) $M^{\mathbb{N}}$ is AS-injective, for any injective module M_R.

Then $(1) \Longrightarrow (2) \Longrightarrow (3) \Longrightarrow (4)$ and if $A_r \subseteq^a R_R$, then $(4) \Longrightarrow (1)$.

Proof. $(2) \Longrightarrow (3)$ and $(3) \Longrightarrow (4)$ are clear.

$(1) \Longrightarrow (2)$. By [13, Theorem 9(1)] and Corollary 2.10.

$(4) \Longrightarrow (1)$. Let $A_r \subseteq^a R_R$ and let $K_1 \subseteq K_2 \subseteq \cdots$ be a chain of right ideals of R with $K_i \subseteq A_r$. Let $E = \bigoplus_{i=1}^{\infty} E_i$, where $E_i = E(R/K_i)$. For every $i \geq 1$, put $M_i = \prod_{j=1}^{\infty} E_j = E_i \oplus \left(\prod_{\substack{j=1\\ j\neq i}}^{\infty} E_j\right)$, thus M_i is injective. By hypothesis, $\bigoplus_{i=1}^{\infty} M_i = (\bigoplus_{i=1}^{\infty} E_i) \oplus \left(\bigoplus_{i=1}^{\infty} \prod_{\substack{j=1\\ j\neq i}}^{\infty} E_j\right)$ is AS-injective. By Proposition 2.4, E is AS-injective. Define $\alpha: \bigcup_{i=1}^{\infty} K_i \longrightarrow E$ by $\alpha(x) = (x + K_i)_i$. Clearly, α is a well-defined homomorphism. By hypothesis, $A_r \subseteq^a R_R$ and hence Lemma 1 in [13] implies that $\bigcup_{i=1}^{\infty} K_i \subseteq^a R_R$. Thus α extends to a homomorphism $\beta: R \longrightarrow E$ and hence $\beta(R) \subseteq \bigoplus_{i=1}^{n} E(R/K_i)$ for some $n \in \mathbb{N}$, since R is finitely generated. Then $\alpha(\bigcup_{i=1}^{\infty} K_i) \subseteq \bigoplus_{j=1}^{n} E(R/K_i)$.

So, if $a \in \bigcup_{i=1}^{\infty} K_i$, then $a \in K_m$ for all $m > n$, and hence $\bigcup_{i=1}^{\infty} K_i = K_{n+1}$. Therefore, the chain $K_1 \subseteq K_2 \subseteq \cdots$ terminates at K_{n+1} and hence A_r is a noetherian module. □

Corollary 2.12. If $A_r \subseteq^a R_R$, then the following are equivalent.

(1) A_r is a noetherian module.

(2) Direct sum of injective modules is AS-injective.

Lemma 2.13. If R satisfies ACC (ascending chain condition) on a-small right ideals of R, then $A_r \subseteq^a R_R$

Proof. Let $\mathcal{H} = \{K \mid K \subseteq^a R_R\}$. Thus $\mathcal{H}$ has a maximal element, say N (by Zorn's lemma). Since $A_r = \sum_{K \in \mathcal{H}} K$, it follows that $A_r = N$ and so $A_r \subseteq^a R_R$. □

Proposition 2.14. For a ring R, the following are equivalent.

(1) R satisfies ACC on a-small right ideals.

(2) A_r is a noetherian R-module.

(3) $M^{\mathbb{N}}$ is AS-injective, for any injective module M_R and $A_r \subseteq^a R_R$.

Proof. $(1) \Longrightarrow (2)$. Let $N_1 \subseteq N_2 \subseteq \cdots$ be a chain of right ideals of R in A_r. By Lemma 2.13, $A_r \subseteq^a R_R$. By [13, Lemma 1], N_i are a-small right ideals. By hypothesis, the chain $N_1 \subseteq N_2 \subseteq \cdots$ terminates and hence A_r is a noetherian R-module.

$(2) \Longrightarrow (1)$. By [13, Theorem 9(1)].

$(2) \Longrightarrow (3)$. By Theorem 2.11 and Lemma 2.13.

$(3) \Longrightarrow (2)$. By Theorem 2.11. □

3. Definability of the class ASI_R

For any class $\mathcal{G}$ of right R-modules, we will set $\mathcal{G}^+ = \{M \in \text{Mod-}R \mid M$ is a pure submodule of a module in $\mathcal{G}\}$ and $\mathcal{G}^{\ominus} = \{M \in R\text{-Mod} \mid M^* = \text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \in \mathcal{G}\}$.

Proposition 3.1. The pair $((ASI_R)^{\ominus}, ASI_R)$ is an almost dual pair over a ring R.

Proof. By proposition 2.4, the class ASI_R is closed under direct summands and direct products. By [11, Proposition 4.2.11, p.72], the pair $((ASI_R)^{\ominus}, ASI_R)$ is an almost dual pair over a ring R. □

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Corollary 3.2. Consider the following conditions for the class ASI_R over a ring R.

(1) The class ASI_R is definable.

(2) $(ASI_R, (ASI_R)^{\ominus})$ is an almost dual pair over a ring R.

(3) $(ASI_R)^* \subseteq (ASI_R)^{\ominus}$.

(4) $(ASI_R)^{**} \subseteq ASI_R$.

(5) The class ASI_R is closed under pure homomorphic images.

Then $(1) \Leftrightarrow (2)$, $(1) \Rightarrow (3)$, $(1) \Rightarrow (5)$ and $(3) \Leftrightarrow (4)$. Moreover, if all a-small right ideals in R are finitely generated, then all five conditions are equivalent.

Proof. $(1) \Leftrightarrow (2)$. By Proposition 3.1 and [11, Proposition 4.3.8, p. 89].

$(1) \Rightarrow (3)$. Since ASI_R is a definable class, it is closed under pure submodules and hence $(ASI_R)^+ = ASI_R$. Since $((ASI_R)^{\ominus}, ASI_R)$ is an almost dual (by Proposition 3.1), it follows from [11, Theorem 4.3.2, p.85], that $(ASI_R)^* \subseteq (ASI_R)^{\ominus}$.

$(1) \Rightarrow (5)$. By [14, 3.4.8, p. 109].

$(3) \Leftrightarrow (4)$. By Proposition 3.1 and [11, Theorem 4.3.2, p.85].

$(4) \Rightarrow (1)$ and $(5) \Rightarrow (1)$. Suppose that all a-small right ideals in R are finitely generated. By Proposition 2.9, the class ASI_R is closed under pure submodules and hence $(ASI_R)^+ = ASI_R$. Thus the results follow from [11, Theorem 4.3.2, p.85]. □

Corollary 3.3. If every AS-injective modules is pure-injective, then the following statements are equivalent for a class ASI_R over a ring R.

(1) ASI_R is definable.

(2) ASI_R is closed under direct sums.

(3) $(ASI_R)^+ = ASI_R$.

(4) Each a-small right ideal in R is finitely generated.

Proof. The equivalence of (1), (2) and (3) follows from Proposition 3.1 and [11, Theorem 4.5.1, p.103].

$(1) \Leftrightarrow (4)$. By Proposition 3.1, Proposition 2.9 and [11, Theorem 4.5.1, p.103]. □

Lemma 3.4. A left R-module $M \in (ASI_R)^{\ominus}$ if and only if $\mathrm{Tor}_1(R/I, M) = 0$, for any a-small right ideal I of a ring R.

Proof. Let M be a left R-module and $I \subseteq^a R_R$. By [5, Theorem 3.2.1, p.75], $\mathrm{Ext}^1(R/I, M^*) \cong (\mathrm{Tor}_1(R/I, M))^*$, so that $\mathrm{Tor}_1(R/I, M) = 0$ if and only if $M^* \in ASI_R$. Hence $({}_RASF, ASI_R)$ is an almost dual, where ${}_RASF = \{M \in R\text{-Mod} \mid \mathrm{Tor}_1(R/I, M) = 0$, for any a-small right ideal I of $R\}$. By [11, Proposition 4.2.11, p.72], $(ASI_R)^{\ominus} = {}_RASF$. □

A right R-module M is called n-presented if there is an exact sequence $F_n \to F_{n-1} \to \cdots \to F_0 \to M \to 0$, with each F_i is a finitely generated free right R-modules [3].

Theorem 3.5. The following statements are equivalent for a class ASI_R over a ring R.

(1) ASI_R is definable.

(2) The class ASI_R is closed under pure submodules and pure homomorphic images.

(3) Every a-small right ideal in R is finitely presented.

(4) A module $M \in ASI_R$ if and only if $M^* \in (ASI_R)^{\ominus}$.

(5) A module $M \in ASI_R$ if and only if $M^{**} \in ASI_R$.

Proof. $(1) \Rightarrow (2)$. By [14, 3.4.8, p. 109].

$(2) \Rightarrow (3)$. Let M be any FP-injective module, thus there is a pure exact sequence $0 \to M \xrightarrow{i} E \xrightarrow{\pi} E/M \to 0$, where E is an injective right R-module. By hypothesis, $E/M \in ASI_R$.

Let $K \subseteq^a R_R$, thus $\mathrm{Ext}^1(R/K, E/M) = 0$. By [6, Theorem 4.4 (4), p. 491], the sequence $0 = \mathrm{Ext}^1(R/K, E/M) \longrightarrow \mathrm{Ext}^2(R/K, M) \longrightarrow \mathrm{Ext}^2(R/K, E) = 0$ is exact and hence $\mathrm{Ext}^2(R/K, M) = 0$. By [13, Theorem 4.4 (3), p. 491], the sequence $0 = \mathrm{Ext}^1(R, M) \longrightarrow \mathrm{Ext}^1(K, M) \longrightarrow \mathrm{Ext}^2(R/K, M) = 0$ is exact, so that $\mathrm{Ext}^1(K, M) = 0$. By hypothesis, ASI_R is closed under pure submodules, so that K is finitely generated by Proposition 2.9 and hence [4, Proposition, p. 361] implies that K is finitely presented.

(3) $\Longrightarrow$ (1). Let $M \in ASI_R$. Let $K \subseteq^a R_R$, thus K is finitely presented (by hypothesis) and hence there is an exact sequence $F_2 \xrightarrow{\alpha_2} F_1 \xrightarrow{\alpha_1} K \longrightarrow 0$, where F_1, F_2 are finitely generated free right R-modules. Let $\beta = i\alpha_1$,where $i: K \longrightarrow R$ is the inclusion mapping, thus the sequence $F_2 \xrightarrow{\alpha_2} F_1 \xrightarrow{\beta} R \xrightarrow{\pi} R/K \longrightarrow 0$ is exact, where $\pi: R \longrightarrow R/K$ is the natural epimorphism. Hence R/K is a 2-presented module, so that from [3, Lemma 2.7 (2)] we have $\mathrm{Tor}_1(R/K, M^*) \cong (\mathrm{Ext}^1(R/K, M))^* = 0$. By Lemma 3.4, $M^* \in (ASI_R)^{\ominus}$ and hence $(ASI_R)^* \subseteq (ASI_R)^{\ominus}$. By hypothesis, every a-small right ideal in R is finitely generated, so that ASI_R is closed under pure submodules by Proposition 2.9. By Theorem 3.2, ASI_R is a definable class.

(1) $\Longrightarrow$ (4). By Corollary 3.2, $(ASI_R, (ASI_R)^{\ominus})$ is an almost dual pair and hence a module $M \in ASI_R$ if and only if $M^* \in (ASI_R)^{\ominus}$.

(4) $\Longrightarrow$ (5). By hypothesis, $(ASI_R)^* \subseteq (ASI_R)^{\ominus}$. By Corollary 3.2, $(ASI_R)^{**} \subseteq ASI_R$. Hence for any right R-module M, if $M \in ASI_R$, then $M^{**} \in ASI_R$. Conversely, if $M^{**} \in ASI_R$, then $M^* \in (ASI_R)^{\ominus}$. By hypothesis, $M \in ASI_R$.

(5) $\Longrightarrow$ (1). Let N be a FP-injective module, thus there is a pure exact sequence $0 \to N \to E \to E/N \to 0$, where E is an injective right R-module. By [18, 34.5, p.286], the sequence $0 \to N^{**} \to E^{**} \to (E/N)^{**} \to 0$ is split. By hypothesis, $E^{**} \in ASI_R$ and hence $N^{**} \in ASI_R$. By hypothesis, $N \in ASI_R$ so that ASI_R is closed under pure submodules by Proposition 2.9. Thus ASI_R is definable class by Corollary 3.2. □

Note that if the class ASI_R is closed under pure submodules, then $(ASI_R)^+ = ASI_R$. Thus we have the following corollary.

Corollary 3.6. The class ASI_R is a definable if and only if it is closed under pure submodules and the class $(ASI_R)^+$ is a definable.

Corollary 3.7. If the class ASI_R is a definable, then the following are equivalent.

(1) The class of flat left R-modules and the class $(ASI_R)^{\ominus}$ are coincide.

(2) Every module in ASI_R is FP-injective.

(3) Every pure-injective module in ASI_R is injective.

Proof. (1) $\Longrightarrow$ (2). Let $M \in ASI_R$, thus $M^* \in (ASI_R)^{\ominus}$ by Corollary 3.2. By hypothesis, M^* is a flat left R-module and hence Proposition 3.54 in [15, p.136] implies that M^{**} is injective. Since M is a pure submodule in M^{**}, we have M is FP-injective by [18, 35.8, p.301].

(2) $\Longrightarrow$ (3). Let M be any pure-injective module in ASI_R. Let $\mathcal{E}: 0 \to M \longrightarrow M' \longrightarrow M'' \to 0$ be any exact sequence. By hypothesis, M is FP-injective. By [16, Proposition 2.6], the sequence $\mathcal{E}$ is pure and hence pure-injectivity of M implies that the sequence $\mathcal{E}$ is split by [18, 33.7, p. 279] . Therefore, M is injective.

(3) $\Longrightarrow$ (1). Let M be a flat left R-module, thus $\mathrm{Tor}_1(N, M) = 0$, for any right R-module N. By Lemma 3.4, $M \in (ASI_R)^{\ominus}$. Conversely, if $M \in (ASI_R)^{\ominus}$, then $M^* \in ASI_R$. By [14, Proposition 4.3.29, p. 149], M^* is a pure injective module. By hypothesis, M^* is injective and hence M is flat by [10, Theorem, p.239]. □

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صنف من المقاسات الأغمارية الصغيرة

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المستخلص:

لتكن R حلقة. في هذا البحث المقاس الايمن M على الحلقة R عرف ليكون اغماري من النمط $AS-$ اذا كان $\mathrm{Ext}^1(R/K\,,M)=0$ لأي مثالي صغير- مبطل ايمن K من الحلقة R. تشخيص الحلقات التي يكون كل مقاس معرف عليها هو اغماري من النمط $AS-$. الشروط التي بموجبها يكون صنف المقاسات الأغمارية من النمط $AS-$ اليمنى على الحلقة R (ASI_R) مغلق تحت القسمة (بالنسبة الى: المقاسات الجزئية النقية، الجمع المباشر) قد اعطيت. اخيراً، ندرس قابلية التعريف للصنف (ASI_R).

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Design and Construct Intelligent Tank "Water Level Sensor"

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Abstract

The system proposed in this paper design a system of intelligent reservoir " water level sensor" helps reduce thewaste percentage of water. Through the automatic control and control of the water level by determining the water proportion of the reservoir in terms of emptiness and fullness and is carried out using the concept of artificial intelligence with micro controls, where the use of a simple and cheap control is arduino and is stored and stored information is written in IDL program to determine the percentage of fullness and lack Depending on the need of the user of the reservoir with the management of electronic parts associated with the Arduino to fill the tank when it is free of water and stop the filling process when reaching the specified level of fullness while monitoring the level of water during use.

This system can be applied at the level of tanks reservoirs Cities and irrigation tanks and reservoirs for agricultural land with the development of electronic parts used in the control system to achieve the desired goal in preserving the amount of safe water for drinking and non-potable.

Keywords: Water Tank , Arduino, microcontroller , Control System, Water Level Sensor

1. Introduction

Because many countries of the world suffer from problems of low water quantity, it is important to manage water level in a modern way in all areas of life in agriculture and industry and reduce waste [1][2].

And to achieve intelligent water management both at the level of dam reservoirs or at the level of reservoirs with local use for individuals in homes or water reservoirs for irrigation in agricultural lands that do not rely on rain water throughout the year[3]. The importance of intelligent control and management of the level of water in the reservoirs, which is of economic importance in preserving water and not wasting it, is mentioned[4]. In addition, this system helps in the industrial side to monitor and follow up the level of different types of dangerous fluids, which are preferred to follow them[5][6].

It is possible to achieve the level of water monitoring in a number of highways, which require many equipment, but in the system used in this paper is explained a simple system of control depends on the tools available and cheap, including sensors and ultrasonic sensor to monitor the water level in the tank Water-level ultrasonic and water level recording on the LCD screen to monitor the water level and use Bluetooth to send information about the water level of the screen by mobile phone, but the use of Bluetooth to monitor a certain distance commensurate with the extent of the Bluetooth broadcast .

It is also possible to control the filling of the tank when the water is connected to the access and stop the flow of water in the tank when it reaches the full extent specified by connecting the water supply and sensor and the screen of the plug and the Bluetooth chip by Arduino, which is fed information about the water level required in the reservoir in terms of fullness The library has been written in c language for each electronic part of the parts associated with Arduino to organize the work between the parts and achieve the desired goal of intelligent reservoir management and knowledge of water level in the reservoir and the optimal use of water and non-waste[7][8].

In this paper will show the following parts . At the first part' the basic concepts of system design'. Second part focuses on' design and implementation'. Third part deals with ' Design and Implementation part '. Fourth part describes conclusion and future work.

2. The Basic Concepts of System Design :

In this section, the basic parts that aggregated together to create the intelligent tank system and water level control will be explained below:

2.1 Water level monitor :

Water level monitor is consist of (LCD and cellular devices by using Bluetooth technique)

- Monitoring by LCD: A liquid-crystal display (LCD) is a flat electronic screen that produces light from liquid crystals that cannot send light directly so it uses a reflector to produce monochrome images. LED screens consist of 7 parts and use technology similar to the technology used in digital clocks. And displays arbitrary images or fixed images with low information content such as on the computer screen. However, other displays have larger elements than arbitrary images with a large number of pixels[9].

LCD interference in many applications such as TV sets, indoor and outdoor signs, and cockpit in aircraft And enter into mobile devices such as smart phones, watches, digital cameras, manual calculator. The sizes of the screens range from small to large, such as small digital clocks and large TV screens.

Because the LCD monitors do not use phosphorus, the fixed image does not burn if exposed for a long time on the screen, such as the inner mark of the aircraft table. LCDs are the best-selling CRT monitors for low power consumption.

Figure 1: Screen Shot of LCD

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- Monitoring by Bluetooth : Bluetoothis a wireless technology found in mobile devices and computer peripherals is easier than WIFI .It is able to access the signal emanating from the devices in a short distance. Anyone can access Bluetooth because it is unauthorized. The frequencies that use in Bluetooth are '2.4 to 2.485 GHz band' .

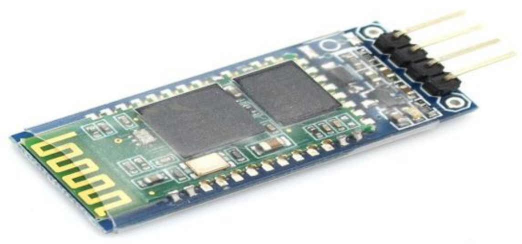

Figure 2: Bluetooth

2.2 Water level Sensor :

Water level Sensor is consist of **(**Ultrasonic Sensor **)**

The Ultrasonic Sensor sends out sound in a high-frequency pulse and then times how long it takes for the echo of the sound to reflect back. The sensor has 2 openings on its front. One opening transmits ultrasonic waves, (like a tiny speaker), the other receives them, (like a tiny microphone) [10].
The speed of sound is approximately 341 meters (1100 feet) per second in air. The ultrasonic sensor uses this information along with the time difference between sending and receiving the sound pulse to determine the distance to an object. It uses the following mathematical equation:
'Distance = Time x Speed of Sound divided by 2'
'Time = the time between when an ultrasonic wave is transmitted and when it is received'
You divide this number by 2 because the sound wave has to travel to the object and back. The HC-SR04 Ultrasonic Sensor (shown in fig.3) is a very affordable proximity/distance sensor that has been used mainly for object avoidance in various robotics projects . It essentially gives your Arduino eyes / special awareness and can prevent your robot from crashing or falling off a table. It has also been used in turret applications, water level sensing, and even as a parking sensor. This simple project will use the HC-SR04 sensor with an Arduino and a Processing sketch to provide a neat little interactive display on your computer screen [11].

Figure 3. HC-SR04 sensor

It has 4 pins (Vcc , Trig , Echo , GND)
Vcc : Connects to 5V of positive voltage for power
Trig : A pulse is sent here for the sensor to go into ranging mode for object detection
Echo : The echo sends a signal back if an object has been detected or not , if a signal is returned an Robjects has been detected , if not , no objects has been detected .
GND: complete electrical pathway of the power

Electrical specifications:

Working Voltage 5V DC
Working Current 15 mA
Working Frequency 40 Hz
Max Range 4 m
Min Range 2 cm
Measuring Angle 15 degree
Trigger Input Signal 10 uS TTL pulse
Echo Output Signal Input TTL level signal and the range in proportion
Dimensions 45*20*15 mm

2.3 Water level controller :

Water level control is consist of **(**Arduino microcontroller **)**

- The Arduino is the best microcontroller because it's easy to use and powerful board. Arduino is like a small computer achieve interact and control electromechanical devices[12]. Arduino works much better than a conventional desktop computer. Technically, arduino uses an open source software platform the system depend on a control board microcontroller (arduino) and Arduino IDEprogramas development environment for the' writing software'.

The strength of Arduino is in noticeable ability to management with other electronic parts, such as 'switches' or 'sensors', and use them to gain different data such as temperature or light intensity, furthermore being very effective in Control of 'motors', 'LEDs', 'lamps' and many other electronic parts. the execute of any Arduino projects via the 'computer-link' and perform a transaction on the device or the software can be run independently[13].

Figure 4:- Arduinouno

The Arduino has Characteristics make its potential is high and the able to control various electronic parts and software The arduinoorder is designed to meet the needs of all, 'professionals, professors and students' .these Characteristics comprise ('Simplicity,'Cheap Price','Open Source hardware','Open Source Software').and there are more types for arduino ('ARDUINO UNO','ARDUINO NANO','ARDUINO LILYPAD','ARDUINO MEGA 2560','ARDUINO MINI','ARDUINO BT').

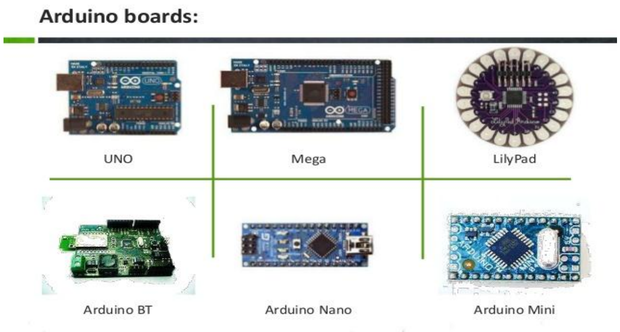

Figure 5:- types of Arduino

In this paper Arduino Uno (shown in Fig.3) was used as the development board to run the intelligent tank because it is a simple, inexpensive board with limited resources can be used to implement complex and intelligent tasks. Here used with pumper ,relay and LCD.

- **Relay**

A relay is defined as an electrically controlled device that opens and closes electrical contacts, or activates and deactivates operation of other devices in the same or another electrical circuit. Two types of relay technology are available, mechanical and solid state. A mechanical relay is essentially a combination of an inductor and a switch, where the

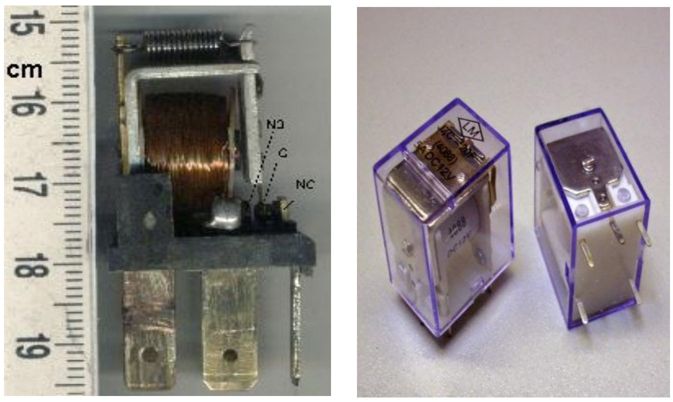

Figure 6:- magnetic relay

electromagnetic force of the inductor causes a switch to change position. A solid state relay accomplishes the same function with semiconductor devices changing impedance to effectively activate or deactivate a circuit open or closed. This document is intended to be a general guide to aid the designer in the appropriate selection of a relay for the intended application. Detailed information on the selection and use of relays can be found in MIL-STD-1346. [14]

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- **Water pump**

It is a device that depends on the mechanical movement of fluid transport. The pumps are divided into three types according to the way the pump moves the liquid: direct lifting, displacement and gravity pumps. Pumps operate by energy consumption to perform mechanical work by moving the liquid. The pumps operate according to many sources of energy, including manual operation, electricity, motors and wind power.It is come in many sizes, from microscopic for use in medical applications to large industrial pumps.
Mechanical pumps serve in a wide range of applications such as pumping water from wells, aquarium filtering, pond filtering and aeration, in the car industry for water-cooling and fuel injection, in the energy industry for pumping oil and natural gas or for operating cooling towers.

Figure 7:- Water pump

3.Implementation and Design :-

The goal of this paper is satisfy intelligent controlling on the electrical parts that's connected by microcontroller 'Arduino' to create smart tank. The part of smart execute by programming code written in 'C programming language' and feed inside 'Arduino'as a kind of artificial intelligence.

3.1 Flowchart for Intelligent tank and Table : This flowchart show the procedure of the work.

Case	Height of the water in the tank	Turn the pump on or off	Tank Cases	Max Capacity
1	2 cm	Pump is on	Tank is empty	10 cm
2	8 cm	Pump is off	Tank is full	10 cm

Table 1:- The cases for Intelligent tank

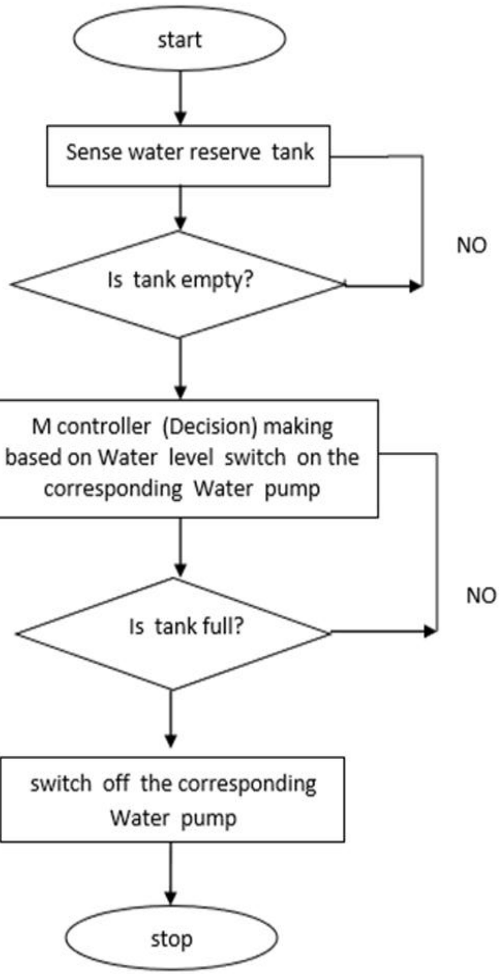

Figure 8:-flowchart for Intelligent tank

3.2 The Modules for Intelligent tank:

The paper contain three primary modules :

1. Water level monitor
2. Water level sensors
3. Water level controller

We use these components to managing the amount of water in the tank depend on the measurement of water level in the tankby using Ultrasonic Sensor that measure the distance between the sensorand Surface of the waterand Follow up readings by the LCD and mobilescreen.This monitor devices connected with Arduino (microcontroller) that control on the level of the water on the tank by matching the measures that come from theUltrasonic Sensorwith the data saved inside Arduino by set of orders written by' C programming language' as a type of artificial intelligence.

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When the tank supervises the water-free level according to the user-defined level, the controller activates the associated water to fill the tank. In case water reaches the specified level of fill, the microcontroller gives the pump order to a stopover. Which helps to solve the problem of waste water and avoid the consequent economic losses.

Figure 9:-Intelligent tank

Here use (HC-SR04Ultrasonic) to send and receive the ultrasonic waves and calculate the speed of waves and the time arriving (receive) by calculate the distance between the sensor and the water to know the water level.

3.2 Integrated development environment

Arduino is programmed through a special program called the Integrated Development Environment Brief for (IDE)

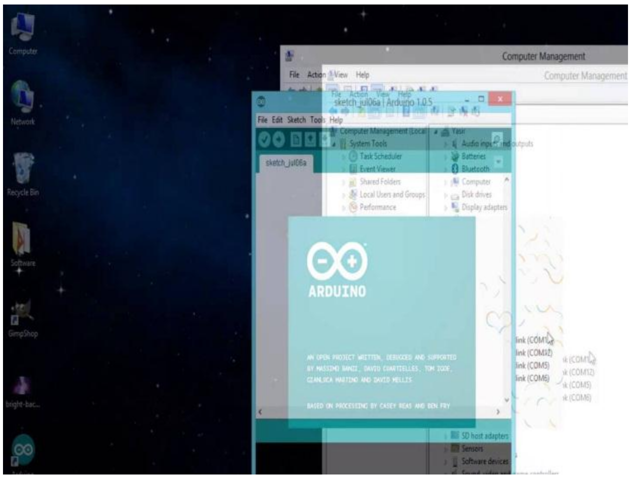

Figure 10:-Arduino Interface

3.3 Link method:

Connects common the variable resistance with pin3(V0) of the crystal and the rest of the sides with the VCC and Gn arduino, lcd pins RSS and RDD is feed from Gn and vccarduino ,also lcd pin4(RS) connected with pin4 for arduino , lcd pin5 (RW) with Gn,lcd pin 6(E) with arduino pin3 ,and lcd pins D4, D5, D6, and D7, respectively with arduino pin4, pin5, pin6, and pin7. In addition, the cathode lcd is connected to the Gn and anode with Vcc. The relay input (IN1) is connected with the pin 8 of the arduino and series with led and the buzzer , either the water pump is connected one wires with 220v source dirctly and other with the relay(NC) and common relay with 220v, finally the HC-SR04 sensor connects the echo input with the pin12, the trig input with the pin11, the vcc and the ground with Feed the Arduino. As well as connect VCC and Bluetooth to the ground with arduino feed and RX ,TX with RX, TX for arduino respectively.

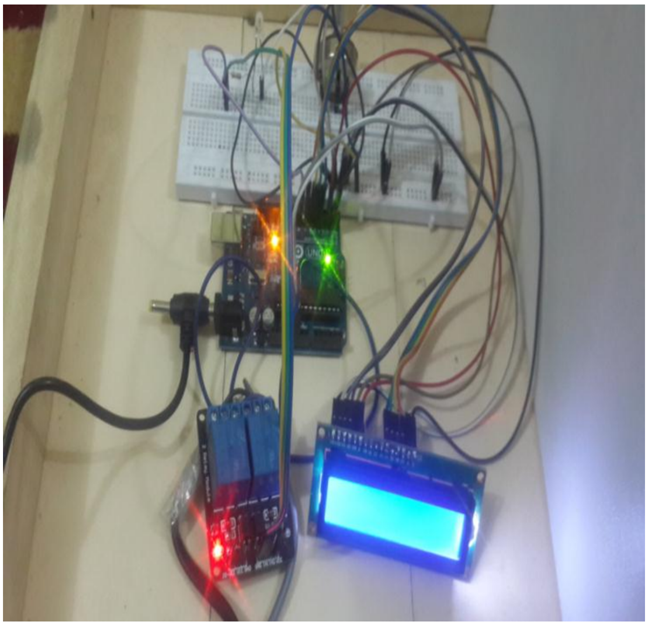

Figure 11:-Link method for components

3.3.2 Pumping and discharge

Perform water pumping and discharge in one

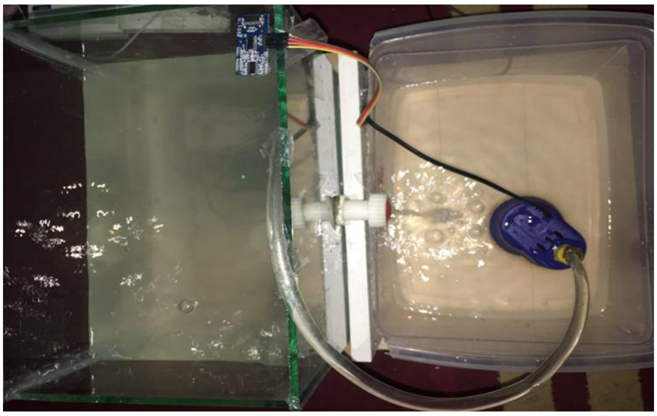

Figure 11:- Water Pumping and Discharge

CONCLUSION

Due to the importance of water in the life of living organisms, can note that this paper focuses on the work of a control system that works accurately and automatically control the level of water in the reservoir, which prevents loss of water and the impact on the life of living organisms, industry, agriculture and affect the economy of countries. The proposed system is cheap and simple and has been successfully tested in the laboratory.

As a future view in this subject we can connect this system with internet and send the information from the Sensor to the database in the server and control on it by using mobile .

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تصميم وبناء خزان ذكي (متحسس مستوى الماء)

براء إسماعيل فرحان
كلية الحاسوب و تكنولوجيا المعلومات
جامعة واسط

المستخلص :

النظام المقترح في هذه الورقة تصميم نظام خزان ذكي متحسس لمستوى المياه الذي يقل لنسبة الهدر في المياه من خلال التحكم التلقائي و السيطرة على مستوى المياه و ذلك بتحديد نسبة الماء في الخزان من حيث الخلو و الامتلاء وينفذ ذلك باستخدام مفهوم الذكاء الاصطناعي مع المسيطرات البسيطة حيث يتم استخدام مسيطر بسيط و رخيص يتمثل بالاوردوينو ويتم خزن معلومات كاملة تكتب في برنامج IDLالتحديد نسبة الامتلاء والخلو حسب حاجة مستخدم الخزان مع إدارة القطع الالكترونية المرتبطة مع الاردوينو لمليء الخزان عند خلوه من الماء و إيقاف عملية المليء عند الوصول إلى المستوى المحدد للامتلاء مع مراقبة مستوى المياه إثناء الاستخدام . ويمكن تطبيق هذا النظام على مستوى خزانات المنازل و خزانات المدن وخزانات الري للأراضي الزراعية مع تطوير القطع الالكترونية المستخدمة في نظام السيطرة لتحقيق الهدف المنشود في الحفاظ على كمية المياه الصالحة للشرب والمياه الغير صالحة للشرب .

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Estimating the state of website security as poorly regulated mechanisms based on fuzzy logic methods

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Abstract

The research is devoted to the analysis of page loading time, which is an important indicator for any web-site. As a rule, the sites are hosted on web-servers with certain characteristics. The users interact with the environment is largely aggressive and uncertain (external threats such as penetration, denial of service, the introduction of code into the language of structured SQL queries, etc.). It should be noted that there are the uncertainties generated by hardware and software are also important. Any site can be sub-is influenced by the influence of the external environment and various subsystems of servicing the functions of web-sites, which leads to occurrence of contingencies and generates uncertainty of his work. The degree of uncertainty is cannot always be assessed on the basis of statistical material alone. This leads to an increase in the number of methods and means of intellectualizing the performance of evaluations on the basis of methods of artificial intelligence, and methods based on the use of fuzzy estimates. The assessment of the security status of Internet sites as poorly formalized objects on the basis of fuzzy logic methods considered in the article is implemented in the form of technology for assessing the states of reliability of a web site. The proposed approach allows a more flexible adaptation to the particular problem and allows diagnosis object already in the step of calculating an integral index of reliability. Flexibility is achieved due to the fact that the state of reliability can be estimated At once on several reliability indicators.

Keywords— fuzzy logic, methods of system analysis, metrics, analysis and features of a web-site.

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Introduction

Currently, information systems (IS) are actively developing, including the intellectual ones. An important link in the development of information technologies are web-sites, which number is more than 4 billion [1-3], and their number is growing rapidly. Therefore, the analysis of the quality and reliability of the created and operated programs, the underlying requirements for the work of web-sites are special requirements. Web-sites depending on the location of elements (pages, sections, navigation) and their relationship with each other can have a different structure [4]. The key ones are linear, hierarchical and lattice. In the design, development and operation of a web site as an information resource, the main criteria and indicators evaluation of its quality, reliability and efficiency.

Description of the metrics of websites

The selection of the best characteristics, which are included in the basis of the web application, implies the use of such a thing as "metrics". Basically, the metric represents a numerical characteristic of the system. When these characteristics are formed, various factors are taken into account [6]. The most common ones are the mean time between failures MTBF (ms), access (%), delay (ms), channel bandwidth (Kb), time to the first byte TTFB (ms), DNS domain server search time (ms), redirection of the universal URL resource pointer, the number of HTTP requests, the size of the main page (Kb), the connection time (ms). There are three types of metrics for assessing web-sites: server, user and network.

Server metrics allow you to determine the system resources used and the possibility of resource conflicts. These metrics are aimed at tracking the resources of the machine level, such as network, memory, processor and disk utilization. They give an idea of the internal conflicts that underlie the computer. Allocate the hardware and software metrics of the server systems (see table 1 and table 2).

Table (1) Hardware metrics of server appliances

Metrics	Calculation / Range	Comment
Latency	**-**	**Waiting time before sending data or the time at the beginning of the transfer. The shorter portion of the transmitted data, the more frequently occurs latency. The transmission speed is characterized by the maximum channel capacity at large portions of the transmitted data, while the costs for latency are reduced**
Mean time between failures MTBF	$\frac{Tu}{NE}$ **, where TU is the total working hours; NE - number of failures**	**The metric of the work of equipment, set by the manufacturer. Due to the reliable operation of modern computer equipment, this metric is missing from some manufacturers or is given as a lifetime warranty**
Network tracking indicator	$\left(\frac{Td}{T}\right)$ **where Td is the operating time; T - total time**	**Describes the time of the system. Similar to the MTBF metric, but only provides network maintenance problems**
Site volume	**-**	**Number in Kb**
Number of I / O operations	**SD * ND, where SD is the I / O disk speed, ND is the number of disks**	
Time to first byte TTFB	**TFB**	**The required wait time before the first byte of the requested resource arrives from the server after sending the HTTPGET request**

Table (2) Software metrics of server systems

Metrics	Calculation / Range	Comment
Number of threads	NTh	Multithreading
Replication delay	Tp	Parameter - time in ms

You can track metrics to determine performance and reliability aspects: It is necessary to be aware of the interdependence between the system indicators and the application load. Probably, the system will need additional hardware resources (real or virtual). In the case of constant load, data values increase metrics. This may be due to external causes: background tasks, constantly running tasks, network activity or input / output (I / O) devices.

WebPageTest.org defines TTFB as the browser wait time until the first byte of the requested resource is received, which begins after the DNS lookup time and connection time [7]. Some sources combine DNS time, connection time, and latency in the TTFB metric, since TTFB represents the amount of time required to respond to the server and create a web page.

Figure (1) shows the TTFB value of 124 ms, which is essentially perfect. Typically, the optimal TTFB should be in the range of 5-180 ms. It is considered that the web page is slow if it has a large TTFB, since the start time of the display will be delayed. This is a form of feedback used to evaluate web sites for effectiveness research. It should be noted that (uodiyala.edu) loads most of the components, such as Java Script JS, CSS style sheets, Flash animation, not immediately, so the website's delay is minimal. The delays of the web page are mentioned in different sources [2, 6]. Metrics (Figure 1) are as follows:

* DNS lookup time- the time to search for the IP address for the corresponding domain;
* Connection time - the time required to establish a TCP connection;
* Timeout - waiting time until the first byte is received after the connection is established;
* Content downloads time - the time it takes to download the entire object.

					Document Complete			Fully Loaded			
Load Time	First Byte	Start Render	User Time	Speed Index	Time	Requests	Bytes In	Time	Requests	Bytes In	Cost
120.050s	0.277s	2.193s	0.526s	29970	-	0	9,169 KB	120.050s	107	9,169 KB	$$$$$

URL: http://www.uodiyala.edu.iq/
Host: www.uodiyala.edu.iq
IP: 94.228.39.10
Error/Status Code: 200
Priority: Very High
Client Port: 65115
Request Start: 0.135 s
DNS Lookup: 27 ms
Initial Connection: 107 ms
Time to First Byte: 124 ms
Content Download: 479 ms
Bytes In (downloaded): 47.6 KB
Uncompressed Size: 47.2 KB
Bytes Out (uploaded): 0.4 KB

Figure (1) Key metrics for web page delay

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Table (3): Hardware metrics of user systems

Metrics	**Calculation / Range**	**Comment**
Requests per second RPS	$\left(\frac{P}{Po}\right) * T_Z^{-1}$	**P - memory, PO - RAM, TZ - preset time**
The amount of memory for programs	**-**	**Measured in MB**

Table (4): Software metrics of user systems

Metrics	**Calculation / Range**	**Comment**
Bandwidth of the channel	$\frac{RWIN}{RTT}$	**RTT - Round-Trip -Time, RWIN - TCP receive window**
Simultaneous Users	**-**	**Number**
Application response time	**100~300**	**Integral performance of IP in terms of user [8]. The amount of time between the appearance of the user's request to the application and the receipt of a response to the request. Depends on the type of user request, from which user and to which server it is accessed, from the current state of the network elements and the settings of the operating systems and DBMS [9]**
Download time	**1–10**	**The time it takes to fully load a web page with a browser. Measured in seconds**

Stages of loading a web page: request→ forwarding→ searching in the DNS cache→ TCP→
receipt→ processing→ reply→ download.
TTFB can be from 1-5 sec to 100-200 ms, but the page loads much faster and will be ready for use in a shorter time. Many web-sites are seeing a general increase TTFB 5-10 times. There are also some disadvantages to Gzip compression:
* increases the total server load during compression;
* Data processing can take a long time, since the first byte is not sent until the compression is completed;
* A large TTFB often causes the user to re-create the current request to the web server, which increases the overall load and the required resources due to consecutive requests.
Network metrics are associated with the emergence of network problems, which are accompanied by a decrease in productivity. Network delays lead to an increase in the duration of query execution (Table 5).

The application response time, as a rule, is formed from the time.
* Preparation of user requests;
* Transfer of requests between the user and the server through network segments and intermediate communication equipment;
* Processing requests on the server and sending responses to the user;
* processing the responses received on the user's device.
To determine the optimal performance of IP in order to determine which metrics are determining, a service level agreement must be drawn up between the various services.
Web pages are often compressed in the Gzip format to reduce the size of the downloaded file, which prevents sending the first byte until compression will not be completed, and greatly increases TTFB.

Table (5): Website network metrics

Metric	Formula / Range	Comment
Network Latency	$\sum(tp+tl+Q(t))$, where tp is the packet delay; tl - propagation delay; Qt - delay in queue	**The performance of the network is extremely important for cloud applications, as it is a conductor through which all information passes**
Bandwidth of the channel	$\frac{1.26*MSS}{RTT*\sqrt{L}}$, where RTT - Round-Trip-Time; MSS - segment size; L - loss of frames	
Packet Throughput PPS	$\frac{Rwin}{RTT}$,where RTT - Round-Trip-Time; Rwin - TCP Receive Window	**Reflects the number of frames transmitted per time unit . It gives an opportunity to assess whether the equipment copes with the load and whether its performance corresponds to the declared**
Frame loss	$\frac{8}{3W^2}$ where W is the segment load	
Network availability	$\frac{MTBF}{MTBF+MTTR}$, where MTBF is the mean time between failures; MTTR - Mean Time To Repair	**It is used to assess the reliability and stability of the network. Displays the time that the network is functioning without fail or need to reboot for administrative or maintenance purposes**
Bandwidth	$(F_{max}$-$F_{min})$, where F_{max}-Max. frequency; and F_{min} -min. frequency	
Response time	$\frac{B_{max}}{F}$, where B_{max}- the maximum bandwidth; F is the number of flows	**The average speed of the full load of the pages of the website. Use a weighted average score for users, servers, and day periods. Time in seconds**

Response time is an important indicator, first of all, for any visitor to the site. Therefore, it is important for the site owner to download the main page of the website. Many users do not have enough speed to quickly download large portals. Waiting for the page to load fully should not exceed 5-10 seconds. To date, for example, MS Windows does not have performance counters to measure the latency of individual application requests.

However, there is a "Resource Monitoring" which is an excellent tool for analyzing network traffic on the local machine. "Resource Monitoring" provides information about lost packets and additional information about the delay of current TCP / IP sessions. The information about the lost packets makes it possible to represent the quality of the connection. The delay describes the time it takes to completely traverse a TCP / IP packet.

Approaches to determining the state

Determining the security status of Internet sites is a complex task, the solution of which requires an integrated or systemic approach. Diagnosis is performed based on a fuzzy set. To do this, it is necessary to determine the degree of fuzziness of all terms relative to the center.

Let X = {x} be a family of objects denoted by x, then the set A in X is A = {x, μA (x)}, x∈X, where μA (x) is the degree of belonging of x to A

When the sets A and B (A ∪ B) are joined,

$\mu A\cup B(x) = \text{Max}(\mu A(x), \mu B(x)), x\in X$.

(1)

The intersection of A and B (A∩B) has the relation $\mu A\cap B(x)= \min(\mu A(x), \mu B(x)), x\in X$.

(2)

The degree of fuzziness is determined from the implication A on B, then we find the inverse implication B on A and compare the obtained implications:

$A \to B = \max(\bar{A}, B) = \max(1 - A, B);$ (3)

$B \to A = \max(\bar{B}, A) = \max(A, 1 - B);$ (4)

$A \equiv B = \min(A, B) = = \min(\max(1 - A, B), \max(A, 1 - B)).$ (5)

Proposed evaluation criteria based on data collected from publicly available sources

Table (6) : Website working parameter's state

	S_0	S_1	S_2	S_3	S_4	S_5	S_6
D	> 780	600 ~ 700	381 ~ 450	321 ~ 380	250 ~ 310	181 ~ 260	5 ~ 190
T_{DNS}	>1200	880~ 1200	580~ 880	380~ 580	300~ 380	200~ 300	<200
P	> 7168	4096~ 7168	1536~ 4096	700 ~ 1536	350 ~ 700	180 ~ 350	>180
R_d	>8	8	7	6	5	4	<3
R_q	>70	54~67	45 ~ 55	34 ~48	27~ 35	19 ~ 29	<20
T_{CON}	>5500	1500~ 5500	1550~750	340~780	285~ 350	248~ 286	<250

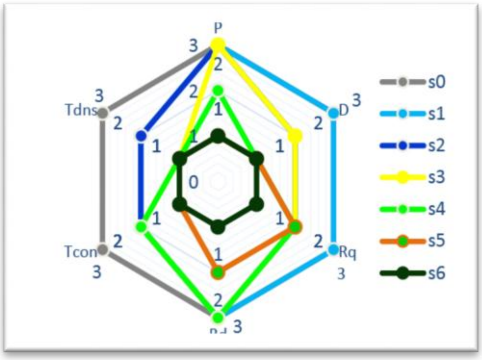

Figure (2) A website condition radar chart for 6-indicators datasets

Based on (Table 6), we construct a diagram (Figure 2), which shows the position of possible states of the web site operation in space. It can be concluded that the closer to the center, the higher the safety of the site.

Conclusions

The assessment of the security status of Internet sites as poorly formalized objects on the basis of fuzzy logic methods considered in the article is implemented in the form of technology for assessing the states of reliability of a web site. The proposed approach allows a more flexible adaptation to the particular problem and allows diagnosis object already in the step of calculating an integral index of reliability. Flexibility is achieved due to the fact that the state of reliability can be estimated At once on several reliability indicators. Diagnostics at the stage of computing the integral index of security states is achieved due to the fact that the calculations can be divided into different stages. Each stage evaluates any of the individual indicators, which ultimately leads to a conclusion about the state of the corresponding element.

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تقدير حالة أمن موقع الويب كآليات ضعيفة التنظيم تستند إلى أساليب منطقية ضبابية

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المستخلص :

تُخصص البحث لتحليل وقت تحميل الصفحة ، وهو مؤشر مهم لأي موقع ويب. وكقاعدة عامة ، يتم استضافة المواقع على خوادم الويب ذات خصائص معينة. يتفاعل المستخدمون مع البيئة بشكل كبير وغير مؤكد (التهديدات الخارجية مثل الاختراق ، الحرمان من الخدمة ، إدخال الشفرة في لغة استعلامات SQL المنظمة ، إلخ). وتجدر الإشارة إلى أن هناك أوجه عدم اليقين التي تولدها الأجهزة والبرمجيات هي أيضا. يمكن أن يتأثر أي موقع بتأثير البيئة الخارجية والأنظمة الفرعية المختلفة لخدمة وظائف مواقع الويب ، مما يؤدي إلى حدوث حالات طارئة ويولد عدم يقين من عمله. لا يمكن دائما تقييم درجة عدم اليقين على أساس المواد الإحصائية وحدها. وهذا يؤدي إلى زيادة في عدد الطرق والوسائل لإضفاء الطابع الذهني على أداء التقييمات على أساس أساليب الذكاء الاصطناعي ، والأساليب القائمة على استخدام التقديرات غير الواضحة. يتم تنفيذ تقييم الوضع الأمني لمواقع الإنترنت كأجسام ضعيفة الشكل على أساس أساليب المنطق الضبابي التي يتم النظر فيها في المقالة في صورة تقنية لتقييم حالات موثوقية موقع الويب. يسمح النهج المقترح بتكييف أكثر مرونة مع المشكلة المعينة ويسمح لكائن التشخيص بالفعل في خطوة حساب مؤشر متكامل للاعتمادية. يتم تحقيق المرونة بسبب حقيقة أن حالة الموثوقية يمكن تقديرها مرة واحدة على عدة مؤشرات موثوقية.

الكلمات المفتاحية - المنطق الضبابي ، طرق تحليل النظام ، المقاييس ، التحليل وخصائص موقع الويب.

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Hybrid Approach to Detect Spam Emails using Preventive and Curing Techniques

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Abstract:

A file that has to be moved between two schemes can be moved through the network. Security needed between the sender and the receiver. Electronic mails are fastest way of communication and information sharing, but in new years, Email system has been changed, which known Spam Mails. Spam is information, which is spread to a big number of receivers without telling them. Now, a number of techniques have been proposed to stop spam. Filters for anti-spam can be worked in two methods: Preventive techniques and Curing Techniques; The Preventive techniques are Stop Spam before delivery which are depend on URL Based and List Based. Such as whitelisting, blacklisting. The Curing Technique is Destination Spam Filtering that used is Content Based Filtering. The Curing Technique, the messages are categorized as Spam or not Spam based on these techniques. Such as Bayesian filtering, keyword-based filtering, heuristic-based filtering, etc. In this study introduce combining preventive techniques and curing techniques to get good algorithm.

Keywords —***Spam, Unsolicited Commercial e-mail, Bayesian Classifier, Black Listing, White Listing, preventive techniques, curing techniques .***

1. INTRODUCTION

Sending messages by the communication network is known Electronic Mail (Email) [1]. Emails are reliable, fastest way of communication and information sharing. E-mails have low transmission costs [2]. E-mail become important topic for huge of persons. One can send information electronically to another one in speedily. However, in current years, Email system has been changed, also affected by Spam Mails. Spam is as unwanted email for a receiver that the user do not required to have in this inbox. Spam is use of messaging system to send unwanted messages randomly [3]. Spam is message, which is send to a number of receivers without inform them. Spam has become huge problem for users of Internet [4].

Spam messages has grown in the recent years. Some researchers consider that spam is becomes from 30 % to 70% of all messages (email) on the Internet [5]. A large number techniques for filtering spam have been proposed such as whitelisting, blacklisting, Bayesian filtering, keyword-based filtering, heuristic-based filtering, etc. Three principles in the following that meet with any email:

1) Anonymity: The address and identity of the sender are concealed.

2) Mass Mailing: The email is sent to large group of people.

3) Unsolicited: recipients do not request the email.

Spam Mail has become an increasing problem in recent years. It has been estimate that around 70% of all emails are spam [6]. The spam classifier makes use of the machine learning to classify web documents as either spam or not spam [7]. The common algorithms are Bayesian Classifier, KNN, NN, Black List, White List [8]. Nowadays, the researchers are working to hybrid two or more filters to develop best classification [9].

This paper introduce merging classifiers (Black List, White List with Bayesian classifier) to get good classification. This paper has been organized in the following parts: Section 2 Related works with this paper. Section 3 Spam Detection techniques. Section 4 Proposed System which is used for this paper. Section 5 Data Set. Section 6 Results of this paper, Section 7 Evaluation the results of this paper and Section 8 conclusion.

2. RELATED WORK

There are large researches existing work to detect spam in E-mails.

[10] A Study in 2013, work on bad URL detection. To classify URLs: spiteful URL and valid URL. In addition, used Bayesian filter to increase the accuracy of the system.

[11] A study in 2015 proposed a spam and bad URLs detection system by stopping spam messages and malicious URLs in Email. And use detect based on Bayesian filter and Decision Tree.

[12], a study, propose hybrid three approaches: (Bayesian, thresholds, probability) working together to detect spam emails.

3. SPAM DETECTION TECHNIQUES

Figure1 showing common techniques using to detect and stop spam from email messages [13].

1) Preventive Techniques (Stop Spam before delivery):

Preventive techniques is better than curing techniques, In the Preventive techniques, the messages that are arrived toward mailbox are checked for legitimacy and then permitted to pass in the mailbox. There are two ways within Preventive Techniques: URL Based and List Based.

URL Based: in this way, spam classifier done based on URL [14]. The arriving URL is first verified to be valid or not. Then accept to email messages entered to the mailbox.

List Based: this is filtering way which is used the network information before a message is received by the receiver in order to classify whether this messages is spam or Ham such as Black Listing[15].

2) Curing Techniques

The common approach that used is Content Based Filtering. Also called Destination Spam Filtering. This approach, message emails are filtered as Spam or Ham. learning techniques and AI ways used to classify Spam.

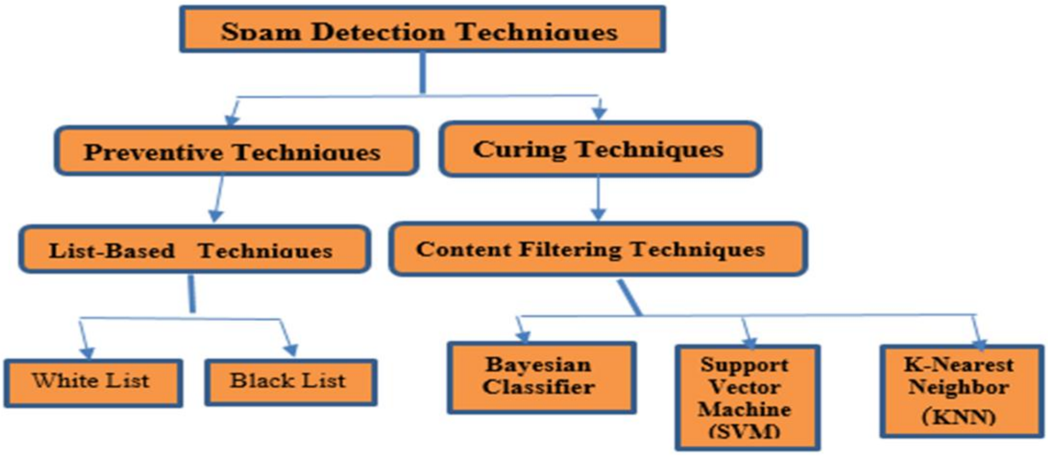

Figure (1) general classification of approaches to spam filtering

Black List: this method is done by use classification principles, the goal of these ways is stopping the unwanted content and do not reach the mailbox. A way to do is on the basis of IP Address.

Blacklists are used to IP addresses [16]. The not strong with this method was that clever spammers frequently change their IP addresses [17]. Black list is the general method of detect spam, since its simple work. The key idea include create simple database and listing (domain names, IP-addresses). Now the messages to arrive from the list that recorded are stopped.

White List: this method is used to categorize users email addresses as valid. Emails addresses are saves automatically in white-listed. Making a database of White lists; which includes domain names and IP-addresses [18].

Bayesian Classifier: is a common method of e-mail filtering. It apply to identify spam e-mail. Classification process apply the Bayesian statistics on the features that drive from these classifications [19].

Bayesian Classification was derived from the Bayes' theorem in probability theory. If the calculated probability value is higher than the preset threshold, the message is classified as a spam, and treated accordingly [20].

Equation (1)

$$P(A|B) = \frac{P(B|A).P(A)}{P(B)}$$

where:

P(A) is the prior probability of A.

P(A|B) is the conditional probability of A, given B.

P(B|A) is the conditional probability of B, given A. It is also called the likelihood.

P(B) is the prior or marginal probability of B, and acts as a normalizing constant.

4. OBJECTIVE OF THE WORK

The aim of the paper is improve spam detection system. A filter is used to organize a message: SPAM or HAM. In this paper, the procedure for the spam detection is summarized under the Figure (2) [21].

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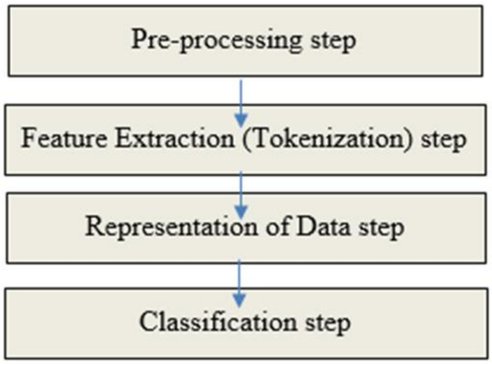

Figure (2) Main Steps in the Spam stopping

The basic steps of spam detection are:

5. Pre-processing

Pre-Processing Steps the purpose for preprocessing is to transfer messages in email into a uniform format that can be understand by the learning algorithm. The basic preprocessing steps of spam detection algorithm are [22]:

1. HTML Removal.
2. The words that have length <=2 are removed. Ex:

Input (x) = ‘I have a list of people you missed!"
Output (x) = “have list people you missed!"

3. All the special characters are removal. Ex: (continue)

Input (x) = “have list of people you missed!"
Output (x) = have list people you missed

4. Stop words are removal. “Words” do not include any useful information. Such as [then,threr,the,was,you,are,by,they,have,has,also,before,both,because,about]. Typically include pronouns, prepositions and conjunction. Ex:(continue)

Input (x) = have list people you missed
Output (x) = list people missed

5. Stemming Algorithm: is used to fetch the basic form of the word (root). This algorithm is used to reduce the words to its root by remove the plural from nouns(e.g. "pens" to "pen"), the suffixes from verbs(e.g. "reading" to "read"). Example (continue)

Input (x) = **list people missed**
Output (x) = **list people miss**

6. Feature extraction

Feature extraction Phase also called, “feature reduction”, “attribute selection”. It **is** the method to choice a subset of relevant features for structure the learning prototyp. This method is used to tokenize the file content into individual word [9]. Feature extraction (Tokenization) is the process that extracting features from email into a vector space [23]. Feature extraction employs to excerpt selective features from the process of pre-processed steps. A feature can be anything in an email message. It can be a word, a phrase, a number, an HTML tag, etc.

7. Feature Selection

This technique must be differentiate from feature extraction. Feature extraction is create new features from the original features, but feature selection select subset of the existing features [6]. Improves the performance of the feature selection by makes training and applying a classifier more efficient by decreasing the size of data set. Second, feature selection enhances accuracy of classifier by eliminating extra features from the data set. An email message contains two parts: a header and a body [24].

There are some approaches used to get features selection[22]:

1) **Chi-square:** Chi-square hypothesis tests may be performed on contingency tables in order to decide whether effects are present. Effects in a contingency table are defined as relationships between the row and column variables; that is, are the levels of the row variable differentially distributed over levels of the column variables. Significance in this hypothesis test means that interpretation of the cell frequencies is warranted.

2) **Gain Ratio** : The various selection criteria have been compared empirically in a series of experiments. When all attributes are binary, the gain ratio criterion has been found to give considerably smaller decision trees. When the task includes attributes with large numbers of values, the subset criterion gives smaller decision trees that also have better predictive performance, but can require much more computation. However, when these many-valued attributes are augmented by redundant attributes which contain the same information at a lower level of detail, the gain ratio criterion gives decision trees with the greatest predictive accuracy. All in all, it suggests that the gain ratio criterion does pick a good attribute for the root of the tree:

7. Representation of Data:

This step is main task of spam detection algorithm because it is very hard to do computations with the textual data. The representation should be show the real statistics of the textual data. The actual statistics of the textual data is converted to suitable numbers. Here are many methods for term weighting that calculate the weight for term differently.

1) Term _ Frequency: counts the number of occurrences of term in a text document. Mathematically it can be represented as:

$$Term\_Frequency\_Wij = tf_{ij} \qquad \text{Equation (3)}$$

Where, tf_{ij} as the frequency of term i in document j

2) In tf-idf, found normalized term frequency, inverse document frequency and tf-idf of each word in document (email). Tf-idf is a statistical measure used to calculate how significant a word is to a document in a feature corpus. Word frequency is established by term frequency (tf) , number of times the word appears in the message yields the significance of the word to the document. The term frequency then is multiplied with inverse document frequency (idf) which measures the frequency of the word occurring in all messages.

The formula is:

$$X_{i,j} = TF_{i,j}.log\frac{|D|}{|\{d_j : t \in d_j\}|} \qquad \text{Equation (4)}$$

Where

i = term.

j = document.

TF i,j = frequency i in the j.

|D| = number of documents [25].

8. Classification

Classification is a task of learning data patterns that are present in the data from the previous known cases and associating those data patterns with the classes. Many techniques used in classification into spam detection algorithm.

The following equations showing the main concepts of the classification.

1- Good message (Ham) = Ham message / Total messages.

2- Bad message (Spam) = Spam message / Total messages [26].

9. Proposed Approach

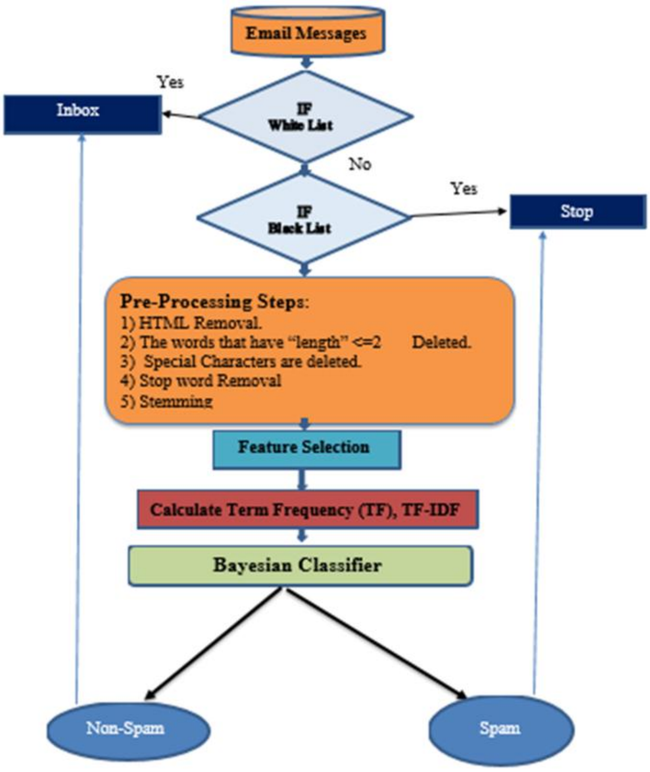

Figure 3. The proposed approach

Algorithm(1) Proposed Spam Email Detection
Step1: Input Email Message. Step2: Apply White List Filter. Step3: Apply Black List Filter. Step4: Using Pre-Processing Steps. Step5: Apply Feature Selection method. Step6: Caculate Term Frequency(TF),TF-IDF. Step7: Classificate using Bayesian classifier.

10. EXPERIMENTS AND RESULTS

10.1 Implementation:

We established four files: The first file used of the White List which is stores the IP addresses and URLs for wanted websites; the system employ the white list to match with the received messages, and this file is updated repeatedly by the user. The second file employ the Black List which is keeps the IP addresses and URLs for unwanted websites; the system uses the black list to match with the received messages, and this file is updated repeatedly. The third file employ to keep the Unsolicited mail List; the filter usages the list to match with the received messages. The four file used keep the Ham List; the filter employ the list to matching with the received messages. This file is updated regularly by the user.

10.2 Data Sets

After collected a new data set, composed by 1424 emails. 1113 are spam emails and 311 emails are HAM messages. Those emails grouped from the mail boxes of some students. Divided the emails in two groups: the training group contains 70 % of the emails. The training 30% email messages; 14 % Ham messages and 16 % unsolicited mail messages. The checking group contains 31 % of the emails: 10. 916 email messages; 3.788 Ham messages and 7.139 unsolicited messages.

10.3 Results

Accuracy of White, Black, and Bayesian filter "with" and "without' pre-processing is shown in table (1).

1) **White listing algorithm**: use preprocessing data is 85% precision.
   Do not use preprocessing data is 40%.

2) **Black listing algorithm**: using preprocessing data is 78% precision. Do not use preprocessing data is 50%.
3) **Bayesian algorithm**: using preprocessing data is 89% precision. Do not use preprocessing data is 48%.
4) **Hybrid approach (proposed approach)**: using preprocessing data is 91% precision. Do not use preprocessing data 66%.

Validate the results using some questions:

Q1/ Can pre-processing benefits to enhancing the results?

As shown in table1, accuracy is better using preprocessing.

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Q2/which algorithm is capable to complete well results?

Hybrid algorithm that is close to 91%

Table1: Accuracy with use and without use preprocessing

Algorithm	Accuracy
White List **With pre-processing data**	%85
White List **Without pre-processing data**	40 %
Black List **With pre-processing data**	78 %
Black List **Without pre-processing data**	50 %
Bayesian Filter **With pre-processing data**	89 %
Bayesian Filter **Without pre-processing data**	48 %
Hybrid Approach **With pre-processing data**	19 %
Hybrid Approach **Without pre-processing data**	66 %

10.4 Estimation

To estimate the performance of the system, following steps are done:
The inbox include 800 email messages:
400 Ham messages arbitrarily selected from the training set.
400 spam messages arbitrarily selected from the training set.
All email messages: 800
Ham messages: 400
Spam messages: 400
Email messages filtered as Ham: 260
Email messages filtered as spam: 240
Accuracy: 89.56%

11. CONCLUSION

More than 70% of emails nowadays is spam. Unsolicited email detection is key part of concern nowadays as it benefits in the finding of spam e-mails. There are a lot of anti-spam techniques, but there is no technology that has processed unwanted messages permanently except the anti-spam techniques that are based on "machine learning" methods. These techniques are basically text classifiers, they classify a email message into two categories (spam or non-spam). This paper described a machine learning approach based on Bayesian analysis to filter spam. The filter learns of what spam and non-spam messages. Before use Bayesian analysis, this study apply white and black listing to classify the email and to stop spam e-mails. Then use Bayesian classifier. You can train it once and after training the classifier, it can filtering spam with high accuracy as shown in the evaluation section.

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طريقة هجينة لكشف رسائل البريد الالكتروني المزعجة باستخدام تقنيات الوقائية وتقنيات المعالجة

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جامعة ديالى

المستخلص :
لنقل ملف ما بين طرفين عبر الشبكة، أمن المعلومات التي يتم تبادلها بين المرسل والمستلم عامل مهم جـــدا. ان البريد الإلكتروني يعتبر أسرع وسيلة للاتصال وتبادل المعلومات، لكن في السنوات الأخيرة، استخدم نظام البريد الإلكتروني بشكل خاطئ من قبل أطراف غير مخولة، ومن هذه الطرق هي رسائل البريد المزعج(سبام) وهـــي المعلومات التي تنتشر إلى عدد كبير من أجهزة الاستقبال بدون طلب سابق منهم. في السنوات القليلة الماضية، تم اقتراح عدد من تقنيات كشف البريد المزعج. من أشهر هذه الطرق:١) الطرق الوقائية وهـــــي مـــنع الرســـائل المزعجة قبل وصولها صندوق البريد. ٢) طرق المعالجة بعد وصول الرسائل المزعجة الى صندوق البريد هي كشف الرسائل المزعجة عند المستلم. هذه الدراسة تقدم الجمع بين الطريقة الوقائية وطريقة المعـــالجة للحصول على نظام كشف ومنع للرسائل غير المرغوب بها بكفاءة عالية.

الكلمات المفتاحية: الرسائل غير المرغوب بها ، البريد الالكتروني غير المرحب به، مصنف بايزن ، القائمة السوداء ، القائمة البيضاء ،تقنيات الوقائية ،تقنيات المعـــالجة.

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Three-dimensional Face Reconstruction using 3D Morphable Model Fitting Method

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Abstract

In this paper, the reconstruction of faces in the form of a 3D model from unconstrained images (different in pose) has been studied. Three-dimensional Morphable Model(3DMM) Fitting method has been used in 3D reconstruction techniques. In this paper propose improvement of the 3DMM fitting procedure to get accurate 3D model by taking best 2D landmark of all images rather than taking only first 2D image landmark (traditional method which proposed by A.Bas et al). The results of the proposed algorithm show its very encouraging as far as execution time and quality of reconstruction as shown in compare the improvement fitting process with the traditional method.

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1.Introduction

Face reconstruction is the way toward making a 3D model of a face from two-Dimension (2D) image(s) [1][2].

It is important to consider in mind that creation human face models should look like as genuine image as possible. This procedure involves a transformation from 2D to 3D spaces[3].

Face reconstruction is a significant application in face recognition, video editing, virtual reality, animation [4], verification, expression recognition and facial animations [5]. For example, exact face models have been appeared to fundamentally progress face recognition [4]. In every one of these applications, the reconstructed face should be compacted and precised, particularly around, and so on. Computer games can be appeared as a good illustration which needs accurate human face models [3]. Fig.1 shows general points identified with the execution of 3D face reconstruction procedures. Most of the 3D reconstruction algorithms share the same basic processing pipeline, and may be not running all the processing steps [4].

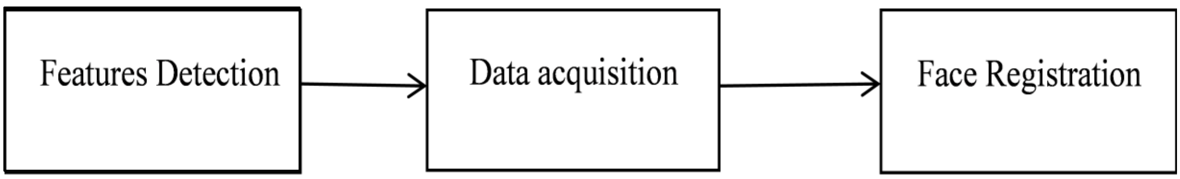

Figure 1: General steps of 3D reconstruction.

The determination of important features in the input 2D image(s) is a significant stage in most 3D face reconstruction procedures. Usually face feature determination has been utilized for [1][6]:

1. Introducing the situating of face models in 3D reconstruction methods based on the model.
2. Finding feature on sideways and forward images so it is conceivable to misshape general 3D face prototypes to accept the form of the specified face.
3. Instating the procedure of point tracing in 3D reconstruction methods based on video.
4. Creating point consistency in faces taken from unlike vantage point

Another important part of the reconstruction process is "data acquisition". It is gathering the 3D information about that object by using one of 3D reconstruction technique. In this paper 3D MM model fitting procedure has been used in reconstruction of 3D faces.

A significant task of 3D reconstruction methods is the face registration. It is bringing the whole generic 3D model vertices as near as could be expected under the corresponding to the relating 3D coordinates of the feature points which computed from images. The opposite is also potential, i.e., transporting the intended 3D points near to the generic 3D model. This corresponding involves scaling, translating and rotating of the one to be moved closer to the other.

The 3D registration approaches can be classified into two distinct collections [4]:

1. **Three-dimensional to 3D registration:** Corresponding is done among two 3D points. One is a base mesh, and other one is the 3D data produced in anyway after the data acquisition procedure.

2. **Three-dimensional to 2D registration:** These methods achieve a 3D model which is utilized to fit 2D data.

In this work, registration approach is applied to perform model fitting. The goal of this paper is reconstruction multiple unconstrained image (different in pose) in the form of 3D model. In the rest of this paper, show a 3D Model Fitting approach in section 2, display a 3D Morphable Model and Fitting with 2D Landmarks and edges in Section 3, section 4 displays the proposed 3D Face Reconstruction, section 5 displays result and Discussion, section 6 shows the conclusions and, finally, a list of the references in section 7

2. Three-dimensional Model Fitting approach

The fitting is actually an optimization process, targeting to discovery the parameters of best model usually by minimizing the difference among the input images and model reconstructed/synthesized image[7]. The 3D model fitting can reconstruct the camera model and 3D shape, lighting texture, from a single image as shown in Fig. 2 [8]. The recovered parameters can then be used for face reconstruction.

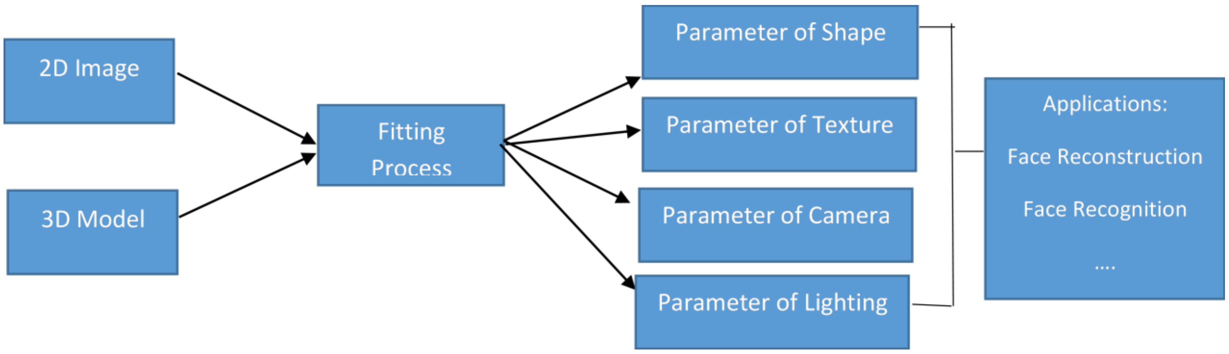

Figure 2: 3D model fitting.

In the proposed system, perform fitting process of a 3D model to a 2D images according to features (landmarks and edges). In the following sections, a detailed introduction of the model fitting approach will be given.

3. Three-dimensional Morphable Model

The Morphable Model is the one applied by The University of Basel in 2009. Its name is *Basel Face Model* [9]. It's a mesh can be deform where shape is specified by the shape parameters $\alpha \in R^S$. Consequently, any face shape can be estimated as[10]:

$$f(\alpha) = C\alpha + \bar{m} \qquad 1$$

where $C \in N^{3R\times S}$ holds the principal components, $\overline{m} \in N^{3R}$, is the shape of mean and the vector f (α) ∈ N^{3R} holds the vertices coordinates (R), accumulated to compose a vector $f = [x_1\, y_1\, z_1\, \ldots\, x_R\, y_R z_R]^T$. Later, the *j*th vertex is specified by $U_j = [f_{3j-2}\ f_{3j-1}\ f_{3j}]^T$. For suitability, indicate the submatrix matching to the *jth* vertex as $C_j \in N^{3\times S}$ and the matching vertex in the shape of mean face as $\overline{m_j} \in N^3$, such that the *j*th vertex is given by $U_j = C_{j\alpha} + \overline{m_\iota}$ also, describes the row equivalent to the element (x) of the jth vertex as C_{jx} (correspondingly for y and z) and describe the element (y) of the jth vertex of mean shape as $\overline{m_{jx}}$ (correspondingly for z and x)[7][11].

3.1. Fitting with 2D Landmarks

Fitting of 3DMM to N detected 2D locations(landmarks) $w_j = [a_j\ b_j]^T$ (j = 1 …N) rising on the projection of matching vertices in the 3DMM. Without wastage of generalization, suppose that the *j*th 2D location matches to the *j*th vertex in the 3DMM. The goal of fitting process is to get the parameters of pose and shape that minimize the reprojection error, E_k, among experiential and projected 2D locations:

$$E_k(\alpha, O, r, l) = \frac{1}{N}\sum_{j=1}^{N}\left\|x_j - [C_j\alpha + \overline{m_j}, R, t, s]\right\|^2 \qquad 2$$

Where O is rotation matrix, r is translation vector and *l* represents Scale.

Pose Estimation represents extraction of O, r and l. let double duplicates of the points (3D), such that $P_{2j-1} = [x_j\ y_j\ z_j\ 1\ 0\ 0\ 0\ 0]$ and $P_{2j} = [0\ 0\ 0\ 0\ x_j\ y_j\ z_j\ 1]$ and compose a vector of the equivalent points(2D) $W = [a_1\ b_1\ \ldots\ a_L\ b_L]^T$. subsequently solve for $d \in R^8$ in Pd = W utilize linear least squares. describe $o_1 = [d_1\ d_2\ d_3]$ and $o_2 = [d_5\ d_6\ d_7]$. Scale is specified by $l = (\|o_1\| + \|o_2\|)/2$ and the vector of translation by $r = [d_4/l\ \ d_8/\ l]^T$. achieve SVD on the matrix established from o_1 and o_2[12]:

$$USV^T = \begin{bmatrix} o_1 \\ o_1 \\ o_1 \times o_1 \end{bmatrix} \qquad 3$$

The matrix of rotation is specified by $O = UV^T$. If $det(O) = -1$ then refuse the third row of (U) and recomput(O). This assurance that (O) is a matrix of valid rotation.

The 2D location of the *j*th vertex as a function of the parameters of shape is assumed by $lO_{1..2}(C_{j\alpha} + \overline{m_j}) + lr$. Later, each detected vertex complements (two) equations of a linear system, for both image compose the matrix $P \in R^{2L\times S}$ wherever

$$P_{2j-1} = l(O_{11}C_{jx}^T + O_{12}C_{jy}^T + O_{13}C_{jz}^T) \qquad 4$$

and

$$P_{2i} = l\left(O_{21}C_{jx}^T + O_{22}C_{jy}^T + O_{23}C_{jz}^T\right) \qquad 5$$

and vector $k \in R^{2L}$ where

$$k_{2j-1} = a_i - l(O_1\overline{m}_j + r_1) \quad \text{and} \quad k_{2j} = b_j - l(O_2\overline{m}_j + r_2) \qquad 6$$

solve $P\alpha = k$ in a logic of least squares topic to an extra limitation to guarantee believability of the explanation[12].

3.2. Iterated Closest Edge Fitting

It's a method to fit a 3DMM to 2D edges. That is, for both vertex of projected model contour, discovery the nearby pixel of image edge and handle this as an identified matching. In combination with the landmark matching, will again perform the process in sections 3.1.

This leads to updated parameters of shape and pose, trying to update edges of model and correspondences.Repeat this procedure for a static amount of repetitions. Denote to this procedure as (ICEF) Iterated Closest Edge Fitting. Discover the pixel of image edge nearby to a vertex of projected contour can be completed powerfully by storage the pixels of image edge in a kd-tree. Then filter the resulting matching using two generally utilized heuristics. Initialy, eliminate 5% of the correspond for which the space to the pixel of neighboring image edge is the biggest. Subsequent, eliminate correspond image space divided by *n* overtake a threshold.

4. The Proposed 3D Face Reconstruction System

The Proposed 3D face reconstruction system begins from obtaining “unconstrained” collection of face images captured under a varied of poses. the ending of proposed procedure is deriving a 3D face model with texture information. The workflow of the proposed approach could be outlined as follows and as shown in figure 3:

- Read 3D general Model, in this paper ‘.ply’ file format are used (Morphable model). Algorithm (1) contains the main steps needed to read 3D model file.
- Load parameters of initial models. The parameters are:
    1. verticesPC is a 3n by k matrix where n is the model vertices number and k the principal components number.
    2. verticesMU is a 3n by 1 vector containing the vertices of the mean shape.
    3. StandardPC is a k by 1 vector containing the sorted standard deviations of each principal component.

fl is an n by 3 matrix containing the face list for the model.

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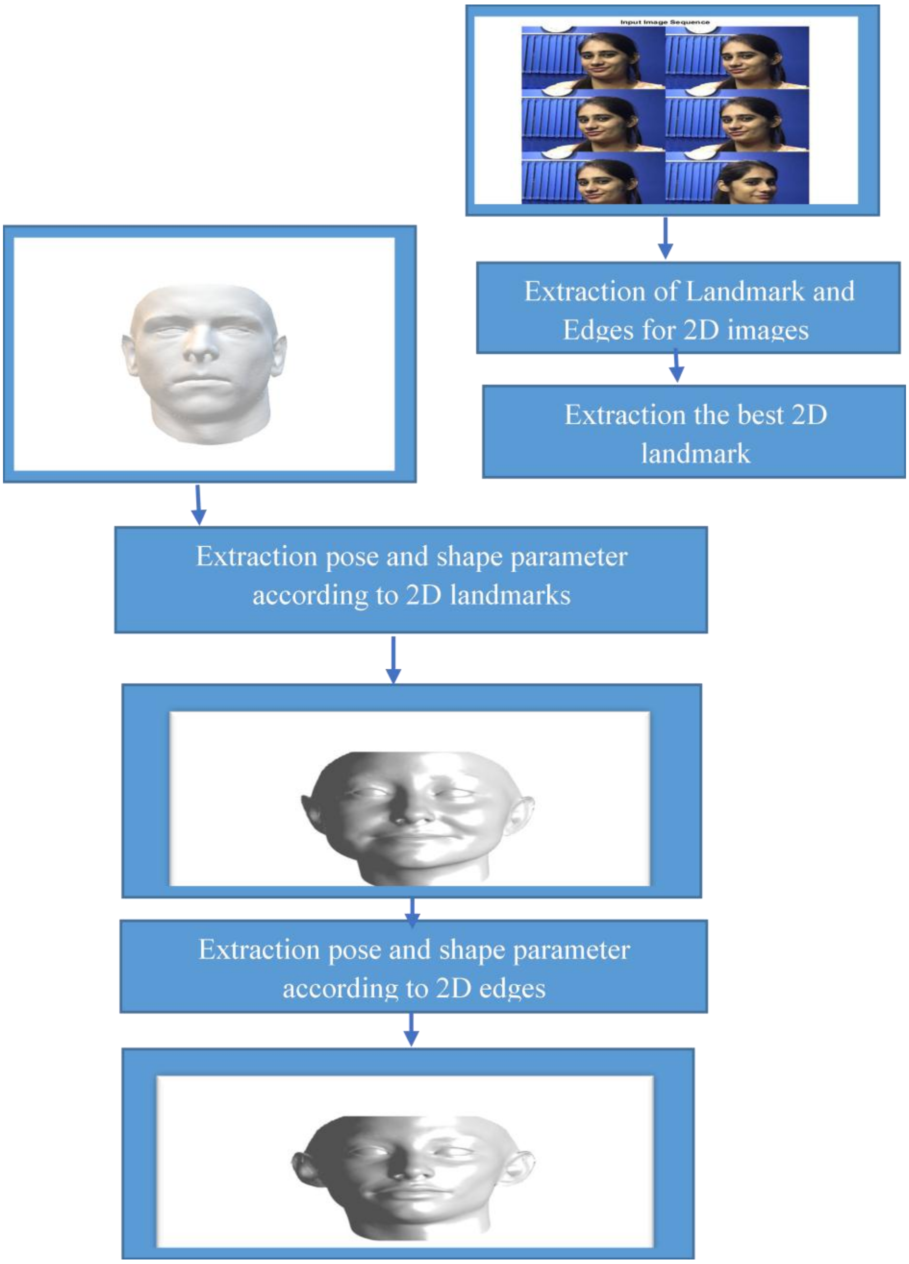

Figure3: The Block Diagram of the improvement 3D MM model fitting Method

Algorithm(1): Read of 3D Model

Input: 3D file
Output: Faces array (F) ,Vertices array(V)

Begin
//Read the size of vertices from 3D file
Vertices no. = size (V)
For i=1: Vertices no // **read the vertices of 3D model**
Read (Vi)
End
// Read the size of faces from 3D file
Faces No =size (F)
For i=1: faces no. // **read the faces of 3D model**
Read (Fi)
End
End

- Load precomputed edge structure for initial models.
- Read sequence of images (2D image).
- Find edges in this 2D image by using Canny methods. The Canny technique discoveries edges by searching for local greatest of the image gradient. The gradient is calculated utilize the derivative of a filter of Gaussian. The method uses two thresholds, to detect strong and weak edges, and includes the weak edges in the output only if they are connected to strong edges. This method is therefore more than the others methods to be "robust" by noise, and more likely to detect true weak edges
- Extraction the landmark of the 2D image by using the method which proposed by [13].
- Initial estimate of pose and shape parameters of finial 3D face model using only location of 2D landmarks as shown in subsection 3.1. In this step, proposed improvement of fitting process to get accurate landmark by taking best 2D landmark of all images rather than taking only first 2D image landmark (traditional method). the step of modifying detection of best landmark shown in the following algorithm (2), while the step of extraction of pose and shape parameters according to 2D landmark shown in the algorithm (3)

Algorithm(2): Detection best of 2D landmark

Input: serial of 2D image, 2D landmark
Output: detection of best 2D landmark

```
Begin
Count=0
for i = 1:imSet.Count // count for number of images
begin
count=count+1
im (i) =      read(image) // read the images
point(i)= landmark(im(i)) // extract the 2d landmark
end
//calculate the difference between each neighbor of 2D image landmark
diff1=landmark{length(landmark)}-landmark{length(landmark)-1}
diff2=  landmark{length(landmark)}-landmark{length(landmark)-2}
diff3=  landmark{length(landmark)}-landmark{length(landmark)-3}
diff4=  landmark{length(landmark)}-landmark{length(landmark)-4}
diff5=  landmark{length(landmark)}-landmark{length(landmark)-5}
 // calculate the average of difference //
   average_diff=(diff1+diff2+diff3+diff4+diff5)/5;
point_thres=4;
[ROW,COL]=size(average_diff);
// calculate the landmark according to average of difference
for RR=1:ROW
   for CC=1:COL
    begin
          if abs(average_diff(RR,CC))>point_thres
            landmarkIM(RR,CC)=(point1(RR,CC)+point2(RR,CC)+point3(RR,CC)
            +   point4 (RR,CC)+point5(RR,CC)+point6(RR,CC))/length(landmark)
        else
          detectededgesIM(RR,CC)=point1(RR,CC)
      end
 end
```

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Algorithm(3): Extraction of Pose and Shape Parameters

Input: landmark of 2D image, (parameter of 3DMM),xyzpoints
Output: pose parameters (R, T) and shape parameter (FACE.vertices)

Begin
mu = verticesMU (u,v,w) // the vertices of the mean shape.
P = verticesPC. // is a 3n by k matrix where n is the model vertices number and k is the principal components number.
b=best 2D landmark using algorithm 2
for k=1:niter // niter is Number of iterations
begin
//Build linear system of equations in 8 unknowns of projection matrix
A_{2i-1} = [u_i v_i w_i 1 0 0 0 0]
A_{2i} = [0 0 0 0 u_i v_i w_i 1]
// Solve linear system
k = (A\b)
// Extract parameter from recovered vector
r1 = k(1:3)
r2 = k(5:7)
S = (norm(R1)+norm(R2))/2 //S is Scale //
r3 = cross(r1,r2)
[U,S,V]=svd([r1; r2; r3])
R = U*transpose(V) // R is rotation matrix//
// Determinant of R must = 1
if (det(R)<0)
begin
// reject the third row of U and precompute R
U(3,:)=-U(3,:)
R = U* transpose(V)
end
sTx = k(4);
sTy = k(8);
T(1,1)=sTx/s; // translation vector //
T(2,1)=sTy/s;
// Obtain initial shape estimate (E)
for j=1:size (P)
begin
$C_{2i-1} = s(R_{11}P_{iu}^T + R_{12}P_{iv}^T + R_{13}P_{iw}^T)$
$C_{2i} = s(R_{21}P_{iu}^T + R_{22}P_{iv}^T + R_{23}P_{iw}^T)$
$h_{2i-1} = x_i - s(R_1 mu + t_1)$
$h_{2i} = y_i - s(R_2 mu + t_2)$
End
solve Cα = h in a least squares problem
E = LSQLIN(C,h) // (norm(c*E-h)^2)
FACE.vertices= verticesPC *E+verticesMU
End
End.

❖ Perform 3DMM fitting using 2D landmark and edges as shown in subsection 3.2. In this stage get the final shape parameter to the final 3D model.

Perform an iterative procedure to change the shape parameter according to 2D edge until get the accurate 3D model as shown in algorithm(4).

Algorithm(4): 3DMM Fitting using 2D Landmark and Edges

Input: edge of 2D image, (parameter of 3DMM)
Output: finial shape parameter

```
Begin
   FV.faces = tri; // Face structure of 3DMM mesh
   //Extract image edges E
   For i=1:niter //  Number of edge fitting iterations
       //Compute vertices lying on occluding boundary(boundary edges)
        //Compute face normal
        fn = normal_face( FV.vertices,FV.faces )
        // Edges with a zero index lie on the mesh boundary, i.e. they are only
       adjacent to one face
         boundaryEdges = Ef(:,1)==0; // Ef : faces adjacent to each edge in the mesh
         //Compute the occluding edges as those where the two adjacent face normal differ in
         the sign of their Z component
         Ev : vertices adjacent to each face in the mesh
         occludingEdges = sign(fn(:,3)) ~= sign(Ef(:,3)) &~boundaryEdges
      //Select the vertices lying at the end of the occluding edges and remove duplicates
      occludingVertices = Ev(occludingEdges)
      occludingVertices = unique(occludingVertices)
      // compute Project occluding boundary vertices
      x2 = R* (FV.vertices(occludingVertices,:) // R is rotation matrix
      x2(1,:) = x2(1,:)+t(1) // t is translation  vector
      x2(2,:) = x2(2,:)+t(2)
      x2 = x2.*s    // s is scale
      // Find edge correspondences
      [idx,d] = knnsearch(E,x2)
      //Filter edge matches
      sortedd=sort(d)
      //Proportion of nearest-neighbour edge matches used at each iteration, may be more
      sensible to threshold on distance to nearest neighbour
      percentile = 1;
      threshold = sortedd(round(percentile*length(sortedd))
      occludingVertices = occludingVertices(d<threshold)
      // Perform 3DMM fit using edge corresponding
      Perform algorithm (3) to get new shape parameters according to edge values
  endfor
End
```

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5. Results and Discussion:

In this work, available comprehensive and popular 2D image database Psychological Image Collection at Stirling (PICS) is used. This dataset contains 687 Colour faces, Between 1 and 18 images of 90 individuals. Its variations in pose, viewpoint, and expression. The database's resolution is varied from 336x480 to 624x544. Fig.4 shows the unconstrained image sequence (different in poses) which represent the input for the system.

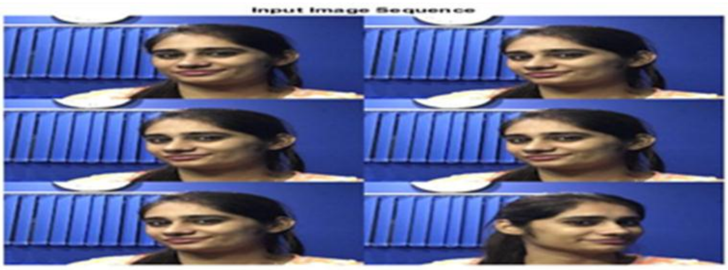

Figure 4: Input image sequence.

Fig.5 (a) represents the generic model, then deforms this model by fitting with the 2D image. First the generic model deforms according to only 2D feature (landmark) as shown in figure 5 (b), then begins in iterative procedure to deform the 3d model with 2D landmark and edges by finding nearest neighbor between model edges and image edges to get the finial 3D model which match with the feature in 2D image as shown in figure 6

(a) (b)

Figure 5 (a): the initial 3D model. **(b):** the initial deform 3D model

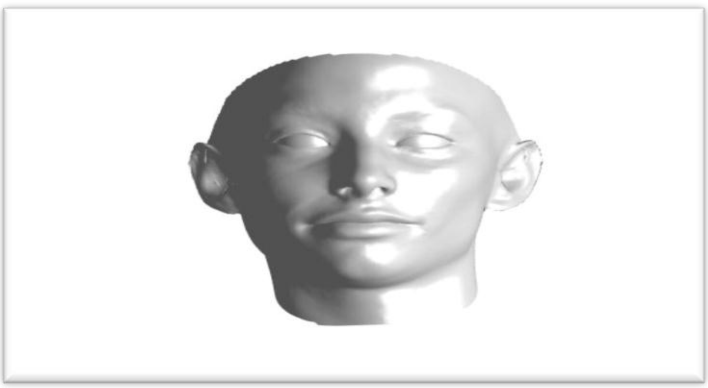

Figure 6 : finial 3D model.

As shown in figure 6 the proposed system can deform the initial 3D model (figure ٥ a) in order to be corresponding with 2D feature, and can change the feature from male in initial model to female feature (figure 6).

5.1. Comparison between traditional fitting procedure and the proposed fitting method:

In this subsection shows the performance the proposed fitting procedure and compare with the classical method as shown in table 1.

Table 1: Comparison between traditional fitting procedure and the proposed fitting method

Deform 3D model according to traditional method	**Deform 3D model according to the proposed method**
	

As shown in above table, the proposed system gives 3D model more accurate and smooth from the traditional method.

6. Conclusions

In this paper new 3D face reconstruction method is designed by improvement fitting procedure to get accurate 3D model by taking best 2D landmark of all images rather than taking only first 2D image landmark (traditional method). this modify is shown by comparing the proposed system with the traditional method. The main conclusions that are achieved from the proposed system are:

1. The proposed system can be applied to any pose of the input image.
2. The proposed 3D reconstruction has achieved good accuracy according to deform general 3D model to correspond the 2D feature.
3. the proposed system can overcome extremely challenge which is deforming the 3D default model to make it compatible with the 2D feature especially female feature.

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إعادة بناء الوجوه الثلاثية الابعاد باستخدام تقنية مطابقة نموذج مورفبل الثلاثي الابعاد

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كلية التمريض

المستخلص :

في هذا البحث، تم دراسة إعادة بناء الوجوه في شكل نموذج ثلاثي الأبعاد للصور غير المقيدة (ذات أوضاع مختلفة). وقد استخدمت طريقة مطابقة نموذج مورفبل ثلاثي الابعاد في تقنيات إعادة الإعمار ثلاثية. في هذا البحث قد تم اقتراح تحسين لطريقة مطابقة نموذج مورفبل ثلاثي الابعاد للحصول على نموذج دقيق ثلاثي الابعاد من خلال اخذ أفضل المعالم الثنائية الابعاد لكل الصور بدلا من اخذ معالم اول صورة (كما في الطريقة التقليدية). أظهرت نتائج مقارنة الطريقة المقترحة مع الطريقة التقليدية نتائج مشجعة من ناحية وقت التنفيذ وجودة إعادة البناء.

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Methods of Secure Routing Protocol in Wireless Sensor Networks

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Abstract:

The Wireless Sensor networks (WSN) consider an emerging technology that have been greatly employed in critical situations like battlefields and commercial applications such as traffic surveillance, building habitat, smart homes and monitoring and many more scenarios. Security is one of the main challenges that face wireless sensor networks nowadays. While an unattended environment makes the deployment of sensor nodes in the networks more vulnerable to vary of potential attacks, the limitations of inherent power and memory for the sensor nodes makes conventional solutions of the security is unfeasible. The sensing technology combined with processing power and wireless communication makes it profitable for being exploited in great quantity in future. The wireless communication technology also acquires various types of security threats. This paper discusses a wide variety of attacks in WSN and their classification mechanisms and different securities available to handle them including the challenges faced.

Keywords-Wireless Sensor Network; Security Goal; Security Attacks; Defensive mechanisms;Challenges

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1. Introduction

This paper presents a thorough picture of Wireless Sensor Network (WSN), grid-based cluster, and role of sensors in WSN. Additionally, the security issue concern in WSN embraced with several kinds of attacks in WSN is described. Then tailed by several authenticated routing protocols and the routing framework. Finally,in the last portion of this paper, the literature survey on intelligent routing for clustered WSN is discussed. An overview of the current paper is depicted in Figure 1.

2.1 Wireless Sensor Network

A WIRELESS sensor Network (WSN) is a self-sufficient, self-arranging, and multi-hop network that encompasses a significant quantity of sensor nodes. The sensor nodes work together with each other to screen the physical condition around them. WSNs are ordinarily unattended and energy compelled in nature. Accordingly, they require energy efficiency and efficientprotocols to expand the system lifetime and tasks [1]. A WSN contains a few sensor nodes, which can be exploited as a mass of various application situations, for example, agriculture, military, health care, and energy. WSN has been used as a piece of various fields such as schools, colleges, battlefields, surveillance and so forth. It has been used as a piece of everyone’s day to day life and its provisions are growing in well-organized fashion. WSN has come in presence as a key solution for a few issues where human mediation is difficult.

The fast advances in wireless technology, implanted chip, incorporated micro-electro-mechanical systems (MEMS), and nanotechnology has engaged the progression of nominal effort, low-control, and, multifunctional sensors. Sensors are of little in size and can perform event identification, data tracking, interacting with each other or with the data sinks. Sensor nodes are associated with each other through the wireless medium, for example, infrared or radio waves and it becomes operational with application data arrival/sensing. Internal memory related to every sensor node stores the data of its related event packets. A group of sensors sharing a wireless medium form a WSN for gathering information and transmitting it to remote clients (sinks). The principle motivation behind the WSN is to screen and gather information from the sensors and after that transmit this information to the sinks [2]. WSNs are more appealing because of their wide range of utilization, like weather monitoring, military surveillance, health care, disaster management etc. The essential reason for WSN growing popularity was to ceaselessly screen such conditions that happen to be extremely difficult or unrealistic by people [3].
The exploration of WSNs is a subset research area of Internet of Things (IoT). The sensors ought to be allocated with a routing protocol such that the information is transmitted to the end-client by means of various intermediate sensors nodes.

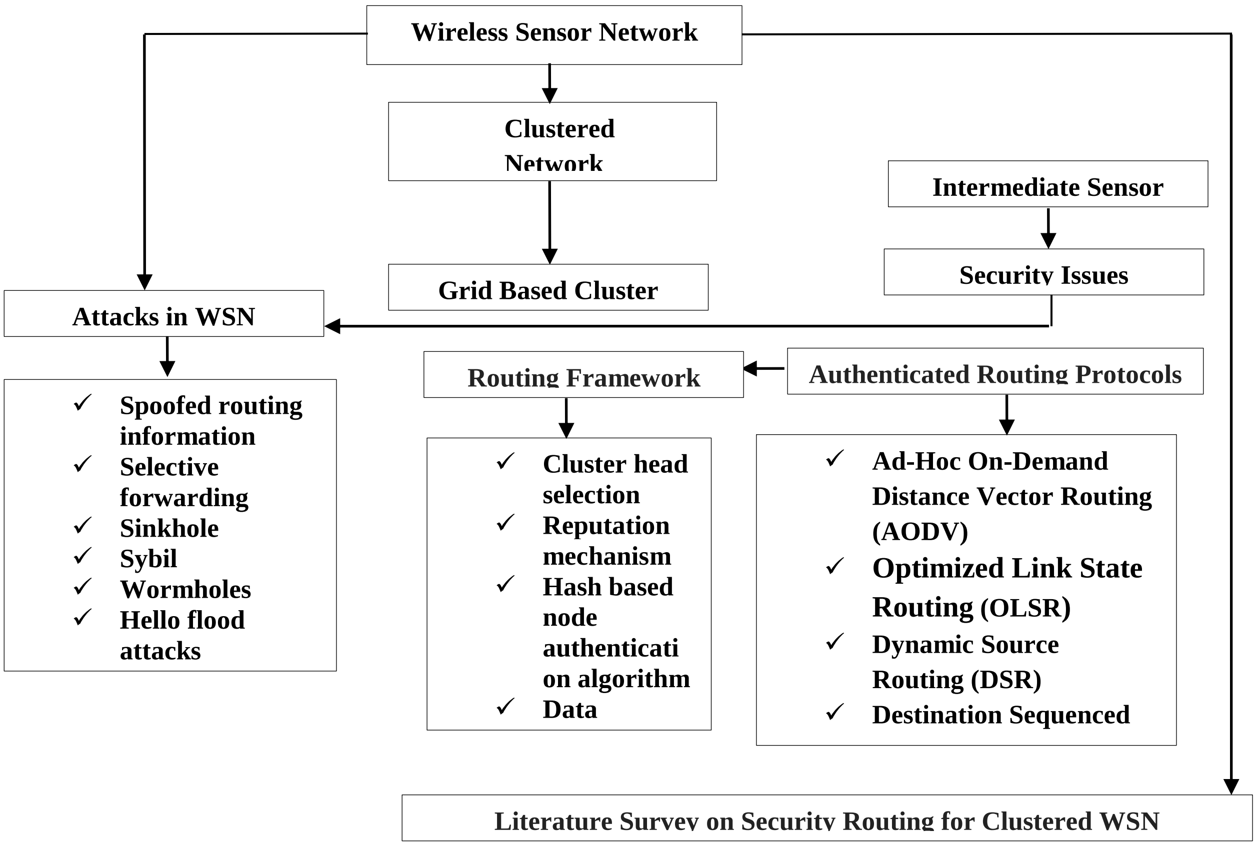

Figure 1: Literature Review Framework

A WSN may comprise of a few heterogeneous sensor nodes to perform dedicated assignments conveyed in a territory of intrigue. There exist numerous proprietary and nonproprietary existing solutions for fetching sensed information from the sensor nodes. In any case, the present pattern of utilizing IP-based sensor organizing arrangements, (for example, 6LoWPAN and IPv6) empowers the WSN to be associated with the web. Therefore, WSNs can be utilized for observing/controlling diverse applications through the web based network, which in turn, builds up the idea of the Internet of Things (IoT) [4]. In sensor nodes, the fault may occur because of the incorrect condition of equipment or a program as an outcome of a sudden fault of any part. Also, any performance degradation for nodes happening due to energy exhaustion is critical and as the time advances these shortcomings may increment, bringing about a non-uniform system topology. The issues that can happen because of the failure of sensor nodes are misfortune in availability, delay because of the misfortune in association and partitioning of the system because of the gap created by the failed sensors [5].

2.2 Clustered Network

Clustering was proposed in early researches [6] as a mean of enhancing WSN lifespan and was immediately embraced in progressive works. In clustered networks, a few nodes move towards becoming Cluster Heads (CHs) and are in charge of directing messages from other cluster CHs and cluster member nodes. Nodes that are not CHs moved toward becoming cluster member nodeand send messages just to CHs. Clustering was introduced to reduce the general power utilization and expand network lifetime. Thought of clustering seems to be self-explanatory and so apparent that it was fixed deeply into WSN theory. Clustering strategies are additionally one of the means to covenant with topology control, which can compose a WSN into a group based system. Task scheduling, data gathering, and Transmission Power Control algorithms can be executed in this structure with a specific end goal to accomplish particular task. A clustering algorithm can partition sensors into various clusters/gatherings/subgroups.

In each cluster, a CH is chosen to be responsible for producing a transmission plan, gathering information from every one of the sensors in the group and transmitting the amassed data back to the Base Station (BS). In light of the grouped structure, the framework can provide extended network life by keeping minimum communication among the sensors within a cluster, without affecting the usefulness of the system. The CH is the developer of a cluster and it is accountable for social affair information and transmitting the information to the BS. The CHs will possess more energy than ordinary cluster member sensors and in this manner, can better control the network for longer period. In many scenarios, CHs also take part to adjust the power utilization on participating sensors. Which sensors should be chosen as CHs requires cautious examination. In general, one of three CH race plans is adopted as : a) Deterministic, b) Random, and c) Adaptive. Subsequent to being chosen, the CHs will publicize themselves by communicating their data to different member sensors. Every sensor will assemble the data from all the CHs inside its radio coverage range and after that choose which CH to join in view of some correspondence properties. A few measurements can be utilized to decide the correspondence properties between a sensor and a CH, for example,

2.3 Grid Based Cluster

Many grid-based algorithms have been developed. The author [7] displayed a calculation called Low Power Grid-based Cluster Routing Algorithm (LPGCRA).It chooses the CH among the typical sensor nodes which have maximum residual energy. Regardless of the fact that LPGCRA focuses to balance the overall load of the network, it has the accompanying limitations. On the off chance that the CH is far from every single other sensors in a group, then all member sensors need to spend more energy including CH itself. Also, CHs transmit accumulated local information specific to the sink node which is moderately less in number than the cumulative amount of sensor nodes, prompting fast battery exhaustion of the CHs. In another framework based approach called Grid Based Routing (GBR), has endeavored to expand the system lifetime by ideal determination and dispersion of the CHs over the objective zone. Nevertheless, they expect a settled number of grids which may not be sensible for a huge region of operation. Besides, in the selection phase, CHs are chosen based on the sensor node’s transmission time to the sink. In this way, the technique chooses the CHs which are close to the sink in every grid and thus the member sensor nodes of the cluster exhaust their energy faster for entire transmission to the CH. However, none of the above algorithms address the energy efficiency, load matching and fault lenient secure routing issues together [8]. Grid network with clustering techniques showed better outcomes in contrast with the grid network without clustering. It requires less energy usage than other [9] available methods.

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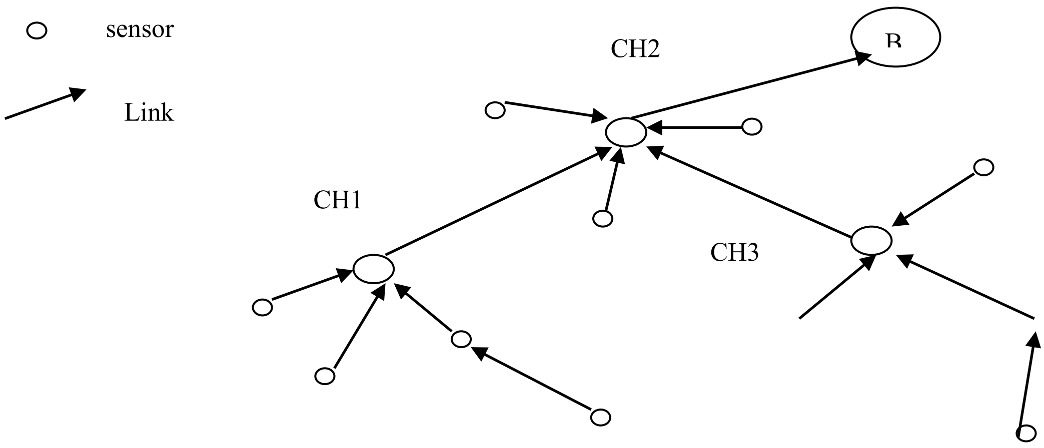

Figure2. Representation of Cluster Based WSN

2.4 Intermediate sensors in WSN

The mix of heterogeneous sensors creates a key innovation for the 21st century named WSN which gives phenomenal open doors in many areas running from military to agriculture and has applications in events like , health monitoring, modern control and home systems.As compared to other type of wireless communication technologies, WSN requires utilization of sensor systems to transport the stream

Security is a vital issue to be considered for the system which is conveyed in threatening condition. Organization of a system in a human uninterruptible place is extremely vulnerable without security. Plan of a security convention is a testing errand because of the nature and accessibility of assets. Security instruments utilized for systems can differ as per the diverse needs ranging from fixed to wireless networks [12].

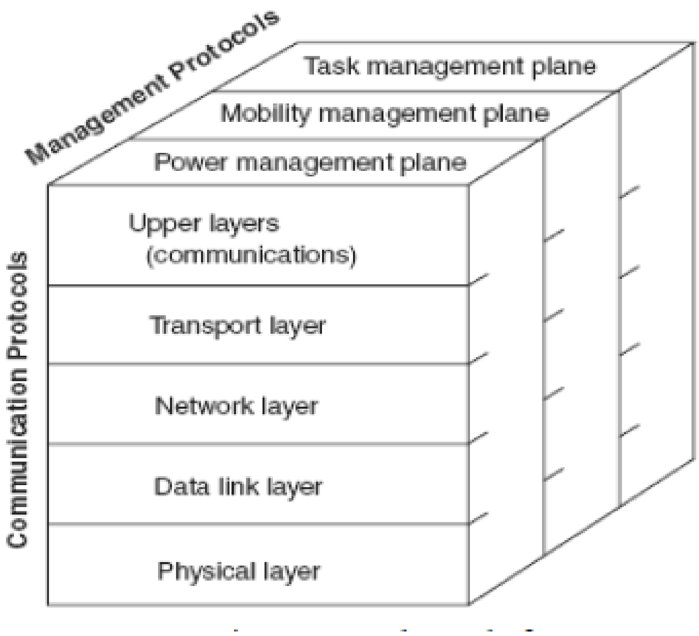

Figure 3. Representation of Generic Protocol Stack for wireless sensor Networks

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of detected information from numerous districts (sources) to a specific sink [10].Sensors are the genuine interface to the physical world, in other words, the modern gadgets that can watch or control physical parameters of the earth.In this context, without the intermediate sensor nodes, which helps to carry forward data en route, a remote sensor system would be irrelevant totally. Any sensor or actuator, which is part of WSN can act as an Intermediate sensor based on their location and data transmission path.Thus, Intermediate sensor nodes are in charge of self-sorting out a suitable system foundation, with multi-hop data forwarding between actual source and destination sensor nodes. In this way, WSNs can operate with little, minimal effort gadgets for an extensive variety of uses and they don't depend on any previous infrastructure [11].

2.5 Security Issues

Security is a vital issue to be considered for the system which is conveyed in threatening condition. Organization of a system in a human uninterruptible place is extremely vulnerable without security. Plan of a security convention is a testing errand because of the nature and accessibility of assets. Security instruments utilized for systems can differ as per the diverse needs ranging from fixed to wireless networks [12].

An arrangement of standards for tending to the issue of securing remote sensor systems was proposed. A solution with regards to these standards, underpins a differential security benefit that can be powerfully arranged to adapt to changing system state. Security of a system is controlled by the security over all layers. In a massively distributed network, safety efforts ought to be agreeable to dynamic reconfiguration and decentralized administration. In a given system, at any given instant, the cost brought about because of the safety efforts ought not to surpass the cost surveyed because of the security dangers around them. If physical security of nodes in a system isn't ensured, the safety efforts must be strong to physical altering nodes in the field of deployment. Since sensor systems posture extraordinary difficulties, the security strategies as a part of conventional systems can't be utilized straightforwardly. On account of different limitations in WSN, the accompanying perspectives ought to be painstakingly considered when planning a security plot: Power efficiency, Node Density and Consistency, Adaptive security, Self configurability, Simplicity and local ID [13]. Security in WSNs is as essential as in different styles of frameworks, particularly in military and security applications (e.g. intruder recognition). Aggressors may endeavor to block movement in destinations (i.e. execute a denial of service attack) or deal information with the expansion of some satirize detected information to arrange (i.e. aggregating invasion). Aggressors from the inside (ruined node already situated into WSN) can confer steering issue by driving information development to parodied sinkholes. In WSNs, different assaults are discovered in the following section that damages the normal operation of the entire system framework.

3 Attacks in WSN

3.1 Spoofed routing information

The most direct attack against a routing protocol in any system is to focus on the routing information itself while it is being traded between nodes. An aggressor may spoof, modify, or replay routing data, keeping in mind the end goal to upset activity in the system. These interruptions incorporate the creation of routing loops, attracting or repulsing system movement from selected nodes, broadening and shortening source routes, producing counterfeit mistake messages, dividing the network, and expanding end-to-end idleness. A countermeasure against spoofing and alteration is to affix a message authentication code (MAC) after the message. Proficient encryption and verification methods can shield satirizing attacks. In this kind of attacks, a malevolent node parodies a MAC address of a node and makes various ill-conceived identities that highly influence the normal operation of wireless sensor network [14].

3.2 Selective forwarding

WSNs are normally multi-bounce systems and consequently in view of the presumption, that the participating nodes will forward the messages dependably. Malicious or attacking nodes can however decline to course certain messages and drop them. In the event that they drop every one of the packets through them, at that point it is known as a Black Hole Attack. In any case, in the event that they specifically forward the packets, is known as selective forwarding. To counter the possible attacks associated with this, Multipath routing can be utilized. Such ways of routing are not direct as they don’t have two consecutive regular nodes, and they utilize implicit acknowledgements, which guarantee that packets are forwarded as they were sent. In networks, attackers focus on two kinds of attacks: an information attack and a routing attack. A selective forwarding attack is a kind of information attack, in which a traded off node specifically drops a packet. In this way, a selective forwarding attack may harm the system proficiency. This kind of attack can also be thought to be a unique instance of black hole attack. Most of such attacks are caused by a malignant node that is situated in the route of information stream. Such nodes can dismiss a certain or specific delicate message that starts from different parts of the network and to be forwarded to the base station. Consequently, a malevolent node can bring down certain nodes near the base station to damage end to end connectivity. Such pernicious node specifically targets dropping delicate packets, for example, a packet that reports foe tank development. Malicious sensor nodes can work as normal sensor nodes. Selective forwarding attacks can be categorized into two classes: drop packets in certain nodes and drop packets of certain types [15].

3.3 Sinkhole attack

Contingent upon the routing algorithm method, a sinkhole attack tries to bait all the activity toward the traded off node, making a figurative sinkhole with the enemy at the middle. Geo-routing protocols are known as one of the directing convention classes that are impervious to sinkhole attacks. Such routing topology is developed utilizing just restricted data, and movement is normally directed through the physical area of the sink node, which makes it hard for an attacker to target somewhere else for making a sinkhole. In a sinkhole attack, the gatecrasher's point is to bait all the activity from a specific zone through a traded off node, to launch an attack. Thereafter, the traded off node tries to pull in all the movement from neighbor nodes in view of the direction measurements utilized as a part of the routing protocol. Sinkhole attack is one of the key routing attacks which is tough to counter on the grounds that the routing data provided by a node in a remote sensor arrangement, is hard to validate [16]. To lunch a sinkhole attack, the gatecrasher first victimize any normal node in the network and then the compromise node attacks the other neighbor nodes based on the data received from the conventional routing protocols. This leads to a serious attack situation due to the traded off nodes location proximity to the sink. As a consequence, the source is unable to send information packets to the sink . This prompts misrouting of information bundles [12].

3.4 Sybil attack

The Sybil attack signifies "damaging device illicitly dealing with several identities". The attacker adopts the procedure of making a few duplicate nodes utilizing a similar identity i.e. indistinguishable node id. For example, an inconvenient node can easily stretch false identities, or may mimic different other trustworthy nodes inside system. In particular, WSNs are more prone to sybil attack due to the nature of open and shared wireless transmission medium. In addition to that , exactly the similar rate of reappearance is being shared among all nodes. In such scenarios, the attacker could make different lacking authenticity characters by manufacturing or maybe taking these personalities with respect to trustworthy nodes. In this manner , the beds base station cannot differentiate between the legitimate and produced nodes. This befuddles the base station alongside different nodes and corrupts the system execution [17]. In this attack, a solitary node exhibits different characters to every single other node in the WSN. This may deceive different nodes, and henceforth routes accepted to be disjoint as a particular node can have a similar enemy node. A countermeasure to sybil attack is by utilizing a remarkable imparted symmetric key for every node to the base station. Thus , in Sybil attacks a noxious gadget wrongfully going up against numerous personalities impact large number of network nodes with a specific end goal to deplete its assets or taking delicate data.

Types of Sybil assaults

1. Direct vs. Indirect Communication: In direct communication, Sybil nodes straightforwardly speak with true legitimate node while in Indirect Communication, no real nodes can discuss specifically with the Sybil nodes.

2. Manufactured versus Stolen Identities: In manufacturing, the vindictive node creates number of characters while in stolen, the node hacks an honest node's authentic node ID.

3.5 Wormholes

This kind of attack is imperative and furthermore, it is a perilous attack for WSN kind of deployment. The attacker could record a packet at a solitary area in the system, burrows them to another area, and resends them into the system. The aggressor can replay messages to any piece of the system. In wormhole attacks, malevolent nodes can make a shrouded channel between sensor nodes. A wormhole attack is a vital risk to a wireless sensor network on the grounds that this kind of attack does not entail that a sensor in the system is traded off. See Figure 4.

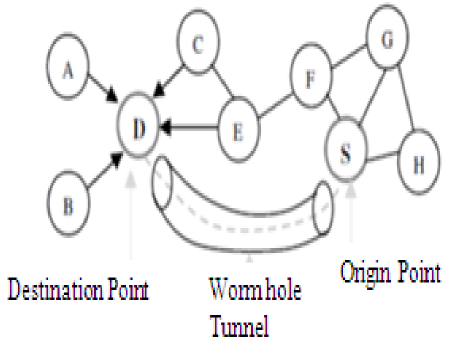

Figure 4. Representation of Worm hole Attack in Network

This kind of attack influences the system layer by constantly hearing and recording information. It can be actualized in the underlying stage when the sensor dispatches to find data [15]. A foe can tunnel all the sensitive messages received into a player node in the system over a low inactivity connection and replay them in another piece of the system. This is normally finished with the coordination of two foe nodes, where the nodes endeavor to downplay their separation from each other, by communicating bundles along an out-of-bound channel accessible just to the attacker. To beat this, the data movement is directed to the base station along a way, which is dependable and topographically shortest or utilize tight time synchronization among the nodes, making the attack situation infeasible in commonsense conditions. The wormhole connection can be set up by numerous kinds, for example, long-range wireless transmission in wireless networks, by using an Ethernet cable, a long-range wireless transmission and an optical connection in a wired medium. Wormhole attack records packets toward one side point in the system and passages them to opposite end-point. These attacks are extreme dangers to MANET and WSN routing protocols. For instance, when a wormhole attack is utilized against an on-request routing convention, for example, AODV/DSR, then all the packets will be transmitted through this malicious path and no other route is found. In the event that the attacker makes the passage sincerely and dependably, then it won't affect the system and furthermore gives the valuable administration in associating the system more proficiently. The attacker can play out the attacks regardless of whether the system correspondence gives privacy and genuineness. A key countermeasure can be integration of the prevention methods into intrusion detection framework. However, it is quite tough to locate the attacker with a software-only approach, since the wormhole nodes send similar packets like their counterpart legitimate nodes. In this case, If single path on-demand routing protocol such as AODV is being deployed in highly dynamic wireless ad hoc scenario, a new route is required to be found in the event of every route break. Each such route discovery demands high overhead and latency. This inefficiency can be overcome with multiple paths route discovery protocol which also addresses the phenomena of all paths breakage in a network. To secure the network against wormhole, a combination of processes, procedures, multipath routing and systems need to be adopted that can ensure confidentiality, authentication, integrity, access control, availability. Few techniques of authentication and integrity mechanism, either by the end-to-end approach or hop-by hop, is useful to provide the accuracy of routing information.

3.6 Hello flood attacks

Hello flood attack has proved to be one of the primary concerns in communication system and it is generally observed in the network layers of WSNs. These attacks are mainly caused by an attacker with high transmission power and that sends or receives the hello packets used for the discovery of neighbor. During these processes, the attacker develops an impression of being a neighbor to other nodes and finds that underlying routing protocol provides disrupting facilities for different types of attacks. Further, the attacker transmits the packets in such a way that large number of nodes are considered as parent node in the WSNs [31]. Figure 5 shows a hello flood attacker broadcasting hello packets with more transmission power than base station. In response to this, a legitimate node has considered the attacker as its neighbor and also an initiator.

In recent years, the research conducted in the field of attacks , ascertains rapid increase in delay while transmitting the message amongst parent node in WSNs. Besides, it has been observed that attacker transmits these hello messages through large number of nodes**.** From this process, the attacker establishes the nodes as the neighbor nodes in the network. Subsequently, considerable amount of energy is dissipated among the nodes while responding to the HELLO message from attackers end , which further leads to confusion state [32].

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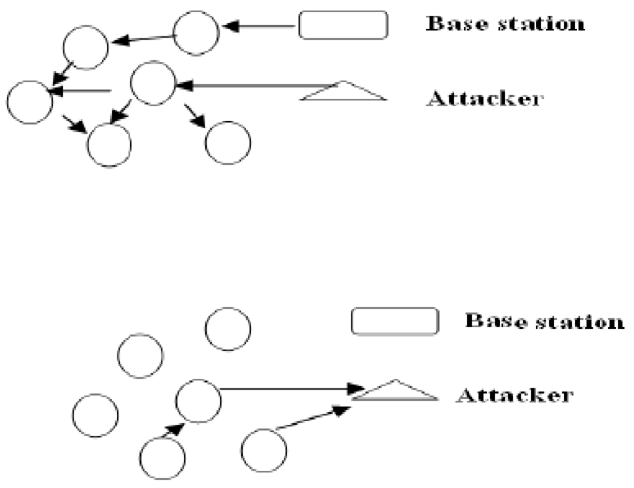

Figure 5. Representation of Hello Flood network

The Hello flood attack recognizes the sensor node, transmits hello message and proclaim itself as their neighbor. Further, any normal neighbor node assumes that the sender/attacker node is in communication range when it receives the hello message and start communicating with that node by providing an entry through its routing table as its neighbor. The communication amongst sensor nodes and base station is provided through the mean of neighbors. The hello message with more power is generated when the attacker tries to capture the legitimate node or develops a fake node in the network and further it produces a dilemma whether the message has been received from neighboring nodes [33]. Moreover, it is observed that hello message has been considered to travel through the shortest path by the nodes from the base station on the basis of assumption that malicious node is the base station and starts communication with the attacker. Through this process, the attacker can gain control over the base station and communication gap raised among the base station and other sensor nodes, which eventually affects the routing process.

The properties of the Hello packet has been categorized into five features and they are as follows,

1. The Hello packet size is less compared to data packet.
2. The tendency of hello packet reaching its destination (receiver) is high compared to data packet in weaker links of the WSNs.
3. The transmission of hello packet is higher at lower bit rate, since basic bit rate is more reliable compared to others.
4. Acknowledgement is not essential while transmitting data in Hello packets.
5. There is no guarantee in bidirectional transmission of hello packets.

Hello packets are capable of providing opportunity for more number of attacks such as flooding, tempering and node capturing, false node replication. These attacks are explained in brief as follows:

Flooding

In this type, a new connection request has been continuously accepted by the neighbor from the attacker for resource capturing. The results observed in legitimate nodes are in terms of severe resource constraints [35].

Tempering and node capturing

Tempering is generally observed in the attacks on the components, which requires alteration in the interior structure of an individual chip. An advisory module can easily recognize it and can be processed for hello flood attack. In this process, the attacker can gain full control over the sensor node through node capture attack. In order to develop node capturing, the attacker should have precise knowledge, efficient equipment with other few resources. The main limitation observed in this technique is complexity in separation or removal of nodes from network [36].

False node replication

In this process, new sensor node has been rooted inside WSNs by an attacker by using the ID of legitimate user. Initially, legitimate ID has been replaced with the false one in the network and replication of this false node along with the support of flood attack can lead to a huge destruction in WSNs. The rate of such damages was predicted to be high and also, the attacker was found to have gained complete access over the network [37].

4.Authenticated Routing Protocols

4.1 Ad-Hoc on Demand Distance Vector Routing (AODV)

The Ad hoc On-Demand Distance Vector (AODV) routing protocol operates with mobile/static sensor nodes for early recognition of the routes and to reach new destination. Here, the nodes are not required to preserve and maintain routes to its destination. The AODV algorithm comprises with features such as autonomous, dynamic, multi hop routing amongst the nodes to establish and maintain route in an ad hoc network. It addresses the issues regarding route breakages and fluctuations in network topology in a well time manner. It is loop free operation and excludes Bellman-Ford "counting to infinity" problem to provide fast convergence rate during the changes in ad hoc network topology, link breakage. Further, the affected set of nodes is identified in order to nullify the routes using the lost link [38].

The demand on various applications of the routing protocol has influenced researchers to modify the basic algorithm and provide better utilization factor in adhoc networking. In this study, various AODV routing protocols have been reviewed, compared to select the effective protocol. Additionally the benefits of ad-hoc frameworks and the present difficulties have been reviewed. Further, this study provides a review on the areas to be enhanced and to understand the capability of the ad-hoc networks. The evaluation parameters such as end to end delay, packet delivery ratio, energy, throughput are often to be considered for analyzing Ad hoc On-demand Distance Vector routing protocol [39] as shown in Figure 6.

Destination	Sequence no	Hop count	Next hop	Time out

Figure 6 Represents the Routing Table structure in AODV

Latest trends show that when a specific team goes for a gathering or meeting, without a doubt they have a handheld PC or laptop instead of scratch pad. The primary reason behind the fact is that the network information system data assumes a dynamic imperative part in sharing data quicker than with settled base stations. Moreover, it conveys the data related information by means of video or voice, starting with one safeguard colleague to the next through means of portable hand held or wearable remote gadget [40]. The challenges arising in deploying ad hoc wireless networks are,

1. Energy management
2. QoS support
3. MAC protocol

It has been observed from the above study that energy management in handheld devices , can genuinely neglect forwarding packets to Ad hoc mobile environment. Henceforth, the routing traffic on the basis of node energy management is the one way approach to recognize stable route and nodes that are more enduring than others. It’s insufficient to consider QoS simply at the system level without considering the fundamental media access control layer. In recent years, the trends in WSNs have been outperformed by QoS approach because of its advantages compared to other approaches. A few ideal models are presented in latest Ad hoc routing methodologies such as reduced time delay, enhanced packet delivery ratio, advanced techniques for path finding and TCP performance to analyze the QoS performance. An adaptable Ad hoc routing protocol can responsively conjure on-demand approaches on the basis of circumstances and communication prerequisites [41]. Additionally, these works are necessary in the future for media access control, and security to understand the capability of Ad hoc networks.

4.2 Optimized Link State Routing (OLSR)

The OLSR commonly acquires the stability of the connection state algorithm into link-state protocol. Individual node recognizes every link in the neighbor nodes and occasionally surges a message containing the entire link connections i.e link State Message [42]. Further, every individual node develops a topology map for the network and freely ascertains the best hop pointing to the destination through shortest path algorithm. Generally, the OLSR routing protocol is an enhanced optimization process to the classical link-state algorithm. The advantages of multipoint relays (MPRs) have been considered as important concept in OSLR [43]. The MPRs mechanism chooses the individual node, which forward transmitted messages during the flooding procedure. This algorithm decreases control packet propagation in the entire network because of the condition that node only selects the subset of the link along with its neighbor MPR selectors. In brief, the packet flooding in the system is considerably decreased because of the fact that exclusive MPRs only create the link state information and communicates the message.

In this way, the MPR nodes may promote just connections among themself and its MPR selectors, thus utilizing partial link-state information for calculating the route.
Only two mobile nodes have been considered in the single hop transmission. A portable mobile node is associated by means of LAN association with Host A, while the other is associated through various LAN associations with Host B. The OLSR routing protocol then spontaneously calculates the link and subsequently creates the routing path. In actuality, every one of the four versatile nodes is utilized in multi hop communication. Single hop experiment also follows similar experimental setup. However, the in-between mobile routers are placed at foreordained areas between two LAN. The chained link between two LANs has been formed by computing the links through OSLR routing protocol and using intermediate routers [44].

4.3 Dynamic Source Routing (DSR)

Dynamic Source Routing protocol is defined as a self-maintaining routing procedure for wireless sensor networks. This protocol can also work with cell phone frameworks and portable systems having up to 200 nodes. A Dynamic Source Routing system can design itself autonomously without the mediation of human overseers. A route on demand can be formed while transmitting the node requests. Self-developed source routing has been utilized at individual intermediate device instead of depending on the routing table. Deciding source routing requires gathering the address of every sensor node between source and destination, during discovery of the route. The captured data is further used by nodes for processing route discovery packets. Moreover, the acquired packets are used in route packets and it comprises with the details regarding the address of the individual device where the packet will traverse [45].
Furthermore, the DSR is considered to be an on demand protocol which is used to monitor and regulate the bandwidth consumption rate by means of control packets in ad hoc wireless networks. Table driven approaches have been considered to do the additional work and updating the table in accordance with changing network conditions. The key comparison amongst the present and on-demand routing protocols is the absence of hello packet transmission, which is used to sense the presence of nearby nodes. The fundamental procedure considered in this protocol is to develop a node's routing database through the process of flooding Route Request packets in the network.
The destination node reacts by replying back with a route reply packet to the source after receiving route request packet. The information regarding traversed route can be obtained through the route reply packet received at the destination node [46].

5 Authenticating Routing Framework

Energy efficient routing protocols have been developed by various authors in order to enhance the life span of Wireless Sensor Networks (WSNs) and balance energy conservation. Different authors reviews along with routing frameworks are as follows,

5.1 Cluster head selection

A priority based load balancing method comprising cluster head selection is used to overcome the limitations arising due to clustering in WSNs [20]. Low Energy Adaptive Clustering Hierarchy (LEACH) technique is deployed to allocate cluster head position amongst member nodes. Later, clustering operation is performed after a particular period of time to return new setup phase and calculate new cluster head. From the experimental study, it has been observed that modified leach protocol comprising cluster head selection provides better results of about 70,000 packets afore the network breakdown, which is much higher compared to other existing techniques. Furthermore, a new clustering approach comprising multi criteria decision making has been developed by author [21] in order to effectively select and manage clusters by considering different criteria. Technique for Order Preference by Similarity to Ideal Solution (TOPSIS) is considered to select the best value and optimize the clusters. From the study, it has been observed the result obtained in terms of number of nodes of 100, initial energy of nodes = 0.5j, shows better performance in terms of lifetime in WSNs compared to other techniques. An enhanced protocol comprising node ranked-LEACH is proposed by author [22] to overcome the limitations such as random process selection in existing techniques and enhance the life span of WSNs. Both past cost and link number amongst nodes is considered to determine the cluster head of the individual cluster and successfully handle shortcomings arises due to extra overhead and high consumption of power. The study on experimental analysis shows that NR-LEACH enhances the lifetime of WSNs compared to other LEACH techniques and provides optimal CH for individual cluster selection. Another research [49], authors proposed HHSRP protocol by developing a mixed hierarchical clustering algorithm that considers greatest value of coordinator node and fitness value to decide CH. The two key algorithm utilized were mixed hierarchical cluster based routing and hybrid hierarchical secure routing. HHSRP showed the capability of packet transmission based on packet priority and without losing the packets due to any malicious network activity.

5.2 Reputation Mechanism

Reputation Based (RB) security scheme has been developed by author [23] to provide reliability and security to beacon nodes which contain the information of sensor nodes in WSNs. Reputation evaluation model is considered to analyze the performance of the individual beacon node and validated to define the credibility of the beacon node. From the study, it has been observed that RB model successfully increase the accuracy of the WSNs in hostile and untrusted environments. The author specified that in future, this technique can be extended to overcome the issues regarding other malicious attacks. Furthermore, a new model comprising Risk-aware Reputation-based Trust (RaRTrust) technique has been developed in order to analyze the issues regarding insider attacks such as node compromise, traditional security and bypassing [24]. The developed technique has the capability to diagnose and separate malicious and faulty nodes. Further risk evaluation is considered to overcome dramatic node spoiling. The results obtained from the study shows that the proposed model reduces the effect of bad mouthing attack and further stated that recommendations can be aggregated securely when dishonest recommendation is 30% more compared to total recommendation.

5.3 Hash based node authentication algorithm

A new memory based approach called hash table has been considered to evaluate the security and performance of energy consumption in WSNs [25]. In this process, the individual node can be distributed over the entire internet and each node can be controlled using whole resource space and related index information. Finally, different experiments have been conducted by author to evaluate the performance and results shows better performance in terms of energy optimization compared to other techniques. Furthermore, the usage of energy consumption using Hash functions such has MD-5, SHA-1, SHA224, SHA512 has been analyzed by author [26] in authentication nodes of WSNs. The hash functions are represented in the form of optimized codes and processed on virtual computer. It is observed from the study that SHA-224 has outperformed other techniques in terms of real time of 0.1263 sec, user time of 0.0087 sec and system time of 0.0634s. A new authentication scheme comprising Hash based DCHST has been proposed by author [27] to analyze data aggregation problems and energy constraints in WSNs. Distributed pseudo random function has been considered to modify the SHA-1 hash function and deliver collision resistant requirements. It has been observed from the study that combined approach of dynamic clustering along with hash technique increases the security and removes redundancy with improved bandwidth utilization and high data privacy security.

5.4Data replication mechanism

WSN comprises with hundreds of sensor nodes and individual node consists of short-distance wireless links. Because of this wireless links, energy constraint has always been a critical issue. Data replication has proved to be an effective and common way to address these issues and improve data management in WSNs [28]. The advantages of virtual grid concept are used to develop Adjustable Data Replication (ADR) to increase the energy of the popular nodes [29]. The data replication based ADR has been used to increase the life time of the data nodes and WSNs. The data replica nodes are repeatedly built using ADR and near to the query node to provide balance between overhead and energy consumption of the sensor nodes. From the study, the results show that ADR can effectively reduce replica nodes by 60%.and reduced energy consumption by 33% compared to other techniques. Furthermore, a low complexity data replication scheme has been proposed by author [30] to increase the rate of data storage and decrease the probability of data loss. Periodical recycling has been considered to limit the memory usage and greedy distribution storage scheme for data loss prevention. From the study, it has been observed that data dissemination scheme is modeled and simulated and results obtained shows that relative improvement in life time, energy usage and balance amongst data storage and neighbor nodes is observed compared to other techniques.

5.5 Security routing for clustered WSNs

The sensed data should be effectively transmitted among the sensor nodes for detection/prevention of attacks after cluster head selection in WSNs. Secure routing protocol has been developed by author [47] for intrusion detection in clustered wireless sensor networks. An energy prediction model has been developed to ensure the node security and prediction flow model is developed to reduce the attacks arise due to the traffic during routing phase. From the study, it is observed that NS2 simulation tool is used for evaluating the security of the developed protocol. The results obtained shows better prediction and deliver higher end security against certain routing attacks such as wormholes, Sybil and hello forwarding attacks. A secure routing protocol comprising LEACH and ESPDA, has been developed by author to address the issues regarding secure data aggregation [48]. Several WSNs security requirements such as confidentiality in data, data integrity and source authentication, availability have been considered during the protocol design. The author has compared the developed protocol with its counterpart namely security context, complexity in computation and communication. The results obtained from the study shows better energy aware with high security compared to other existing techniques. Furthermore, an efficient technique on the basis of cluster based hybrid hierarchical secure routing protocol has been developed to analyze the issues arising due to clustering algorithms [49]. An unique technique has been considered for co-ordinator head selection through a combination based cluster algorithm and the co-ordinator head selection has proved to be the highest value of co-ordinator node with fitness value. The description of the packet is received by co-ordinator node through the source node followed by shortest pathway selection on the basis of value generated amongst the intermediate and sensor node. From the study on experimental results, it is observed that HHSRP approach provides better transmission capability between the source to destination without the loss of packet to malicious node.

Table 1. Comparative analysis of the existing techniques

Author	Techniques/Terminologies	Description	Comments
Devi et al., (2017)	Low Energy Adaptive Clustering Hierarchy (LEACH) technique	LEACH technique has been deployed for cluster head selection	Cluster head is used to provide energy balance among all the low priority and high priority nodes. Efficient cluster head selection leads to better energy utilization which tends to higher lifespan of WSNs. It is observed that 70,000 packets have been transferred afore the breakdown of the network
Al Baz et al., (2018)	node ranked-LEACH technique	Advancement of LEACH has been deployed foe cluster head selection and provide balance amongst the nodes	Node rank algorithm which depends on the cost of the path and node link number is used to select the cluster head Different type of LEACH protocol is considered for experimental analysis and result shows that NR LEACH is found to be effective and enhance the lifetime of the system.

Nunoo - Mensah et al., (2015)	HASH functions	Different categories of HASH such as (MD-5, SHA-1, SHA-224, SHA-256, SHA-384 and SHA-512) are considered to evaluate security and energy consumption	Optimized codes have been developed to represent the HASH functions and the same is processed through virtual computer Results show that SHA-224 outperformed other techniques in terms of processing time of around 0.1263 sec
Chen et al., (2016)	Adjustable Data Replication (ADR) Technique	It is used to increase the lifetime of the nodes and WSNs	Results obtained demonstrates that ADR successfully reduces the energy consumption rate by 33% by identifying and neglecting 60% of the duplicated nodes.
Singh et al., (2010)	HELLO flood attack	It is found to be vulnerable type of attack which is used to break the security of WSNs	signal strength and client puzzle method has been used for solving the issues The result obtained shows that it requires less computational energy and power to solve security issues
Ukey et al., (2013)	HELLO flood attack	Recognition is achieved through signal strength	Nodes are categorized as stranger and friend on the basis of signal strength If the received reply is within the predefined time, then it is said to be normal node, if it exceeds, then considered it as malicious node Results obtained shows better performance with higher packet delivery ratio compared to AODV technique
Madhavi, S et al., (2013)	HELLO flood attack	Flooding Attack Aware SAODV (FAA-SAODV) technique is deployed to solve the flood attack	Simulation is carried out using NS-2 simulator It is observed that percentage of control overhead is increased compared to SAODV Furthermore, slight improvement has been achieved in terms of PDR ratio and throughput

6.Proposed Method

The methodologies planned for the proposed work are as following:

- Our approach views the WSN network as a square grid and utilizes a grid based clustering method to arrange nodes and to afford an optimized route to the destination, taking advantage of the geometric properties of the grid.
- Design & development of a secure and fast cluster head (CH) selection algorithm. This algorithm will be extended to select a backup CH node, in case the cluster head is not there.
- Design & development of network reputation based system for sensor nodes, in order to authenticate them.

- Design & development of a hash based robust authentication technique that any sensor node will use to provision a set of well reputed neighbours (which are determined by the network reputation based system) with a one-time password and the associated hash function.

6.1 Proposed Secure Routing Framework

The proposed routing protocol consists of couple of functional modules that make it highly efficient to choose the bona fide sensor nodes in network. The high level block diagram of the routing framework is shown in Figure 6. The entire routing engine will be built on the top of already existing routing protocols for WSN, such as aodv ,olsr etc. The lower level interface module will collaborate data from all other modules in order to convey the sensor node validation decision taken by higher level modules.

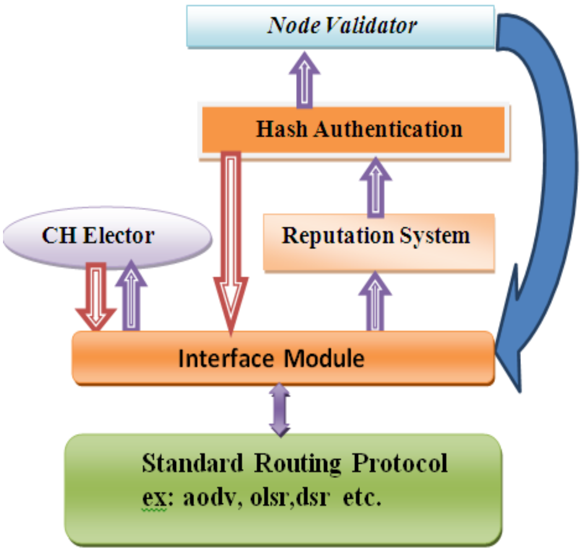

Figure 6: High Level Block Diagram of Routing Framework

7.Discussion

In this research, a wide range of literature overview is presented in terms of wireless sensor network deployment, protocols and security loopholes, which are found to be important aspects and latest trends in the field of communication systems. Different techniques, protocols and algorithms have been considered to address the issues in WSNs such as data clustering and various attacks such as spoofed routing information, selective forwarding, sinkhole, Sybil based attack, wormholes, hello flood attack and acknowledged spoofing is ascertained in this research.

From the aforementioned review as shown in Table 1., it is observed that number of authors have developed different algorithms to address and solve the issues in WSNs. Due to the digitization and increase in the amount of data for communication purpose, considerable amount of work is required to reduce the errors and overcome the attacks. Furthermore, security acts a key constraint and it is to be considered as theprimary factor during the advancement of the communication technologies. From Table. 1 it is shown that HASH based function technique; particularly SHA-224 outperformed other techniques in terms of energy consumption and security with less processing time. The output parameters such as PDR ratio and throughput should be improved to achieve effective and efficient protocol for wireless sensor networks.

The study on routing framework is found to increase the energy consumption rate and enhance the lifespan of the system by providing energy balance between the nodes. From the study, it is observed that HASH based technique is found to be effective in selecting the better cluster head and providing proper communication among the sensor nodes. Furthermore, a study is conducted to address issues arising in Ad hoc mobile environment such as energy management, QoS support and MAC protocol. This study conducted helps to find some research areas and gaps in the field of WSNs in which by developing an efficient algorithm can improve the performance of the system. The future studies comprise with several research topics in the sensor technology field of providing better communication among sensor nodes and issues regarding the security.

8. CONCLUSION

The network vulnerability is more possible with the sensor nodes in an unattended environment. Wireless Sensor networks are gradually increased used by military, health, environmental and commercial applications. Wireless Sensor networks are inherently different from traditional wireless networks as well as ad-hoc wireless networks. Security is a significant aspect for the deployment of Wireless Sensor Networks. In this paper we review the attacks and their taxonomies in wireless sensor networks. Furthermore, an overview has been made to explore the security approach that widely used to handle those attacks. Wireless Sensor Networks challenges are also briefly discussed. In this survey we hopefully motivate the researchers in the futures by using this survey to bring more effective and robust security mechanisms that makes their network more safe.

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طرق بروتوكولات التوجيه الآمنة في شبكات الاستشعار اللاسلكية

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المستخلص:

تنظر شبكات المستشعرات اللاسلكية (WSN) في تكنولوجيا ناشئة تم استخدامها بشكل كبير في الحالات الحرجة مثل ساحات المعارك والتطبيقات التجارية مثل مراقبة حركة المرور، ومباني البناء، والبيوت الذكية والرصد والعديد من السيناريوهات الأخرى. يعد الأمان أحد التحديات الرئيسية التي تواجه شبكات المستشعرات اللاسلكية في هذه الأيام. في حين أن البيئة غير المراقبة تجعل نشر عُقد أجهزة الاستشعار في الشبكات أكثر عرضة للهجمات المحتملة، فإن القيود المفروضة على الطاقة المتوفرة والذاكرة لعقد أجهزة الاستشعار تجعل الحلول التقليدية للأمان غير قابلة للتطبيق. تجعل تكنولوجيا الاستشعار المدمجة مع طاقة المعالجة والاتصالات اللاسلكية من المربح للاستغلال بكمية كبيرة في المستقبل. تكنولوجيا الاتصالات اللاسلكية أيضا الحصول على أنواع مختلفة من التهديدات الأمنية. تناقش هذه الورقة مجموعة واسعة من الهجمات في WSN وآليات تصنيفها وأليات الحماية المختلفة المتاحة للتعامل معها بما في ذلك التحديات التي تواجهها.

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Identification based Dental Image

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Abstract

Identification (matching) system is a recognition and classification form that helps in identifying the identity depending on his/her dental X-Ray image. This work displays a new method to identify a person from his/her dental X-Ray image that is widely used in forensic, border organization, parenting selecting and investigations. The goal of this paper is to design an efficient identification model depending on dental X-Ray image of identities and that can be useful to identify the unknown Individuals who died. The techniques that used were image processing and features extraction techniques which had been added to improve the goal. The best result attained from the dental X-Ray model was 89% as an identification rate.

Keywords: dental X-Ray; identification; recognition; image processing; features extraction

1. Introduction

The goal in identification systems is to determine the identity of an individual from a large set of possible identities. The system depends on the templates of the users that have been inserted into the database at the enrollment process to extract the desired features to be used for the identification process. In this process, the system decides whether there is a matched identity or not [1][2][3]. Biometric identification system became an industrial implement for person's matching. The increasing needs for this system has a significant importance for the security because there is massive information that the identities wanted to be protected from the other damages or hacking. Because of its reliability and accuracy, biometrics became an important tool for the security [1][4][5]. Dental identification process applied in case of the totally damaged, burned bodies. This biometric measurement is useful tool in forensic identification in case that the system couldn't recognize the identity from his/her face if it was smashed or burned, so it would be difficult to be identified by face. Here dental biometrics would be represented as a positive tool for identifying a person from his/her dental besides another biometrics such as finger-print, iris…etc. Besides the forensic identification, dental biometrics can also be used in security authentication [6][7]. The aim of paper is identifying a person from many persons by collecting their biometric traits in a database. The biometric trait that employed on in this paper is Teeth measurement. the aim of this paper comes true when the designed identification system verify persons depending on his/her biometrics traits without humans entry in decision making. this decision making based on a table that contains the calculations of the similarity or the differences between the traits of each person which is got through different stages to calculate these results. each stage has its importance to get the desired result. There are some of the past researches that are related to the subject of the paper and it is explained as the following:

[M.Cr. 2016], Proved an accurate edge detection method for dental X-Ray which is canny edge detection algorithm. It has been tasted of different bite wing dental X-Ray images for the upper, lower teeth and it also took in consider the missing teeth as an object to ensure an accurate result. The stages of the system included an image enhancement, teeth segmentation and edge detection with feature extraction. Three features are selected to be the identity for each tooth individually: Area, Euler Number and Standard Deviation.

The preprocessing stage works on bite wing X-Ray images and gives an accuracy results. The results that obtained were for normal and missing teeth. The problem was dental X-Ray suffers from the accuracy due to different issues, such as segmentation and teeth edge detection. Additionally, dental X-Ray images could be changed according to the shoot and weather conditions [7].

[S.Dh. 2017], Presented a full system for human identification by using dental biometric traits with full actions such as searching, matching and insertion. The database was a collection of a bite wing dental X-Ray with high resolution to be compared with the other types. The system performance evaluated according to its capacity, accuracy, and time complexity by using different dental X-Ray images samples [10].

[L. Ka.2017], Proposed a dental biometric to identify human. It is an important tool for forensic identification in case of the totally damaged face, in this case biometric identification is the most promising way to authenticate humans with high level of accuracy rate. The data tested on two types of database such as dental radiographs and colored teeth images. One of the problems that the researchers was face is the bad quality of the Images which create difficulties at every stages of features extraction and matching [6].

2. Proposed Method

The typical design of verification and identification model consists of many stages as shown in fig (1).

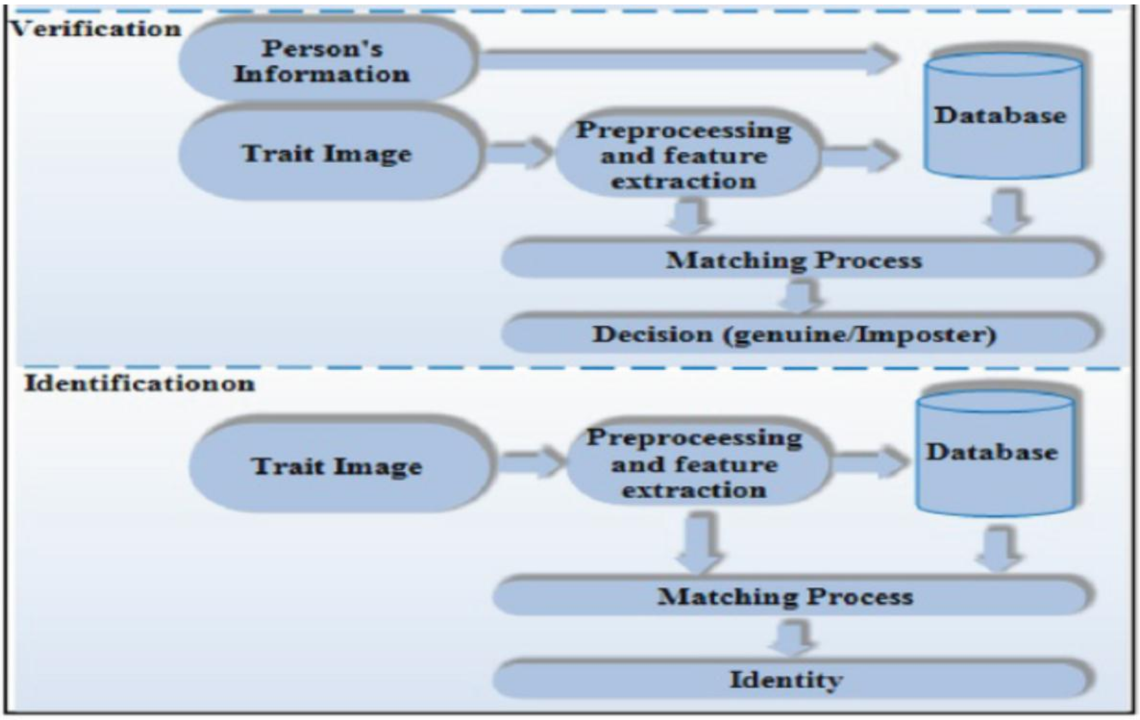

Figure (1): Verification and Identification Process

- **Preprocessing Stage**: It has an effective role on the system performance.
  This role contributes in minimizing the problems that will be found in the image which is enhancing the image quality by filtering the image from noise and dirt that might happen in the capturing stage (the acquisition device). The preprocessing methods differ from one trait to another and according to the traits nature.

- **Feature Extractions**: This stage contributes in extracting the outstanding features of the biometric measure using different methods to find out the distinguishable feature of these measurements.

- **Matching**: The result of the matching phase can be gotten from the ability of the minimum distance which is used to calculate the lower value of the test sample and compare it with the saved one to get the nearby matching result.

- **Experiment Results**: It is depending on the used biometric measurements which is employee some as a training sample and another as a testing one, so it differs from one system to another in computing the recognition rate.

In our proposed system the database constructed by Microsoft SQL server 2008 that contains the person's info (name, mother name and age), besides the extracted features of the dental X-Ray images which are mean($\bar{g}$), Standard Deviation (STD), variance, PCA feature that we got it from Principal Component Analysis algorithm (PCA) and the minimum distance. The proposed dental model divided into two phases: enrollment phase that contains preprocessing stage, feature extraction stage, while the identification phase including preprocessing stage, feature extraction stage and the matching /identification stage. Fig(2)illustrates the proposal dental model structure.

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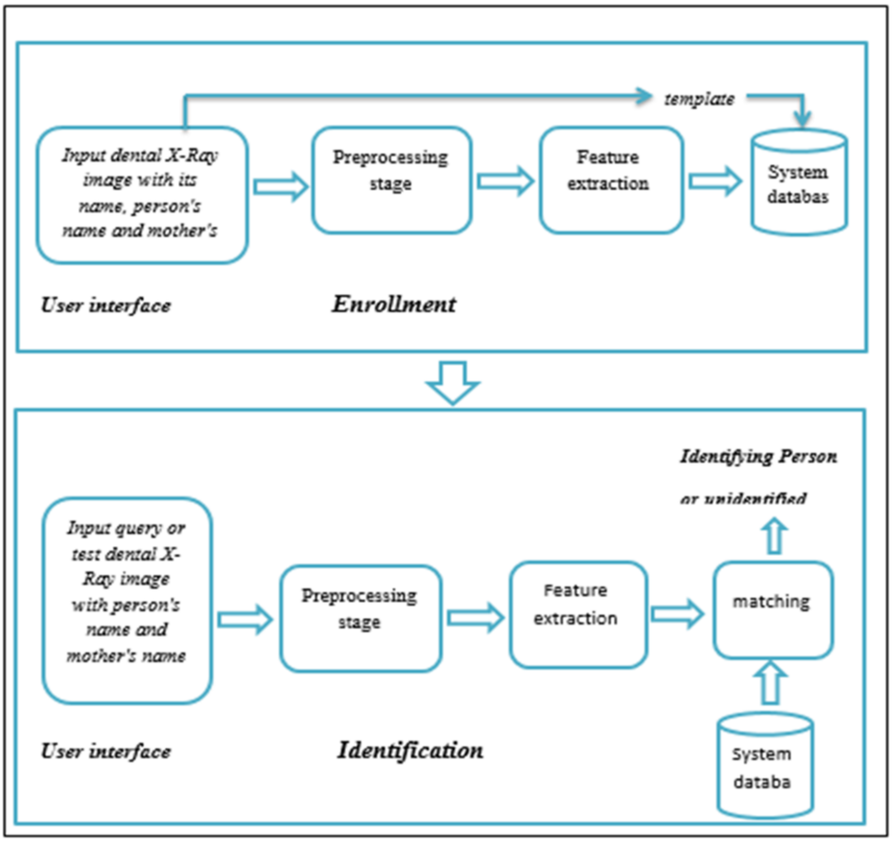


Figure (2): Stages of Two Phases of Identification Dental Model

The enrollment phase is sequence of processes, firstly the user reads the X-Ray image with Joint Photographic Experts Group (JPEG) format, then passes through the preprocessing stage that contains some steps such as image resizing by scale factor, converting into gray scale, filtering the image with median filter and image histogram with Contrast Limited Adaptive Histogram Equalization (CLAHE), then passes through features extraction stage that consists of steps like PCA feature, $\bar{g}$, STD, variance and minimum distance where these extracting features are inserted into the Dental Features Table (DFT) database.

The work of this process inserts the features into the database only without identification. Fig (3) illustrates the dental enrollment phase.

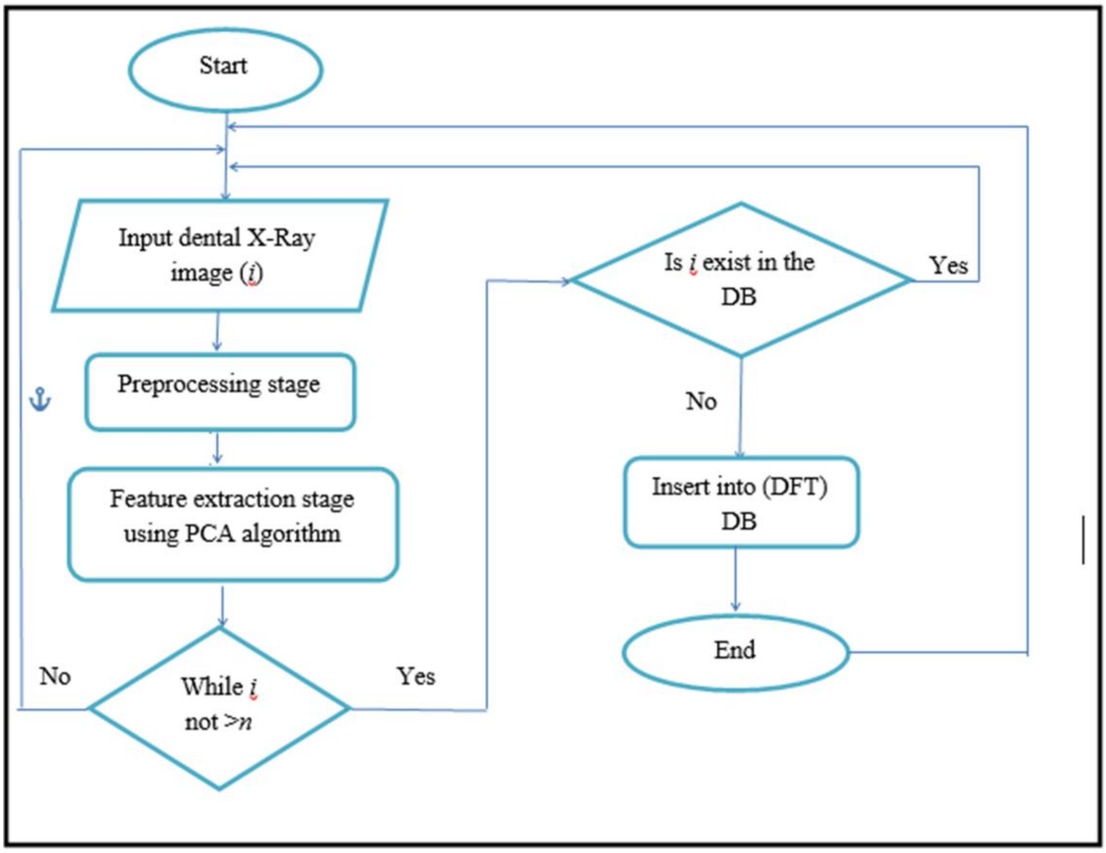
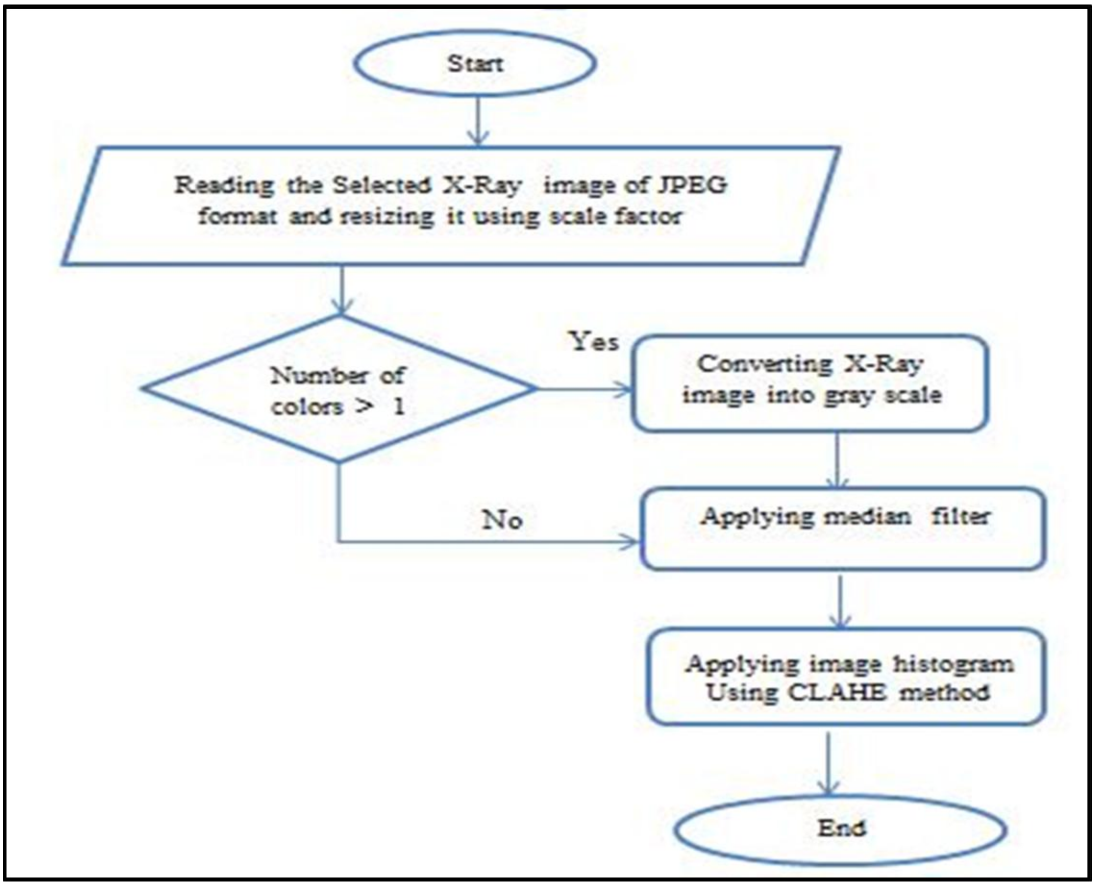


Figure (3): The Dental Enrollment Phase

The preprocessing Phase summarized as the following:

- The input X-Ray image will be preprocessed starting from resizing all the selected X-Ray 's at the same size by scale factor.
- Checking the X-Ray to convert it into gray scale image.
- Apply median filter to filter the image from salt and pepper noise by using "imnoise" function in matlab program, if it exists in the image.

Apply image histogram using CLAHE method Which help to get best results in the features extraction stage. Fig (4) illustrates the preprocessing stage.



Figure(4): Dental Preprocessing Stage

The output result of the previous stage treated as an input for the features extraction stage, to extract the outstanding feature of the specific X-Ray by using PCA algorithm which is used to select a collection of X-Ray images as a training matrix. It would select the prominent traits of each X-Ray in the training matrix, and then eliminate these traits to the strongest one to be the wanted feature after comparing the training matrix with the test X-Ray image to decide which one is the strongest trait to be added to the PCA feature column in the database. PCA algorithm summarized as the following:

- Reading n of 2D images to convert it into 1D images (vector).
- Put them in the training matrix which is contains all the selected X-Ray images vectors as shown in the formula below:

Training matrix= $[x_1, x_2, x_3, \ldots, x_n]$

- Calculating mean value (ḡ) for the training matrix rows as shown in eq (1):

$$\bar{g} = 1/k \sum_{i=0}^{k} x_i \quad \text{....(1)}$$

- Calculating Covariance equation (Cov) depending on the previous results from the formula and eq (1) as shown in the eq (2). Fig (5) illustrates PCA algorithm process.

$$\text{Cov} = (x_k - \bar{g})(x_k - \bar{g})T \quad \text{....(2)}$$

Each **x** represents an input X-Ray image vector (1D) with **k** length. **n** represent the number of the input X-Ray images. **ḡ** represent the mean value of the training matrix for each row.
K represents the row length of the input vector. **Cov** is the covariance matrix that used to make the dimensions of the training matrix in the above formula symmetrical to use the diagonal as variance of x. **T** is the transpose of the matrix $(x_k - m)$which is made the row as a columns and columns as rows.

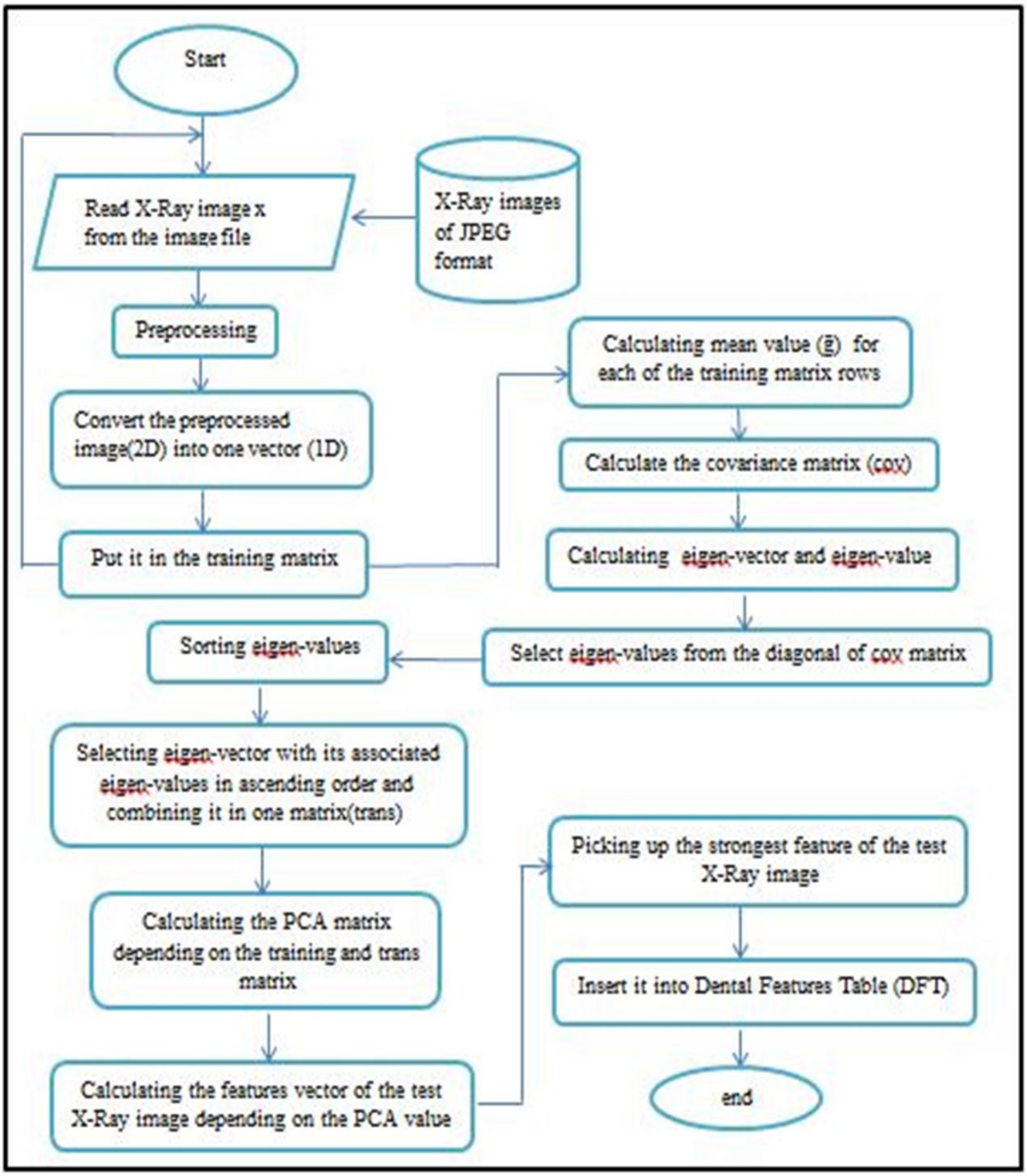


Figure (5): Features Extraction Stage using PCA Algorithm

Features extraction stage that depending on the PCA algorithm to extract the prominent feature to insert it into DFT is not the only feature to be extracted, but also there is more such as extracting the mean of the whole X-Ray image as illustrated in eq (1), the variance of the image is also one of these features. The variance defined as the squared difference from the mean as illustrated in eq (3), STD defined as the measure of how spread out pixels numbers are, as shown in eq (4) and the Euclidian distance of the test image is illustrated in eq (5).

$$\sigma^2 = \frac{1}{N}\sum_{i=1}^{N}(x_i - \bar{g})^2 \qquad \text{.........(3)}$$

$$\sigma = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(x_i - \bar{g})^2} \qquad \text{.........(4)}$$

$$\text{Distance} = \frac{1}{N}\sum_{i=1}^{N}(x_i - y_i)^2 \qquad \text{.........(5)}$$

σ^2 Represents variance, σ represents STD, X_i displays each pixel value of the image, $\bar{g}$ represents the mean value, and **N** shows the total number of pixel values. x_i , y_i represents different pixel point in the X-Ray image to give the minimum distance between two outstanding pixels in the image.

DFT was constructed by using Microsoft SQL Server 2008. This table not specified with a specific size which means that the data space of the table is unlimited to a specific size. It contains 10 columns including person-id and age of numeric datatype of size 4 digits, person's name, mother's name and image name of char datatype of size 10 digits, pca value, mean value, variance value, STD value and distance value of float datatype of different sizes according to the image feature using Microsoft SQL Server 2008. These values considered as the features of the dental X-Ray image of each person which are inserted by the user into the DFT as shown in fig (6). Besides the ability of inserting to the table, there are ability of searching, deleting and editing functions. Algorithm (1) shows the way of calculating the test X-Ray other features such as mean, STD, variance, distance.

personid	name	mothersname	imgname	age	pca	mean	variance	STD	dis
1	g	gg	g3	47	16078.3656770...	0.51456747373...	0.07972479503...	0.28235579511...	4.47213595499...
2	a	aa	a1	56	3211.90074716...	0.51865555231...	0.08105294432...	0.28469798791...	84.7230783199...
3	a	aa	a2	56	7407.33959735...	0.64237373739...	0.13620530104...	0.36906002363...	93.2308961664...
4	a	aa	a3	56	18330.7178177...	0.51866754827...	0.08128367452...	0.28510291917...	123.065023463...
5	a	aa	a5	56	3405.69103111...	0.51696001725...	0.08094532944...	0.28450892682...	84.7230783199...
6	l	ll	l1	50	4993.35961957...	0.48010738713...	0.08510322910...	0.29172457747...	4.12310562561...
7	b	bb	b1	21	5245.83994389...	0.51441145114...	0.07728642405...	0.27800435977...	4.12310562561...
8	b	bb	b2	21	38.0803201623...	0.51296262777...	0.07746991979...	0.27833418725...	2
9	b	bb	b3	21	5231.10429058...	0.51550895510...	0.07702292860...	0.27753004991...	3.60555127546...
10	b	bb	b4	21	12172.5545155...	0.55044049690...	0.09934207955...	0.31518578577...	2.23606797749...
11	b	bb	b5	21	5069.22345655...	0.51556931184...	0.07566638005...	0.27507522618...	2.23606797749...
12	d	dd	d5	52	5200.96265744...	0.51769971045...	0.08068258303...	0.28404679726...	6
23	e	ee	e2	41	3333.44503034...	0.52583515054...	0.07568774295...	0.27511405445...	2.23606797749...
19	g	gg	g1	47	4066.57353408...	0.51494023387...	0.07954596312...	0.28203893901...	2
20	g	gg	g2	47	3936.68453426...	0.51405114396...	0.07999990367...	0.28284254219...	2.82842712474...
21	g	gg	g4	47	4076.96921236...	0.51539929365...	0.07969149495...	0.28229682065...	2.23606797749...
22	g	gg	g5	47	4579.83189605...	0.53145727917...	0.08759863476...	0.29597066537...	2.23606797749...
24	e	ee	e3	41	69.3411615477...	0.51759370676...	0.07477849197...	0.27345656322...	3.60555127546...
26	e	ee	e5	41	4043.64346527...	0.52339180243...	0.07698624534...	0.27746395323...	9.05538513813...
25	e	ee	e4	41	3968.99230212...	0.51866999987...	0.07557742977...	0.27491349507...	9.05538513813...
28	d	dd	d2	52	4929.95450410...	0.51548740812...	0.08069091219...	0.28406145848...	3.60555127546...
31				57	13727.8341332...	0.51507805054...	0.08165928394...	0.28576088596...	90
13	a	aa	a4	56	4274.60067008...	0.51512656353...	0.08001568990...	0.28287044721...	85.1645466141...
27	d	dd	d1	52	5195.52427990...	0.51773861028...	0.08060943119...	0.28391800083...	149.013422214...
14	l	ll	l5	50	159.722779996...	0.48010806575...	0.08510297238...	0.29172413748...	4.12310562561...
15	l	ll	l4	50	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

Figure (6): Dental Features Table (DFT)

Algorithm (1): Mean, Variance, STD, Distance Features of the Test X-Ray

Input: Read the test X-Ray image from the preprocessing stage
Output: mean, STD, variance, distance

Begin:
Step one: Read X-Ray image from the preprocessing stage
1.1 Read row and column of the image
1.2 Calculate mean value from the equation

Mean = $\sum_{r=0}^{row}\sum_{c=0}^{col} image_{(r,c)} / (row * col)$

Step two: Calculate variance, STD from equations

Var= $\sqrt{\frac{(\sum_{r=0}^{row}\sum_{c=0}^{col} image_{(r,c)} - Mean)^2}{(row * col)}}$

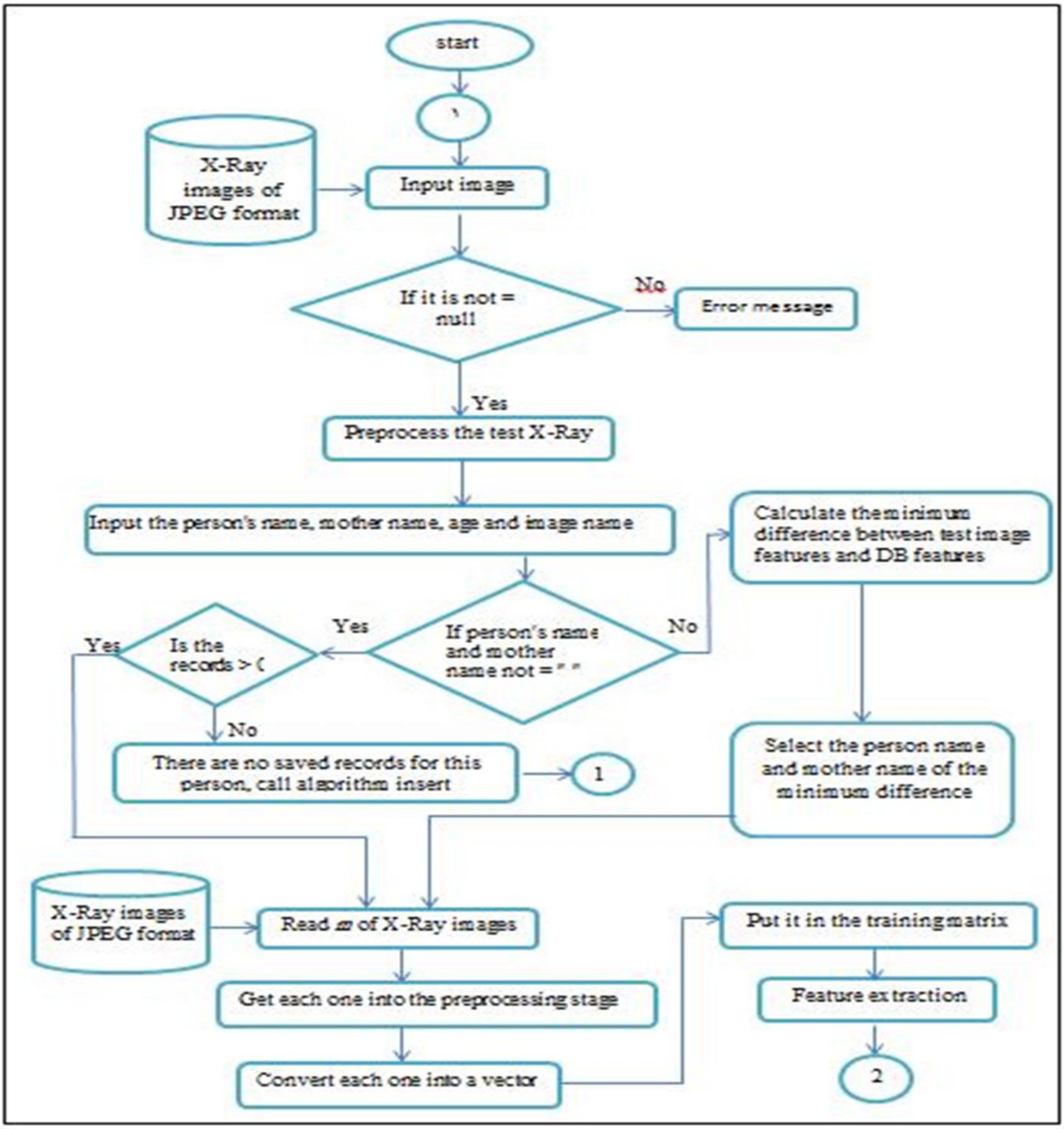
STD= $\sqrt{Var}$

Step three: Detect boundary for each object in the image
3.1 Read each pixel point (x,y) in each boundary
3.2 Calculate the distance (dis) between each point in the disjoint boundary objects (obj1, obj2) from the equation

distance= $\sqrt{(obj_{1x} - obj_{2x})^2 + (obj_{1y} - obj_{2y})^2}$

Select the minimum distance (dis)
Step Four: return Mean, STD, variance, distance
End

After inserting the dental features into the DFT, now continue to the identification stage. Firstly, the system input a test X-Ray image from the X-Ray images database that is containing all the dental X-Ray images of all the persons with all of its cases (original, darken, lighten, noisy and rotated one). And pass through all the stages (preprocessing, features extraction) where the system asked for the person's name and mother's name. Then, reading five X-Rays, preprocessed to get them into the training matrix. checking if there are five records for this person's X-Ray in the DFT, if so, it will bring all the features that belong to that person as average which is bringing all the five rows of the pca value to calculate the average pca value and all the other features of the test image and the training matrix to calculate the minimum distance between the average features of test X-Ray five records and the features of the training matrix. Algorithm (2) displays the dental identification rate calculation, Identification process structure illustrated in fig (7).


start
X-Ray images of JPEG format
Input image
If it is not = null
No
Error message
Yes
Preprocess the test X-Ray
Input the person's name, mother name, age and image name
Calculate the minimum difference between test image features and DB features
Yes
No
If person's name and mother name not = " "
Yes
Is the records > 0
No
There are no saved records for this person, call algorithm insert
1
Select the person name and mother name of the minimum difference
X-Ray images of JPEG format
Read m of X-Ray images
Put it in the training matrix
Get each one into the preprocessing stage
Feature extraction
Convert each one into a vector
2

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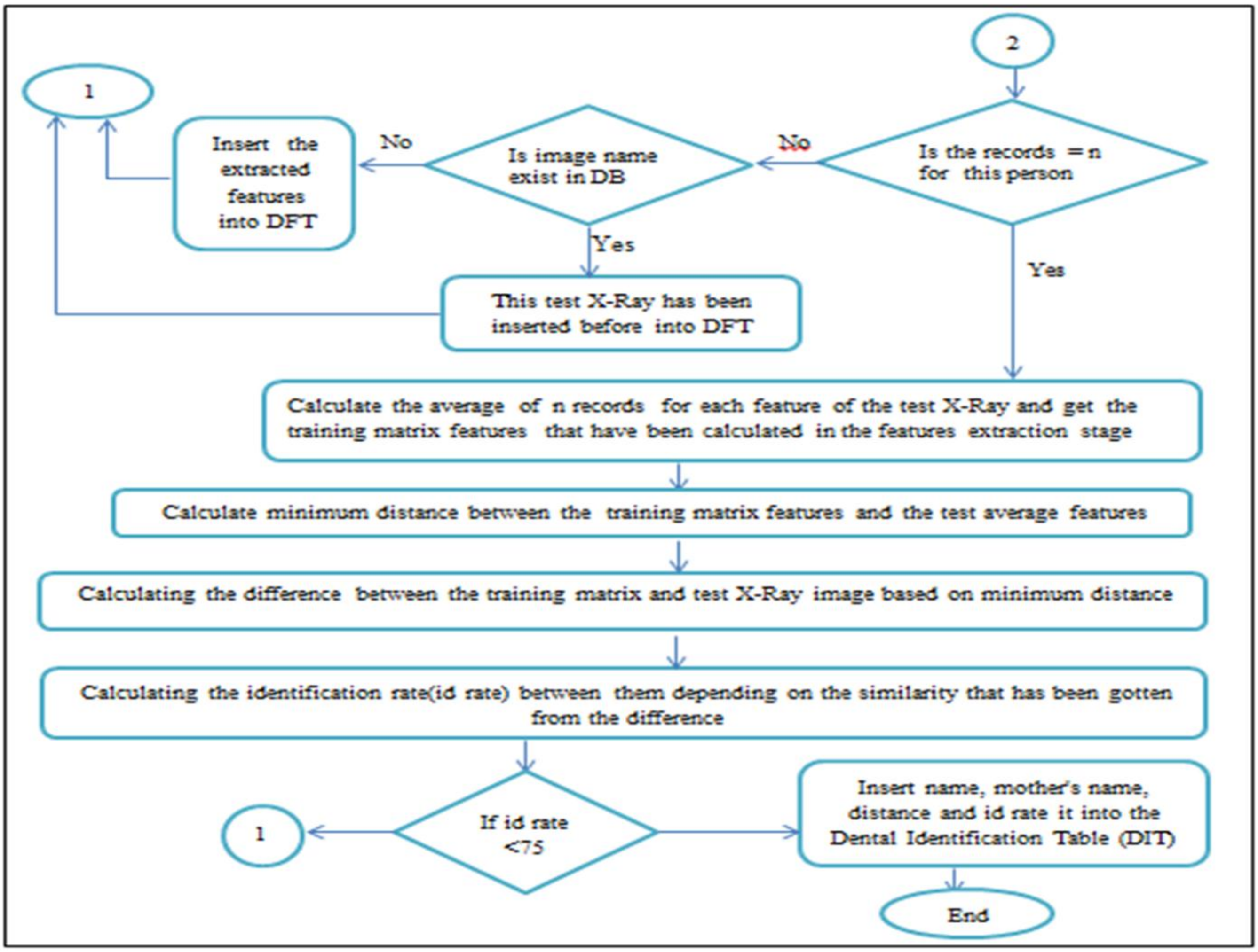

Figure (7): Structure of the Identification Process

Algorithm (2): Dental Identification Rate Calculation

Input: Read test X-Ray image and input person name (name), mother name (mname), age and image name (imgn)

Output: Identification rate

Begin:

Step One: Get the test image into preprocessing stage

Step Two:

2.1 if name and mname not = " "

2.2 calculate the difference (minimum distance) between the test features and each row of the DFT features

For each row (i) do:

$$D=\sqrt{(PCA_i - PCA_{test})^2 + (Mean_i - Mean_{test})^2 + (STD_i - STD_{test})^2 + (var_i - var_{test})^2 + (dis_i - dis_{test})^2}$$

If D < min

Min= D

ii=i

End if

End for

Select person name and mother name of the Min (ii) and go to step 3.2

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Step Three:

3.1 if records > 0

3.2 read m of X-Ray images and get into preprocessing stage

Convert each image into a vector (1D) and put it in the training matrix

Get the training matrix into features extraction stage

3.3 if records = n for the query person

3.3.1 Calculate average of n records for each feature of the test image as Equations:

avgPCAtest= $\sum_{i=1}^{n} pca_i$ / n

avgMeantest= $\sum_{i=1}^{n} mean_i$ / n

avgSTDtest= $\sum_{i=1}^{n} STD_i$ / n

avgvartest= $\sum_{i=1}^{n} var_i$ / n

avgdistest= $\sum_{i=1}^{n} dis_i$ / n

Calculate the training matrix features as equations:

PCAtr= $\sum_{i=1}^{n} pca_i$ / n

Meantr= $\sum_{i=1}^{n} mean_i$ / n

STDtr= $\sum_{i=1}^{n} STD_i$ / n

vartr= $\sum_{i=1}^{n} var_i$ / n

distr= $\sum_{i=1}^{n} dis_i$ / n

Calculate the minimum distance between test and training features As equation:

$$Dis=\sqrt{(PCA_{tr-}avgPCA_{test})^2+(Mean_{tr\,-}avgMean_{test})^2+(STD_{tr-}\,avgSTD_{test})^2+(var_{tr-}\,avgvar_{test})^2+(dis_{tr-}\,avgdis_{test})^2}$$

Calculate the length of the integer number in Dis select Num from the formula:

Num = 10, 100, 1000, 10000, 100000

Calculate the difference (Diff) from equation: Diff= Dis / Num

Calculate identification rate (id rate) from equation:

3.3.2 Id rate= 100 – Diff, check if it is less than 75, don't insert it into DIT otherwise insert it into DIT with personal information.

Else if the imgn exist in the DFT, then show a message that it has been previously inserted the features into DFT.

Else show a message that there are no saved records in DFT for this person Insert records by pressing button "insert"

Step Four: return Id rate

End

Discussion of the Results

The preprocessing stage can be considered as a useful tool for enhancing the image because it is taken from radiograph and that may have poor quality therefore it needs to be adjusted in case of image contrast, brightness, removing noise and threshold to show the unclear important features. While, Features extraction is used to extract all the important distinguishable features that make each one unique than the other one. In this paper, the features that have been extracted from the image are mean (ḡ), Principal Component Analysis (PCA), variance, distance and Standard Derivation (STD). Training phase take five images for each person to be processed and extract their features and inserted into the dental features table. Then testing phase compare the extracted feature of the test image with the training phase to give the decision (matched or rejected). There are some measurements that are calculated to evaluate the quality of the image, compare and evaluate the performance of the system. Such measurements are Mean Square Error (MSE) and Peak Signal to Noise Ratio (PSNR). MSE is calculated to give the average of the squared intensity of the original image and the output image pixels as illustrated in eq(6). PSNR is a quality measure that calculated from the difference pixels between two images (original image pixels and the reconstructed image pixels) which is depending on MSE to give the result of PSNR as illustrated in eq (7). Table (1) displays the differences of the MSE and PSNR on different images sizes.

$$\mathbf{MSE} = \frac{\sum_{M,N}[I1(m,n)-I2(m,n)]^2}{MXN} \quad ....(6)$$

$$\mathbf{PSNR} = 10\log_{10}\frac{1}{MSE} \quad ....(7)$$

Table (1): MSE and PSNR on Different Images Sizes

	Original	Processed
Image Size	286 KB	164 KB
Mean	0.7736	0.5187
MSE	0.0650	
PSNR	27.3323	
Image Size	435 KB	164 KB
Mean	0.6652	0.5144
MSE	0.0227	
PSNR	37.8363	
Image Size	269 KB	164 KB
Mean	0.8605	0.5215
MSE	0.1149	
PSNR	21.6362	

These measurements would let us being sure that the preprocessing stage enhances the image or not. The dental model training dataset includes 75 images belong to 15 persons (five for each one of them). The best result displays 89% as an identification rate.

4. Conclusions

The usage biometric measures to identify unknown person would achieve flexibility to the system which means that an identifying him/her from the dental record. The usage of PCA algorithm increased the accuracy of the system because it extracted the most important trait (unique) that is existed in the dental image to distinguish between them. Besides the other extracted feature that helps to accurate the result. It can resist the changes that may be found in the image, so it enhances the image (remove the damage and show the important features) to identify the person even though the image is damaged. The maximum identification rate in dental model is 89%.

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تحديد الهوية بالاعتماد على صور الاسنان

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المستخلص :

نظام تحديد الهوية (مطابقة) هو نموذج للتمييز والتصنيف يساعد في تحديد الهوية اعتمادًا على صورة الأسنان السينية الخاصة به / بها. يعرض هذا العمل طريقة جديدة للتعرف على شخص من صور الأشعة السينية التي تستخدم على نطاق واسع في الطب الشرعي ، وتنظيم الحدود ، واختيار الأبوة والأمومة والتحقيقات. الهدف من هذه الورقة هو تصميم نموذج تحديد فعال يعتمد على صورة الهويات السينية للأسنان والتي يمكن أن تكون مفيدة في التعرف على الأفراد المجهولين الذين ماتوا. التقنيات المستخدمة هي معالجة الصور وتتميز بتقنيات الاستخراج التي تمت إضافتها لتحسين الهدف. كانت أفضل نتيجة تم الحصول عليها من نموذج الاشعة السينية للأسنان ٨٩٪ كمعدل لتحديد الهوية.

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Security Systems Based On Eye Movement Tracking Methods

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Abstract

This survey presents an overview of eye-tracking techniques for various security systems, provides insights into using eye-tracking techniques in security systems. This paper classifies the techniques using some researches. Researches depend on characteristics of eye tracking data, which includes viewer and stimuli aspects, also it does depend on characteristics related to visualization methods. This work contributed in explain some of the security systems that depend on eye gaze tracking, the consequences, the positive and negative aspects resulting from the use of eye gaze tracking in the security system, presents the strengths and weaknesses to keep in mind in future.

The work finally presents a comparison between those security systems for facilitating choose the more accurate and effective system.

Keywords. Real-time eye tracking, real-time eye detection, password, gaze, authentication.

1. Introduction

One of the security requirements for general terminal authentication systems is to be easy, fast and secure as people face authentication mechanisms every day and must authenticate themselves using conventional knowledge-based approaches like passwords. But these techniques are not safe because they are viewed by malicious observers who use surveillance techniques such as shoulder-surfing (observation user while typing the password through the keyboard) to capture user authentication data. Also there are security problems due to poor interactions between systems and users. As a result, the researchers proposed eye tracking systems, where users can enter the password by looking at the suitable symbols in the appropriate order and thus the user is invulnerable to shoulder surfing. Eye tracking is a natural interaction method and security systems based on eye movement tracking provide a promising solution to the system security and usability. The aim of this paper is to review techniques or solutions to dealing with eye movement tracking in security systems.

2. Related Work

This paper has performed a study of existing security systems that based on eye movement tracking developed by different researchers according to their area of expert. In the following paragraphs are given several of the published researches related to the goals of this work.

- Alexander, Martin and Heinrich (2009) present “Eye-PassShapes Method”, Eye-PassShapes extends and develops two authentication approaches via combining them, PassShapes and EyePIN. In PassShape the users must paint shapes (that consist of strokes) in a certain order, this method increase memorability but doesn’t improve security in comparison with PIN or password entry. EyePIN is focused on security instead of usability.
-

The user’s PIN is still the token of authentication, and the security is improved when the input method is changed. Rather than inserting numbers, an eye movement is performed by the the user representing the associated digits. Eye-PassShapes can be considered simpler to be detected than the exact location of the user's look and can work with cheap devices. [1].

- Alain, Sonia and Robert (2010) present “CGP enhancements”, CGP which is an abbreviation for Cued Gaze-Points can be considered as a system of graphical pass-word defending from such attacks with the use of eye-gaze pass-word input, rather than mouse-clicks, but it requires certain approaches for improving the accuracy of gaze. This study presented two improvements: a nearest-neighbor gaze-point aggregation method and a one-point calibration prior to entering the pass-word. They reached the conclusion that those improvements made a significant enhancement to the precision of users’ gaze and system efficiency [2].
- Justin, Kenrick and Bogdan (2011) present “EyeDent System”, which is an improvement to the present authentication systems that depends on eye-tracking which requires pressing a trigger by the user when looking at any symbol. Rather than that, EyeDent, gaze points are being clustered in an automatic way for determining the character chosen by the user; this method is beneficial in allowing the user the authentication at their preferred pace, instead of a predetermined dwell time. In addition, not having visible trigger does not reveal the number of symbols in the pass-word [3].

- Andreas, Florian and Albrecht (2012) presented "a novel gaze-based authentication scheme", this scheme uses cued-recall graphical pass-word on all images. This approach uses a computation of visual attention for masking these image parts which will probably be focused on. They created a realistic threat-model concerned with the attacks which could happen in public places, like recording user's actions throughout the process of drawing money from an ATM [4].
- David Rozado (2013) present "the subjectspecific gaze estimation parameters" using this parameter which has been gathered throughout a calibration process to render impractical to another individual to input a pass-word by gazing even in the case where the impostor is aware of the correct pass-word [5].
- Mihajlov, Trpkova and Arsenovski (2013) present "eye tracking study of ImagePass", ImagePass can be considered as a graphical authentication system that is based on recognition. The aim of the study was discovering the users perception and reaction to graphical authentication [6].
- Mohamed, Florian, Mariam, Emanuel, Regina and Andreas (2016) present "Gaze-TouchPass Scheme", it's a multi-modal method combining touch and gaze regarding shoulder-surfing resistant authentication in mobiles. Gaze-TouchPass accepts pass-words with several switches between input modalities throughout the process of authenticating [7].
- Zhenjiang, et al. (2017) present "iType System", a system which utilizes eye gaze to type private input on commodity mobile platforms.

  The idea faced 3 main issues: 1) quite low precision of gaze tracking for mobiles; 2) issues in the correction of the input errors because of a lack in comparing of the value the true text-entry value; and 3) the movement of the device along with other noises which could be interfering the precision of the gaze tracking and therefore the efficiency of the iType [8].

3. Typical Model Of Eye Tracking Movement

Eye tracking can be defined as the procedure of the detection of the eye place throughout video frames for the determination of the position of the gaze. The movement of the eye according to the head could also have some impact [9]. This method will be helpful for disabled people in communicating with their voluntary motions such as movements of eyes and nose. People that have unadorned disabilities could also get benefits from computer access to do their mundane activities such as play games and surf the internet [10]. The concept of eye tracking is used which continuously track the eye movement of a person by using a simple webcam and moves the mouse curser accordingly. The whole process is divided into four stages such as face detection, eyes detection, pupil detection and eye tracking [11] as shown in figure (1). This system utilize a USB (Universal Serial Bus) or built-in camera for capturing and detecting the movements of the user's face [10].

3.1 Face Detection: Face detection is the most important part of the eye tracking process. Featured base and image base methods are the two ways of face detection. [11].

A. **Feature-based method:** In this method, facial properties are detected (such as, Nose, eyes and so on), then assess their efficiency by observing position and distance from one another. This method can reach high speed in facial detection. Mainly, it is known for its speed pixel precision [10].

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This technique divided by Hjemal and Low into three categories [15]:

- **Low level analysis:** It handles segmentation of visual features using pixel properties, grayscale, and animation information.
- **Feature analysis:** It uses additional knowledge about the face and eliminates ambiguity resulting from low level analysis. The first involves strategies for searching for serial features based on the relative situation of individual facial features. Highlighted facial features are initially identified and allow less visible features.
- **Active shape models:** Used to make physical and higher level appearance of features. It works in two steps: Look in the image around each point to get a better position for that point, and update the form parameters to better match to these new placements.

B. Image-based Method: This method performs scanning of the particular image with a window looking for faces at every scale and location. By the study of Hjelman this algorithm is essentially just on exhaustive search of the input images for possible face locations at every scale [10]. It contains various approaches such as neural networks, example based learning, support vector machine [15].

3.2 Eye Detection: It consists of four projections for detecting of eye that are Edge-Projection, Luminance Projection, Chrominance Projection, and Final Projection. Viola Jones algorithm is used to use object Detector it is used for detecting object [1]. In the pattern of “Between-the-eyes”, eyes are found and tracked with updated pattern matching. After that, from the taken images correct image is chosen based on the distance “Between-the-eyes” [10]. Eye detection approaches [16]:

A. Regression approach: Minimize the distance between the expected and actual eye positions by understanding the job assignment from the image input to the eye sites.

B. Bayesian approach: Learn model of the appearance of the eye and the appearance of non-eye. Use the Baye’s principle to build an "eye prospect". Outputs formulas around each pixel of the input image, from which prediction will be extracted.

C. Discriminative approach: The problem is treated as one of the classifications. A classifier is trained to produce a positive result for spots around the eye and negative elsewhere.

3.3 Pupil Detection: In this section the actual pupil of eye is detected first. After detecting eyes it will start its own processing and one type of mark is form on eye portion, after that image is converted into binary form [11].

3.4 Eye tracking: Eye tracking is the last section of this process, in this stage the mouse little bit start moving from its own location. Gradually it starts up its process and start working according to eye movement [11]. Eye tracking techniques [16]:

A. Limbus Tracking: The limbus is the boundary between the white sclera of the eye and the dark iris. Since the sclera is white and the iris is darker, these borders can be easily detected visually as well as tracked. This technique is negatively affected by the eyelid and often hides all or part of the limbus. This makes its uses limited to horizontal tracking. This technique usually does not involve the use of infrared light.

B. **Pupil tracking:** There are several reasons for this; but the main advantage is the idea of "bright spot". Such as the red-eye mode when capturing flash images at night, infrared can be used to detect the pupil to create a high intensity bright spot that is easy to find with image processing.

Electrooculography: Depends on electrodes connected to human skin. Because of the high rate of metabolism in the retina compared to cornea, the eye maintains a constant voltage with respect to the retina.

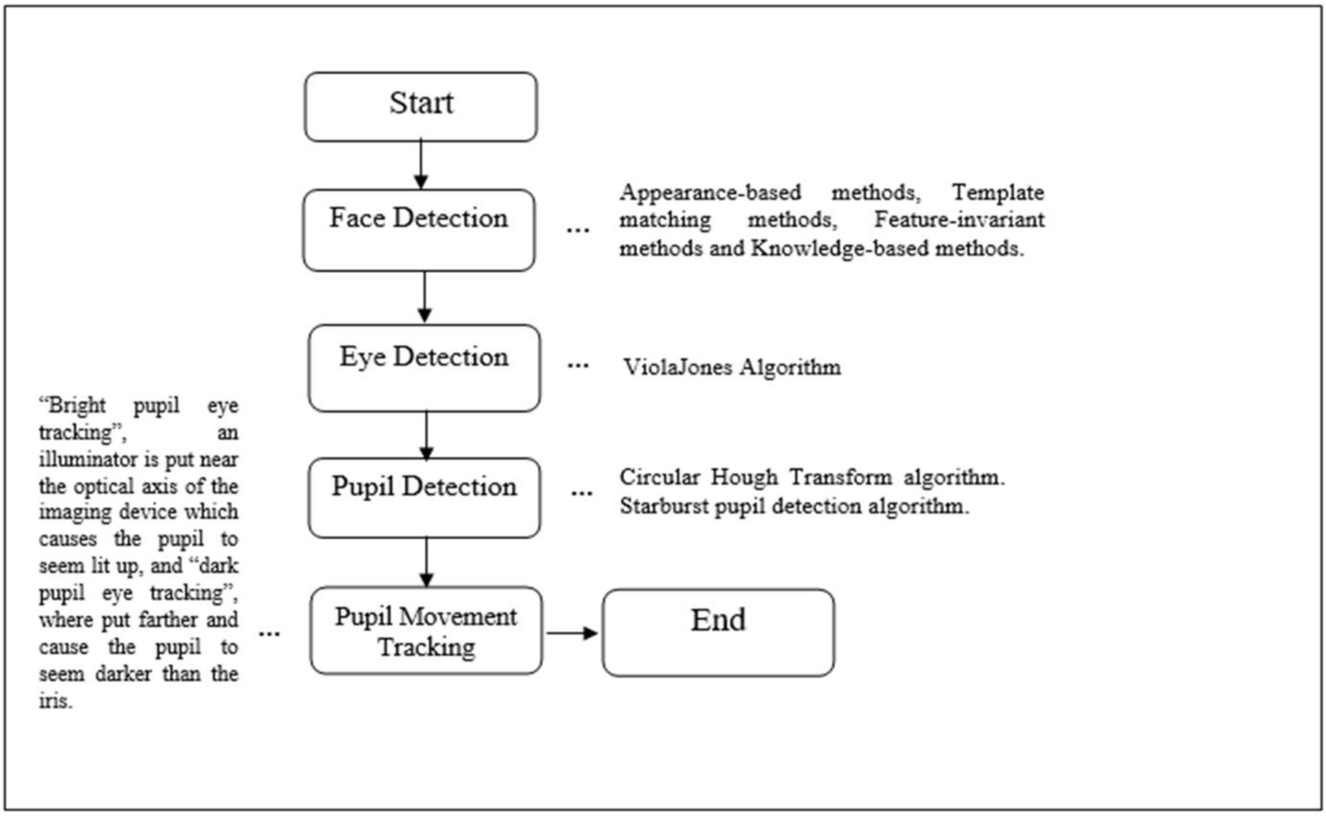

Figure (1): Typical Model eye movement tracking System

4. Basic Measurement Units Used in Pupil Movement Tracking

a) Fixation. It is gathering the gaze points, those points are gathered according to a certain region and time-span. The region of gathering is typically in a range between 20 and 50 pixels, and the time span between 200 and 300 millisecond. Typical fixation measurements are fixation location given as x- and y-coordinates in pixels, fixation duration in milli-seconds and the fixation count (in other words, number of fixations) [12].

b) Saccade. It describes a fast eye motion from a fixation to another one. Their duration is usually between 30 and 80 milliseconds and are the fastest motion the human can do. The visual information is suppressed during this time span. Usual measurements are the amplitude of the cascade (which is the distance that the saccade has traveled), the duration of the saccade in MS, and the velocity of the cascade in degrees per second [12].

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c) **Smooth Pursuit.** Throughout the process of presenting dynamic stimuli smooth pursuits could happen. This could be unintentional and only in the case where viewers follow a motion in a presented stimulus. Eye velocity throughout smooth pursuits is in the range between 10 and 30 degrees each second [12].

d) **Scan-path.** Which is defined as a series of alternating fixations and saccades [12]. Scan-paths explain the path of eye movements on a monitor or a book page [13]. A scan-path can provide data concerning the participant's searching behavior. The optimal scan-path would be a straight line to a certain target. Deviance from this optimal scan-path could be understood as inefficient search [12]. Regarding the efficiency of the task, it has been suggested that the fixations number and duration, and certain scan-paths patterns can be associated with differences in effectiveness in the efficiency of the task [13].

e) **Stimulus.** It is any visual content which is presented to participants throughout the process of eye tracking. Usually, static and dynamic stimuli, with active or passive content are categorized. Typically, two-dimensional stimuli are shown to participants. Nevertheless, recently, three-dimensional stimuli have also become a focus of study [12].

f) **Area of Interest (AOIs), or regions of interest (ROIs)** could be defined as parts of a stimulus which are highly important for hypothesis. Generally, depending on the semantic information regarding the stimulus, Area of Interest are being created. A movement from one Area of Interest to another is called a transition. Common metrics of Area of Interest are Area of Interest hit that determines if the fixation is in the Area of Interest or not, the Area of Interest's dwell time in milliseconds, and the transition count (the number of transitions between two Areas of Interest) [12].

g) **Blinks.** Rapid bilateral eyelid closure and co-occurring eye movement [14].

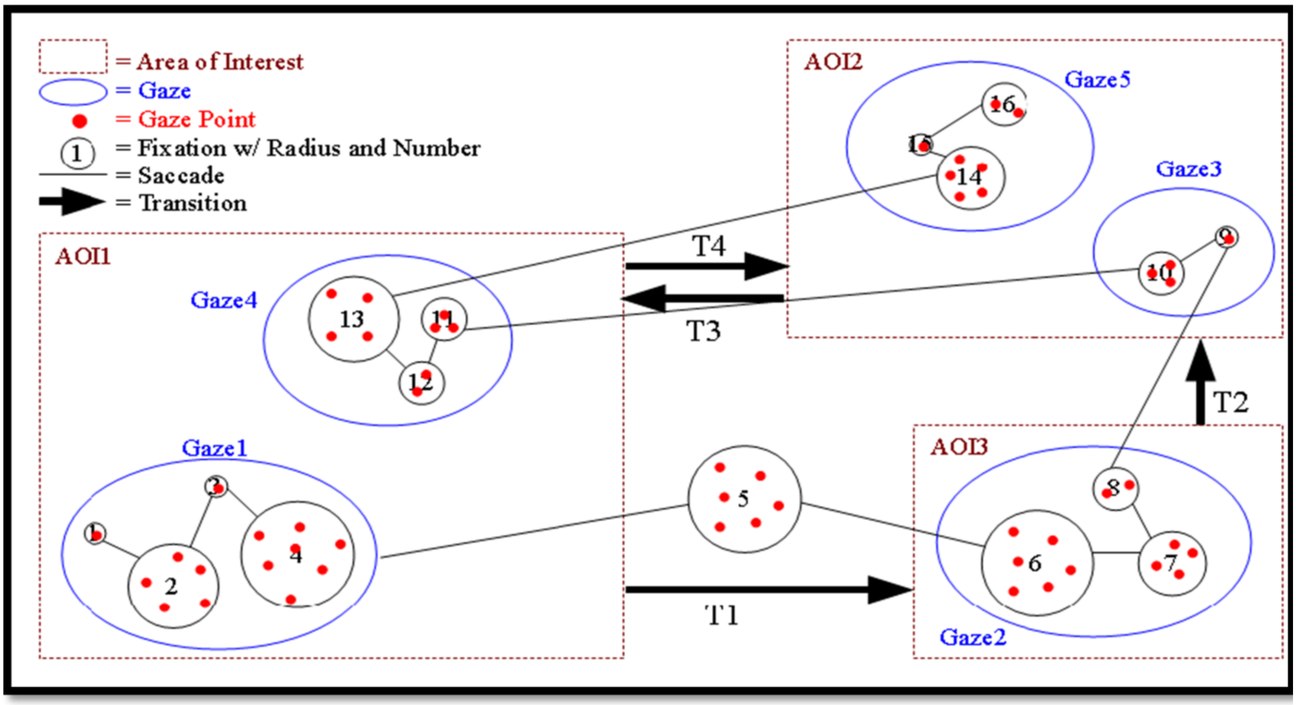

Figure (2): The Gaze points are temporally and spatially aggregated in the fixations. Saccades work as a connection for the fixations, also the fixations have a specific duration that is being represented through radius. A temporal order of fixations is a gaze, but, just if the fixations are within the Area of Interest. An Area of Interest is a region of certain interest on the stimulus. A saccade from one AOI to the next is called a transition. A complete sequence of fixations and saccades is called a scanpath.

5. Comparison Among Some Researches:

This section introduce some details about some works that related of the security system based on eye movement tracking system that shown in table [1].

Table 1: A list concerned with all the references which presented an adoption, enhancement or new existing security systems that based on eye-tracking techniques.

Researcher Name	Year	Technique	Details
Alexander, Martin and Heinrich [1]	2009	Eye-PassShapes Method	-EyePassShapes is considered to be simpler to utilize than EyePIN. -EyePassShapes is considered to be faster than EyePIN. -EyePassShapes can be considered slower than the standard PIN-entry. -EyePassShapes have higher security the than standard PIN-entry. -EyePassShapes have higher security than the PassShapes. -PassShapes utilizing EyePassShapes are as memorable as PassShapes utilizing touchpad. -PassShapes utilizing EyePassShapes with the repeated input strategy is considered to be simpler to remember than without.
Alain, Sonia and Robert [2]	2010	CGP enhancements	-CGP-2 perform lower rates of error and more success through re-entering password than CGP-1. -CGP-2's 1-point calibration retain gaze accuracy at the edge of the images and not result in more errors at the edges of the images.
Justin, Kenrick and Bogdan [3]	2011	EyeDent System	-improvement authentication process by allow single errors such as a single character missing, inserted, or substituted.
Andreas, Florian and Albrecht [4]	2012	a novel gaze-based authentication scheme	-Image-based graphical passwords are considerably have higher security than the PIN-based passwords, but it rated as lower usability.
David Rozado [5]	2013	the subjectspecific gaze estimation parameters	-error rates are lower than traditional text based password. -faster to perform than inputting a password by means of a keyboard. -increasing the security by using a database of gaze estimation models associated to each user.
Mihajlov, Trpkova and Arsenovski [6]	2013	eye tracking study of ImagePass	-through the process of selecting passwords, the Passimage area have the most recognition with 55% attraction, comes next the Selected password box with 25% attraction. -through the process of password confirmation task, Passimage area have the most recognition with 65% attraction.
Mohamed, Florian, Mariam, Emanuel, Regina and Andreas [7]	2016	GazeTouchPass Scheme	-especially have security against the side attacks (only 15%-21% success rate). -iterative attacks are considered to be complicated, however they are possible in ideal conditions (23%-46%). -authentication time is 3.1 seconds. -faster and more secure than gaze-based schemes.
Zhenjiang, Mo Li, Prasant, Jinsong and Shuaiyu [8]	2017	iType System	- address a series of design challenges, covering accuracy, latency and mobility several aspects. -high typing accuracy within reasonable short latency in variant environments.

According to the comparison, we have graphical passwords and graphical user interface. A quick login and passwords that are easy to use and remember are offered by these systems. These systems face several challenges, one of them gaze accuracy. We found in the paper [2] a solution through 2 recent improvements which increase the accuracy of the gaze without compromising the security and the usability of the system. These enhancements get a good results with low rate of errors. Also we have enhancements in paper [4], but it focused on security and succeed in increase it but rated as low usability. There is also graphical user interface systems, these technique achieved using an eye tracker were shoulder surfing become practically impossible. These system developed in paper [6] and enhancement in paper [3] were it allow single errors such as a single character missing, inserted, or substituted. in these competition there is two systems that depend the method of combining multi-techniques, EyePassShapes in paper [1] get good rates in security and usability but a little slower than PIN-entry, GazeTouchPass in paper [7] that combine between gaze and touch is faster and more secure. in the case of eye tracking there is some enhancements in paper [8] that make eye tracking more efficient in case accuracy and limit some sign in errors, for more secure sign in there is some enhancements in paper [5].

6. Conclusion

In this paper, that the techniques was used to obtain the security of different systems was identified depending on the eye pupil tracking, and also the used algorithms were classified to achieve the pupil movement tracking. In addition to that, various measurements were reviewed for the purpose of achieving facial recognition and eye recognition and then distinguish the pupil movement. As well as the show of advantage and disadvantage of the techniques that used to track the eye movement.

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الأنظمة الأمنية القائمة على طرق تتبع حركة العين

إسراء احمد عبد الله شيماء حميد شاكر إقباس علي
الجامعة التكنولوجية / قسم علوم الحاسوب

المستخلص :

هناك بعض المزايا والعيوب في جميع طرق تتبع العين، ويعتمد اختيار نظام تتبع العين على الأخذ بعين الاعتبار التطبيق والتكاليف. يقدم هذا البحث نظرة عامة على تقنيات تتبع العين لأنظمة الأمن المختلفة. تصنف هذه الورقة التقنيات باستخدام بعض الأبحاث. تعتمد الأبحاث على خصائص بيانات تتبع العين، والتي تشمل جوانب المشاهد والمحفزات، كما أنها تعتمد على الخصائص المتعلقة بأساليب التمثيل البصري. يقدم البحث في النهاية مقارنة بين تلك الأنظمة الأمنية لتسهيل اختيار نظام أكثر دقة وفعالية.

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Proposed aspect extraction algorithm for Arabic text reviews

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Abstract :

Opinion mining from reviews is a very crucial area in NLP. This area has many applications in social networks, business intelligence, and decision making. Aspect extraction is the main step to achieve opinion mining. This paper proposed an algorithm for aspect extraction from reviews in the Arabic language, to determine the aspects that the reviewers are described in their comments. The proposed algorithm begins with analyzing the comments dataset using latent Dirichlet analysis (LDA) to identify the aspects and its essential representative words, then extracting nouns and its' adjectives as a possible aspect phrase in a review. After that the categorizing process to categorize the extracted phrases according to the words specified from LDA analysis. The proposed approach has been tested by using two standard Arabic reviews datasets. The result was auspicious in spite of the difficulties if the Arabic natural language processing.

Key Words. Natural language processing, opinion mining, sentiment analysis, aspect extraction, latent Dirichlet analysis, LDA.

1. Introduction.

Opinion mining means extracting and analyzing people's opinion about an aspect of an entity, while sentiment analysis (SA) is analyzed people's emotions towards entities such as products, services, and topics. SA can be classified into three levels: documents level assumes that each document holds opinions about one entity. Sentence level SA, which aims to classify the sentiment in a clause to positive or negative. The third class is aspect level an SA where the system finds what the writer like or dislike about the entity. It's also known as feature-based or attribute based sentiment analysis[1]. The goal of aspect level sentiment classifications is to specify aspects along with their sentiment. For example, "the food is delicious, but the service is very slow", reflects the opinion of the reviewer about two aspects: the food and the service, the sentiment toward the food is positive while service sentiment is negative. Aspect level SA includes many tasks, aspect extraction, aspect categorization, and aspect sentiment classification [2]. The aspect extraction process works to find the aspect and opinions about them before sentiment analysis step. Unfortunately, the opinion mining in the Arabic language did not receive enough attention, due to the limited number of tools and other challenges that relate to the nature of the language. This paper would manage two issues; the first is aspect extraction from the Arabic sentence, the second issue is aspect categorization.

2. Contributions

This work contributes to the field of Arabic sentiment analysis, firstly; by proposing a method to identify the aspect terms, and opinion words related to them. This target is accomplished by using latent Dirichlet analysis(LDA) to specify the most critical aspects mentioned by the reviewers and the words that used to describe these aspects. The second contribution is design a specific Arabic parsing system that works to extract the nouns and adjectives or in Arabic (الصفة و الموصوف), which is chunking system for the Arabic language that chunk bigrams and trigrams for a specific pattern like noun phrases patterns, then categorize the extracted phrase using specified words from LDA step.

3. Related works

Al-Samadi et al., worked on his first paper to classify sentiment of Arabic news by extracting aspect to classify news topics. For aspect, extraction he used N-gram feature pruning and Stanford POS tagger. His work relies on searching for nouns to classify news more than sentiment aspect extraction [3]. The second paper of Al-Samadi et al. was about aspect-based sentiment analysis for hotel reviews. His work depends on using SVM classier to classify extracted noun phrases. The system uses a training data set that has XML form extracted noun aspect by training SVM classifier. XML is a structured document while our work is more complicated as it uses unstructured text reviews about hotels and books[4]. Manahel et al. have built an aspect based sentiment analyzer for Arabic tweets depending on the parsing system to extract noun and adjectives from n-gram, then categorize this aspect using lexicon made by her [4]. Shima et al. worked on developing a root lexicon to lemmatize sentiment words in Arabic by collecting patterns that used to be sentiment words like (جميل = فعيل) and (افضل = افعل) but this work has a weakness mentioned by the author. The reason for this weakness is that the word orientation depends on the subject and the context that guide the sentiment of the word. For example, the word (big =كبير) has a positive orientation when talking about the hotel but negative orientation when talking about technology or an electronic device [6]. Abdul-Majeed et al. used the SVM approach for subjectivity and Arabic sentiment analysis. The feature used in that study are POS tagging, gender, and lemma as features, and polarity from a lexicon. The highest accuracy for sentiment classification was about 71% [7]. Al-Subaihin et al. built a system in two steps; the first is gaming an aa approach to build the lexicon through player annotation. The second step is a sentiment analyzer through word segmentation then calculate the accumulated score for the sentence. The precision reached 6,0.32 [8].

4. Challenges in the Arabic language

There are three types of Arabic language: classical Arabic, which is not used in our daily life, modern standard Arabic (MSA) and dialect Arabic [1]. MSA is a standard form that is used in official letters and schools. MSA is a simplified form of classical Arabic which is the language of the Quran and the old Arabian scriptures [9]. The dialect Arabic are local dialects used in Arabic countries, and this dialect not standard enough, and differs from each other in many idioms.

For example, the English word (many), in Tunisian dialect (Barsha = برشة) to represent adjective many, while the same word in Iraqi dialect (hwaya=هواية) and in Egyptian dialect (keteer =كتيير) and son on [9]. The second challenge is the lack of Arabic lexicon for sentiment and also the absence of reliable tools for part of speech tagger, for example, tashfeen tagger can tag only verbs and nouns which is not much useful in the case of sentiment analysis [9]. Stanford tagger has some issues in its accuracy and sense of the right tag of some words. Also, there is a technical issue is that until now there are no reliable NLP tools for this language. There are also differences between researchers about how to deal with the Arabic corpus of text. For example, Abdulraheem, and Al-khlaifan recommend stemming to reduce the size of the lexicon corpus [10], while Rushdie Saleh et al., does not recommend stemming for the task of opinion mining, because stemming may alter the meaning of the word in context and may alter its POS tagging [11]. The other obstacle that most of the reviews in Arabic social networks written in Arabic dialect form, dialect form causes many problems in that it required different lexicon corpus for each local, national dialect [1].

5. Latent Dirichlet analysis

Latent Dirichlet analysis is a generative statistical model that let sets of observations in the text to be explained by unobserved groups, to explain the reason of behind some chunks of text is similar. LDA is a topic modeling method. LDA assumes that each review is a mixture of a small number of topics and each word participates in one of the reviews topics [12]. This method is identical to probabilistic latent semantic analysis, except in LDA topic distribution is assumed to have sparse Dirichlet prior. Dirichlet priors encode the intuition that reviews cover only a small set of topics and that topics used just a small set of words frequently. This method tries to find a statistical distribution for topics inside the document and a model for each topic in a document. Figure (1) explain the LAD topic word distribution model[12].

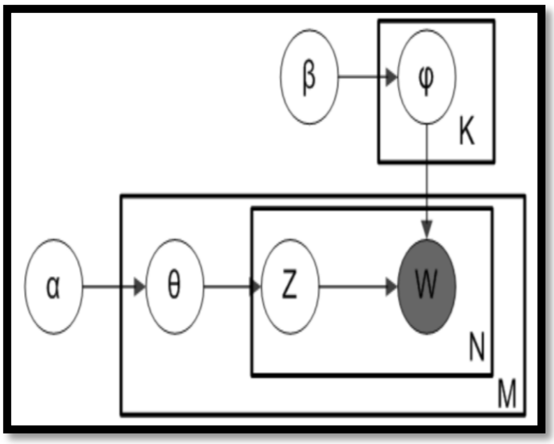

Figure (1) LDA documents analysis model

The probabilistic model in figure (1) represents the dependencies among the variables. The outer plate represents documents (reviews), while the inner plate represents the repeated word positions in a specific document. Each word position is associated with a choice of topic and word. M is the number of documents, and variables are: α is the parameter of the Dirichlet prior on the per-document topic distributions, β is the parameter of the Dirichlet prior on the per-topic word distribution, θm is the topic distribution for document, φk is the word distribution for topic k, Zmn is the topic for Nth word in document m and Wmn is specific word. Entities represented by θ and φ are matrices coming from decomposing the original document word matrix. θ Consist of rows of documents (reviews) and columns defined by topics. φ Consist of rows of topics and columns of words, so $\varphi \ldots \ldots \ldots \varphi k$ refers to set of rows, each of which is distribution over topics .**خطأ! لم يتم العثور على مصدر المرجع**. Now the fully generative procedure for LDA: Assume that $\bar{X}$ is a document or a review in the case study of this paper, Gr topic, and t is a term

Then:

1. Start
2. S= number of topics
3. Generate the n tokens in ith document form a Poisson distribution
4. Generate relative frequencies $\Theta = (\theta 1, \theta 2. \ . \ . \theta k)$ of different topics in ith document from an Dirchilet distribution. This step is like generating$\theta r = p(Gr|\bar{x})$ for all topics r for a specific document (review). Note that$\theta r = p(Gr|\bar{x})$ probability of topic given document.
5. For each of the nth tokens in the document, first select rth latent component with probability $P\ (Gr|Xi)$ and then generate jth term with probability $P\ (tj|Gr)$, $P\ (tj|Gr)$ where the probability of term given topic.

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6. Proposed system

The proposed system consists of two parts, the first part is the aspect extraction algorithm and the second part is the aspect categorization process. Flowchart (2) is a general view of these two processes:

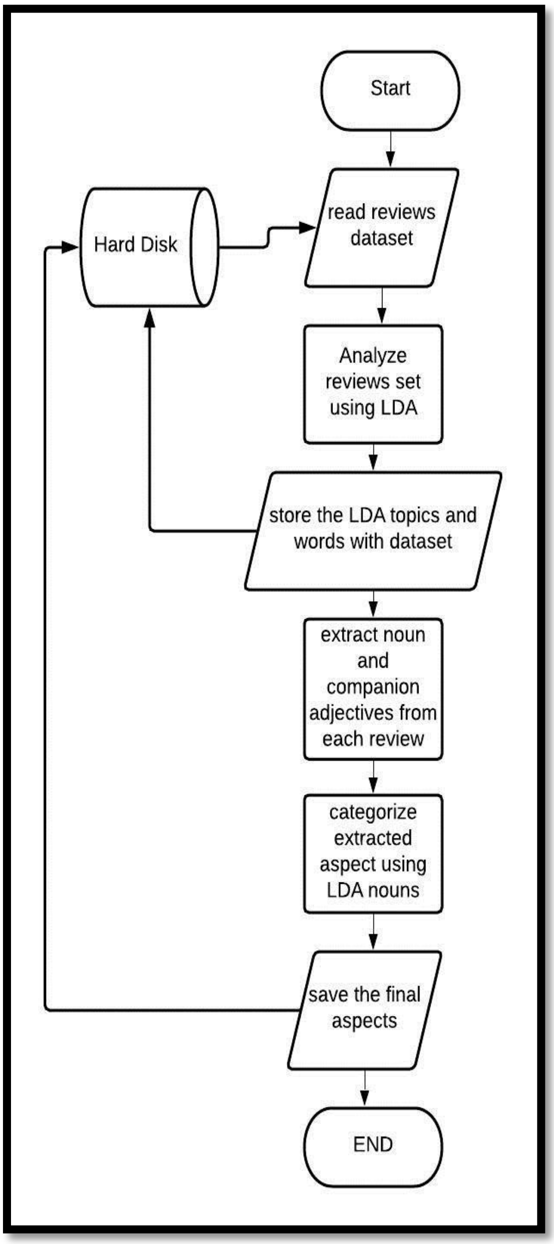

Figure (2) proposed system flow chart

The aspect extraction algorithm is a probabilistic approach that depends on the latent Dirichlet analysis model to identify the aspects, and a parser that parses the reviews to extract pattern of (described and description) or in Arabic (الصفة و الموصوف) by using the patterns of (NN, JJ) or (NN, NN, JJ), note that NN for noun and JJ for adjective. The proposed approach for aspect extraction exploit a specification in Arabic language which is that the attribute (adjective) come adjacent to the described noun like (الفندق رائع = hotel is wonderful), or adjacent to a composite noun in case of (NN, NN, JJ) in (خدمة الغرف جيدة) which is mean (room service is good). The Arabic language does not use auxiliary verbs, so the JJ (adjective) comes adjacent to a noun or composite noun. In the English language, the descriptor may be blocked from the noun by either auxiliary verb as in (Hotel is good) or by auxiliary verb and exaggeration formula (hotel was extremely good), so this would make aspect extraction process is little harder. After aspect extraction the step of aspect categorization is coming, firstly LDA must be done on the data to extract the central aspect mentioned by all reviewers, and specify the representative words for each aspect. The use of LDA analysis makes the proposed method probabilistic. There is one obstacle to extract a demanded pattern from Arabic text, is that there are no chunking tools to extract demanded pattern. NLTK has chunking tools for the English language only, which can extract noun phrases or any pattern effectively. For this reason, there is a need to build an Arabic chunking system or shallow parser that take bigrams, or trigrams and parse each word in these pieces and looking for a pattern of (NN+JJ) and trigrams with (NN+NN+JJ) as in

The following algorithm1:

```
Algorithm 1: Arabic chunk parser algorithm

Input: Arabic reviews corpus
Output: corpus of bigrams and trigrams of aspect phrases

  1. Start
  2. For  all reviews in the corpus DO
  3. Read a review
  4. Clean text review from punctuation marks and non-Arabic text and
     numbers
  5. Divide the text review into bigram chunks
     For i in range (0, length of review -1) DO
            Take every two adjacent words   Bigram = [text [i], text [i+1]]
            Check the bigram to observe if hold aspect and opinion word
            IF bigram [0] in ['DTNN','NN','NNS','NNP'] AND bigram [1] like ['JJ']
     THEN:  Save bigram to final list
  6. Divide the text review into Trigram chunks
     For I in range (0, length of review-2) DO
            Take every three adjacent words   Bigram = [text [i], text [i+1], text [i+2]]
            Check the Trigram to observe if hold aspect and opinion word
            IF Trigram[0] in ['DTNN','NN','NNS','NNP'] AND Trigram[1] in
     ['DTNN','NN']  AND Trigram[2] in ['JJ','ADJ']          THEN :  Save Trigram to
     final list
  7. Save the extracted chunks to the data frame
  8. END.
```

Note that DTNN is DT for determiner followed by a noun, NN noun, NNS means noun singular, and NNP means noun plural. Now the second part of the process classifies the aspects according to the most aspects that the user has focused on his reviews about the hotel and from human experience that are:

- The hotel: contain general opinion about the hotel
- Rooms: user opinion about the room
- Staff: one of the most important aspects that most of the reviewers mention.
- Services: an aspect of general services like Wi-Fi internet, taxi, swimming pools, and spa.
- Price: is another crucial aspect of choosing the hotel
- Food: also is an aspect always described by visitors about the breakfast, restaurant, and bars.
- Location: location aspect consists the location of the hotel and its closeness from the city center and markets and other famous tourist places.

To specify the words that represent these aspects, we use the probabilistic approach by using latent Dirichlet analysis. LDA used to analysis the corpus reviews to determine the essential keywords through topics. To reach to the optimal number of topics, we have to run the LDA with a various number of topics on the same dataset with measuring coherency each time to choose the best amount of topics that give an excellent coherency of words. As in the following figure (3), the highest portion of coherency 0.45 when the number of topics reaches to 35 topics.

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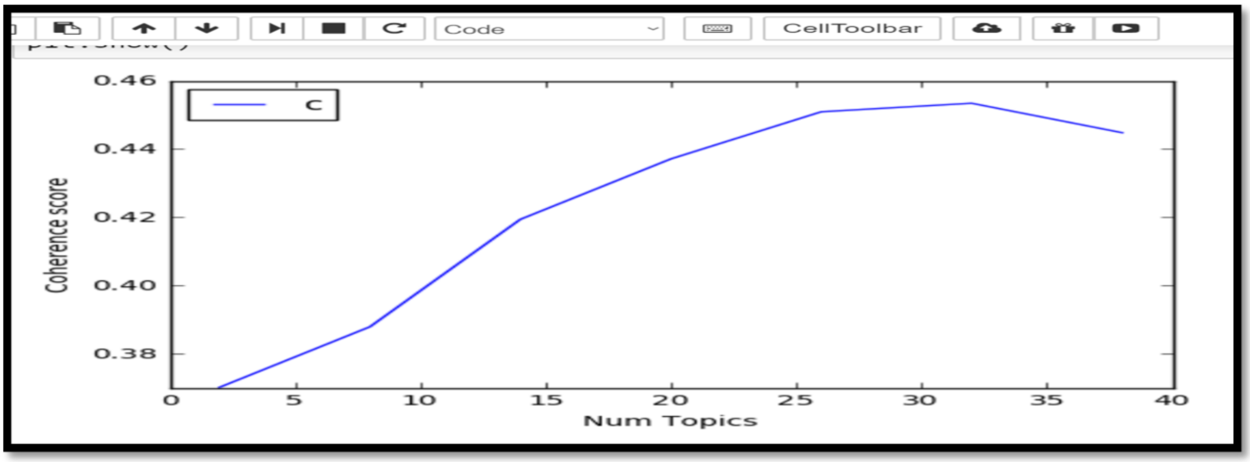

Figure (3) coherency chart for the different number of topics in LDA process

And the most important words to represent each aspect are:
Staff: 'وموظفي ' ,'عاملين' ,"مدير" ,'الطاقم' ,'مضيافين .... etc.
Rooms: أثاثا' ,'وديكوره' ,'وديكورة' ,'فيلا' ,'الشقه' ,'الشقة' ,'سرير ... etc.
Price:
رخيص' ,'مجاني' ,'ومجاني' ,'مجانيه' ,'أسعار' ,'واسعار' ,'وأسعار ...etc.
Location: 'وموقع' ,'سيرا' ,'قدميك' ,'الاقدام' ,'بجانب ...etc.
Hotel:
فندق' ,'الفندق' ,'تجربه' ,'المكان' ,'مكان' ,'تجربة' ,'منزل' ,'نزل ... etc.

Food: الافطار' ,'ألافطار' ,'وجبه الافطار' ,'وجبة الافطار' ,'وجبه ...etc.
Services:
نقل' ,'مكيف_الهوا' ,'مكيف_الهواء' ,'دش' ,'الانترنيت اللاسلكي' ,'
...etc.
These extracted words from 35 topics would be used to categorize the extracted aspects.

The general chart bar of the most important words in the hotel reviews dataset is clear in the figure below (4):-

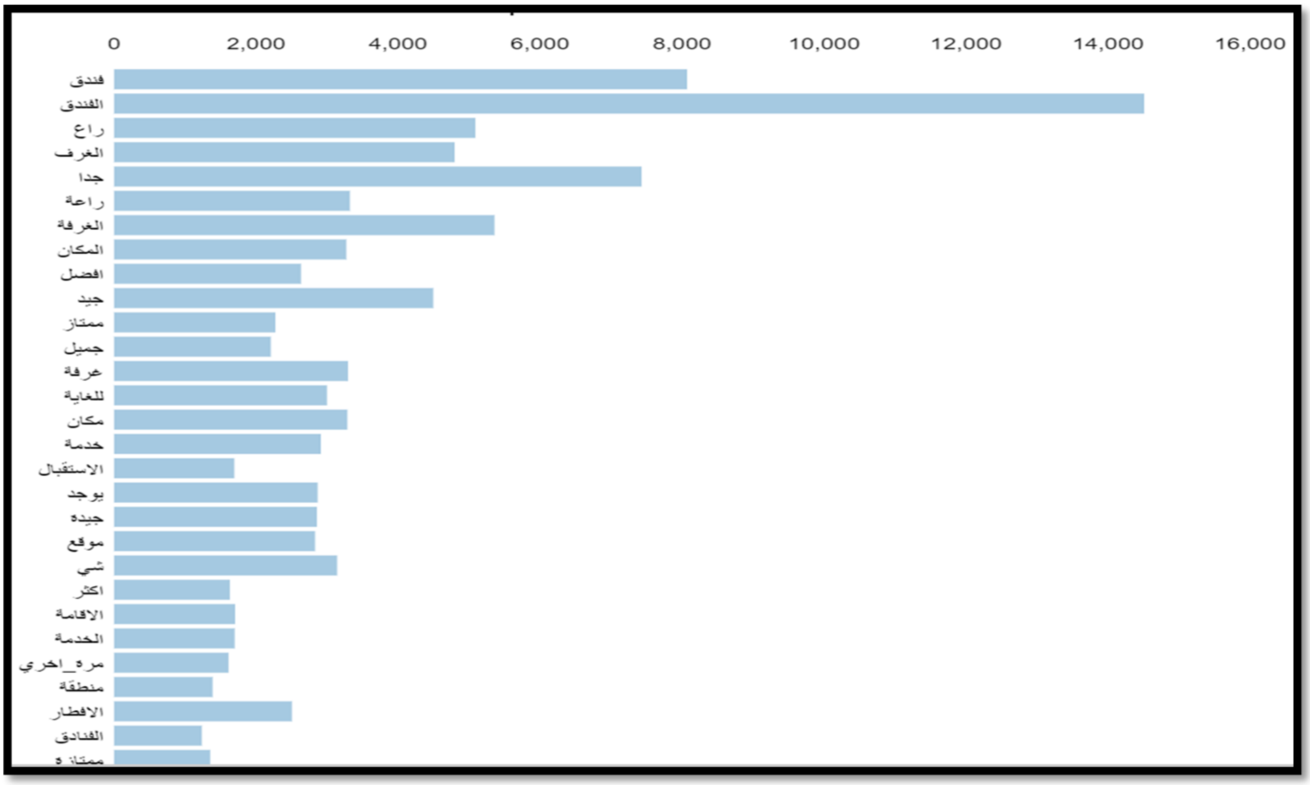

Figure (4) chart of the most essential words in the hotel's dataset using LDA

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The results of the LDA refers that there is overlapping between aspects inside these topics. This overlapping is useful, and it's powerful for our approach, it's not a weakness. The process of aspect Categorization of aspects phrases by using the words specified from LDA analysis process. This

categorizing process is done by matching the noun in the extracted aspect phrase with the set of words that represent each aspect as in algorithm 2 as follows:-

```
Algorithm 2: aspect categorizing algorithm
Input: set of aspect phrases from reviews dataset
Output: corpus of categorized dataset
   1. Start
   2. For all extracted aspect phrases (i) :
            a. Clean phrase (i) from punctuation marks
            b. Split the words in phrase (i) to list x
            c. IF first word in list X[0] in hotel aspect word list THEN
                 Save phrase X (i) in hotel column

               IF first word in list X [0] in rooms aspect word list THEN
                 Save phrase X (i) in rooms column

               IF first word in list X [0] in service aspect word list THEN
                 Save phrase X (i) in service column

               IF first word in list X [0] in staff aspect word list THEN
                 Save phrase X (i) in staff column

               IF first word in list X [0] in prices aspect word list THEN
                 Save phrase X (i) in prices column

               IF first word in list X [0] in food aspect word list THEN
                 Save phrase X (i) in foods column

               IF first word in list X [0] in location aspect word list THEN
                 Save phrase X (i) in location column
   3. END
```

Now we use the same LDA analysis for the Hady al-Sahar hotel reviews dataset and calculate the coherency to see the optimal number of topics that are yield the highest coherency which is 40 topics as in figure (5) below:

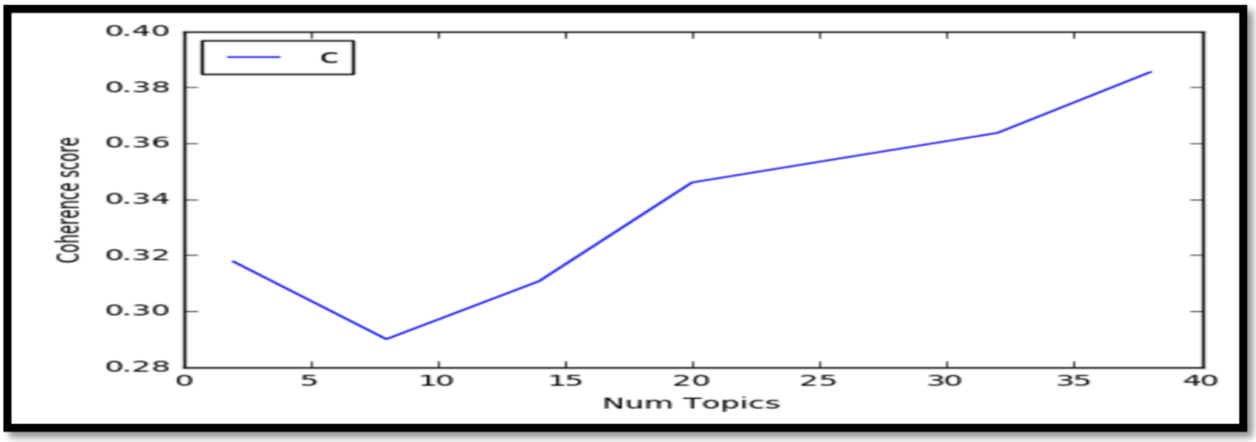

Figure (5) number of topics suitable for hotel dataset

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After that, the most important words are extracted using the LDA, but in books reviews dataset we don't need to categorize the aspect, the need only to find the

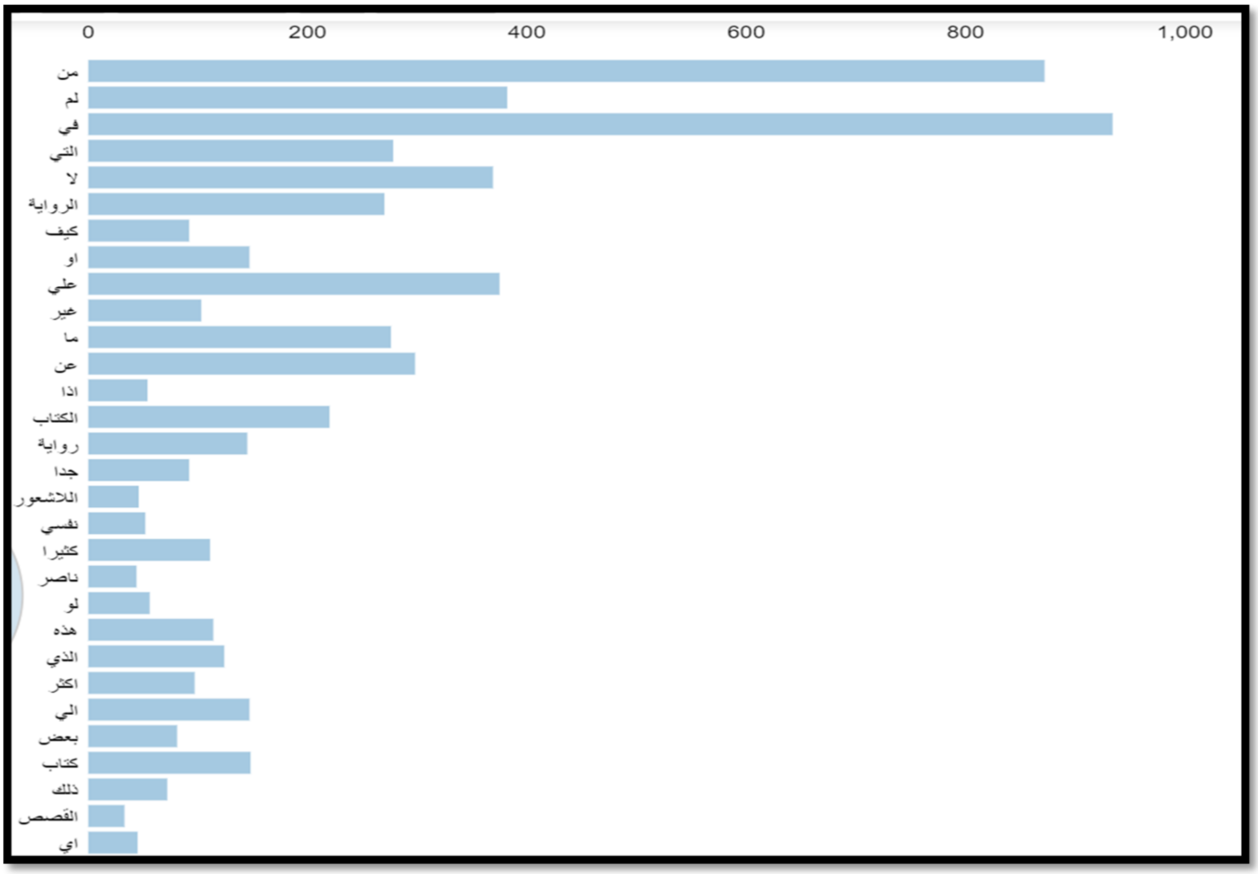

Figure (6) the most significant words in the 40 topics of hotels dataset

The result showed that the important words are:
['
الرواية','الكتاب','رواية','كتاب','القصص','قصص','الكاتب','كات
ب','والحبكة','الحبكة','حبكة','الحزن','حزن','روايات','الحياه','ال
حياة','حياة','والحياة','الكاتبة','سرد','السرد','التاريخ','شخصية','
حواسه','الافكار','الكتب','الادب','مقالات','اسلوب','اسقاطات','ال
سرد','الادب','المقالات','الروايه','كتب','روايات','الروايات','الش
عور','شعور','النص','مشاعر','تفاصيل','التفاصيل','اللغة','لغة'،
اللغه','الوصف','الاسلوب']

The aspect categorization process for books dataset is different from hotels dataset because the aspect in hotels case is more detailed than books aspects. Hotels reviewers described detailed part of service or room, while the aspect of books is more general, for example, its extract aspect as (poem, novel, etc.), so it seems to be aspect name more than aspect description for an aspect of the product or service.

For obvious reason the process first is extracted aspect using LDA process, then find the important aspect by seeking for the noun with part of speech (DTN = determiner). For example, a noun like "AL-ketab" AL in Arabic is equivalent to "THE" determiner in English and "ketab" noun means "book") and if the percentage of this noun in the TFIDF table is less than 0.09 this noun would be accepted as an aspect name. The number 0.09 was specified from LDA as a threshold to specify the common nouns mentioned by reviewers as an indicator of its generality among them as an aspect.

Now all these explained in details in algorithm 3:-

```
ALGORITHM 3: Books reviews aspect extraction
Input: Extracted aspect phrases
Output: nouns as aspect about books
    1. Start
    2. For i in range( 0 to length (aspect phrases)) :
           a. For noun in phrase(i):
                  1- IF noun in TFIDF table THEN
                         A. IF TFIDF (NOUN) <=0.09 or noun in LDA books aspect
                                List THEN
                                    Save NOUN
                         b. ELSE: ignore Noun
    3. For I (0 to length (reviews)):
            For each word in reviews:
                  IF part of _speech [word] ==DTNN AND TFIDF (word)≤0.09 THEN
                         Save Noun
    4. END
```

But in many cases, the Arabian reviewer is not using the traditional formal approach to, the reviewer may use the narrative way to declare his opinion using verb phrase as in (" كلكم بحبكم وبشعر أني بيتي التحية خيي وسيم صالحة "). Or in English ("I love you all, and I feel like at my home, my brother "), so, this makes a misleading for any parsing system, causing an open problem.

7. Results

To test the proposed algorithm for extracting aspects from Arabic reviews text, two datasets have been used, the first al-smadi* Dataset for books reviews in the Arabic language, and the second is hady-al-sahaar**Arabic reviews dataset about hotels. This target is accomplished by calculating true positive, true negative, false positive, and then precision and recall. The true positive (TP) means the number of intersections between aspects tagged by the proposed algorithm and identified in the dataset. The false positive (FP) represent the number of aspects term occurrences specified by the proposed algorithm but not mentioned in the dataset. False negative (FN) represents the number of aspect terms occurs in the dataset but not have been identified by the proposed algorithm. Precision and recall then calculated

$$precision = \frac{TP}{TP+FP} \text{ ------------- (1)}$$

$$Recall = \frac{TP}{TP+FN} \text{ --------------- (2)}$$

The testing dataset consists of 400 reviews from the al-samadi dataset. We calculate the precision and recall for each row, then plotting the result in figure (7) and figure (8) as below:

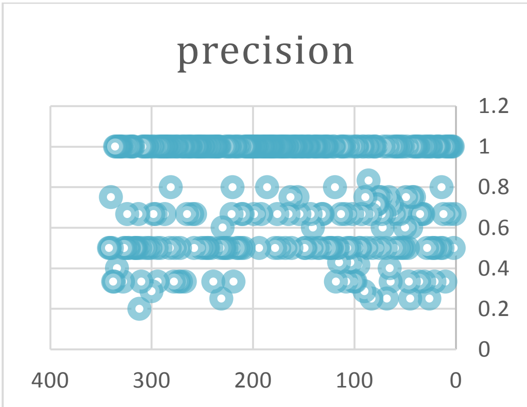

Figure 7: the precision plotting for each review in books datasets

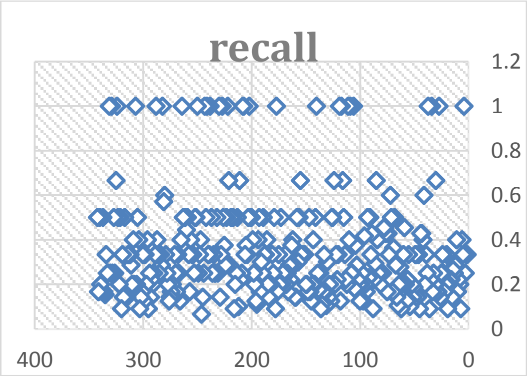

Figure 8: recall plotting for each review in books reviews dataset

From figure 7 and figure 8, the reader can notice that precision is between 0.2 and 1 while the recall taking values between 0.1 and 1. The system catch aspects from 28 reviews (8% from reviews) with accuracy reach to 100%. We can see that 106 reviews recall between 0.1 and 0.5, which make about 30% from reviews. So the precision and recall result is in the table (1) below:

Dataset	Precision	Recall	f-score
Books reviews	0.78	0.35	0.48

Table (1) accuracy of the al-samadi dataset

Hotel reviews dataset consist of 367 reviews have been analyzed, and a more detailed graph for precision and recall are in the figures (9) and (10):

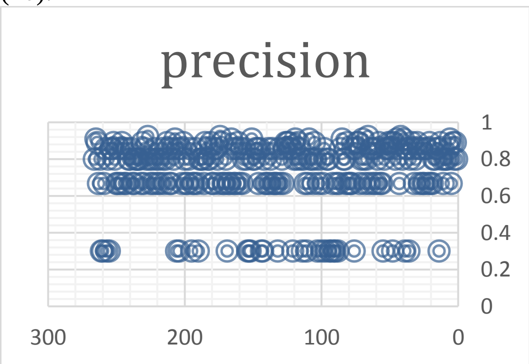

Figure (9) precision for each review in hotels dataset

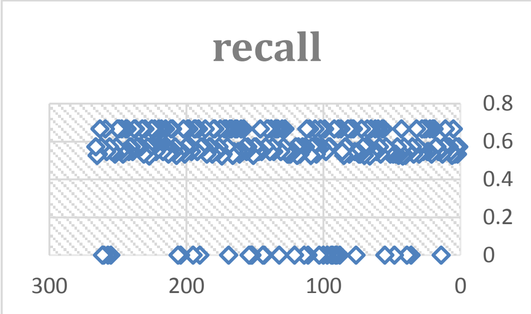

Figure (10) recall for each review in the hotel's dataset

The precision and recall for hady al_sahar hotels review dataset is on the following table (2):

Dataset	Precision	Recall	f-score
hotel reviews dataset	0.73	0.52	0.61

Table (2) accuracy of hotels dataset

From the table, the average of precision is 0.73, and most of the aspects have been identified with precision between 0.6 and 0.9 as in figure (9). In hotels dataset, the f-score is higher than f-score in books dataset because the precision and recall are more consistent.

8. Conclusions

Aspect extraction for the Arabic language is a new scope in Arabic NLP. The work on this paper revealed many issues; first, the need for a standard lexicon to specify sentiment orientation of words for different Arabic dialects, second Arabic NLP tools need more accurate taggers and consider the context of the text. During the work, we also specify the need for a shallow parser mechanism for Arabic language tools or (chunking), which can extract bigrams and trigrams with a specific pattern. The chunking software has been built in this work. Now let discuss our results, from results it's clear that precision is very good and it's varying according to the topic of the dataset. The researchers in aspect extraction should take into their consideration that there are two kinds of aspects, a general aspect that determines what is the text is talking about which is found in the books reviews dataset (AL-samadi dataset). The second type of aspect a more detailed aspect that appeared in hotel or restaurant dataset which is dealing with more deep points about the object like price, food, decoration,…etc.

The precision for books reviews dataset is reach to 0.78 and its good accuracy for the Arabic language in the absence of lemmatizes. The recall is 0.35 in books reviews aspect extractor because the false negative is small, as the proposed method can catch aspects more than aspects (and deeper) than aspects gotten by human annotator in the dataset, and this is not a weakness, but it is a positive indicator for the proposed aspect extractor efficiency. The precision for hotel reviews dataset is 0.73 and recall 0.52 for the same reason as in books dataset, but the categorization process was much easier in the hotel because the aspect was apparent in LDA analysis through the topics.

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خوارزمية استخلاص جوانب الحديث المطلوبة لتحليل الاراء العربية

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المستخلص :

برز في السنوات الاخيرة مجال تنقيب الاراء من تعليقات الاشخاص كمجال مهم للدراسة. وقد تعددت تطبيقات هذا المجال لتشمل تحليل الاراء في الشبكات الاجتماعية، انظمة الاعمال الذكية ، وأنظمة اتخاذ القرار.للقيام بتنقيب الاراء فأن عملية استخلاص جوانب الحديث و تحديد ما تطرق اليه المعلق عن خصائص منتج او خدمةٍ ما احد المراحل الأساسية لتنقيب الرأي وتحديد شعور المعلق من الخدمة أو المنتج. يقترح هذا البحث خوارزمية لأستخلاص جوانب الحديث للغة العربية وذلك بدءاً بعملية تحليل الموضوعات الاحتمالي latent dirichlet analysis لجميع التعليقات ،وذلك لتحديد أهم الجوانب المشتركة التي ركز عليها الجمهور المعلق حول خدمة او منتوجٍ ما . يلي ذلك عملية استخراج جمل الاراء بنظام استخراج الانماط الاعرابية مستفيداً من تتابع الصفة و الموصوف فيٍ اللغة العربية . وأخيرا تتم تبويب كل عبارة مستخرجة حسب أهم الجوانب المحددة من عملية LDA و الكلمات الممثِلة لهذهِ الجوانب . تم قياس دقة الخوارزمية المقترحة بواسطة مجموعتي بيانات قياسيتين ، الاولى تعليقات عن الكتب و الثانية عن الفنادق باللغة العربية وكانت النتائج جيدة رغم عقبات معالجة اللغة العربية .

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Design of keystream Generator utilizing Firefly Algorithm

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Abstract :

Stream cipher is one of encryption procedures for sending data in internet; stream cipher is suitable in telecommunications and real-time apps. The robustness measurement of stream cipher is according to the randomness of keystream that is utilized. If the random series of keystream generator is low, the keystream of stream cipher can be read and encrypted data by stream cipher become vulnerable to attackers. This paper utilizes Firefly Algorithm based Local Key Generation for generation keystream. The generated keystream is independent of original messages. The randomness of keystream series of Firefly passing the five standard criteria. The suggested keystream generator is word-established appropriated to fast real-time apps than are bit-established linear stream ciphers. Furthermore, the suggested keystream generator satisfies the three demands of benchmarks such as maximum correlation, robust randomness and huge complexity.

Key Words. Stream Cipher, keystream Generator, Firefly Algorithm, Local Keys Function**.**

1. Introduction

Cryptography is an exercise together with analyze procedures for safe communication in the presence of adversaries. Generally, it is about founding and studying protocols which overcome the technique of attackers and that are linked to a diversity of parts in data security [1]. The mammoth issues of secure telecommunication is keeping the confidential information from interception. In cipher frameworks , It is popular that the experts of encryption procedures required techniques to discover an orderly procedure in checking their ciphers to guarantee that they are protected from attackers [2][3]. One of a popular symmetric procedure is stream cipher, all portion of original message and keystream are encrypted together. The keys of Stream cipher utilized for encryption procedure is altered randomly. Consequently, the cipher that's created is mathematically very hard to breach. .The altering of random keystream will not permit any sample to be frequent that allocate a guide to attacker to breach cipher data [4], [5].

Stream cipher is suitable on equipment and programs, and in some situations obligatory in telecommunications and real-time apps chiefly with restricted memory [6], [7]. Stream cipher is less apt to cryptanalysis due to identical portions of original messages are encipher with various portions of the keystream [8]. The essential idea of stream cipher is one-time pad cipher which is refer to Vernam cipher, necessity that is true random series with generated keystream , Both transmitter and recipient are participated keystream , and keystream only utilized once [9]. The cons of stream cipher are the total volume of the keystream and the original message should be the identical. Subsequently a huge quantity of keystream have to be kept and transmitted. Furthermore, if the random series is discover , the keystream utilized for encryption procedure can be keep track readily [10].

Many different papers utilized to evolve Stream Cipher generator for Encryption according to SI and several techniques, In[11] suggested keystream procedure for encryption data according to Ant Colony and the allocation of letters in the original message. In [12] suggested keystream generator according to a Particle swarm optimization for encryption data. The con of these articles is that enumeration the appearance of letters of keystream in the original messages. Of course, that it excess the overall time consumption for encryption, when the volume of keystream selected is huge.

In [13] suggested keystream generator according to 3D chaotic maps and Particle swarm optimization to create a random number generator and tested by five standard criteria. In [14] suggested keystream generator according to on dynamic the thought of updates composite LFSR stream cipher every update in message that indicates to has a huge complexity encryption procedure and tested by standard criteria
.

Concerning , the stream cipher and Firefly Algorithm that is debated at the previously, the contribution of this paper is suggested Keystream generator utilizing Firefly Algorithm based Local Key Generation where the generated keystream is independent of original messages .

2. Firefly Algorithm

Optimization is an arithmetic procedure for getting either highest or smallest value of a target function by selecting a best solution from a set of solutions [15]. There are a huge number of optimization problems, which were complicated and toughed to discover a solution at a sensible time in different areas like business, operations research and computer science [16].

These optimization problems can be successfully solved by applied Biology-Nature-Inspired-Metaheuristic-Algorithms (BNIMAs) mostly that according to swarm intelligence (SI) which is a subset of Artificial Intelligence. Birds, Wasps, Ants, Bugs ,Bees and Firefly are different models of the family of BNIMAs which are using the behavior of SI based on target function [17][18].

In optimization problems, firefly algorithm is a one of SI clans of algorithms that lately exhibited magnificent performances by presenting best solutions [19] as shown in Algorithm (1). Firefly algorithm utilizes the subsequent three essentials [20][21]:

- All firefly would be going to other fireflies that have a high attractiveness irrespective of their gender.
- The attraction of a firefly is apt to its lighting that is minify as distance of different firefly rises. Fireflies shall move arbitrarily, when no it's lighter one than a specific firefly.
- The aim of target function is locating the illumination of firefly.

Three critical criteria in Firefly algorithm can be summarized as [22]:

- The 1st critical criteria is attraction (attractiveness N) that is specified by its lighting strength as Eq.1 follows:

$N=N_0 e^{-vt^2}$ Eq.1

N_0 is an attraction at distance t equal to zero, while v is refer to lighting absorption in weather.

- The 2nd critical criteria is distance that is specified by utilizing the Cartesian distance between two fireflies as Eq.2 follows:

$R_{i,j} = \sqrt{\sum_{s=1}^{n}(F_{I,S} - F_{J,S})^2}$ Eq.2

In the n-dimensional space, $F_{i,s}$ is s-th component of coordinate F_i of i-th Firefly.

- The 3rd critical criteria is movement that is specified by brightness for instance, firefly (a) would be going to firefly (b) that has a high attractiveness as Eq.3 follows:

$M_a=M_a + N_0 e^{-vR^2 ij} (M_b - M_a) + P\ (r\text{-}0.4)$ Eq.3

Where P is refer to randomization value, r is refer to a random value that its range between 0 and 1 a, M_a is refer to movement firefly (a) and M_b is refer to movement firefly (b). Furthermore Firefly is equal to particle swarm optimization when v is equal to zero [22].

Algorithm (1): Firefly Algorithm
Input: number of firefly population ,light absorption coefficient v,max- iteration, objective
Output: set of keystream
Begin $Step_1$: Determine the objective function. $Step_2$: Generate initial population of fireflies. $Step_3$: Calculate the Light intensity at fireflies based on objective function. $Step_4$: iteration=1 $Step_5$: A=1,B=1 $Step_6$: if fitness function of firefly (A) less than fitness function of firefly (B) then Move firefly (A) towards firefly (B) $Step_7$: modify attraction differs with distance through e^{-vt^2} $Step_8$: Evaluate modern resolutions and modify brightness intensity. $Step_9$:B=B+1 $Step_{10}$: if (B less than or equal to number of fireflies) Goto $Step_6$ $Step_{11}$: A=A+1, B=1 $Step_{12}$: if (A less than or equal to number of fireflies) Goto $Step_6$ $Step_{13}$: Rank fireflies and find the current best $Step_{14}$: iteration= iteration +1 $Step_{15}$: if (iteration less than or equal to max-iteration) Goto $Step_5$ **End**

3. Methodology of Designing Stream Cipher utilizing Firefly Algorithm

The main essentials of Firefly Algorithm and stream ciphers have been learned and analyzed. In this Methodology, Firefly Algorithm based Local Key Generation (FAbLKG) is suggested for generation keystreams as exhibited in Algorithm (2). In FAbLKG, a Firefly is utilized to assign a keystream. Each Firefly have many keys inside that keystream, the keys in the keystream are set of bits can be 0 or 1. For instance, if the length of keys of each Firefly is equal to 512 consequently it is depicted by Eq.4.

Firefly (keystream) key1, key2 , . , key512 Eq. 4

Algorithm (2): Firefly Algorithm based Local Key Generation
Input: number of firefly population , max-iteration
Output: set of keystream
Begin $Step_1$: iteration=1 $Step_2$: For each firefly in the population, randomly generate the initial keystream utilizing the Local Key Generation. $Step_3$: Calculate the Light intensity based on fitness function of keystream of each firefly $Step_4$: A=1,B=1 $Step_5$: if fitness function of firefly (A) less than fitness function of firefly (B) then $Step_{5.1}$: Compute hamming distance between firefly (A) and firefly (B) $Step_{5.2}$: Determine number of swap of keys in the keystream of firefly (A) between 1 and Hamming Distance $Step_{5.3}$: Making swap operation on different locations of keys in the keystream of firefly (A) according to the possible range of $Step_{5.2}$. $Step_{5.4}$: update fitness function of keystream of firefly (A) and modify brightness intensity. $Step_6$: B=B+1 $Step_7$: if (B less than or equal to number of fireflies) Goto step5 $Step_8$:A=A+1, B=1 $Step_9$: if (A less than or equal to number of fireflies) Goto step5 $Step_{10}$: Rank fireflies and check fitness function of them, if its equal to five then store the keystream obtained. $Step_{11}$: iteration= iteration +1 $Step_{12}$: if (iteration less than or equal to max-iteration) Goto step4 **End**

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3.1 Representation of Fireflies

For generating a keystream, Firefly Algorithm utilize the binary code as a solution. The Firefly Algorithm initiates with an elementary population encompasses set of fireflies (keystream) according to Local Key Generation.

3.1.1 Local Key Generation

Local Key Generation **(LKG)** is a novel word-established NLFSR stream ciphers which is offer numerous volumes of keystream per round than bit-established LFSRs stream ciphers, according to the word volume. Thus, in LKG an equilibrium is accomplished between security and efficiency. LKG can be referred as a development of the outcome feedback mode. The outcome of keystream is furthermore the feedback to the inner state. LKG has initially seed of 512 bits inner state sets by initialization vector. These bits are splitted into sixteen 32-bit words categorized W_1 to W_{16} and making XOR and Local Function feeding among some of them as clarified in Figure (1).

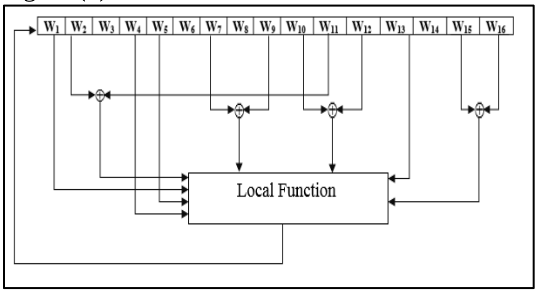

Figure (1): Local Key Generation

The Local Function that is called on one occasion per iteration, impacts the inner state to produce 512 bits of keystream per iteration. The input of Local Function is 512 bits (eight variable, the size of each variable is 32 bits). In Local Function, A network of summation mod 2^{32} and XOR are utilized for diffusion and it conclude of three procedures as clarified in algorithm (3). The 1st procedure of Local Function is Pre-confusion. The 2nd procedure of Local Function is combination (M and Q Functions) according to two s-box ($O_1$and O_2)as clarified in algorithm (4). The 3rd procedure of Local Function is Post-confusion.

Algorithm (3): Local key Function

Input: Sixteen 32-bit word (W1 to W16), n=number of keystream

Output: 512- bit (keystream)

Begin
Step1: iteration=1
Step2: // Map initialization
$R_1=W_1$ $R_2=W_7 \oplus W_9$ $R_3=W_4$
$R_4=W_{15} \oplus W_{16}$ $R_5=W_5$ $R_6=W_{13}$ $R_7=W_{10} \oplus W_{12}$ $R_8=W_2 \oplus W_{11}$
Step3: // Pre-confusion. 2^{32}= 4294967296
$R_1 = (R_1 + R_8) \text{ Mod } 2^{32}$ $R_2=R_2 \oplus R_7$
$R_3 = (R_3 + R_6) \text{ Mod } 2^{32}$ $R_4=R_4 \oplus R_5$
$R_5 = (R_5 + R_1) \text{ Mod } 2^{32}$ $R_6=R_6 \oplus R_2$
$R_7= (R_7 + R_3) \text{ Mod } 2^{32}$ $R_8=R_8 \oplus R_4$
Step4:// combination functions = M_1, M_2, M_3,M_4,Q_1, Q_2 ,Q_3 Q_4
$R_1 = R_1 \oplus M_1(R_2)$ $R_2 = R_2 \oplus Q_3(R_5)$
$R_3 = R_3 \oplus M_2(R_8)$ $R_4 = R_4 \oplus Q_4(R_6)$
$R_5 = R_5 \oplus M_3(R_3)$ $R_6 = R_6 \oplus Q_1(R_7)$
$R_7 = R_7 \oplus M_4(R_4)$ $R_8= R_8 \oplus Q_2(R_1)$
Step5: // Post-confusion
$R_1=R_1 \oplus R_6$ $R_2 = (R_2 + R_4) \text{ Mod } 2^{32}$
$R_3=R_3 \oplus R_5$ $R_4 = (R_4 + R_8) \text{ Mod } 2^{32}$
$R_5=R_5 \oplus R_2$ $R_6= (R_6 + R_4) \text{ Mod } 2^{32}$
$R_7=R_7 \oplus R_1$ $R_8= (R_8 + R_6) \text{ Mod } 2^{32}$
Step6: //Update inner state
$W_i= R_i$ $1 \le i \le 8$
$W_i= R_{i-8}$ of Step_2 $9 \le i \le 16$
Store the update sixteen 32-bit words called W as keystream
iteration = iteration +1
Step6: if (iteration less than or equal to n) Goto Step_2
End

Algorithm (4): M and Q Functions

Input: 32-bit , N=number of M and Q function,O1= 32 × 32 and O2= 32 × 32

Output: Update 32-bit

Begin
Step1: Break 32 –bit to four variable (y_1,y_2,y_3,y_4)
Step2:
if (N==1) then
$M_1(y) = O_1(y_1) \oplus O_1(y_2) \oplus O_1(y_3) \oplus O_2(y_4)$
$Q_1(y) = O_2(y_1) \oplus O_2(y_2) \oplus O_2(y_3) \oplus O_1(y_4)$
Else if (N==2) then
$M_2(y) = O_1(y_1) \oplus O_1(y_2) \oplus O_2(y_3) \oplus O_1(y_4)$
$Q_2(y) = O_2(y_1) \oplus O_2(y_2) \oplus O_1(y_3) \oplus O_2(y_4)$
Else if (N==3) then
$M_3(y) = O_1(y_1) \oplus O_2(y_2) \oplus O_1(y_3) \oplus O_1(y_4)$
$Q_3(y) = O_2(y_1) \oplus O_1(y_2) \oplus O_2(y_3) \oplus O_2(y_4)$
Else
$M_4(y) = O_2(y_1) \oplus O_1(y_2) \oplus O_1(y_3) \oplus O_1(y_4)$
$Q_4(y) = O_1(y_1) \oplus O_2(y_2) \oplus O_2(y_3) \oplus O_2(y_4)$
End if
End

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3.2 Fitness Function of Fireflies

For each firefly, the fitness function is utilize to measure the robustness of keystream as brightness (Light Intensity) which is computed by testing the five standard criteria of generated keystream as interpreted in Table (1). If the generated keystream is exceeding all five standard criteria therefore robustness of Firefly (keystream) is five and if the generated keystream is exceeding four of standard criteria therefore robustness of Firefly is four and so on.

3.3 Moving of Fireflies

The Firefly Algorithm initiates with an elementary population randomly according to Local Key Generation consequently the fireflies dispersed in state space. If the robustness of keystream (lighting) of Firefly (A) less than the firefly(B) consequently Firefly (A) move in the direction of the Firefly (B) by swapping the keys in the keystream of the Firefly (A). The range of swapping keys in the keystream of Firefly (A) is determined randomly from one to Hamming Distance between Firefly (A) and Firefly (B) as depicted by Eq2. Indeed, the swapping keys in the keystream creates modern diffusions (permutations). For example, assume the fitness (robustness of keystream) of Firefly (A) and Firefly (B) are 2 and 4 respectively, and the Hamming Distance between Firefly (A) and Firefly (B) is 5. Consequently, Firefly (A) move in the direction of the Firefly (B) by swapping keys in the keystream of the Firefly (A). The range of swapping keys in the keystream of Firefly (A) is randomly selected between 1 and 5 (Hamming Distance). Assume the selected the range of swap is 3 that is led to making at most three swap of keys (different swap locations) in the keystream of Firefly (A).

Table (1): Five Standard Criteria Equations Information [23]

Five Standard Criteria Equations	**Information on Five Standard Criteria**
$T1 = \frac{(M0 - M1)^2}{M}$	M0: number of 0's in keystream. M1: number of 1's in keystream. M: total size of keystream.
$T2 = \frac{4}{M-1}\left((M11)^2 + (M00)^2 + (M01)^2 + (M10)^2\right) - \frac{2}{M}(M1^2 + M0^2) + 1$	M11: number of 11's in keystream. M00: number of 00's in keystream. M01: number of 01's in keystream. M10: number of 10's in keystream
$T3 = \frac{2^N}{P}\left(\sum_{j=1}^{2^N} {M_j}^2\right) - P$	M_j: number of appearance of the j[th] of length N $P = \frac{M}{N}$ $\frac{M}{N} \geq (5 * 2^N)$
$T4 = \left(\sum_{j=1}^{N} \frac{(B_j - P_j)^2}{P_j}\right) + \left(\sum_{j=1}^{N} \frac{(G_j - P_j)^2}{P_j}\right)$	N :maximum j for which $P_j \geq 5$. B_j: Amount of blocks (subsequences runs of 1's) of length j in M. G_j: amount of gabs (subsequences runs of 0's) of length j in M. $P_j = \frac{M - j + 3}{2^{(j+2)}}$
$T5 = \frac{2\left(A(k) - \frac{(M-k)}{2}\right)}{\sqrt{(M-k)}}$	k : $1 \leq k \leq [m/2]$ $A(k) = \sum_{j=0}^{M-k-1} (S_j + S_{+k}) \text{ Mod } 2$

4. Case Study of Suggested Keystream Generator

Suppose, the number of firefly population is two and max- iteration is one. The following steps illustrated the suggested keystream generator according to Algorithm (2), Algorithm (3) and Algorithm (4).

- S_1: Suppose the seed of two fireflies:

Firefly(A)→"5d413d691dd67b3449bf371ac15968afeebe6361f83cf5c359ae7fc83178ab8ce0157eba9a38655da4abe24f359f664ad6036972c002764f3373894d49a6df77"

Firefly(B)→"27d591c45f02e8c6d0624a6a20d9b4bb5f59a6e8e497a2e235feb4ad5d45ec8aca348c8f24a269ae9c608d099a652cd973f9098e69360fab3c31880567ca82e4".

Each one of fireflies is represented by 128 hexa-number that is equal to 512 bits.

- S_2: At first, each firefly is splitted to sixteen 8-hexa-number (W_1 to W_{16}) ,for instance Firefly(A) is splitted to (W_1= "5d413d69", W_2="1dd67b34",…,W_{16}= "49a6df77").
- S_3: For Map initialization of keystream for Local key Function utilized according to **Step**$_2$ of Algorithm (3), R_1= W_1→ R_1= "5d413d69" → R_2= $W_7 \oplus W_9$ → R_2="59ae7fc8" ⊕ "e0157eba"="b9bb0172" and so.
- S_4: For Pre-confusion of keystream for Local key Function utilized according to **Step**$_3$ of Algorithm (3), $R_1 = (R_1 + R_8)$ Mod 2^{32} →R_1="16bed6e4", R_2=$R_2 \oplus R_7$ → R2="161c0265"and so.
- S_5: For combination functions of keystream for Local key Function utilized according to **Step**$_4$ of Algorithm (3), $R_1 = R_1 \oplus M_1(R_2)$ At first, calculating M_1 (R_2) according to Algorithm (4),y=R_2. y will be splitted to y1,y2,y3and y4, considering as indexing of O_1; O_2 (two S-box in Appendix) so, $M_1(y) = O_1(y_1) \oplus O_1(y_2) \oplus O_1(y_3) \oplus O_2(y_4)$ → $M_1(R_2)$="5ddc0340" then make ⊕ with "16bed6e4"→ R_1 = "5bc6f670" and so on.
- S_6: For Post-confusion of keystream for Local key Function utilized according to **Step**$_5$ of Algorithm (3),

R_1=$R_1 \oplus R_6$ → R_1="b80d61b4" ⊕ "e3cb97c4" →R_1="b80d61b4" and so on.

- S_7: For Update inner state of keystream for Local key Function utilized according to **Step**$_6$ of Algorithm (3), $W_i= R_i$, $1 \le i \le 8$, $W_i= R_{i-8}$ of **Step**$_2$ of Algorithm (3), $9 \le i \le 16$, W1="b80d61b4",W2="a71ed0fa", and so on.
- S_8: The output of local key function of two fireflies:
  Firefly(A)→="b80d61b4a71ed0fa97c5bc4c202b6aebfa54589b03f702af5cf8116440967f365d413d691dd67b3449bf371ac15968afeebe6361f83cf5c359ae7fc83178ab8c"
  Firefly(B)→"b94b365494d6e6db29b3a3d6efa21ef6cda2534b5e42a326606cba30407549e227d591c45f02e8c6d0624a6a20d9b4bb5f59a6e8e497a2e235feb4ad5d45ec8a".
- S_9: for each firefly, calculating Light intensity According to five standard criteria that is illustrated in Table (1). The fitness function of firefly (A) is 5, i.e. firefly (A) is passing five standard criteria. However, the fitness function of firefly (B) is 3,i.e. firefly (B) is failing in serial test and run test.
- S_{10}: According to Step$_5$ of Algorithm (2), compute hamming distance between firefly (A) and firefly (B) by convert them to binary and count the matching number as maximum number of making swap operation on different locations.
- S_{11}: check fitness function of two firefly, already, firefly (A) is passing five standard criteria early then discovering firefly (B) is also passing five standard criteria too due to iterated swap processing. So the final result of two fireflies in binary form.
  firefly(A)="10111000000011010110000110110100101001110001111011010000111110101001011111000101101111000100110000100000001010110110101011101011111110100101010001011000100110110000001111110111000000101010111101011100111110000001000101100100010000000100101100111111110011011001011101010000001001111010110100100011101110101100111101100110100010010011011111100110111000110101100000010101100101101000101011111110111010111110011000110110000111111000000111100111101011100001101011001101011100111111111001000000110001011110001010101110001100".
  Firefly(B)="101111000100101110010110011101000000101001101011011100011010110110010100110110110101000110101010011001111110100010010111111111011011001001

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١٠٠٠١٠١٠٠٠٠١٠٠١١٠١٠٠١٠١١٠١٠٠١١١٠٠٠
٠١٠١١٠١٠١١٠٠١١٠٠٠٠٠١١٠٠١٠٠٠٠٠٠٠٠١١٠
١٠٠٠١٠١١١٠١٠٠١١١٠٠٠٠٠١٠٠٠١٠٠٠٠١١١٠١
٠١٠١٠٠١٠٠١١١١٠٠٠١٠٠٠١٠١٠١١١١١٠٠١١١
١٠٠١٠٠٠١١٠٠٠٠٠١٠١٠١٠١١١٠١٠٠٠٠٠٠٠١٠١١
١٠١٠٠٠١١١٠٠١١٠١١٠١٠٠٠٠٠٠١١٠٠٠١١٠١١٠
١٠١٠١٠١٠١٠١١٠٠١١٠٠٠٠٠١١٠١١٠٠١٠٠١١٠١
٠٠١٠١١١٠١١١١٠١١١١٠٠١١١١٠٠٠١٠٠٠١١١٠
١١١٠١٠٠٠٠١١١001001001011110100010111
0001000110101111111110101101001010110
10101110101000101111011001000101 0" .

The implementation of above case study of suggested keystream Generator for generating two fireflies as two Keystream as showing in Figure (2) utilizing JavaScript language

Figure (2): Implementation of sample Case Study

5. Results of Suggested Keystream Generator

This paper utilizing one of SI procedures called firefly with local key generation for generation keystream. Through implementation of suggested key generator, the procedure has several variables as illustrates in Table 2.

Table 2. Variables chosen of suggested keystream generator

Variables	Range
number of firefly population	2 to 1000
max- iteration	1 to 1000
Length of firefly	512 bits
Number of swap of keys in the keystream of firefly.	1 to Hamming Distance

If utilize only firefly algorithm for generation keys will be stuck in local optimal and some of generated keystream will be poor. While, the generated keystream of suggested FAbLKG has high randomness by passing the five-benchmark tests due to utilizing local key generation and firefly algorithm. Local key generation is prevent the firefly algorithm from stuck in local optimal and generated keystream

The suggested FAbLKG is implemented on various entire size of data utilizing JavaScript language. The FAbLKG performance is utilized on many keystream series of Firefly of length 512 bits. Each iteration, discovering the best keystream if it found then store it, when the robustness of keystream series of Firefly is five that means passing the five benchmark tests.
For instant, the result of one keystream series of Firefly that is passing the five standard criteria as interpreted in Table (3).

Table (3): Five Standard Criteria Performance

5 - benchmark Tests	Test Value	Threshold	Test value< Threshold
Frequency T1	0.007	3.841	pass
Serial T2	4.941	5.991	pass
Poker T3	11.813	24.995	pass
Runs T4	5.028	12.591	pass
Autocorrelation T5	0.531	1.96	pass

5. Conclusion

This paper suggests FAbLKG with robust infrastructure with more diffusion of the generated keystream, From examining the best keystream of rounds by utilizing FAbLKG and rely on randomness standard, it is simple to observe that the suggested FAbLKG have a huge keystreams solutions which are satisfied the three demands of benchmarks such as maximum correlation, robust randomness, huge complexity. Attacker want to 2^{512} prospective trails to breach of the generated keystream of FAbLKG, consequently in this situation, a brute-force attacking seems unwieldy step. FAbLKG is Word-established may be better appropriated to fast real-time apps than are bit-established linear stream ciphers, FAbLKG is offer numerous volumes of keystream per iteration than bit-established LFSRs.

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Appendix A

S1 and S2 are generated randomly as follows:

S1=[0xB2C420F8,0x3218BD4E,0x84590D94,0xD51D3A8C,0xA3AB3D24,0x2A339E3D,0xFEE67A
23,0xAF844391,0x17465609,0xA99AD0A1,0x05CA597B,0x6024A656,0x0BF05203,0x8F559DDC,0
xEE171139,0x99B9E75F,0x8829A7ED,0x2C511CA9,0xD89BF75,0xF2F8CDD0,0x2DA2C498,0x48
314C42,0x922D9AF6,0xAA6CE00C,0xAC66E078,0x7D4CB0C0,0x5500C6E8,0x23E4576B,0x6B36
5D40,0x88C28D4A,0xCA6A8992,0xB40726AB,0x508C65BC,0xBE87B3B9,0x4A894942,0x9AEEC
C5B,0x6CA6F10B,0x303F8934,0xD7A8693A,0x7C8A16E4,0xB8CF0AC9,0xAD14B784,0x819FF9F
0,0x336BE860,0x5DBEEEFE,0x0E945776,0xD4D52CC4,0x0E9BB490,0x376EB6FD,0x6D891655,0
xD4078FEE,0xE07401E7,0xA1E4350C,0xABC78246,0x73409C02,0x24704A1F,0x478ABB2C,0xA0
849634,0x9E9E5FEB,0x77363D8D,0xD350BC21,0x876E1BB5,0xC8F55C9D,0xD112F39F,0xDF1A
0245,0x9711B3F0,0xA3534F64,0x42FB629E,0x15EAD26A,0xD1CFA296,0x7B445FEE,0xB8FC2E0
D,0x80168E15,0x0D7DEC9D,0xC5581F55,0xBE4A2783,0xD27012FE,0x53EA81CA,0xEBAA07D2
,0x54F5D41D,0xABB26FA6,0x41B9EAD9,0xA48174C7,0x1F3026F0,0xEFBADD8E,0x387E9014,0
x1505AB79,0xEADF0DF7,0x67755401,0xDA2EF962,0x41670B0E,0x0E8642F2,0xCE486070,0xA4
7D3312,0x4D7343A7,0xECDA58D0,0x1F79D536,0xD362576B,0x9D3A6023,0xC795A610,0xAE4D
F639,0x60C0B14E,0xC6DD8E02,0xBDE93F4E,0xB7C3B0FF,0x2BE6BCAD,0xE4B3FDFD,0x7989
7325,0x3038798B,0x08AE6353,0x7D1D20EB,0x3B208D21,0xD0D6D104,0xC5244327,0xF20DCDF
A,0xB7CB7159,0x58F3199F,0x9855E43B,0x1DF6C2D6,0x46114185,0xE46F5D0F,0xAAC70B5B,0
x48590537,0x0FD77B28,0x67D16C70,0x75AE53F4,0xF7BFECA1,0x6017B2D2,0xD8A0FA28,0x98
93F59F,0xE976832A,0xB1EB320B,0xA409D915,0x7EC6B543,0x66E54F98,0x5FF805DC,0x599B2
23F,0xAD78B682,0x2CF5C6E8,0x4FC71D63,0x08F8FED1,0x81C3C49A,0xE4D0A778,0xB5D369C
C,0x2DA336BE,0x76BC87CB,0x957A1878,0xFA136FBA,0x894A1911,0x909F21B4,0x6A7B63CE,
0xE28DD7E7,0x4178AA3D,0x4346A7AA,0xA1845E4C,0x166735F4,0x639CA159,0x58940419,0x4
E4F177A,0xD17959B2,0x12AA6FFD,0x1D39A8BE,0x7667F5AC,0xED0CE165,0xF1658FD8,0x28
B04E02,0x8F3C0E7B,0x7A1FF157,0x598324AE,0xFFBAAC22,0xD67DE9E6,0x3EB52897,0x4E07
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ADDAC,0x141E9B1C,0x8B937233,0x9B59DCA7];

تصميم مولد مفاتيح انسيابي بواسطة خوارزمية اليراعة

محمد صالح مهدي
جامعة تكنولوجيا المعلومات والاتصالات
كلية معلومات الاعمال

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كلية علوم الحاسوب

المستخلص :

التشفير الانسيابي هو احدى طرق التشفير للبيانات المرسلة عبر الانترنت. التشفير الانسيابي ملائم في الاتصالات وتطبيقات الوقت الفعلي. ان قوة قياس التشفير الانسيابي يعتمد على مدى عشوائية مولد المفاتيح المستخدم. اذا كانت عشوائية المفتاح الناتجة من مولد المفاتيح ضعيفة ، فان مفاتيح التشفير الانسيابي ممكن قراءتها والبيانات المشفرة بواسطة التشفير الانسيابي تكون غير محصنه للمهاجمين. هذا البحث يقترح اعتماد خوارزمية اليراعة (ذُبَاب يَطِير بِاللَّيْل يُضِيء ذَنَبُهُ) ودالة مفاتيح محلية لتوليد المفاتيح. المفاتيح المتولدة تكون مستقلة من النصوص الاصلية. ان نتيجة سلسلة المفاتيح من اليراع نجحت في الاختبارات القياسية الخمسة. ان مولد المفاتيح المقترح هو مبني على مفهوم الكتلة حيث يكون أفضل استخدامه للتطبيقات الوقت الفعلي بدلا من بناءة على مفهوم البت. من ناحية أخرى، ان مولد المفاتيح المقترح يحقق المتطلبات الثلاثية القياسية من ارتباط عالي، عشوائية قوية وتعقيد عالي.

الكلمات المفتاحية: التشفير الانسيابي، مولد مفاتيح، خوارزمية اليراعة، دالة مفاتيح محلي .

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Forecast the exchange rate of the Iraqi dinar against the US dollar using different versions of GARCH models.

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Abstract

There are many time series that are characterized by large variability, which makes them suffer from the problem of heterogeneity of contrast clearly, Where the time series analysis requires the homogeneity of variance for this purpose was the study and review of some important models used in dealing with the time series heterogeneous in contrast, a GARCH, ARMA-GARCH,TGARCH, EGACH , When the distribution of errors follows the normal distribution which was discovered by Engle since 1982, The aim of this study was to forecast . This study aimed at forecasting the exchange rates of the Iraqi dinar against the US dollar for the period from 2010 to 2018 through an analysis of fluctuations in the exchange rate series. The application of the studied data showed that the best model for predicting volatility is ARMA (0-1) -GARCH 2.1) based on some criteria for selecting the AIC, SIC, H-QIC and the significance of the estimated model parameters.

Keywords: heterogeneity of variance, yield chain, GARCH, EGARCH, TGARCH, ARMA-GARCH, AIC, SIC, H-QIC

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Review of Literature

The first model was proposed by the world (Robert F. Engle) in 1982 when he studied the variability of monetary inflation in the UK. It has been found that the proposed model is more accurate in prediction.

In 1986, Bollerslev presented a more generalized model called Generalized Autoregressive Conditional Heteroscedasticity, which became known as the GARCH model. It includes the conditional variation in the model when the error follows normal distribution as well. Determine the terms of the stability and self-correlation of this model.

In 1987, the researchers (Engle, RF) proposed the GARCH-in-Mean (GARCH-M) model, since this model adds the term heteroscedasticity to the arithmetic mean equation, which allows the average revenue to be defined as a linear function of time.

In 1991 Nelson presented the EGARCH model with an error following the distribution of the general error and used it to estimate the exceptional risk in the weighted value index market for 1962-1978 and concluded that the EGARCH model was better than the GARCH model.

In 1994, Zakoian first proposed the Threshold GARCH (TGARCH) model, which allows the asymmetric effects of positive and negative shocks to fluctuate and enables it to be applied to the pattern of volatility in stock returns in the French market.

In 2008, the researcher, Abdullah Suhail star, the study included models (ARCH and GARCH) and the use of method (Conditional MLE) to estimate and then studied the predictions of subsequent views you and use the style simulation to generate data and calculate a general formula to calculate the torque coefficient splaying models GARCH and ARCH from the lower grades.

In 2011, the researcher Mohammed, Mohamed Jassim, studied the use of GARCH models to predict the Saudi stock market index. He studied the process of diagnosis and estimation of the appropriate model and found that the best model for data representation is GARCH (1,1) t-distribution errors.

1. Introduction:

Some researchers focus on time series topics because they are important in studying the behavior of different phenomena over specific time periods through their analysis and interpretation, The topics of the time series include many areas (medical, environmental, economic, etc.) ,The definition of the time series (it is a series of observations that are arranged according to time of occurrence) There are two types, the first is Discrete Time Series and the second is the continuous time series, The aim of the time series analysis is to obtain an accurate description of the features of the phenomenon from which the time series is produced, and to construct a model to explain the behavior of that phenomenon , and predict future observations of the phenomenon studied based on what happens in the past. The most important models applied to time series data are the ARMA models used in many different fields. To be able to use the ARMA model, there must be three conditions for the random error of the model:

$$i)\ E(\varepsilon_t) = 0$$
$$ii)\ V(\varepsilon_t) = E(\varepsilon_t^2) = \sigma^2$$
$$iii)\ E(\varepsilon_t \varepsilon_s) = 0 \quad for \quad t \neq s$$

In the event of a breach of those conditions for the existence of a particular factor may be externally or an emergency on time series must look for other models can adapt to those factors that led to the existence of differences in terms of this time series, and in particular in the time series on financial transactions.

2. Problem Search:

The problem for the search in the presence of fluctuations in the prices of a series of Iraqi dinar against the US dollar, which led to the instability of the US dollar and thus the use of regular ARMA models would get irrational future predictions. The plans based on these results are therefore useless.

3. The aim:

The aim is to build the best model for forecasting the Iraqi dinar price series versus the US dollar daily for the period from 2010 to 2018 by applying a number of different models that are used to predict in time series of volatility, including GARCH model, TGARCH model, ARMA-GARCH model, and EGARCH model.

4. Autoregressive Conditional Heteroscedasticity models ($ARCH_{(p)}$): [5][4]

It was the first model proposed by Robert Engle in 1982. The ARCH model is a return series with a conditional average and a conditional variation. The conditional mean of the return series μ_t is constant, the conditional variance of the return series is in the form of a model that contains an error limit and a non-stability equation, the equations of the ARCH model are as follows.

$y_t = \mu + x_t$ …… (1)

$x_t = \sigma_t * \varepsilon_t$ … … (2) , $\varepsilon_t \approx iidN(0,1$ and

$\sigma_t^2 = \Omega + \alpha_1 x_{t-1}^2 + \alpha_2 x_{t-2}^2 + \cdots + \alpha_j x_{t-p}^2$ …… (3)

Where σ_t^2 is the equation of volatility, which can be written in the formula below.

$\boldsymbol{\sigma_t^2 = \Omega + \sum_{j=1}^{p} \alpha_j x_{t-j}^2}$ **…… (4)**

Whereas $\Omega > 0$, $\alpha_j \geq 0$, $j = 1,2, \ldots, p$

and, Ω, α_j, represent the parameters of the model. The process is in the case of stability if and only if the total parameters of the Autoregressive are positive and less than one.

5.Generalized Autoregressive Conditional Heteroscedasticity Model ($GARCH_{(p,q)}$): [5][4]

GARCH models (p≥1) and (q≥1) can be defined as follows:

$\boldsymbol{y_t = \mu + x_t}$ **and ,** $\boldsymbol{\varepsilon_t \approx iidN(0,1)}$

$\boldsymbol{x_t = \sigma_t * \varepsilon_t}$

$\boldsymbol{\sigma_t^2 = \Omega + \alpha_1 x_{t-1}^2 + \alpha_2 x_{t-2}^2 + \cdots + \alpha_p x_{t-p}^2 + \beta_1 \sigma_{t-1}^2 + \beta_2 \sigma_{t-2}^2 + \cdots + \beta q\, \sigma_{t-q}^2}$ **….. (5)**

As the y_t series represents a stable return series and uncorrelated

and μ represents the average of the stable return series

And that they are independent ε_t series and similar distribution (independent identically distribution) and keep track of the standard normal distribution with mean 0 and variance 1.

And σ_t^2 is the equation of volatility, which can be written in the formula below.

$\sigma_t^2 = \Omega + \sum_{j=1}^{p} \alpha_j x_{t-j}^2 + \sum_{i=1}^{q} \beta_i \sigma_{t-i}^2$ ... (6)

whereas.

$\Omega > 0$, $\alpha_j \geq 0$, $j = 1,2, \ldots, p$, $\beta_i \geq 0$, $i = 1,2, \ldots, q$

And, Ω, αj, βi represent the parameters of the model.

6. Exponential Generalized Autoregressive Conditional Heteroscedastic Models (EGARCH) [1][4]

This model suggested by Nelson in (1991) and on the contrary, the classic model GARCH which assumes symmetry oscillations around the shock. As well as the positive constraint imposed on parameters, Because the EGARCH model describes the relationship between the previous values of the random error and the conditional variation logarithm, with no restrictions on transactions that ensure that there are no negative effects of conditional variation, which allows avoiding positive transaction constraints ($\boldsymbol{\beta_j}$ & $\boldsymbol{\alpha_i}$), As follows:

Let us have the EGARCH model of the class (p, q) (p ≥1) & (q ≥1). Therefore, this model can be written as follows:

$$\boldsymbol{y_t = \mu + x_t}$$

$$\boldsymbol{x_t = \sigma_t\ \varepsilon_t \qquad \varepsilon_t \sim iid\ N(0,1)}$$

$$\ln(\sigma_t^2) = \Omega + \sum_{j=1}^{q} \beta_j \ln(\sigma_{t-j}^2) + \sum_{i=1}^{p} \alpha_i \left\{ \left| \frac{x_{t-i}}{\sigma_{t-i}} \right| - \sqrt{\frac{2}{\pi}} \right\} + \lambda_i \frac{x_{t-i}}{\sigma_{t-i}} \quad ..(7)$$

or

$$log(\sigma_t^2) = \Omega + \sum_{i=1}^{p} \alpha_i\, g(Z_t) + \sum_{j=1}^{q} \beta_j \log(\sigma_{t-j}^2) \quad ...(8)$$

whereas

$$\mathbf{g(Z_t) = \theta Z_t + ?(|Z_t| - E(|Z_t|)} \quad \& \quad \mathbf{Z_t = x_t/\sigma_t}$$

$$\mathbf{E(x_t/\sigma_t) = E\left\{\frac{|x_{t-i}|}{\sigma_{t-i}}\right\} = \sqrt{\frac{2}{\pi}}}$$

(Ω) (&) j=1,2,…,q، i=1,2,…,p , β_j α_i

Represent the model parameters is not required to be positive, while Z_t may be a normal standard variable, or come from the generalized error distribution, and that the equation g (Z_t) allows the signal size Z_t to be discrete effects from fluctuations, , And that the z_t limits are positive if the g(z_t) is linear with parameters ($\boldsymbol{\theta}$ + λ) If z_t is negative, the g(z_t)is linearized by parameters ($\boldsymbol{\theta}$ - λ) This situation allows for asymmetry on the rise and fall in the share price, which in turn is very useful, especially in the context of bond pricing.

7. (Threshold Generalized Autoregressive Conditional Heteroscedastic Models) (TGARCH) [11],[1]

The idea behind the TGARCH model is that it is better to capture negative shocks because they have a greater impact on fluctuations than positive shocks. To be able to capture these movements, a model must be studied that allows this model to determine the conditional standard deviation by referring to the previous lag. Allow models (GARCH) with threshold ((TGARCH access to different functions depending on fluctuations in this signal and shock value.

The models can be defined (TGARCH) class (p, q) (P ≥1) & (q ≥1) the following formula:

$$y_t = \mu + x_t$$

$$x_t = \sigma_t \varepsilon_t \qquad \varepsilon_t \sim \text{iid } N(0,1)$$

$$\sigma_t^2 = \alpha_\circ + \sum_{i=1}^{p} (\alpha_i + \gamma_i d_{t-i}) x_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2 \quad \dots (9)$$

Where

$$d_{t-i} = \begin{cases} 1 & \text{if } x_{t-i} < 0 \quad \text{bad news} \\ 0 & \text{if } x_{t-i} \geq 0 \quad \text{good news} \end{cases}$$

As the ($\alpha_0 > 0$) and (i = 1,2, ..., p, j = 1,2, ..., q for $\gamma_i \geq 0$, $\beta_j \geq 0, \alpha_i \geq 0$) represent the model parameters (Parameters), (d_{ti}) variable placebo (Dummy variable).

Known as γ or variable balance financial leverage. In this model, good news ($x_{t-i} > 0$) and bad news ($x_{t-i} < 0$) have a different effect on conditional variation. Good news has an effect on α_i,

While the bad news affecting αi and γi. Thus, if it is large and positive γ, the negative shocks have a greater impact on σ_t^2 of positive shocks and that the abolition of positive transaction restrictions allow for taking into account the phenomenon of symmetry or asymmetry that characterize the fluctuations, and thus become a shock (x_{t-i}) The conditional variation depends on both volume and shock signal.

8. Model ARMA (n, m) - GARCH (p, q) [4], [6]:

We know that the ARMA (n, m) models have a conditional modulus of previous information that is not constant and the conditional variance of a fixed error. The GARCH (p, q) models have a conditional average of the previous information constant and the conditional variance of the error is not constant. If both conditional and conditional conditions depend on the past (not fixed), then the two models are combined with a model known as ARMA (n, m) - GARCH (p, q) where it becomes as follows:

$$y_t = \emptyset_0 + \sum_{i=1}^{n} \emptyset_i y_{t-i} + x_t - \sum_{j=1}^{m} ?_j x_{t-j} \quad \dots (10)$$

$$x_t = \sigma_t \varepsilon_t \quad , \qquad \varepsilon_t / F_{t-1} \sim \text{iid}(0,1)$$

$$\sigma_t^2 = \Omega + \sum_{i=1}^{p} \alpha_i x_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2$$

And y_t represents ARMA model (n, m), except that the error limit (white noise) x_t not be independent, but symmetric distribution, which is not linked to the process, so has the same function properties of autocorrelation (ACF) process independent error, as well as the y_t possesses the same function autocorrelation model ARMA (n, m) normal (symmetric error and independent distribution). Also, x_t^2 series will be subject to the model ARMA (p, q) error is weak.

9. Augmented Dickey-Fuller Test [5][8]:

Augmented Dickey Fuller test uses (ADF) to detect the presence of the root of the unit in univariate test any time series whether stable series or not, The ADF test has a regression in the first difference in the string against the string with the time offset (p).

Using the following equation:

$$\Delta y_t = \mu + \lambda_t + \emptyset y_{t-1} + \sum_{j=1}^{k} \delta_j \Delta y_{t-j} + x_t \qquad \dots (11)$$

As the y_t represents the time series to be tested, k number of offsets time, Δ represents the first differences in a series of return, x_t represents an error $x_t \sim$ iid (0, σ²), and ($\mu, \lambda, \delta_j, \emptyset$) Symbolizes the parameters of its appreciation. The hypothesis can be tested

$H_\circ : \emptyset = 0$ Series yield has unit root (Series yield is unstable)

$H_1 : \emptyset < 0$ A series yield does not has a unit root (Series yield is stable)

Using statistics:

$$\tau = \frac{\widehat{\emptyset}}{se(\widehat{\emptyset})} \qquad \dots \dots (12)$$

The null hypothesis is rejected if the t-statistic value is greater than the statistical value of t-statistic and vice versa.

10. Ljung - Box Test [10][7]

The researchers (Ljung & Box) proposed this test in 1978, To test random errors of the time series by calculating the autocorrelation coefficients of a series of displacements, the following hypothesis is tested:

$H_\circ : \rho_1 = \rho_2 = \cdots = \rho_k \ldots = \rho_m = 0$

; $k = 1, 2, \ldots, m$

$H_1 : \rho_k \neq 0$ for some values of k

Using statistics:

$$Q_{(m)} = n(n+2)\sum_{k=1}^{m}\frac{\hat{\rho}_k^2}{n-k} \sim \chi^2_{m-p} \quad \ldots\ldots(13)$$

And that each of:

n: Represents the sample size (number of views of the time series).

m: It represents the number of shifts to self-link.

p: Number of parameters estimated in the model

$\hat{\rho}_k^2$: Represent the capabilities of the self-correlation boxes of the model's residual series.

then for string $x_t = y_t - \mu$ and x_t^2.

If p-value ≥ 0.05 means not rejecting the hypothesis H_0, the errors $x_t = y_t - \mu$ are random (Identically Independent Distribution) and there is (no effect ARCH) or (heteroscedasticity), and vice versa.

11. Lagrange Multiplier (*ARCH – Test*):[7][8]

It was proposed by Engle in 1982 and is used to determine whether the errors follow the ARCH process or not, which is based on the estimation of the equation under study in the form of the smallest squares and then the estimation of errors and squares for previous periods. This means that we estimate the following equation:

$$x_t^2 = \Omega + \alpha_1 x_{t-1}^2 + \alpha_2 x_{t-2}^2 + \ldots + \alpha_p x_{t-p}^2 \quad \ldots(14)$$

$x_t = y_t - \mu$, To test (ARCH (P)) we calculate the product of the coefficient of determination resulting from this estimate used the sample size of any amount TR^2 , Which is followed by χ^2_p, Of the degree of freedom (p) under the premise of the nuisance that the errors are homogeneous (Conditional Homoscedasticity) The small values of R^2 mean that the errors of the previous periods do not affect the current error and therefore there is no trace of the ARCH effect.

$$\text{archtest} = T\,\hat{R}^2 \sim \chi^2_{(P)} \quad \ldots\ldots(15)$$

12. Estimation [5]:-

Can be used Maximum likelihood Method To estimate GARCH parameters (p, q) as follows:

$$f(x_t/F_{t-1}) = \frac{1}{\sqrt{2\pi\sigma_t^2}}\exp\left(-\frac{1}{2}\frac{x_t^2}{\sigma_t^2}\right) \quad \ldots.(16)$$

The natural logarithm (L) function of vector parameters $\vartheta = (\Omega_\circ, \alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q)'$

We can write as follows:

$$L(\vartheta) = \sum_{t=1}^{n} I_t(\vartheta) \quad \ldots\ldots \quad (17)$$

the conditional logarithm of the parameter vector ϑ is

$$I_t(\vartheta) = Ln\, f(x_t/F_{t-1})$$

$$I_t(\vartheta) = \frac{1}{2}Ln(2\pi) - \frac{1}{2}Ln(\sigma_t^2) - \frac{1}{2}\left(\frac{x_t^2}{\sigma_t^2}\right) \quad \ldots \quad (18)$$

The following derivatives are calculated:

$$\frac{\partial I_t}{\partial \vartheta} = \frac{\partial I_t}{\partial \sigma_t^2}\frac{\partial \sigma_t^2}{\partial \vartheta}$$

The logarithm of the conditional probability density function is derived for the variable y_t for Ω , α_i , β_j

13. Model selection criteria: [10]

To choose the best model among those proposed models for assessment and prediction of the studied data, developed choice of model data, which ideally criteria and selection of the most common model standards are:

I - Akaikes Information criterion (AIC) [10][5]

Akaike (1974) presented a standard of information known as (AIC) When the time series models in (L) are reconciled with the parameters of the time series data under consideration and to assess the suitability of those models, the AIC is calculated for each model and the model that gives the lowest value is selected. The AIC formula can be written as follows:

$$AIC = n\ln(\hat{\sigma}_e^2) + 2L \quad \ldots\ldots(19)$$

n: represents the sample size.

$\hat{\sigma}_e^2$: The variance of the model is calculated as follows:

$$\hat{\sigma}_e^2 = \frac{1}{n-L}\sum_{t=1}^{n}(y_t - \hat{y}_t)^2 \quad \ldots\ldots(20)$$

L: is the rank of the model.

II - Schwarz Information criterion (SIC): [7][5]

In 1978, Schwarz introduced a new standard known as the Schwarz standard.

$$SIC = n\, ln(\hat{\sigma}_e^2) + L \ln(n) \quad \ldots\ldots(21)$$

n: represents the sample size.

$\hat{\sigma}_e^2$: The variance of the model is calculated as follows:

$$\hat{\sigma}_e^2 = \frac{1}{n-L}\sum_{t=1}^{n}(y_t - \hat{y}_t)^2$$

L: is the rank of the model.

This criterion addressed the problem of over-estimation in the AIC standard, And make the Penalty of the additional parameters stronger than the penalty in the AIC standard). The penalty for this criterion is L (In) n. One of the advantages of the SIC is that it estimates the rank of a model consistently that p , q is less or equal (p_{max}, q_{max}, respectively). It is stated that with reference to the chosen AIC or SIC, $\hat{P}(SIC) \le \hat{P}(AIC)$ Remain constant even in cases of small samples. Therefore, the use of SIC results leads us to models with minimal parameters.

iii- H-Q Hannan- Quinn Criterion: [5][7]

The researchers Quinn and Hannan (1979) proposed a new criterion for determining the rank of the studied model called Hannan-Quinn Criterion (H-Q (h)) and its mathematical formula:

$$H - Q = \ln \hat{\sigma}_e^2 + 2L\, C\, ln\left(\frac{ln(n)}{n}\right) \quad , \quad C > 2 \quad \ldots(22)$$

As the second limit above decreases as quickly as possible at the stability of the rank due to the repeated logarithm.

14. Forecasting:

Prediction is one of the most important objectives of model construction in time series. It represents the last stage of time series analysis that can’t be accessed without passing all tests and diagnostic tests to validate the model used in prediction. The following is a forecast prediction of the GARCH model. In the same way for all models (EGARCH, GARCH-M)

The prediction of the GARCH model (p, q)) (where p = 1, q = 1, GARCH $_{(1.1)}$) is as follows:

$$\sigma^2{}_t = E\,(x^2{}_t\,|I_t) = \hat{\Omega} + \hat{\alpha}_1\; x^2{}_{t-1} + \hat{\beta}_1\,\sigma^2{}_{t-1}$$

Predicting one future value

$$\sigma^2{}_{t+1} = E(x^2{}_{t+1}|I_t) = \hat{\Omega} + \hat{\alpha}_1\, E(x^2{}_t|I_t) + \hat{\beta}_1\,\sigma^2{}_t$$

$$\sigma^2{}_{t+1} = \hat{\Omega} + \hat{\alpha}_1\,\sigma^2{}_t + \hat{\beta}_1\,\sigma^2{}_t$$

$$\sigma^2{}_{t+1} = \hat{\Omega} + (\hat{\alpha}_1 + \hat{\beta}_1)\sigma^2{}_t$$

Prediction of value L

$$\sigma^2{}_{t+l} = E(x^2{}_{t+l}|I_t) = \hat{\Omega} + \hat{\alpha}_1 E(x^2{}_{t+l-1}|I_t) + \hat{\beta}_1\, E\,(\sigma^2{}_{t+l-1}\,|I_t\,)$$

$$\hat{\Omega} + \hat{\alpha}_1\,\sigma^2{}_{t+l-1} + \hat{\beta}_1\,\sigma^2{}_{t+l-1} = \sigma^2{}_{t+l}$$

$$\sigma^2{}_{t+l} = \hat{\Omega} + (\hat{\alpha}_1 + \hat{\beta}_1)\sigma^2{}_{t+l-1}$$

Thus, the general formula for predicting GARCH (p, q) models is as follows:

$$\sigma^2{}_{t+l} = \hat{\Omega} + \sum_{i=1}^{p}\hat{\alpha}_i\sigma^2{}_{t+l-i} + \sum_{j=1}^{q}\hat{\beta}_j\sigma^2{}_{t+l-j}$$

15. Forecasting accuracy measures:

to measure prediction accuracy developed standards are called the chosen model prediction accuracy standards is the most important.

i-: Root Mean Square Error (RMSE) :- [3][11]

This criterion is defined as the square root of the squared difference between both the actual variance and the prediction variability σ_t^2, Due to the absence of significant real variation, the time series observations were used x_t^2.

Thus, the RMSE formula is given as follows:

$$RMSE = \sqrt{\frac{1}{n}\sum_{t=1}^{n}(x_t^2 - \widehat{\sigma^2}_t)^2} \quad ; \; t = 1,2,\ldots,n \quad \ldots.(23)$$

whereas

$\widehat{\sigma^2}_t$ **represents the estimated variance.**

x_t^2 **represents the actual contrast**.

ii-: Mean Absolute Error (MAE) [3][10]

This standard is defined as the absolute difference between actual and forecast variability, and the formula of the standard is given as follows:

$$MAE = \frac{1}{T}\sum_{t=1}^{T}\left|x_t^2 - \widehat{\sigma^2}_t\right| \quad \ldots(24)$$

16.Applied side

The exchange rate series of the Iraqi dinar against the US dollar was analyzed for the period from 9/4/2010 to 10/5-2018 on a daily basis except for non-trading days, and the views were 2864 daily. Y_t returns were calculated using the natural logarithm of the data according to the following equation:

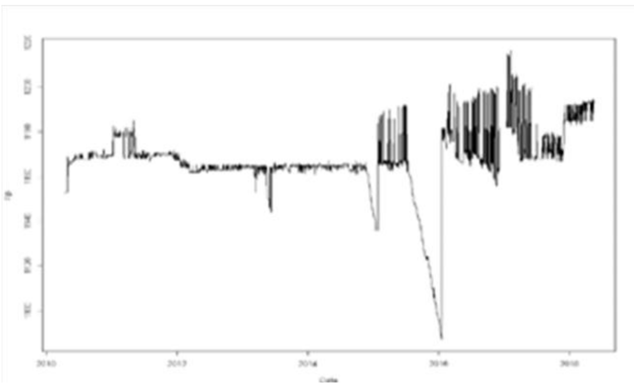
$$y_t = \ln(P_t) - \ln(P_{(t-1)}) \qquad ..... (25)$$

whereas:

P_t: is the exchange rate of the Iraqi dinar against the US dollar at the period t.

$P_{(t-1)}$: The exchange rate of the Iraqi dinar against the US dollar at the period t-1.

Figure (1) shows the format of the time series of the Iraqi dinar against the dollar.



Note from Figure (1) that the time series of the dinar exchange rate against the US dollar is unstable and has high volatility, indicating that there are fluctuations in the variance.

Dicky Fuller Developer's Series Test Exchange Rates for Iraqi Dinar against US Dollar:

For the purpose of detecting the stability of the time series of the exchange rate of the Iraqi dinar daily was calculated test Dicky Fuller developer and the test results as shown in Table (1)

Table (1) shows the developer's Dicky Fuller test to test the stability of the time series

Augmented Dickey-Fuller Test		
alternative: stationary		
Type 1: no drift no trend	Type 2: with drift no trend	Type 3: with drift and trend
Lag ADF p.value	Lag ADF p.value	Lag ADF p.value
[1,] 0 -0.0574 0.627	[1,] 0 - 12.72 0.01	[1,] 0 - 12.78 0.0100
[2,] 1 0.0658 0.663	[2,] 1 -8.05 0.01	[2,] 1 - 8.09 0.0100
[3,] 2 0.1420 0.685	[3,] 2 -6.17 0.01	[3,] 2 - 6.21 0.0100
[4,] 3 0.1754 0.694	[4,] 3 -5.52 0.01	[4,] 3 - 5.55 0.0100
[5,] 4 0.2259 0.709	[5,] 4 -4.67 0.01	[5,] 4 - 4.71 0.0100
[6,] 5 0.2596 0.719	[6,] 5 - 4.23 0.01	[6,] 5 - 4.27 0.0100
[7,] 6 0.2852 0.726	[7,] 6 -3.93 0.01	[7,] 6 - 3.96 0.0106
[8,] 7 0.2958 0.729	[8,] 7 -3.82 0.01	[8,] 7 - 3.85 0.0159
[9,] 8 0.2941 0.729	[9,] 8 - 3.85 0.01	[9,] 8 -3.88 0.0145

Table 1 shows that the values of p-value are greater than 5%, so we cannot reject the null hypothesis that there is a root alone in the time series, that is mean, the time series is unstable.

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Test the existence of a Autocorrelation between the errors of the series of the Iraqi dinar against the US dollar

Box-Ljung test according to Equation (13) The results as built in Table (2)

Table (2) shows the value of the Q-Stat test in the Iraqi dinar price series against the US dollar

Box-Ljung test	Box-Ljung test
data: residual	data: residual^2
X-squared = 59299, df = 40, p-value < 2.2e-16	X-squared = 57562, df = 40, p-value < 2.2e-16

As shown in table (2), the correlations of the self-correlations according to Q and p-value values less than 5% indicate that there is a significant correlation between the errors and the absence of homogeneity of the discrepancies of the observed series. U.S. dollar.

Homogeneity of variance test series returns

For the purpose of detecting the stability returns a string variation was calculated ARCH Test and referred to in the theoretical side of the equation (15) The test results as shown in Table 3.

Table 3 shows the test to see arch and determine the homogeneity of variances for a series of Iraqi dinar rate against the US dollar

ARCH LM-test; Null hypothesis: no ARCH effects
data: residual
Chi-squared = 2524.5, df = 12, p-value < 2.2e-16

From Table (3) we note that the value of p-value is less than 5% where we can't reject the null hypothesis that provides homogeneity of discrepancies of the original data series

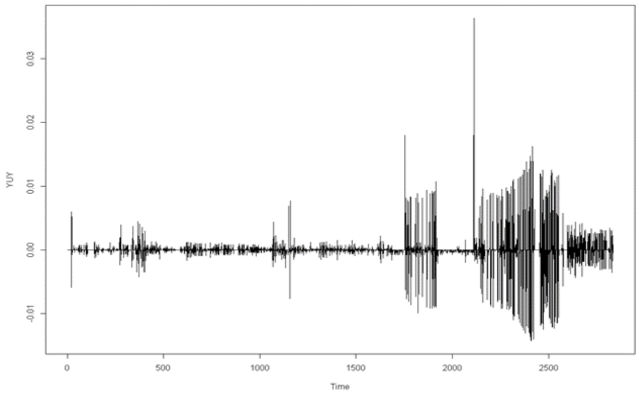
Series Returns: returns were awarded a series by taking the first difference of natural logarithm data series exchange rates of the Iraqi dinar against the US dollar and in the daily summary below some measures series returns.

Table 4 shows some statistics about the series returns

Min.;	1st Qu.;	Median;	Mean;	3rd Qu.	Max.
-1.432e-02	-9.324e-05	0.0	5.290e-06	0.00	3.639e-02
Jarque Bera Test					
data: return series					
X-squared = 73341, df = 2, p-value < 2.2e-16					

Is evident from the above indicators calculated that the smallest value in the revenue chain was (-1.432e-02) and the largest value was (3.639e-02) and that the average time series equal to 5.290e-06)), and calculable (Jarque-Bera) which indicates That these residues do not follow the law of natural distribution at a significant level (5%). Can be illustrated by the graph of the series returns, as shown in Figure (2)

Figure (2) shows the series of returns of the Iraqi dinar exchange rate against the dollar for the period 2010-2018



We note from the figure above that the string contains periods of volatility followed by periods of relative stagnation in the twists and turns, and so as we proceed in time.

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Table 5 shows Dickey Fuller test developer to test the stability of returns series

Augmented Dickey-Fuller Test alternative: stationary		
Type 1: no drift no trend	Type 2: with drift no trend	Type 3: with drift and trend
lag ADF p.value	lag ADF p.value	lag ADF p.value
0 -84.3 0.01	0 -84.3 0.01	0 -84.2 0.01
1 -58.5 0.01	1 -58.5 0.01	1 -58.5 0.01
2 -43.9 0.01	2 -43.9 0.01	2 -43.9 0.01
3 -39.7 0.01	3 -39.7 0.01	3 -39.7 0.01
4 -35.0 0.01	4 -35.0 0.01	4 -35.0 0.01
5 -31.3 0.01	5 -31.3 0.01	5 -31.3 0.01
6 -27.7 0.01	6 -27.7 0.01	6 -27.7 0.01
7 -24.4 0.01	7 -24.4 0.01	7 -24.4 0.01
8 -21.1 0.01	8 -21.1 0.01	8 -21.1 0.01

We note from the above table that the values of p-value less than 5%, which indicates the rejection of the null hypothesis, which states that the series returns are unstable and this avoids that the predictions are inaccurate to appear.
Table (6) below shows the test Q as well as the values of P-Value of less than 5%, indicating a significant self-correlation between the residuals, which confirms the existence of homogeneity of errors of the chain of returns shown by the box-Ljung test

Table 6 shows the effect of the test box-Ljung in a series of price returns of the dinar against the US dollar values

Box-Ljung test	Box-Ljung test
data: Residual	data: residual squared
□-squared = 784.17, df = 40, p-value < 2.2e-16	□-squared = 1653.9, df = 40, p-value < 2.2e-16

Homogeneity of variance test series returns

For the purpose of detecting the stability returns a string variation was calculated multiplier test Lagrange (ARCH Test) and referred to in the theoretical side of the equation (15) The test results as shown in Table 6.

Table (7) shows the arch test to determine and determine the homogeneity of the variances of the series of returns of the Iraqi dinar against the US dollar

Lagrange-Multiplier test:		
order	LM	p.value
[1,]	4 6436	0
[2,]	8 2096	0
[3,]	12 1341	0
[4,]	16 923	0
[5,]	20 730	0
[6,]	24 602	0

Table No. (7) Note that the values of p-value less than 5% for more than one shift where I reject the null hypothesis which states that the homogeneity of variances series returns and accept alternative hypothesis which provides an impact for ARCH in returns string data.

Estimation:

At this stage, the parameters of the studied models (GARCH, TGARCH, EGARCH, ARMA-GARCH) are estimated for the purpose of determining the best model for forecasting the Iraqi dinar price series against the US dollar using the greatest possible method.

Estimation of the GARCH model

MOLEL / Estimate	TGARCH(1,1)	TGARCH(1,2)	TGARCH(2,1)	TGARCH(2,2)
µ	-0.000019	-0.000015	-0.000013	-0.000027
Ω	0	0	0	0
α1	0.052333	0.051286	0.026413	0.025701
α2			0.024866	0.02506
ϒ1	0.032921	0.085427	-0.02668	-0.000957
ϒ2			0.056003	0.088272
β1	0.907894	0.442637	0.911187	0.440264
β2		0.438672		0.440877
AIC	-11.009	-11.014	-10.985	-11.003
BIC	-10.999	-11.001	-10.97	-10.987
H-QIC	-11.006	-11.01	-10.98	-10.997

By studying the functions of self-correlation and partial depending on the tests used in the diagnosis of the degree of the specimen described in the preceding paragraphs could be four models diagnosed as shown in Table 8 was used model GARCH was estimated models described parameters and calculating the criteria for selection of the specimen is best as shown in the table (8)

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Table (8) shows the studied GARCH models and the normal distribution of errors is classified according to the criterion of choice of the best model.

MOLEL Estimate	GARCH(1,1)	GARCH(1,2)	GARCH(2,1)	GARCH(2,2)
μ	-0.000015	-0.000014	-0.000009	0.000048
Ω	0	0	0	0
α1	0.067387	0.078758	0.031976	0.049032
α2			0.028076	0.033707
β1	0.912577	0.453834	0.916645	0.471176
β2		0.439405		0.430349
AIC	-11.025	-11.008	-10.986	-10.998
BIC	-11.017	-10.998	-10.975	-10.985
H-QIC	-11.022	-11.005	-10.982	-10.993

From Table (8) we find that the best model according to the selection criteria of AIC, SIC, H-QIC is GARCH (1.1). The estimated equation is:

$$y_t = -0.00002 + \sqrt{0.0674\, x_{t-1}^2 + 0.9126\, \sigma_{t-1}^2} * \varepsilon_t$$

Estimation of the TGARCH model

The TGARCH model was applied to the four sample models described above. Model parameters were estimated and the criteria for selecting the best model were calculated. As shown in Table 9.

Table (9) shows the TGARCH models studied and the normal distribution of errors is determined according to the criterion of choice of the best model.

From Table (9) we find that the best model studied in GARCH-M models will be TGARCH (1,2) according to AIC, SIC, H-QIC selection criteria.

$$y_t = -0.00002 + \sqrt{(0.0513 + 0.0854 d_{t-i})\, x_{t-i}^2 + 0.4426\sigma_{t-1}^2 + 0.4387\sigma_{t-2}^2} * \varepsilon_t$$

Estimation of the EGARCH model

Table (10) shows that the best model estimated in the EGARCH models applied according to the appropriate model selection criteria is EGARCH (2.2), which can be written as follows:

$$y_t = -0.000002 + \sqrt{e^{-1.1754+0.994\, ln(\sigma_{t-j}^2)+0.0241\left\{\left|\frac{x_{t-i}}{\sigma_{t-i}}\right|-\sqrt{\frac{2}{\pi}}\right\}-0.1197\left\{\left|\frac{x_{t-i}}{\sigma_{t-i}}\right|-\sqrt{\frac{2}{\pi}}\right\}+0.2611\frac{x_{t-1}}{\sigma_{t-1}}+0.3478\frac{x_{t-2}}{\sigma_{t-2}}}} * \varepsilon_t$$

Table (10) shows the studied EGARCH models, and the normal distribution of errors is classified according to the criterion of choice of the best model

MOLEL Estimate	EGARCH(1,1)	EGARCH(1,2)	EGARCH(2,1)	EGARCH(2,2)
μ	-0.000002	-0.00006	-0.000005	-0.000002
Ω	-1.194879	-0.059614	-1.171493	-1.175397
α1	0.061135	-0.204231	0.127602	0.024047
α2			-0.164868	-0.11967
λ1	0.494138	0.041791	0.271619	0.261054
λ2			0.28956	0.347751
β1	0.900554	1	0.900745	0.449306
β2		-0.006076		0.45087
AIC	-10.313	-10.256	-10.334	-10.476
BIC	-10.303	-10.243	-10.32	-10.459
H-QIC	-10.309	-10.251	-10.329	-10.47

Estimation of the ARMA-GARCH model

The ARMA-GARCH model has been applied to the scores of the four previously diagnosed models, model parameters have been estimated and the criteria for selecting the best model have been calculated. As shown in Table 11,

Table (11) shows the studied ARMA-GARCH models, which have the normal distribution of errors

MOLEL Estimate	ARMA(0,1)-GARCH(1,1)	ARMA(0,1)-GARCH(1,2)	ARMA(0,1)-GARCH(2,1)	ARMA(0,1)-GARCH(2,2)
ma	-0.637184	-0.638666	-0.63795	-0.6397
Ω	0	0	0	0
α1	0.050525	0.050367	0.025117	0.025
α2			0.025077	0.025
β1	0.901086	0.449921	0.900758	0.45
β2		0.450173		0.45
AIC	-10.975	-11.048	-10.957	-11.028
BIC	-10.967	-11.038	-10.947	-11.016
H-QIC	-10.972	-11.045	-10.954	-11.024

From Table (11) we observe that the best model chosen is the ARMA (0,1) -GARCH (1,2) model according to the criteria for selecting the best model and its estimated equivalent as shown below:

$$y_t = -0.6387\, x_{t-1} + \sqrt{(0.0504)x_{t-1}^2 + 0.45\sigma_{t-1}^2 + 0.45\, \sigma_{t-2}^2} * \varepsilon_t$$

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And to choose the best model for the series of returns of exchange rates of the Iraqi dinar against the US dollar for the period 2010 to 2018 by comparing the estimated models as shown in table (12)

Table (12) shows the best GARCH family models that have been adopted and the normal distribution of errors is classified according to the criterion of choice of the best model.

MOLEL / Estimate	GARCH (1,1)	TGARCH(1,2)	EGARCH (2,2)	ARMA (0,1)-GARCH (1,2)
μ	-0.000015	-0.000015	-0.000002	-0.638666
Ω	0	0	-1.175397	0
α1	0.067387	0.051286	0.024047	0.050367
α2			-0.11967	
ϒ1		0.085427	0.261054	
ϒ2			0.347751	
β1	0.912577	0.442637	0.449306	0.449921
β2		0.438672	0.45087	0.450173
AIC	-11.025	-11.014	-10.476	-11.048
BIC	-11.017	-11.001	-10.459	-11.038
H-QIC	-11.022	-11.01	-10.47	-11.045

From Table (11), the ARMA-GARCH model is superior to the other models. The best model of the proposed models is ARMA (0,1) -GARCH (1,2) according to the selection criteria of AIC, SIC, H-QIC

Check the appropriate model

After diagnosing the model, determining its grade and estimating its parameters for a series of price returns, the model's efficiency and accuracy in interpreting the behavior of the time series should be ascertained. This is done by the Test and Box-Ljung for standard and standard square

Table 13 shows the arch-test of the gears

Weighted ARCH LM Tests		
	Statistic	P-Value
ARCH Lag[4]	0.001832	0.9659
ARCH Lag[6]	0.003999	0.9999
ARCH Lag[8]	0.004658	1.0000

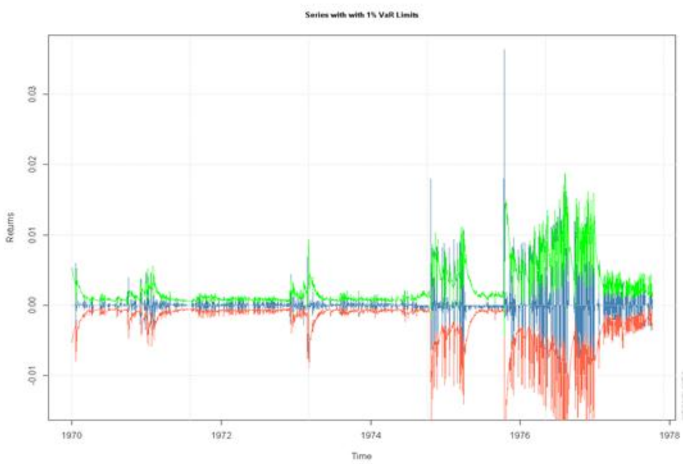
Note from the table above that the value of p-value is greater than 5%. We can't reject the null hypothesis that the errors are homogeneous

Table (14) shows the Box-Ljung test to detect random errors of standard random square error errors

Weighted Ljung-Box Test on Standardized Residuals		
	statistic	p-value
Lag[1]	10.25	1.365e-03
Lag[2*(p+q)+(p+q)-1][2]	10.25	6.829e-10
Lag[4*(p+q)+(p+q)-1][5]	11.48	2.434e-04
d.o.f=1		
H0 : No serial correlation		
Weighted Ljung-Box Test on Standardized Squared Residuals		
	statistic	p-value
Lag[1]	0.001033	0.9744
Lag[2*(p+q)+(p+q)-1][8]	0.005800	1.0000
Lag[4*(p+q)+(p+q)-1][14]	0.010925	1.0000
d.o.f=3		

Note that the standard errors in the displacements (1,4,6) mean that the standard errors are not normal distributed. Note that the standard error boxes are not significant in the displacements (1,4,6). This means that the standard errors are distributed naturally

Figure 3 illustrates the prediction of the values of the series returns and confidence limits for those predictions and the estimated variation



We note from Figure (3) confidence limits for the values of the real and predictive values of the series returns, as well as the estimated variance.

Table (15) shows a comparison between the ARMA-GARCH and GARCH model according to the criterion of choice of the best model

MODEL	RMSE	MAE
ARMA (0,1)-GARCH (2,1)	0.0006355353	0.0003258179
GARCH (1,١)	0.0006560953	0.0002865495

We note from the above table as well as superior specimen ARMA-GARCH on the specimen by GARCH standards (MAE, RMS,) and this in turn suggests that the specimen ARMA (0,1) - GARCH (2,1) flour to a large extent and is therefore the best specimen to predict prices Exchange of the Iraqi dinar against the US dollar. The GARCH (1,1) model was chosen for comparison because it is very close to the ARMA model (0.1) -GARCH (2.1) according to the criteria for model selection (AIC, SIC, H-QIC).

Conclusions and recommendations

Conclusions

1 - The exchange rate of the Iraqi dinar against the US dollar was unstable in the middle and contrast.
2. The series returns have turned stable chain in the center by Dickey Fuller test.
3. The series returns are unstable by arch test contains a serial link (moral links).
4. The best model is the ARMA (0-1) -GARCH (2.1) model, which is superior to the other models studied according to AIC, SIC, H-Q
5. The selected model is superior to RMSE, MAE, compared with GARCH (1,1).
6. The models of autoregression conditional on heterogeneity of variance are more efficient in predicting fluctuations.

Recommendations

1. Use other models for comparison such as models, GARCH-M IGARCH, NGARCH
2. Use other methods to estimate model parameters such as QMLE.
3. Using the GARCH family models to predict other financial time series to estimate and study the behavior of these strings because they have the ability to explain the behavior of these strings characterized by heterogeneity of variance.

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التنبؤ بأسعار سعر الصرف للدينار العراقي مقابل الدولار الامريكي باستعمال صيغ مختلفة من نماذج GARCH.

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المستخلص:

هناك العديد من السلاسل الزمنية التي تتصف بتقلباتها الكبيرة مما يجعلها تعاني من مشكله عدم تجانس التباين بشكل واضح، حيث ان تحليل السلاسل الزمنية تشترط تجانس التباين ولهذا الغرض تم دراسة واستعراض بعض النماذج المهمة المستخدمة في التعامل مع السلاسل الزمنية غير المتجانسة في التباين وهي

GARCH, TGARCH, EGACH, ARMA-GARCH عندما يتبع توزيع الأخطاء التوزيع الطبيعي والتي اكتشفت من قبل Engle منذ عام ١٩٨٢, وقد هدفت هذه الدراسة الى التنبؤ بأسعار الصرف للدينار العراقي مقابل الدولار الامريكي للفترة من ٢٠١٠ ولغاية ٢٠١٨ من خلال تحليل بالتقلبات لسلسلة عوائد أسعار الصرف حيث تبين من خلال التطبيق على البيانات المدروسة ان افضل انموذج للتنبؤ بالتقلبات هو ARMA (0,1)-GARCH (2,1) بالاعتماد على بعض معايير اختيار الانموذج AIC, SIC, H-QIC ومعنوية معلمات الانموذج المقدر .

قواعــد النشــر

- تعنى مجلة القادسية لعلوم الحاسوب والرياضيات بنشر البحوث العلمية الرصينة ذات العلاقة بعلوم الحاسبات ،الرياضيات ،الإحصاء والمعلوماتية والفيزياء الحاسوبية والتي لم تنشر أو تقدم للنشر سابقا .
- تخضع البحوث المقدمة للتقييم العلمي من لدن اختصاصين من داخل القطر وخارجه .
- يقدم البحث مطبوعا بنظام العمودين على ورق ابيض جيد قياس (A4) وبمسافة مضاعفة وبنظام الـ word حصرا وان يكون نظام office 2010 وان يكون حجم الخط المستخدم في طباعة البحث (١٠) ونوعه Time New Roman ماعدا العنوان واسم الباحث يكون حجم الخط (١٢)bold ونوعه Time New Roman ، أما الجداول والإشكال فيكون الخط bold ونوعه Time New Roman وعند وجود المعادلات في البحث يجب إضافتها باستخدام محرر المعادلات .
- على الباحث (أو الباحثيين) تقديم ملخص لبحثه باللغتين العربية والانكليزية يتضمن عنوان البحث واسم الباحث أو الباحثيين وعناوينهم بحدود (١٥٠-٢٠٠) كلمة .
- على الباحث (أو الباحثيين) ادراج البريد الالكتروني ويفضل ان يكون بريد رسمي .
- استخدام الباحث (أو الباحثيين) ذات البيانات الخاصة به (اسم الباحث ، المرتبة العلمية ، جهة الانتساب ، البريد الالكتروني الرسمي) والمستخدمة في بحوثه السابقة .
- يرتب البحث كما يأتي الخلاصة ، المقدمة ، المواد وطرائق العمل ، النتائج والمناقشة ، الخلاصة باللغة الثانية تتضمن عنوان البحث، اسم الباحث ومكان عمله .
- يتم ذكر المصادر في البحث بإتباع أسلوب الترقيم حسب أسبقية ذكر المصدر وتذكر المصادر في النهاية على الوجه الأتي :
  اسم الباحث (أو الباحثين) عنوان البحث اسم المجلة ، المجلد ، العدد ، رقم صفحتي بدء وانتهاء البحث ، سنة النشر بين قوسين .
- تنشر البحوث باللغة الانكليزية فقط وان يقدم الباحث أربع نسخ من البحث (ورقية + اقراص CD) .
- بعد الانتهاء من عملية التقييم والتصويبات وعند القبول النهائي يقدم البحث على قرص CD (office 2010 + pdf) مع نسخة ورقية نهائية .
- أن لا تزيد صفحات البحث المقدم للنشر عن عشر صفحات وبنظام العمودين وفي حالة تجاوز عدد صفحات البحث اكثر من ذلك يتم دفع خمسة الاف دينار عراقي لكل صفحة زيادة وان لايتجاوز العدد الاجمالي للبحث ٢٠ صفحة .
- تعتمد المجلة تصنيف (Mathematics Subject Classificatio) في نشرها للبحوث العلمية .
- يقدم الباحث التصنيف المعتمد في المجلة لموضوع البحث .
- اجور التقييم والنشر للمجلة كالاتي :
  اولا :- اجور التقييم (٣٠٠٠٠) الف دينار عراقي .
  ثانيا :- اجور النشر حسب اللقب العلمي للباحث وكالاتي :
  ١- المدرس المساعد والمدرس (٥٠٠٠٠) الف دينار عراقي .
  ٢- الاستاذ المساعد (٧٥٠٠٠) الف دينار عراقي .
  ٣- الاستاذ (١٠٠٠٠٠) الف دينار عراقي .

ملاحظة : عند تقديم البحث يدفع الباحث مبلغ (٣٥٠٠٠) الف دينار عراقي غير قابل للرد وفي حالة قبول نشر بحثه في المجلة يدفع بقية الاجور حسب لقبه العلمي .
كما ويدفع مبلغ (١٠٠٠٠) الاف دينار عراقي غير قابل للرد اجور استلال ، وفي حالة اعادة فحص الاستلال للبحث مرة اخرى يعاد دفع المبلغ (١٠٠٠٠) كأجور اعادة استلال .

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