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T-essentially Quasi-Dedekind modules

Farhan Dakhil Shyaa
Department of Mathematics
College of Education
University of Al-Qadsiyah
farhan.shyaa@qu.edu.iq

Shukur Neamah Al-aeashi
Department Of Urban Planning
College of Physical Planning
University Of Kufa
shukur.mobred@uokufa.edu.iq

Inaam Mohammed Ali Hadi
Department of Mathematics
College of Education for Pure Sciences(Ibn-Al-Haitham)
University Of Baghdad
innam1976@yahoo.comq

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Abstract:
In this paper, we introduce and study type of modules namely (t-essentially quasi-Dedekind modules) which is generalization of quasi-Dedekind modules and essentially quasi-Dedekind module. Also, we introduce the class of t-essentially prime modules which contains the class of t-essentially quasi-Dedekind modules.

Keywords: quasi-Dedekind modules, essentially quasi-Dedekind modules, t-essentially quasi-Dedekind modules, essentially prime modules, t-essentially prime modules.

Mathematics Subject Classification: 2010:16 D10, 16D20, 16 D50.

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1. Introduction

Let R be a commutative ring with unity and M be a right R-module. A submodule N of M is called quasi-invertible if $Hom\left(\frac{M}{N},M\right)=0$ [10]. M is called quasi-Dedekind if every nonzero submodule N of M is quasi-invertible, that is $Hom\left(\frac{M}{N},M\right)=0$ for each nonzero submodule N of M. Equivalently M is quasi-Dedekind if for each $f\in End(M), f\neq 0$, then $Ker(f)=0$ [10]. As a generalization of quasi-Dedekind modules. Tha'ar in [14] introduced the concept essentially quasi-Dedekind (briefly, ess.q-Ded.) by restricting the definition of quasi-Dedekind on essential submodules, where a submodule N of M is called essential in M (denoted by $N\leq_{ess} M$) if $N\cap W\neq 0$ for each nonzero submodule W of M[7]. However, the concept essentially quasi-Dedekind is equivalently to k-nonsingular which is introduced by Roman C.S[12], that M is ess-q-Ded. Module if for each $f\in End(M), Ker(f)\leq_{ess} M$ implies $f=0$.

In [3] ″introduced the concept t-essential submodule, a submodule N of M is called t-essential submodule (denoted by $N\leq_{tes} M$) if $N\cap W\leq Z_2(M)$, then $W\leq Z_2(M)$, where $Z_2(M)$ is the second singular submodule of M and defined by $Z(\frac{M}{Z(M)})=\frac{Z_2(M)}{Z(M)}$, $Z(M)=\{m\in M{:}mI=0$ for some $I\leq_{ess} R\}$[7]. It is clear that $Z(M)=\{m\in M{:}\, ann(m)\leq_{ess} R\}$.Also,$Z_2(M)=\{m\in M{:}\, mI=0$ for some $I\leq_{tes} R\}=\{m\in M{:}\, ann(m)\leq_{tes} R\}$″. It is obvious; every essential submodule is t-essential, but not conversely.

In section two, we define t-essentially quasi-Dedekind module, where an R-module M is called t-essentially quasi-Dedekind if every nonzero t-essential submodule is quasi-invertible, that is $Hom\left(\frac{M}{N},M\right)=0$ for each $(0)\neq N\leq_{tes} M$.

Analogus characterization of ess.q-Ded. module we have . An R-module M is t-ess.q-Ded. if for each $f\in End(M), Ker(f)\leq_{tes} M\ implies\ f=0$. We study t-essentially quasi-Dedekind module. It is clear that every t-essentially quasi-Dedekind module is essentially qusi-Dedekind but not conversely (Remarks and Examples 2.2(2) and every quasi-Dedekind module is t-essentially quasi-Dedekind, but the converse may be not true (Remarks and Examples 2.2(4)). Also we see that every nonsingular module is t-essentially quasi-Dedekind(Remarks and Examples 2.2(3)).

The property of t-essentially quasi-Dedekind is inherited by direct summand (Proposition 2.3); however it is not inherited by direct sum. So we provide necessary and sufficient conditions for a direct sum of t-essentially quasi-Dedekind to be t-essentially quasi-Dedekind.

Beside these some connections between t-essentially quasi-Dedekind modules and other types of modules are investigated.

It is known that every quasi-Dedekind module M is a prime module (that is $annM=annN$ for each $(0)\neq N\leq M$) but the converse may be not true [11]. However implies that every prime modules is ess.q.Ded.. Also, every essentially quasi-Dedekind module M is essentially prime module (that is $annM=annN$ for each $N\leq_{ess} M$) and the converse is not true in general [14, Proposition 2.1.8]. We notice that every t-ess.q.Ded. module M implies $annM=annN$ for each $(0)\neq N\leq_{tes} M$, so this note lead us in section three to introduce and study the concept of t-essentially prime module (that is $annM=annN$ for each, $(0)\neq N\leq_{tes} M$). Thus for a module M, we have the following implications.

t-ess.q-Ded. $\Rightarrow$ t-ess.prime $\Rightarrow$ ess.prime.

But none of these implications is reversible(Remarks and Examples 3.3(2),(3)). The concepts essentially prime module and t-essentially prime module are equivalent, under certain conditions(Propositions 3.4,3.7). Also we have that for an R-module , with $annM = ann\bar{M}$($\bar{M}$ is the quasi-injective hull of M) then M is t-essentially prime if and only if $\bar{M}$ is t-essentially prime (Proposition 3.9). Beside these many other properties of t-essentially prime modules, also several connections between this type of modules and other modules are presented.

We list some known results, which will be needed for future use.

Proposition 1.1:[3, Proposition 2.2]. The following statements are equivalent for a submodule A of an R-module M:

(1) A is t-essential in M;

(2) $\frac{(A+Z_2(M)}{Z_2(M)}$ is essential in $\frac{M}{Z_2(M)}$;

(3) $(A + Z_2(M)$ is essential in M;

(4) $\frac{M}{A}$ is Z_2-torsion.

Remark 1.2: [2, Corollary 1.3] Let A_λ be a submodule of M_λ for each $\lambda \in \wedge$

(1) If $\wedge$ is a finite set and $A_\lambda \leq_{tes} M_\lambda$ then $\cap_{\lambda\in\wedge} A_\lambda \leq_{tes} \cap_{\lambda\in\wedge} M_\lambda$;

(2) $\oplus_{\lambda\in\wedge} A_\lambda \leq_{tes} \oplus_{\lambda\in\wedge} M_\lambda$ if and only if $A_\lambda \leq_{tes} M_\lambda$ for each $\lambda \in\wedge$.

Proposition 1.3: [2, Corollary1.2] Let $A \leq B \leq M$. Then $A \leq_{tes} M$ if and only if $A \leq_{tes} B$ and $B \leq_{tes} M$.

2. T-essentially Quasi-Dedekind modules

Definition 2.1: An R-module M is called t-essentially quasi-Dedekind (brifly t-ess.q.Ded.) if every nonzero t-essential submodule N of M is quasi-invertible, that is M is t-ess.q-Ded. if $Hom\left(\frac{M}{N}, M\right) = 0$ for all nonzero t-essential submodule N of M. A ring R is t-ess.q-Ded. if it is t-ess.q-Ded R-module.

Remarks and Examples 2.2:

(1) It is clear that every simple is t-ess.q-Ded. module.

(2) Every t-ess.q-Ded. module is ess.q-Ded. module, since every essential submodule is t-essential. However the converse may be not true, for example: Let $M = Q\oplus Z_2$ as Z-module. M is ess.q-Ded. let $N = Q\oplus(0)$. Then $N + Z_2(M) = (Q\oplus(0)) + ((0)\oplus Z_2) = Q\oplus Z_2 = M \leq_{ess} M$ and so by Proposition 1.1, $N \leq_{tes} M$. It follows that $Hom(\frac{M}{N}, M) \simeq Hom(Z_2, Q\oplus Z_2) \neq 0$ and hence M is not t-ess.q-Ded.

(3) Every nonsingular module is t-ess.q-Ded.

Proof: Let M be a nonsingular module. Then by [11, Proposition 3.13], every essential submodule is quasi-invertible. Hence every t-essential submodule is quasi-invertible by Remark 1.2, and so M is t-ess.q-Ded.. ☐

(4) It is obvious that every quasi-Dedekind is t-ess.q-Ded, but the converse is not true in general, for example: The Z-module $Z\oplus Z$ is nonsingular, so it is t-ess.q-Ded. (see part (3)), but M is not quasi-Dedekind since $Hom(\frac{M}{Z\oplus(0)}, M) \simeq Hom(Z, Z\oplus Z) \neq 0$.

Similarly each of the Z-module $Q\oplus Z, Q\oplus Q$ is t-ess.q-Ded., but not quasi-Ded.

(5) Let R be a ring. Then the following are equivalent:

(1) R is t-ess.q.-Ded.;

(2) R is ess. Q-Ded.

(3) $R$is a nonsingular(R is a semiprime)ring.

Proof: (1)⇒(2) It follows by Remarks and Examples 2.2(2).

(2)⇒(3) It follows by [14, Proposition 2.2.6]

(3)⇒(1) It follows by Remarks and Example 2.2(3). ☐

(6) For R-module M, $\frac{M}{C}$ is t-ess.q-Ded. for each t-closed submodule C of M, where a submodule C of M is called t-closed if C has no proper t-essential extension in M [3].

Proof: If C is a t-closed submodule, then by [3, Proposition 2.6] $\frac{M}{C}$ is nonsingular. Hence by Remarks and Examples 2.2(4), $\frac{M}{C}$ is t-ess.q-Ded. ☐

In particular, $\frac{M}{Z_2(M)}$ is t-ess.q-Ded. for any R-module M.

(7) Let M be a t-uniform module (that is, for submodule of M is t-essential[8] . Then M is t-ess.q-Ded. if and only if M is ess.q-Ded.

(8) A homomorphic image of t-ess.q-Ded. need not be a t-ess.q-Ded. for example : Z as a Z-module is t-ess.q-Ded. let $\pi: Z \mapsto \frac{Z}{<4>} \simeq Z_4$ be the natural projection, hence $\pi(Z) = Z_4$ is not t-ess.q-Ded. since $Hom(\frac{Z_4}{(\bar{2})}, Z_4) \neq 0$ and $(\bar{2}) \leq_{tes} Z_4$.

(9) Let M and M'be two isomorphic R-module. Then M is t-ess.q.Ded. if and only if M' is t-ess.q-Ded.

(10) If M is t-ess.q-Ded., then $annM = annN$ for each $N \leq_{tes} M$ and $N \neq 0$

Proof: Since M is t-ess.q-Ded., every $N \leq_{tes} M$, $N \neq 0$ is quasi-invertible submodule. Hence $annM = annN$ for each $0 \neq N \leq_{tes} M$ by [11] ☐

(11) Let M be an R-module such that $Z_2(M) \leq N$ for all $N \leq M$. Then M is t-ess.q.Ded. if nd only if M is ess.q-Ded.

Proof:⇒ It is clear.

⇐ Let $N \leq_{tes} M$. Then by Remark 1.2, $N + Z_2(M) \leq_{ess} M$, hence $N \leq_{ess} M$ (since $Z_2(M) \leq N$). As M is ess.q-Ded., thus $Hom\left(\frac{M}{N}, M\right) = 0$. ☐

The property of t-ess.q-Ded. is inherited by direct summand.

Proposition 2.3: A direct summand of t-ess.q-Ded. module M is t-ess.q-Ded.

Proof: Let N be a direct summand of $M (N \leq^{\oplus} M)$. To prove N is a t-ess.q.Ded. Let (0)≠ $K \leq_{tes} N$. As $N \leq^{\oplus} M, M = N \oplus W$, for some $W \leq M$. Since $K \leq_{tes} N$ and $W \leq_{tes} W$, then $K \oplus W \leq_{tes} N \oplus W = M$. By t-essentially quasi-Dedekind of M, $Hom\left(\frac{M}{K \oplus W}, M\right) = 0$; thus , $Hom\left(\frac{N}{K}, M\right) = 0$. Suppose , $Hom\left(\frac{N}{K}, N\right) \neq 0$ that is there exist $f: \frac{N}{K} \mapsto N, f \neq 0$. Hence $i \circ f: \frac{N}{K} \mapsto M, i \circ f \neq 0$, where i is the inclusion mapping. Thus $Hom\left(\frac{M}{K}, M\right) \neq 0$, which is a contradiction. It follows that $Hom\left(\frac{N}{K}, N\right) = 0$ and N is t-ess.q-Ded.

☐

Thaa'r in [14, Theorem1.2.3] an R-module is ess.q.Ded. if and only if M is K-nonsingular that is for each $f \in End(M)$ implies $f = 0$.

By similar proof of this result, we get the following.

Theorem 2.4: Let M be an R-module. Then M is t-ess. Q-Ded., if and only if for each $f \in End(M)$, $0 \neq Kerf \leq_{tes} M$ implies $f = 0$.

Note 2.5: Every semisimple module is ess.q-Ded. [14, Proposition 1.2.4]. However semisimple module may not t-ess. Q-Ded., since $Hom(\frac{Z_6}{<3>}, Z_6) \simeq Hom(Z_3, Z_6) \neq 0$ and $(\bar{3}) \leq_{tes} Z_6$(because $(\bar{3}) + Z_2(Z_6) = (\bar{3}) + Z_6 = Z_6 \leq_{ess} Z_6$).

"Asgari in [4] introduced t-semisimple module, where an R-module M is called t-semisimple if for each $N \leq M$, there exists $K \leq^{\oplus} M$ such that $K \leq_{tes} N$. It is clear that every semisimple is t-semisimple but the converse may be not true " [4].

Proposition 2.6: Let M be t-semisimple module and t-ess.q-Ded. module. Then t-closed submodule of M is t-ess.q-Ded.

Proof: Let N be t-closed submodule of M. Then by [3, Lemma 2.5(1)]$N \geq Z_2(M)$, and so[4, Theorem 2.3], N is direct summand . Thus by Proposition 2.3, N is a t-ess. Q-Ded. □

Corollary 2.7: Let R be a t-semisimple ring and t-ess.q-Ded.. Then R is semisimple.

Proof: Since R is t-ess. Q-Ded, R is nonsingular by Remarks and Examples 2.2(5). But R is nonsingular and t-semisimple ring implies R is semisimple. □

"Recall that a module M over a commutative ring R is called scalar module if for each $f \in End(M)$, there exists $0 \neq r \in R$ such that $f(x) = xr$ for each $x \in M$" [13].

" An R-module M is called quasi-prime if $ann(m)$ is a prime ideal of R, for each $m \neq 0$ and $m \in M$" [1].

Theorem 2.8: Let M be a scalar quasi-prime module. Then M is t-ess.q-Ded.

Proof: Let $f \in End(M)$ and suppose that $\neq 0$. Since m is a scalar module, there exists $0 \neq r \in R$ and $f(x) = xr$ for each $x \in M$. Assume $Ker(f) \leq_{tes} M$, hence $Ker(f) + Z_2(M) \leq_{ess} M$ by Proposition 1.1. So that for any $m \in M$, there exist $a \in R$ such that $0 \neq ma \in Ker(f) + Z_2(M)$. It follows that $ma = m_1 + m_2$ for some $m_1 \in Kerf, m_2 \in Z_2(M)$. Thus $f(ma) = mar = f(m_1) + f(m_2) = f(m_2) \in Z_2(M)$. If $mar = 0$, then $ar \in ann(m)$. But $ann(m)$ is a prime ideal of R since M is quasi-prime, so either $a \in ann(m)$ or $r \in ann(m)$. If $a \in ann(m)$, then $ma = 0$, which is a contradiction. If $r \in ann(m)$ then $mr = 0$ for each $m \in M$ and $Mr = f(M) = 0$ (that is $f = 0$) which is a contradiction. Thus $0 \neq mar \in Z_2(M)$ which implies that $Z_2(M) \leq_{ess} M$ and so $Z_2(M) \leq_{tes} M$ which a contradiction is since $Z_2(M)$ is t-closed by [3, Corollary 2.7(1)]. Therefor $Ker(f) \not\leq_{tes} M$. Thus M is t-ess.q-Ded. □

Remark 2.9: If M is a t-ess.q-Ded. module, then either $\bar{M}$ or $E(M)$ (quasi-injective hull or injective hull of M) is t-ess.q-Ded. The following example explain this: Let $M = Z_3$ as Z-module. M is t-ess.q-Ded, but $\bar{M} = E(M) = Z_{3^\infty}$ is not t-ess q-Ded.

The converse of Remark 2.8 follows directly by the following result, which is an analogous to [14, Proposition 1.2.15].

Proposition 2.10: Let M be a t-ess. q-Ded R-module and it is quasi-injective. If $N \leq_{tes} M$, then N is a t-ess. Q-Ded R-module.

Proof: It is similar to the proof of [14, Proposition 1.2.15] and so is omitted. □

Corollary 2.11: Let M be an R-module. If $\bar{M}$(or $E(M)$ is a t-ess.q-Ded R-module. Then M is tes.q-Ded.

Proof: Since $M \leq_{ess} \bar{M}(M \leq_{ess} E(M))$, so $M \leq_{tes} \bar{M}(M \leq_{tes} E(M))$, the result follows by Proposition 2.10. □

Now we turn our attention to the direct sum of t-ess.q-Ded modules. First we notice that the direct sum of two t-ess.q-Ded modules need not be t-ess.q-Ded, as the following example: The Z-module Z_2 and Z_3 are t-ess.q-Ded. module, but $Z_2 \oplus Z_3 \simeq Z_6$ is not t-ess.q-Ded.

Definition 2.12: Let M and W be R-module. M is said to be t-ess.q-Ded relative to W for all $f \in Hom(M,W), f \neq 0$ implies $Kerf \not\leq_{tes} M$.

Remarks and Examples 2.13:

(1) Let M be an R-module. M is a t-ess.q-Ded module if and only if M is a t-ess. Q-Ded relative to M.

(2) Let M be a t-ess.q-Ded . Then M is a t-ess. q-Ded. relative to N, for each $N \leq M$.

(3) Z_6 is not t-ess. q-Ded relative to Z_2, since there exists $f: Z_6 \longmapsto Z_2$ defined by $f(\bar{0}) = f(\bar{2}) = f(\bar{4}) = \bar{0}_{Z_2}$, $f(\bar{1}) = f(\bar{5}) = f(\bar{3}) = \bar{1}_{Z_2}$

Thus $Ker(f) = \{\bar{0}, \bar{2}, \bar{4}\} \leq_{tes} Z_6$ and $f \neq 0$.

The following Theorem is analogous to [14, Theorem 1.3.5].

Theorem 2.14: Let $\{M_i\}_{i\in\wedge}$ be a family of R-modules. Then $M = \{M_i\}_{i\in\wedge}$ is t-ess. q-Ded if and only if M_i t-ess. q-Ded relative to M_j for $i, j \in \wedge$.

Proof: It is similar to Theorem 1.3.5 in [14] and so is omitted. □

3. t-essentially prime Modules

Ali Saba in [11] prove that: If M is a prime module, then for each $f \in End(M)$ and $Ker(f) \leq_{ess} M$ then $= 0$; that is every prime module is ess. q-Ded module. However prime module does not imply t-ess. q-Ded. for example : Let M be the Z-module $Z_2 \oplus Z_2$. M is a prime module but M is not t-ess. q-Ded since M is singular and so every submodule N of M, $N \leq_{tes} M$. Take $N = Z_2 \oplus (0)$. Then $Hom(\frac{M}{N}, M) \neq 0$.

We have the following:

Proposition 3.1: Every faithful prime module is t-ess. q-Ded.

Proof: First we shall show that M is nonsingular. Let $x \in Z(M)$ and suppose that $x \neq 0$. Then $ann(x) \leq_{ess} R$. Hence there exists $x \in R, r \neq 0$ and $r \in ann(x)$ and so $xr = 0$. As M is a prime module and $x \neq 0, r \in annM = 0$ which is a contradiction. Thus $Z(M) = 0$ (M is nonsingular) and so by Remarks and Examples 2.2(3), M is t-ess. q-Ded. □

Notice that the condition M is faithful is necessary in Proposition 3.1 as we have seen $M = Z_2 \oplus Z_2$ as Z-module is prime, not faithful and M is not t-ess. q-Ded.

Now it is known by [14, Proposition 2.1.8], every ess. q-Ded module is an essentially prime module (that is $ann_R M = ann_R N$ for each $N \leq_{ess} M$). Also, by Remarks and Examples 2.2(9), if M is a t-ess. q-ded module, then $ann_R M = ann_R N$ for each $(0) \neq N \leq_{tes} M$. This leads us to introduce the following.

Definition 3.2: An R-module is called t-essentially prime (briefly t-ess.prime) if $ann_R\, M = ann_R N$ for each $(0) \neq N \leq_{tes} M$.

Remarks and Examples 3.3:

(1) It is clear that every prime module is t-ess. prime is, but the converse is not true in general (see part(3), II).

(2) Every t-ess. prime module is ess. prime, since every essential submodule is t-essential. But the converse may not be true in general, for example. The Z-module Z_6 is ess. prime module, but it is not t-ess. prime since $ann_Z Z_6 \neq ann_Z(\bar{2})$ and $(\bar{2}) \leq_{tes} Z_6$.

(3) A t-ess. prime module need not be t-ess. q-Ded module, as the following examples show :

(I) Let M be the Z-module $Z_2 \oplus Z_2$. M is t-ess. prime, but M is not t-ess. q-Ded as we have seen in the beginning of section three.

(II) Let $M = Z_2 \oplus Z_2$ as Z-module . M is not t-ess. q-Ded , since if $N = Z \oplus (0)$, then $N + Z_2(M) = M \leq_{ess} M$ and so by Proposition 1.1, $N \leq_{tes} M$. But $Hom(\frac{M}{N}, M) \simeq Hom(Z_2, Z \oplus Z_2) \neq 0$. On the other hand, we can show that M is t-ess. prime as follows: Let $W \leq_{tes} M$ then $W + Z_2(\mathrm{M}) \leq_{ess} M$(by Proposition 1.1). As M is an ess. prime module by [14, Example 2.1.12], hence $ann_Z(W + Z_2(M)) = ann_Z M = (0)$. It follows that $ann_Z W \cap ann_Z Z_2(M) = 0$ and so $ann_Z W \cap 2Z = 0$. (since $Z_2(M) = (0) \oplus Z_2$ and $ann_Z Z_2(M) = 2Z$). Since $2Z \leq_{ess} Z$ then $ann_Z W = 0$. This implies $ann_Z W = ann_Z M$ and M is t-ess. prime. Also, note that M is not prime module.

(4) Let M be a nonsingular module. Then M is an ess. prime if and only if M is a t-ess. prime module.

Proposition 3.4: Let M be a faithful R-module such that $ann_R(Z_2(M)) \leq_{ess} R$. Then M is an ess. prime module if and only if M is t-ess. prime.

Proof: $\Leftarrow$ It is clear.

$\Rightarrow$ Let $0 \neq N \leq_{tes} M$. Then $N + Z_2(M) \leq_{ess} M$. As M is ess. prime, $ann(N + Z_2(M)) = annM = (0)$. Hence $annN \cap ann(Z_2(M)) = 0$. By hypothesis, $ann(Z_2(M)) \leq_{ess} R$, so that $annN = 0 = annM$. It follows that M is t-ess. prime. □

"Recall that an R-module M is bounded if there exists $x \in M$ such that $ann_R M = ann_R(x)$ " [6].

Proposition 3.5: Let M be a bounded module with $ann_R M$ is a prime ideal of R and $ann_R M < ann(Z_2(M))$. Then M is t-ess. prime.

Proof: Let $(0) \neq N \leq_{tes} M$. Then $N + Z_2(M) \leq_{ess} M$ by proposition 1.1. Since M is bounded with $annM$ is a prime ideal, then by [14, Lemma 2.1.11], M is ess. prime. Hence $ann_R(N + Z_2(M)) = ann_R M$. It follows that $ann_R \cap ann_R(Z_2(M)) = ann_R M$. As $ann_R M$ is a prime ideal, either $ann_R N \leq ann_R M$ or $ann_R Z_2(M) = ann_R M$. Thus either $ann_R N \leq ann_R M$ or $ann_R(Z_2(M)) = ann_R M$. But by hypothesis $ann_R M \neq ann_R(Z_2(M))$, so that $ann_R N = ann_R M$ and so M is t-ess. prime. □

Corollary 3.6: Let M be a bounded quasi-prime R-module with $ann_R M \subsetneq ann_R(Z_2(M))$. Then M is t-ess. prime.

Proof : As M is a quasi-prime module, then $ann_R M$ is a prime ideal of R and so by [14, Lemma 2.1.11] M is an ess. prime module. Then by the same procedure of Proposition 3.5, M is a t-ess. prime module. □

As application of Corollary 3.6, $M = Q \oplus Z_2$ as Z-module is t-ess. prime module since M is bounded (where $ann_Z M = ann_Z(1, \bar{1})$, also it is easy to check that M is quasi-prime, and 0=$ann_Z M \subsetneq ann_Z(Z_2(M)) = ann_Z Z_2 = 2Z$.

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" Recall that an R-module is called multiplication if for each $N \leq M$, $N = MI$ for some ideal I of R" [5].

Proposition 3.7: Let M be a faithful multiplication R-module. Consider the following statements:

(1) M is a t-ess. prime .
(2) M is t-ess.q-Ded.
(3) M is ess.prime;
(4) R is t-ess. q-Ded;
(5) R is ess. q-Ded;
(6) $End_R(M)$ is t-ess.q-Ded.

Then (1) ⇔ (2) ⇔(3)⇔ (5) ⇔ (6) and (4) ⇔ (6) if M is a finitely generated module.

Proof: (1) ⇒(2) Since M is t-ess. prime, M is ess. prime. Hence by [14, Proposition 2.1.16], R is ess. q-Ded and so R is nonsingular by [14, Proposition 1.2.6]. On the other hand, M is faithful mulitiplication implies $Z(M) = MZ(R)$ by [5, Corollary 2.1.4]. It follows that $Z(M) = M(0) = 0$; that is M is nonsingular and hence by Remarks and Examples 2.3(3), M is t-ess. q-Ded.

(2)⇒(1) It follows by Remarks and Examples 3.3(3).

(2)⇒(3) M is t-ess.q-Ded implies M is t-ess. prime and hence M ess. Prime (see Remarks and Examples 3.3(2),(3)).

(3)⇒(5) Since M is an ess. prime faithful module then by [14,Lemma 2.1.16], R is ess. q-Ded.

(5)⇒(2) Since R is ess. q-Ded, R is nonsingular which implies M is nonsingular because $Z(M) = MZ(R) = 0$. Thus M is t-ess q-Ded by Remarks and Examples 2.2(3).

(4)⇔(5) It follows by Remarks and Examples 2.2(5).

(4)⇒(6) Since M is a finitely generated multiplication module, then M is scalar R-module [13]. Hence by [10], $E(M) \simeq \frac{R}{annM} \simeq \frac{R}{(0)} \simeq R$. Thus $End(M)$ is t-ess. q-Ded if and only if R is t-ess. q-Ded. □

Remark 3.8: The condition M is a multiplication module cannot be dropped from Theorem 3.7. The following example explains this:

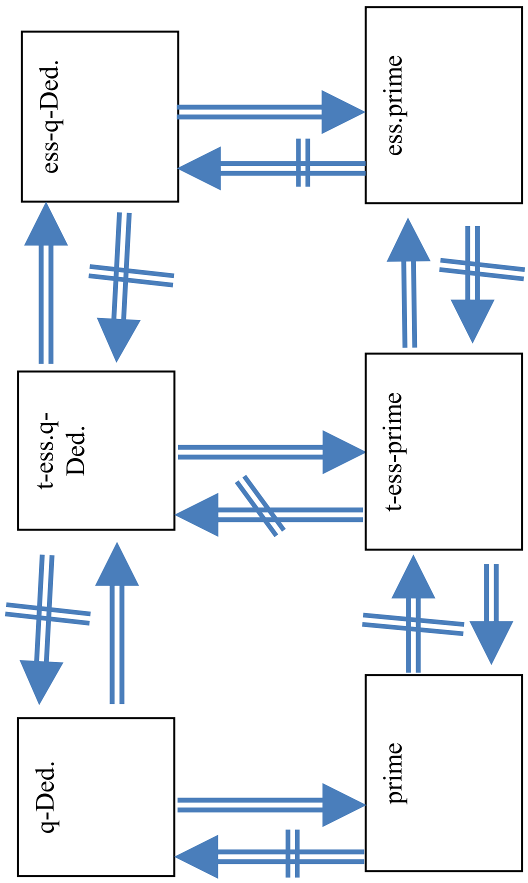
Let $M = Z \oplus Z_2$ as Z-module but not multiplication module. However, M is t-ess. prime Z-module and it is not t-ess. q-Ded (see Remarks and Examples 3.3(3(II)). Also note that R is t-ess. q-Ded.

Proposition 3.9: Let M be an R-module. Then M is t-ess prime and $annM = ann\bar{M}$ if and only if $\bar{M}$ is tes- prime. Where $\bar{M}$ is the quasi-injective hull of M.

Proof: ⇒ Let $(0) \neq N \leq_{tes} \bar{M}$. To prove $ann_R N = ann_R \bar{M}$. Since $M \leq_{ess} \bar{M}$, then $M \leq_{tes} \bar{M}$ and so $N \cap M \leq_{tes} \bar{M}$ by Proposition 1.3. Let $B \leq M$ and $(N \cap M) \cap B \subseteq Z_2(M) - - - - - - - (I)$. Then $N \cap B \subseteq Z_2(M) \subseteq Z_2(\bar{M})$. It follows that $B \subseteq Z_2(\bar{M})$, since $N \leq_{tes} \bar{M}$ and $B \leq M \leq \bar{M}$. Thus $B \subseteq Z_2(\bar{M}) \cap M = Z_2(M)$; and so by (I) implies $N \cap B \leq_{tes} M$. On the other hand M is t-ess. prime, which implies that $ann_R(N \cap M) = ann_R(M) = ann_R(\bar{M})$. Since $ann_R(N \cap M) \supseteq ann_R(N)$(because $(N \cap M) \leq N$, hence $ann_R(\bar{M}) \supseteq ann_R N$. But $ann_R(\bar{M}) \subseteq ann_R N$. Thus $ann_R(\bar{M}) = ann_R(N)$ and so $\bar{M}$ is t-ess. prime.

⇐ Since $M \leq_{ess} \bar{M}$, then $M \leq_{tes} \bar{M}$. So that by t-essentially prime of M, $ann_R(M) = ann_R(\bar{M})$. Now, let $(0) \neq N \leq_{tes} M$, hence $N \leq_{tes} M \leq_{tes} \bar{M}$ which implies $N \leq_{tes} \bar{M}$. It follows that $ann_R(N) = ann_R(\bar{M})$ (since $\bar{M}$ is t-ess. prime), but by the proof $ann_R(\bar{M}) = ann_R(N)$. Thus $ann_R N = ann_R M$ and M is t-ess. prime. □

Remark 3.10: The condition $ann_R M = ann_R \overline{M}$ can't be dropped from Proposition 3.9 and the following example explains this: Let M be the Z-module Z_P (where P is a prime number). M is a prime module, so it is t-ess. prime, but $\overline{M} = Z_{P^\infty}$ is not t-ess. prime (since $(0) = ann_Z \overline{M} \neq ann_Z\left(\frac{1}{P} + Z\right) = PZ$. Also notice that $PZ = ann_Z M \neq ann_Z \overline{M} = 0$.



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المقاسات شبه الديدكاندية الواسعة من النمط t

فرحان داخل شياع شكرنعمه العياشي انعام محمد علي هادي

قسم الرياضيات
كلية التربية
جامعة القادسية

قسم التخطيط الحضري
كليةالتخطيط العمراني
جامعة الكوفة

قسم الرياضيات
كلية التربية-ابن الهيثم
جامعة بغداد

المستخلص :

في هذا البحث قدمنا و درسنا صنف من القاسات اطلقنا عليه المقاسات شبه الديدكاندية الواسعة من النمط t وهي تعميم للمقاسات شبه الديد كاندية الواسعة والمقاسات شبه الديكاندية.كذلك قدمنا صنف المقاسات الاولية الواسعة من النمط t والذي يحتوي على صنف المقاسات شبه الديدكدية الواسعة من النمط t.

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The study of new iterations procedure for expansion mappings

Zena Hussein Maibed **Reyadh.D.Ali**

Department of Mathematics

College of Education for Pure Science Ibn Al-Haithem
University Of Baghdad
Mrs_zena.hussein@yahoo.com

College of Education for Pure Science
University of Karbala
reyadhdelphi@gmail.com

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Abstract:

In this research, we introduce new iteration process for different types of mappings and introduce a concept of expansion mapping, it is independent of non – expansive mapping. Also, we study the convergences for these iterations to common fixed point in real Hilbert spaces.

Keywords: asymptotic fixed point, strong convergence, non-expansive mapping, maximal monotone , weak convergence.

Mathematics Subject Classification: 46S40.

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1. Introduction

Let H be a real Hilbert space $\emptyset \neq C \subseteq H$, and $T: C \longrightarrow C$ is non − expansive. Mapping. That is, if $\|a - b\| \geq \|f(a) - f(b)\|$ for each $a, b \in C$. Also any multivalued operator A is called monotone if the following condition hold:

$< a_1 - a_2, d_1 - d_1 > \geq 0 \quad \forall a_i \in D(A),\ d_i \in A(z_i)$. And it is called maximal monotone if for all $(a, h) \in H \times H, \langle a - b.,.h - d \rangle \geq 0$ and for all $(b, d) \in gph(A)$ then we get, $h \in A(z)$. The monotone operators has an important role in different branches of mathematics, see.([1]-[5]). On other hand, The convergence of the iteration method studied by many researchers see ([6]-[16]). Define the following mapping as follows:

$J_{r_n} = (I + r_n A^{-1})(a)$ this mapping is called resolvet mapping where $< r_n >$ be a sequence of positive real numbers. Also, the metric projection $P_c(a)$ from H onto C is defined as follows:

For any $a \in H$ there exists a unique element $P_c(a) \in C$ satisfies the following:

$\|a - P_c(a)\| \leq \|a - b\|$, for all $b \in C$.That is, for each $a \in X$, $P_c(a) = b$ iff $b \in C$ and $\|a - b\| = \inf\{\|a - c\|; c \in C\}$.

Now, the following definitions and lemmas are interesting to area of research:

Lemma(1.1) [16]

Let $< \alpha_n >$ and $< \beta_n >$ are sequences of nonnegative real number such that $\alpha_{n+1} \leq \alpha_n + \beta_n$, for each. $n \geq 1$. If $\sum_{n=o}^{\infty} \alpha_n$ converge,then $\lim_{n\to\infty} \alpha_n$ exists.

Definition(1.2) : [17]

Let $\Gamma: C \longrightarrow C$ be a mapping then every $p \in C$ is called asymptotic fixed point of Γ if there exists $\langle \alpha_n \rangle$ is sequence in C such that $\alpha_n \rightharpoonup p$ and $\|\alpha_n - \Gamma(\alpha_n)\| \longrightarrow 0$.

Lemma (1.3) : [18]

Let C be a nonempty convex closed subset of real Hilbert space H and Γ is non-expansive multivalued mapping such that $Fix(\Gamma) \neq \emptyset$.Then Γ is demi-closed, i.e., $\alpha_n \rightharpoonup p$ and $\lim_{n\to\infty} d\big(\alpha_n, \Gamma(\alpha_n)\big) = 0$.Then $p \in \Gamma(p)$.

Lemma(1.4) : [19]

If $\langle \alpha_n \rangle$ be a sequence in H and $\|\alpha_{n+1} - \alpha\| \leq \|\alpha_n - \alpha\|$ for all $\alpha \in C$. Then $\langle P_c(\alpha_n) \rangle$ converges strongly to a point in C.

Now, we introduce the concept of expansion mapping

Main Results

In this section, we define a new iterations for sequence of expansion mapping. Also, we study the convergence for these iterations.

Definition(2.1)

Any mapping f is called expansion mapping if for each sequence, $\langle z_n \rangle$ in $(0,1)$ converges to zero then there exists a nonnegative real number z such that

$$(1 - z_n)||x - w||^2 + z\,\langle x - f_x, w - f_w \rangle^{k+1} \geq ||fx - fw||^2, \text{for all } k > 0 \text{ and } x, w \in C$$

The concept of expansion mapping is independent of non − expansive mapping. As shown by the following examples:

Example (2.2)

If $f:(0,\infty) \longrightarrow (0,\infty)$ be a mapping such that $f(x) = x$.Then the mapping f is not non-expansive but it is expansion,mapping. Since, for each sequence $\langle z_n \rangle$ in $[0,1]$ converges to zero then there exists z such that,

$$z = \frac{4}{\langle x - f_x, w - f_w \rangle^{k+1}} ||x - w||^2 \text{ and satisfy}$$

$$(1 - z_n)||x - w||^2 + z\,\langle x - f_x, w - f_w \rangle^{k+1}$$

Example (2.3)

Let $f: H \longrightarrow H$ be a mapping such that $f(x) = x$.

It is clear that the mapping f is not expansion mapping but it is non − expansive.

Theorem (2.4):

Let $A_1, A_2, \ldots, A_m$ are maximal monotone multivalued mapping C nonempty convex closed in H, $\langle f_n \rangle$ be a sequence of non-expansive mapping and $\langle T_n \rangle$ is bounded sequence of expansion mapping on C. Let $\langle a_n \rangle, \langle b_n \rangle$ are sequences in $(0,1]$ converges to 0, such that $a_n + b_n = 1$ and $\sum_{i=1}^{m} \gamma_{n,i} = 1$. Define the iteration process$\langle x_n \rangle$ as follows:

$$w_n = b_n v_n + (1 - b_n) \sum_{i=1}^{m} \gamma_{n,i}\, J_{r_{n,i}}^{i}\, v_n$$

$$v_{n+1} = a_n T_n v_n + (1 - a_n) f_n w_n$$

If $\bigcap_{n=1}^{\infty} Fix\left(J_{r_{n,i}}^{i}\right) \cap \left(\bigcap_{n=1}^{\infty} Fix(T_n)\right) \cap \left(\bigcap_{n=1}^{\infty} Fix(f_n)\right) \neq \emptyset$.Then $\langle x_n \rangle$ has converges weakly to an asymptotic common fixed point of T_n for each $n \in N$. Moreover $\langle P_C(v_n) \rangle$ converges strongly to a point in C.

Proof :

Let $p \in \bigcap_{n=1}^{\infty} Fix\left(J_{r_{n,i}}^{i}\right) \cap \left(\bigcap_{n=1}^{\infty} Fix(T_n)\right) \cap \left(\bigcap_{n=1}^{\infty} Fix(f_n)\right)$

$$\|w_n - p\|^2 = \left\| \begin{array}{c} b_n(v_n - p) + (1 - b_n). \\ \left(\sum_{i=1}^{m} \gamma_{n,i}\, J_{r_{n,i}}^{i}\, y_n\,(v_n) - p \right) \end{array} \right\|^2$$

$$\leq b_n \|v_n - p\|^2 + (1 - b_n) \sum_{i=1}^{m} \gamma_{n,i}\, \|v_n - p\|^2$$

$$\leq b_n \|v_n - p\| + (1 - b_n)\|v_n - p\|^2$$

$$= \|v_n - p\|^2$$

Now, for any sequence $\langle z_n \rangle$ in $[0,1]$ converges to zero then there exists a nonnegative real number z such that

$$\|v_{n+1} - p\|^2 = \|T_n v_n + (1 - a_n) f_n w_n - p\|^2$$

$$\leq a_n \|T_n v_n - p\|^2 + (1 - a_n)\|f_n w_n - p\|^2$$

$$\|v_{n+1} - p\|^2 \leq a_n \|T_n v_n - p\|^2 + (1 - a_n)\|w_n - p\|^2$$

$$\|v_{n+1} - p\|^2 \leq a_n (1 - z_n)\|v_n - p\|^2 + b_n z_n \left\|(T_n p - p T_n)(v_n - v_n - (T_n p - p))\right\| + b_n z\,\langle v_n - f_x, p - f_p \rangle^k + (1 - a_n)\|v_n - p\|^2$$

$$\|v_{n+1} - p\|^2 \leq a_n\{(1 - z_n)\|v_n - p\|^2\} + (1 - a_n)\|v_n - p\|^2$$

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$$\le a_n\|v_n - p\|^2 + (1-a_n)\|v_n - p\|^2 = \|v_n - p\|^2$$

By lemma (1.1), we get $\lim_{n\to\infty}\|v_n - p\|$ exists and hence $\langle f_n\rangle$ is also bounded. So by lemma (1.4) we get $\langle P_C(v)\rangle$ converges strongly to the point in C.

$$\|v_n - T_n v_n\| \le \|a_{n-1}(b_{n-1}T_{n-1}v_{n-1} + (1-b_{n-1})f_{n-1}w_{n-1} - T_n v_n) + (1-a_{n-1})f_{n-1}v_{n-1} - T_n v_n\|$$
$$\le a_{n-1}\|b_{n-1}T_{n-1}v_{n-1} + (1-b_{n-1})T_n w_{n-1} - T_n v_n\| + b_{n-1}\|f_{n-1}v_{n-1} - T_n v_n\|$$

Since $\langle f_n\rangle$ and $\langle T_n\rangle$ are also bounded and $\langle a_n\rangle, \langle b_n\rangle$ are sequences in $(0,1]$ converges to zero. As $n \longrightarrow \infty$ we get, $\|v_n - T_n v_n\| \longrightarrow 0$.

Now, since $\langle v_n\rangle$ is bounded then there exists subsequence $\langle v_{nk}\rangle$ of v_n such that $v_{nk} \rightharpoonup z$ and $\|v_n - T_n v_n\| \longrightarrow 0$. Then we get z is an asymptotic common fixed of T_n, for each $n \in N$.Then the iteration,$\langle v_n\rangle$ has converges weakly to an asymptotic common fixed point of T_n , for each $n \in N$. ■

Now, we consider property $\mathcal{P}$ for any sequence as follows:

Let $\langle T_n\rangle$ be a sequence, of mapping we say that $\langle T_n\rangle$ has property $\mathcal{F}$ if $\langle T_n\rangle$ satisfies the condition:

$\|T_n - z\|^2 \le \|T_n\|^2$,for each $z \in (\cap_{n=1}^{\infty} Fix(T_n))$.

In the following theorem we study the convergence for the iteration process

$$w_n = \acute{b}_n\left[\acute{a}_n v_n + (1-\acute{a}_n)\sum_{i=1}^{m}\gamma_{n,i}\, J^i_{r_{n,i}}\; v_n\right] + (1-\acute{b}_n)g_n v_n$$

$$v_{n+1} = \acute{a}_n[a_n T_n v_n + b_n f_n v_n + c_n f_n g_n v_n] + \acute{b}_n g_n w_n \qquad (2.1)$$

where $\langle \acute{a}_n\rangle$ and $\langle \acute{b}_n\rangle$ are sequences in $[0,1]$ such that $\langle \acute{a}_n\rangle, \langle \acute{b}_n\rangle$ converges to zero, $\acute{a}_n \ge \acute{b}_n$. Such that $\acute{a}_n + \acute{b}_n = 1$, $a_n + b_n + c_n = 1$, $\sum_{i=1}^{m}\gamma_{n,i}$.

Theorem (2.5) :

Let A_1, A_2 , ... , A_m are maximal monotone multivalued mapping and $\emptyset \ne C$ convex closed in X, $\langle T_n\rangle$ is bounded,sequences of expansion mappings on C and $\langle f_n\rangle$, $\langle g_n\rangle$ are sequences of non-expansive mapping on C. If the iteration process defined as (2.1) and $\left(Fix\left(J^i_{r_{n,i}}\right)\right) \cap (\cap_{n=1}^{\infty} Fix(T_n)) \cap (\cap_{n=1}^{\infty} Fix(f_n)) \cap (\cap_{n=1}^{\infty} Fix(g_n)) \ne \emptyset$.Then $\langle x_n\rangle$ has converges weakly to an asymptotic common fixed point of T_n , for each $n \in N$. Moreover $\langle P_C(x_n)\rangle$ converges strongly to a point in C.

Proof :

Let

$$p \in \left(Fix\left(J^i_{r_{n,i}}\right)\right) \cap \left(\bigcap_{n=1}^{\infty} Fix(T_n)\right) \cap \left(\bigcap_{n=1}^{\infty} Fix(f_n)\right) \cap \left(\bigcap_{n=1}^{\infty} Fix(g_n)\right)$$

$$\|w_n - p\|^2 \le \left\|\acute{b}_n\left[\acute{a}_n(v_n - p) + (1-\acute{a}_n)\left(\sum_{i=1}^{m}\gamma_{n,i}\, J^i_{r_{n,i}}\; v_n - p\right)\right] + (1-\acute{b}_n)g_n v_n - p\right\|^2$$

$$\le \acute{b}_n\left\|\acute{a}_n(v_n - p) + (1-\acute{a}_n)\left(\sum_{i=1}^{m}\gamma_{n,i}\, J^i_{r_{n,i}}\; v_n - p\right)\right\|^2 + (1-\acute{b}_n)\|P_c g_n v_n - p\|^2$$

$$\|w_n - p\|^2 \le \acute{b}_n \left[\begin{array}{l} \acute{a}_n \|v_n - p\|^2 + (1-\acute{a}_n) \\ \sum_{i=1}^{m} \gamma_{n,i} \left\| J^i_{r_{n,i}} v_n - p \right\|^2 \end{array} \right] + \left(1-\acute{b}_n\right)\|v_n - p\|^2$$

$$\le \acute{b}_n[\acute{a}_n\|v_n - p\|^2 + (1-\acute{a}_n)\|v_n - p\|^2] + \left(1-\acute{b}_n\right)\|v_n - p\|^2$$

$$\|w_n - p\|^2 = \acute{b}_n\|v_n - p\|^2 + \left(1-\acute{b}_n\right)\|v_n - p\|^2 = \|v_n - p\|^2$$

Hence, $\|w_n - p\|^2 \le \|v_n - p\|^2$

Now, by (2.1) then we have

$$\|v_{n+1} - p\|^2 \le \acute{a}_n\|a_nT_nv_n + b_nf_nv_n + c_nf_ng_nv_n - p\|^2 + \acute{b}_n\|g_nw_n - p\|^2$$

$$\|v_{n+1} - p\|^2 \le \acute{a}_na_n\|T_nv_n - p\|^2 + \acute{a}_nb_n\|f_nv_n - p\|^2 + \acute{a}_nc_n\|f_ng_nv_n - p\|^2 - \acute{a}_na_nb_n\|T_nv_n - f_nv_n\|^2 - \acute{a}_nb_nc_n\|f_nv_n - f_nT_nv_n\|^2 - \acute{a}_nc_na_n\|f_nT_nv_n - T_nv_n\|^2 + \acute{b}_n\|g_nw_n - p\|^2$$

$$\le \acute{a}_na_n\|T_nv_n - p\|^2 + \acute{a}_nb_n\|f_nv_n - p\|^2 + \acute{a}_nc_n\|f_ng_nv_n - p\|^2 + \acute{a}_na_nb_n\|T_nv_n - f_nv_n\| + \acute{a}_nb_nc_n\|f_nv_n - f_nT_nv_n\|^2$$
$$-\acute{a}_nc_na_n\|f_nT_nv_n - T_nv_n\|^2 - \acute{a}_na_nb_n\|T_nv_n - f_nv_n\| - \acute{a}_nb_nc_n\|f_nv_n - f_nT_nv_n\|^2 - \acute{a}_nc_na_n\|f_nT_nv_n - T_nv_n\|^2 + \acute{b}_n\|g_nw_n - p\|^2$$

For any sequence $\langle z_n \rangle$ in [0,1] converges to zero there exists a nonnegative real number z such that

$$\|v_{n+1} - p\|^2 \le \acute{a}_na_n[(1-z_n)\|v_n - p\|^2 + z_n\|p - T_np\|.\|(p - T_np)(_cv_n - T_nv_n - (p - T_np))\| + z(\langle p - T_np, v_n - T_nv_n\rangle)^{k+1}] + \acute{a}_nb_n\|v_n - p\|^2 + \acute{a}_nc_n\|v_n - p\|^2 + \acute{b}_n\|w_n - p\|^2$$

Now,

$$\|v_{n+1} - p\|^2 \le \acute{a}_na_n \|v_n - p\|^2 + \acute{a}_nb_n\|v_n - p\|^2 + \acute{a}_nc_n\|v_n - p\|^2 + \acute{b}_n\|v_n - p\|^2$$

$$\|v_{n+1} - p\|^2 = \acute{a}_n\|v_n - p\|^2 + (1-\acute{a}_n)\|v_n - p\|^2 = \|v_n - p\|^2$$

By lemma (1.1),

we get $\lim_{n\to\infty}\|v_n - p\|$ exists. Hence, $\langle v_n \rangle$ is bounded sequence, so that $\langle g_n \rangle$ and $\langle f_n \rangle$ are also bounded sequences.

So, by lemma (1.4) we deduce $\langle P_C(x_n) \rangle$ converges strongly to the point in C.

$$\|v_n - T_nv_n\| = \|a'_{n-1}[a_{n-1}T_{n-1}v_{n-1} + b_{n-1}f_{n-1}v_{n-1} + c_{n-1}f_{n-1}g_{n-1}v_{n-1}] + b'_{n-1}[a_{n-1}b_{n-1}(T_{n-1}v_{n-1} - f_{n-1}v_{n-1}) + b_{n-1}c_{n-1}(f_{n-1}v_{n-1} - f_{n-1}T_{n-1}v_{n-1}) + c_{n-1}a_{n-1}(f_{n-1}T_{n-1}v_{n-1} - T_{n-1}v_{n-1}) + d_{n-1}\,g_{n-1}w_{n-1}] - T_nw_n\|$$

$$\|v_n - T_n v_n\| \leq a'_{n-1}\|a_{n-1}T_{n-1}v_{n-1} + b_{n-1}f_{n-1}v_{n-1} + c_{n-1}f_{n-1}T_{n-1}v_{n-1} - g_n w_n\| + b'_{n-1}\|a_{n-1}b_{n-1}(T_{n-1}v_{n-1} - f_{n-1}v_{n-1}) + b_{n-1}c_{n-1}(f_{n-1}v_{n-1} - f_{n-1}T_{n-1}v_{n-1}) + c_{n-1}\alpha_{n-1}(f_{n-1}T_{n-1}v_{n-1} - T_{n-1}v_{n-1}) + d_{n-1}\, g_{n-1}w_{n-1} - T_n w_n\|$$

Since $\acute{a}_n, \acute{b}_n \longrightarrow 0$ and $\langle T_n\rangle, \langle f_n\rangle$ and $\langle g_n\rangle$ are bounded then we get

$$\|v_n - T_n v_n\| \longrightarrow 0 \ as\ n \longrightarrow \infty$$

Now, since $\langle v_n\rangle$ is bounded sequence then there exists subsequence $\langle v_{nk}\rangle$ of $\langle v_n\rangle$ such that $v_{nk} \rightharpoonup z$ and since $\|v_n - T_n v_n\| \longrightarrow 0$,then we get,

z is asymptotic common fixed point of T_n, for all $n \in N$.

Then the iterations $\langle v_n\rangle$ has converges weakly to an asymptotic common fixed point of T_n, for all $n \in N$. ■

In the following theorem we give a new iteration process and we study the convergence for this iteration to an asymptotic common fixed point.

Theorem (2.6) :

If $\langle f_n\rangle$ be a sequence of non-expansive mapping on C and $\langle T_n\rangle$ be a bounded sequence of expansion mappings on C. Define the iteration $\langle v_n\rangle$ as follows:

$$w_n = \acute{a}_n f_n v_n + (1 - \acute{a}_n)(T_n v_n)$$

$$v_{n+1} = a_n \sum_{i=1}^{m} \gamma_{n,i}\, J^i_{r_{n,i}}\, v_n + (1 - a_n) f_n w_n \qquad (2.2)$$

where $\langle \acute{a}_n\rangle, \langle \acute{b}_n\rangle, \langle a_n\rangle, \langle b_n\rangle$ are sequences in [0,1] such that $\langle a_n\rangle, \langle b_n\rangle$ converges to 0 such that . If $\left(\bigcap_{n=1}^{\infty} Fix\left(J^i_{r_{n,i}}\right)\right) \cap \left(\bigcap_{n=1}^{\infty} Fix(T_n)\right) \cap \left(\bigcap_{n=1}^{\infty} Fix(f_n)\right) \neq \emptyset$. Then the iteration process $\langle v_n\rangle$ has converges weakly to an asymptotic common fixed point of T_n , for all $n \in N$. Moreover $\langle P_C(v_n)\rangle$ converges, strongly to a point in C.

Proof :

Let $p \in \left(Fix(P_c)\right) \cap \left(\bigcap_{n=1}^{\infty} Fix(T_n)\right) \cap \left(\bigcap_{n=1}^{\infty} Fix(f_n)\right)$

Since $w_n = \acute{a}_n f_n v_n + (1 - \acute{a}_n)\left(\acute{b}_n P_c T_n v_n + (1 - \acute{b}_n) f_n P_c T_n v_n\right)$ then we have,

$$\|w_n - p\|^2 \leq \acute{a}\|f_n v_n - p\|^2 + (1 - \acute{a}_n)[\|T_n v_n - p\|^2]$$

$$\|w_n - p\|^2 \leq \acute{a}_n\|v_n - p\|^2 + (1 - \acute{a}_n)[\|T_n v_n - p\|^2] = \acute{a}_n\|v_n - p\|^2 + (1 - \acute{a}_n)\|T_n v_n - p\|^2$$

For any sequence $\langle z_n\rangle$ in [0,1] converges to zero there exists a nonnegative real number z such that

$$\|w_n - p\|^2 \leq \acute{a}_n\|v_n - p\|^2 + (1 - \acute{a}_n)[(1 - z_n)\|v_n - p\|^2 + z_n\|(p - T_n p)(v_n - T_n v_n - (p - T_n p))\| + z(\langle v_n - T_n v_n, p - T_n p\rangle)^k]$$

$$\|w_n - p\|^2 \leq \acute{a}\|v_n - p\|^2 + (1 - \acute{a}_n)\|v_n - p\|^2 = \|v_n - p\|^2$$

Hence, $\|w_n - p\|^2 \leq \|v_n - p\|^2$

$$\|v_{n+1} - p\|^2 \leq a_n \sum_{i=1}^{m} \gamma_{n,i} \left\|J^i_{r_{n,i}}\ v_n - p\right\|^2 + (1 - a_n)\|f_n w_n - p\|^2$$

$$\|v_{n+1} - p\|^2 \le a_n \sum_{i=1}^{m} \gamma_{n,i} \; \| \; v_n \; - p\|^2$$
$$+ (1 - a_n)\|w_n - p\|^2$$
$$\|v_{n+1} - p\|^2 \le a_n\| \; v_n \; - p\|^2$$
$$+ (1 - a_n)\|v_n - p\|^2$$
$$= \|v_n - p\|^2$$

By lemma (1.1), we get $\lim_{n\to\infty}\|v_n - p\|$ exists

Hence, the iteration $\langle x_n \rangle$ is bounded sequence. So $\langle f_n \rangle$ and $\langle g_n \rangle$ also bounded sequences. And hence, by lemma (1.4) we deduce $\langle P_C(v_n) \rangle$ converges strongly to a point in C. ∎

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دراسة إجراءات التكرارات الجديدة للتطبيقات التوسعية

زينة حسين معيبد

قسم الرياضيات
كلية التربية للعلوم الصرفة ابن الهيثم
جامعة بغداد

رياض دلفي علي

قسم الرياضيات
كلية التربية للعلوم الصرفة
جامعة كربلاء

المستخلص :

في هذا البحث سنقدم عمليات تكرارية جديدة لانواع مختلفة من التطبيقات وسنقدم مفهوم التطبيقات التوسعية والتي تكون مستقلة عن التطبيقات الغير توسيعية .ايضا سندرس التقارب لهذا النوع من التكرارات الى نقطة صامدة مشتركة في فضاء هلبرت

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Connectedness in Čech Fuzzy Soft Closure Spaces

Lina H. Maibed **Rasha N. Majeed**

Department of Mathematics , College of Education for pure sciences Abn Al-Haitham , University of Baghdad , Baghdad, Iraq
Linah8324@gmail.com rashanm6@gmail.com

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Abstract:
The notion of Čech fuzzy soft closure spaces was defined and its basic properties are introduced very newly by Majeed [1]. In the present paper, we define the notion of fuzzy soft separated sets in Čech fuzzy soft closure spaces and prove some properties concerning to this notion. By using the notion of fuzzy soft separated sets we introduce and study the concept of connected in both Čech fuzzy soft closure spaces and their associative fuzzy soft topological spaces. Then we introduce the concept of feebly connected, and discuss the relationship between the concepts of connected and feebly connected. Finally, we introduce several examples to clarify our results.

Keywords. Fuzzy soft set, Čech fuzzy soft closure operator, Fuzzy soft separated sets, Connected Čech fuzzy soft closure space, Feebly connected Čech fuzzy soft closure space.

Mathematics Subject Classification: 54A40, 54B05, 54C05.

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1. Introduction

It is known that Zadeh [2] in 1965 introduced the principal idea of fuzzy sets, which is supply a natural basis for handling mathematically the fuzzy phenomena which exist in our real world, and for constructing new branches of fuzzy mathematics. Later in 1999, Molodtsov [3] initiated the concept of soft set theory, which is a purely new way for modeling uncertainty. Molodtsov [3] established the main results of this new theory and successfully applied the soft set theory into several directions, such as theory of probability, Riemann integration, smoothness of functions, operations research and game theory. The concept of fuzzy soft sets was defined by Maji et al. [4] as fuzzy generalizations of soft sets. Then in 2011, Tanay and Kandemir [5] were gave the concept of topological structure based on fuzzy soft sets. The study of fuzzy soft topological spaces was pursued in recent years by some others [6, 7, 8, 9, 10, 11].

$\check{C}$ech [12] in 1966, introduced the notion of $\check{C}$ech closure spaces $(X, \mathcal{C})$, where $\mathcal{C}: P(X) \to P(X)$ is a mapping satisfying $\mathcal{C}(\emptyset) = \emptyset, A \subseteq \mathcal{C}(A)$ and $\mathcal{C}(A \cup B) = \mathcal{C}(A) \cup \mathcal{C}(B)$, the mapping $\mathcal{C}$ called $\check{C}$ech closure operator on X. After Zadeh introduced the concept of fuzzy sets, in 1985 Mashhour and Ghanim [13] put the concept of $\check{C}$ech fuzzy closure spaces when they exchange sets by fuzzy sets in the definition of $\check{C}$ech closure space. In 2014, Gowri and Jegadeesan [14] using the concept of soft sets to introduced and investigation soft $\check{C}$ech closure spaces, the soft closure operator in that sense was defined from the power set $P(X_{F_A})$ of X_{F_A} to itself (where F_A is a soft set over the universe set X with the set of parameter K, and $A \subseteq K$). Also, in the same year, Krishnaveni and Sekar [15] introduced and study $\check{C}$ech soft closure spaces (where the soft closure operator here defined from the set of all soft sets over X to itself). Very recently Majeed [1] employ the fuzzy set theory to define and study the notion of Čech fuzzy soft closure spaces which is a generalization to $\check{C}$ech soft closure spaces that given by Krishnaveni and Sekar [15]. Also, Majeed and Maibed [16] introduced some structures of Čech fuzzy soft closure spaces. They show that every Čech fuzzy soft closure space gives a parameterized family of Čech fuzzy closure spaces, and defined and studied fuzzy soft exterior (respectively, boundary) in Čech fuzzy soft closure spaces.

On the other hand, the notion of connectedness in closure spaces is introduced and studied. Čech [12] defined the notion of connected spaces in closure spaces. According to Čech a subset A of a closure space X is said to be connected in X if A can not be represent as the union of two nonempty semi-separated subsets of X, that is $A = A_1 \cup A_2$, $(\mathcal{C}(A_1) \cap A_2) \cup (A_1 \cap \mathcal{C}(A_2)) = \emptyset$ implies $A_1 = \emptyset$ or $A_2 = \emptyset$. Plastria [17] studied connectedness and local connectedness of simple extension. Gowri and Jegadeesan [18] introduced the concept of connectedness in soft Čech closure spaces.

In the present paper, we extend the notion of connectedness in Čech fuzzy soft closure spaces. In Section 3, we define the concept of fuzzy soft separated sets in Čech fuzzy soft closure spaces and give some of its basic properties. Then we introduce the notion of disconnected in both Čech fuzzy soft closure spaces and their associative fuzzy soft topological spaces based on fuzzy soft separated sets. In Section 4, we present the concept of feebly disconnected Čech fuzzy soft closure space. We show that the concept of disconnected and feebly disconnected are independent (see Examples 4.11 and 4.12).

2. Preliminaries

In this section we review some basic definitions and results related of fuzzy soft theory and Čech fuzzy soft closure spaces that will be needed in the sequel, and we foresee the reader be familiar with the usual notions and most basic ideas of fuzzy set theory. Throughout our paper, X will refer to the initial universe, $I = [0,1]$, $I_0 = (0,1]$, I^X be the set of all fuzzy sets of X, and K the set of parameters for X.

Definition 2.1 [9, 10, 19, 20] A fuzzy soft set (fss, for short) λ_A on X is a mapping from K to I^X, i.e., $\lambda_A: K \to I^X$, where $\lambda_A(h) \neq \bar{0}$ if $h \in A \subseteq K$ and $\lambda_A(h) = \bar{0}$ if $h \notin A \subseteq K$, where $\bar{0}$ is the empty fuzzy set on X. The family of all fuzzy soft sets over X denoted by $\mathcal{F}_{ss}(X, K)$.

In the next definition, the basic operations between fuzzy soft sets are given.

Definition 2.2 [9, 10, 20] Let $\lambda_A, \mu_B \in \mathcal{F}_{ss}(X, K)$, then

1. λ_A is said to be a fuzzy soft subset of μ_B, denoted by $\lambda_A \subseteq \mu_B$, if $\lambda_A(h) \leq \mu_B(h)$, for all $h \in K$.
2. λ_A and μ_B are said to be equal, denoted by $\lambda_A = \mu_B$ if $\lambda_A \subseteq \mu_B$ and $\mu_B \subseteq \lambda_A$.
3. The union of λ_A and μ_B, denoted by $\lambda_A \cup \mu_B$ is the fss $\sigma_{(A \cup B)}$ defined by $\sigma_{(A\cup B)}(h) = \lambda_A(h) \vee \mu_B(h)$, for all $h \in K$.
4. The intersection of λ_A and μ_B, denoted by $\lambda_A \cap \mu_B$ is the fss $\sigma_{(A\cap B)}$ defined by $\sigma_{(A\cap B)}(h) = \lambda_A(h) \wedge \mu_B(h)$, for all $h \in K$.

Definition 2.3 [9, 11, 20] The null fss, denoted by $\bar{0}_K$, is a fss defined by $\bar{0}_K(h) = \bar{0}$, for all $h \in K$.

Definition 2.4 [9, 11, 20] The universal fss, denoted by $\bar{1}_K$, is a fss defined by $\bar{1}_K(h) = \bar{1}$, for all $h \in K$, where $\bar{1}$ is the universal fuzzy set of X.

Definition 2.5 [20] The complement of a fss $\lambda_A \in \mathcal{F}_{ss}(X, K)$, denoted $\bar{1}_K - \lambda_A$, is the fss defined by $(\bar{1}_K - \lambda_A)(h) = \bar{1} - \lambda_A(h)$, for each $h \in K$, Its clear that $\bar{1}_K - (\bar{1}_K - \lambda_A) = \lambda_A$.

Definition 2.6 [21] Two fss's $\lambda_A, \mu_B \in \mathcal{F}_{ss}(X, K)$ are said to be disjoint, denoted by $\lambda_A \cap \mu_B = \bar{0}_K$, if $\lambda_A(h) \cap \mu_B(h) = \bar{0}$ for all $h \in K$.

Definition 2.7 [5, 20] A fuzzy soft topological space (fsts, for short) (X, τ, K) where X is a nonempty set with a fixed set of parameters and τ is a family of fuzzy soft sets over X satisfying the following properties:

1. $\bar{0}_K, \bar{1}_K \in \tau$,
2. If $\lambda_A, \mu_B \in \tau$, then $\lambda_A \cap \mu_B \in \tau$,
3. If $(\lambda_A)_i \in \tau$, then $\bigcup_{i\in J\in}(\lambda_A)_i \in \tau$.

τ is called a topology of fuzzy soft sets on X. Every member of τ is called open fuzzy soft set (open-fss, for short). The complement of open-fss is called a closed fuzzy soft set (closed-fss, for short).

Definition 2.8 [1] An operator $\theta: \mathcal{F}_{ss}(X, K) \to \mathcal{F}_{ss}(X, K)$ is called $\check{C}$ech fuzzy soft closure operator ($\check{C}$-fsco, for short) on X, if the following axioms are satisfied.

(C1) $\theta(\bar{0}_K) = \bar{0}_K$,
(C2) $\lambda_A \subseteq \theta(\lambda_A)$, for all $\lambda_A \in \mathcal{F}_{ss}(X, K)$,
(C3) $\theta(\lambda_A \cup \mu_B) = \theta(\lambda_A) \cup \theta(\mu_A)$, for all $\lambda_A, \mu_B \in \mathcal{F}_{ss}(X, K)$.

The triple (X, θ, K) is called a Čech fuzzy soft closure space ($\check{C}\mathcal{F}$-fscs, for short).

A fss λ_A is said to be closed-fss in (X, θ, K) if $\lambda_A = \theta(\lambda_A)$. And a fss λ_A is said to be an open-fss if $\bar{1}_K - \lambda_A$ is a closed-fss.

Proposition 2.9 [1] Let (X, θ, K) be a $\check{C}\mathcal{F}$-scs, and $\lambda_A, \mu_B \in \mathcal{F}_{ss}(X, K)$ such that $\lambda_A \subseteq \mu_B$, then $\theta(\lambda_A) \subseteq \theta(\mu_B)$.

Definition 2.10 [1] Let (X, θ, K) be a $\check{C}\mathcal{F}$-scs, and let $\lambda_A \in \mathcal{F}_{ss}(X, K)$. The interior of λ_A, denoted by $Int(\lambda_A)$ is defined as $Int(\lambda_A) = \bar{1}_K - (\theta(\bar{1}_K - \lambda_A))$.

Definition 2.11 [1] Let V be a non-empty subset of X, then $\bar{V}_K$ denotes the fuzzy soft set V_K over X for which $V(h) = \bar{1}_V$ for all $h \in K$, (where $\bar{1}_V: X \to I$ such that $\bar{1}_V(x) = 1$ if $x \in V$ and $\bar{1}_V(x) = 0$ if $x \notin V$).

Theorem 2.12 [1] Let (X, θ, K) be a $\check{C}\mathcal{F}$-scs, $V \subseteq X$ and let $\theta_V: \mathcal{F}_{ss}(V, K) \to \mathcal{F}_{ss}(V, K)$ defined as $\theta_V(\lambda_A) = \bar{V}_K \cap \theta(\lambda_A)$. Then θ_V is a $\check{C}\mathcal{F}$-sco. The triple (V, θ_V, K) is said to be $\check{C}$ech fuzzy soft closure subspace ($\check{C}\mathcal{F}$-sc subspace, for short) of (X, θ, K).

Theorem 2.13 [1] Let (X, θ, K) be a $\check{C}\mathcal{F}$-scs and let $\tau_\theta \subseteq \mathcal{F}_{ss}(X, K)$, defined as follows

$$\tau_\theta = \{\bar{1}_K - \lambda_A : \theta(\lambda_A) = \lambda_A\}.$$

Then τ_θ is a fuzzy soft topology on X and (X, τ_θ, K) is called an associative fsts of (X, θ, K).

Definition 2.14 [22] Let (X, τ_θ, K) be an associative fsts of (X, θ, K) and let $\lambda_A \in \mathcal{F}_{ss}(X, K)$. The fuzzy soft topological closurer of λ_A with respect to θ, denoted by $\tau_\theta\text{-}cl(\lambda_A)$, is the intersection of all closed fuzzy soft super sets of λ_A. i.e.,

$$\tau_\theta\text{-}cl(\lambda_A) = \cap\{\rho_C : \lambda_A \subseteq \rho_C \text{ and } \theta(\rho_C) = \rho_C\}. \quad (2.1)$$

And, The fuzzy soft topological interior of λ_A with respect to θ, denoted by $\tau_\theta\text{-}int(\lambda_A)$ is the union of all open fuzzy soft subset of λ_A. i.e.,

$$\tau_\theta\text{-}int(\lambda_A) = \cup\{\rho_C : \rho_C \subseteq \lambda_A \text{ and } \theta(\bar{1}_K - \rho_C) = \bar{1}_K - \rho_C\}. \quad (2.2)$$

The next theorem give the relation between the $\check{C}$-fsco θ (respectively, interior operator Int) and the fuzzy soft topological closure $\tau_\theta\text{-}cl$ (respectively, interior $\tau_\theta\text{-}int$).

Theorem 2.15 [22] Let (X, θ, K) be $\check{C}\mathcal{F}$-scs and (X, τ_θ, K) be an associative fsts of (X, θ, K). Then for any $\lambda_A \in \mathcal{F}_{ss}(X, K)$

$$\tau_\theta\text{-}int(\lambda_A) \subseteq Int(\lambda_A) \subseteq \lambda_A \subseteq \theta(\lambda_A) \subseteq \tau_\theta\text{-}cl(\lambda_A). \quad (2.3)$$

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3.Connected Čech Fuzzy Soft Closure Spaces

In this section we introduce and study fuzzy soft separated sets in $\check{C}\mathcal{F}$−scs, then we use it to introduce the notion of connectedness in $\check{C}\mathcal{F}$−scs's.

Definition 3.1 Let (X,θ,K) be a $\check{C}\mathcal{F}$-scs. If there exist non-empty proper fss's λ_A, $\mu_B \in \mathcal{F}_{ss}(X,K)$, such that $\lambda_A \cap \theta(\mu_B) = \bar{0}_K$ and $\theta(\lambda_A) \cap \mu_B = \bar{0}_K$, then the fss's λ_A and μ_B are called fuzzy soft separated sets.

In other words, two non-empty fuzzy soft set λ_A, μ_B of $\check{C}\mathcal{F}$-scs (X,θ,K) are said to be fuzzy soft separated sets if and only if $(\lambda_A \cap \theta(\mu_B)) \cup (\theta(\lambda_A) \cap \mu_B) = \bar{0}_K$.

Remark 3.2 It is clear that if λ_A and μ_B are fuzzy soft separated sets in (X,θ,K), then λ_A and μ_B are disjoint fuzzy soft sets. The following example shows that the converse is not true.

Example 3.3 Let $X=\{a,b,c\}$, $K=\{h_1,h_2\}$ and let $\rho_C = \{(h_1,b_{0.5}),(h_2,b_{0.5})\}$. Define $\theta: \mathcal{F}_{ss}(X,K) \to \mathcal{F}_{ss}(X,K)$ as follows:

$$\theta(\lambda_A) = \begin{cases} \bar{0}_K & if\ \lambda_A = \bar{0}_K, \\ \{(h_1, a_{0.5} \vee b_{0.5}), (h_{0.5}, a_{0.5} \vee b_{0.5})\} & if\ \lambda_A \subseteq \rho_C, \\ \bar{1}_K & otherwise. \end{cases}$$

Then θ is $\check{C}$-fsco on X. Here we have $\lambda_A = \{(h_1,b_{0.5})\}$ and $\mu_B = \{(h_1,a_{0.5}),(h_2,c_{0.5})\}$ are non-empty disjoint fuzzy soft sets but λ_A and μ_B are not fuzzy soft separated sets.

Theorem 3.4 Let (X,θ,K) be a $\check{C}\mathcal{F}$-scs. Then every fuzzy soft subset of fuzzy soft separated sets are also fuzzy soft separated sets.

Proof. Let λ_A and μ_B are fuzzy soft separated sets in (X,θ,K), and let $\rho_C \subseteq \lambda_A$ and $\eta_D \subseteq \mu_B$. Since $\rho_C \subseteq \lambda_A$ and $\eta_D \subseteq \mu_B$, then by Proposition 2.9, we have $\theta(\rho_C) \subseteq \theta(\lambda_A)$ and $\theta(\eta_D) \subseteq \theta(\mu_B)$. This implies $\theta(\rho_C) \cap \eta_D \subseteq \theta(\lambda_A) \cap \mu_B$ and $\theta(\eta_D) \cap \rho_C \subseteq \theta(\mu_B) \cap \lambda_A$. But λ_A and μ_B are fuzzy soft separated sets, it follows $\theta(\rho_C) \cap \eta_D \subseteq \theta(\lambda_A) \cap \mu_B = \bar{0}_K$ and $\theta(\eta_D) \cap \rho_C \subseteq \theta(\mu_B) \cap \lambda_A = \bar{0}_K$. Hence $\theta(\rho_C) \cap \eta_D = \bar{0}_K$ and $\theta(\eta_D) \cap \rho_C = \bar{0}_K$. Thus ρ_C and η_D are fuzzy soft separated sets. ■

Theorem 3.5 Let (V,θ_V,K) be a $\check{C}\mathcal{F}$-sc subspace of (X,θ,K) and let $\lambda_A, \mu_B \in \mathcal{F}_{ss}(V,K)$, then λ_A and μ_B are fuzzy soft separated sets in (X,θ,K) if and only if λ_A and μ_B are fuzzy soft separated sets in (V,θ_V,K).

Proof. Let (X,θ,K) be a $\check{C}\mathcal{F}$-scs and (V,θ_V,K) be a $\check{C}\mathcal{F}$-sc subspace of (X,θ,K). Assume that that λ_A and μ_B are fuzzy soft separated sets in (X,θ,K), this implies that $\lambda_A \cap \theta(\mu_B) = \bar{0}_K$ and $\theta(\lambda_A) \cap \mu_B = \bar{0}_K$. Which means $(\lambda_A \cap \theta(\mu_B)) \cup (\theta(\lambda_A) \cap \mu_B) = \bar{0}_K$.
Now,

$$(\lambda_A \cap \theta_V(\mu_B)) \cup (\theta_V(\lambda_A) \cap \mu_B) = (\lambda_A \cap (\bar{V}_K \cap \theta(\mu_B))) \cup ((\bar{V}_K \cap \theta(\lambda_A)) \cap \mu_B)$$

$$= ((\lambda_A \cap \bar{V}_K) \cap \theta(\mu_B)) \cup ((\bar{V}_K \cap \mu_B) \cap \theta(\lambda_A))$$
$$= (\lambda_A \cap \theta(\mu_B)) \cup (\mu_B \cap \theta(\lambda_A))$$
$$= \bar{0}_K.$$

Therefore, λ_A and μ_B are fuzzy soft separated sets in (X,θ,K) if and only if λ_A and μ_B are fuzzy soft separated sets in (V,θ_V,K). ■

Definition 3.6 A $\check{C}\mathcal{F}$-scs (X,θ,K) is said to be disconnected Čech fuzzy soft closure space (disconnected-$\check{C}\mathcal{F}$-scs, for short) if there exist fuzzy soft separated sets λ_A and μ_B such that $\theta(\lambda_A) \cap \theta(\mu_B) = \bar{0}_K$ and $\theta(\lambda_A) \cup \theta(\mu_B) = \bar{1}_K$.

Definition 3.7 A $\check{C}\mathcal{F}$-scs (X,θ,K) is said to be connected Čech fuzzy soft closure space (connected-$\check{C}\mathcal{F}$-scs, for short) if it is not disconnected-$\check{C}\mathcal{F}$-scs.

Now we give two examples one is disconnected-$\check{C}\mathcal{F}$-scs and the other is connected-$\check{C}\mathcal{F}$-scs.

Example 3.8 Let $X=\{a,b\}$, $K=\{h_1,h_2\}$. Define $\theta: \mathcal{F}_{ss}(X,K) \to \mathcal{F}_{ss}(X,K)$ as follows:

$$\theta(\lambda_A) = \begin{cases} \bar{0}_K & if\ \lambda_A = \bar{0}_K, \\ \{(h_1, a_1 \vee b_1)\} & if\ \lambda_A \subseteq \{(h_1,a_1)\}, \\ \{(h_2, a_1 \vee b_1)\} & if\ \lambda_A \subseteq \{(h_2,a_1)\}, \\ \bar{1}_K & other\ wise. \end{cases}$$

Then (X,θ,K) is disconnected-$\check{C}\mathcal{F}$-scs. To explain that taking $\lambda_A = \{(h_1,a_{0.5})\}$ and $\mu_B = \{(h_2,a_{0.2})\}$. It is clear that λ_A and μ_B are fuzzy soft separated sets such that $\theta(\lambda_A) \cap \theta(\mu_B) = \bar{0}_K$ and $\theta(\lambda_A) \cup \theta(\mu_B) = \bar{1}_K$.

Example 3.9 Let $X=\{a,b\}$, $K=\{h_1,h_2\}$. Define $\theta: \mathcal{F}_{ss}(X,K) \to \mathcal{F}_{ss}(X,K)$ as follows:

$$\theta(\lambda_A) = \begin{cases} \bar{0}_K & if\ \lambda_A = \bar{0}_K, \\ \{(h_1, a_1 \vee b_1)\} & if\ \lambda_A \subseteq \{(h_1, a_1 \vee b_1)\}, \\ \bar{1}_K & otherwise. \end{cases}$$

Then (X,θ,K) is connected-$\check{C}\mathcal{F}$-scs.

Remark 3.10 Connectedness in $\check{C}\mathcal{F}$-scs is not hereditary property. The following example explain that.

Example 3.11 Let $X=\{a,b,c\}$, $K=\{h_1,h_2\}$ and let $(\lambda_A)_1, (\lambda_A)_2 \in \mathcal{F}_{ss}(X,K)$ such that

$(\lambda_A)_1 = \{(h_1, a_1 \vee b_1 \vee c_{0.4})\}$ and $(\lambda_A)_2 = \{(h_2, a_1 \vee b_1 \vee c_{0.7})\}$.

Define $\theta: \mathcal{F}_{ss}(X,K) \to \mathcal{F}_{ss}(X,K)$ as follows:

$$\theta(\lambda_A) = \begin{cases} \bar{0}_K & if\ \lambda_A = \bar{0}_K, \\ \{(h_1, a_1 \vee b_1 \vee c_{0.4})\} & if\ \lambda_A \subseteq (\lambda_A)_1, \\ \{(h_2, a_1 \vee b_1 \vee c_{0.7})\} & if\ \lambda_A \subseteq (\lambda_A)_2, \\ \theta((\lambda_A)_1) \cup \theta((\lambda_A)_2) & if\ \lambda_A \subseteq (\lambda_A)_1 \cup (\lambda_A)_2, \\ \bar{1}_K & otherwise. \end{cases}$$

Then (X,θ,K) is connected-$\check{C}\mathcal{F}$-scs. Let $V=\{a,b\}$, then $\theta_V: \mathcal{F}_{ss}(V,K) \to \mathcal{F}_{ss}(V,K)$ defined as

$$\theta_V(\lambda_A) = \begin{cases} \bar{0}_K & if\ \lambda_A = \bar{0}_K, \\ \{(h_1, a_1 \vee b_1)\} & if\ \lambda_A \subseteq \{(h_1, a_1 \vee b_1)\}, \\ \{(h_2, a_1 \vee b_1)\} & if\ \lambda_A \subseteq \{(h_2, a_1 \vee b_1)\}, \\ \bar{V}_K & otherwise. \end{cases}$$

Then (V,θ_V,K) is disconnected-$\check{C}\mathcal{F}$-sc subspace of (X,θ,K). Since there exist $\lambda_A = \{(h_1, a_1 \vee b_1)\}$ and $\mu_B = \{(h_2, a_1 \vee b_1)\}$ are fuzzy soft separated sets such that $\theta_V(\lambda_A) \cap \theta_V(\mu_B) = \bar{0}_K$ and $\theta_V(\lambda_A) \cup \theta_V(\mu_B) = \bar{V}_K$.

Now, we introduce the concept of fuzzy soft separated sets in the associative fsts's of $\check{C}\mathcal{F}$-scs's.

Definition 3.12 Two non-empty fss's λ_A and μ_B are said to be fuzzy soft separated sets in the associative fsts (X,τ_θ,K), if $\lambda_A \cap \tau_\theta\text{-}cl(\mu_B) = \bar{0}_K$ and $\tau_\theta\text{-}cl(\lambda_A) \cap \mu_B = \bar{0}_K$.

Theorem 3.13 If λ_A and μ_B are fuzzy soft separated sets in the associative fsts (X,τ_θ,K), then λ_A and μ_B are also fuzzy soft separated sets in (X,θ,K).

Proof. Let λ_A and μ_B are fuzzy soft separated sets in (X,τ_θ,K). Then $\lambda_A \cap \tau_\theta\text{-}cl(\mu_B) = \bar{0}_K$ and $\tau_\theta\text{-}cl(\lambda_A) \cap \mu_B = \bar{0}_K$. By Theorem 2.15, we get, $\lambda_A \cap \theta(\mu_B) = \bar{0}_K$ and $\theta(\lambda_A) \cap \mu_B = \bar{0}_K$. This implies λ_A and μ_B are fuzzy soft separated sets in (X,θ,K). ■

Definition 3.14 An associative fsts (X,τ_θ,K) of $\check{C}\mathcal{F}$-scs (X,θ,K) is said to be disconnected fsts, if there exist two fuzzy soft separated sets λ_A and μ_B in (X,τ_θ,K) such that $\tau_\theta\text{-}cl(\lambda_A) \cap \tau_\theta\text{-}cl(\mu_B) = \bar{0}_K$ and $\tau_\theta\text{-}cl(\lambda_A) \cup \tau_\theta\text{-}cl(\mu_B) = \bar{1}_K$.

Definition 3.15 An associative fsts (X,τ_θ,K) of $\check{C}\mathcal{F}$-scs(X,θ,K) is said to be connected fsts, if it is not disconnected fsts.

Theorem 3.16 If (X,τ_θ,K) is a disconnected fsts, then (X,θ,K) is disconnected-$\check{C}\mathcal{F}$-scs.

Proof. Let (X,τ_θ,K) be disconnected fsts, then there exist two fuzzy soft separated sets λ_A and μ_B in (X,τ_θ,K) such that $\tau_\theta\text{-}cl(\lambda_A) \cap \tau_\theta\text{-}cl(\mu_B) = \bar{0}_K$ and $\tau_\theta\text{-}cl(\lambda_A) \cup \tau_\theta\text{-}cl(\mu_B) = \bar{1}_K$. Since $\tau_\theta\text{-}cl(\lambda_A)$ and $\tau_\theta\text{-}cl(\mu_B)$ are closed-fss's, then $\theta(\tau_\theta - cl(\lambda_A)) = \tau_\theta\text{-}cl(\lambda_A)$ and $\theta(\tau_\theta\text{-}cl(\mu_B)) = \tau_\theta\text{-}cl(\mu_B)$. Let $\rho_C = \tau_\theta\text{-}cl(\lambda_A)$ and $\eta_D = \tau_\theta\text{-}cl(\mu_B)$. Then we have ρ_C and η_D are fuzzy soft separated sets in (X,θ,K) such that $\theta(\rho_C) \cap \theta(\eta_D) = \rho_C \cap \eta_D = \bar{0}_K$ and $\theta(\rho_C) \cup \theta(\eta_D) = \rho_C \cup \eta_D = \bar{1}_K$. Hence, (X,θ,K) is disconnected-$\check{C}\mathcal{F}$-scs. ■

Corollary 3.17 If (X,θ,K) is connected-$\check{C}\mathcal{F}$-scs, then (X,τ_θ,K) is a connected fsts.

Proof. The proof follows by suppose (X,τ_θ,K) is disconnected fsts. From Theorem 3.16, we get (X,θ,K) is disconnected-$\check{C}\mathcal{F}$-scs which is a contradiction with hypothesis. Hence, the result. ■

Remark 3.18 The converse of Theorem 3.16 and its corollary is not true in general. That is, if (X,θ,K) is disconnected-$\check{C}\mathcal{F}$-scs, then (X,τ_θ,K) need not to disconnected fsts. The following example shows that.

Example 3.19 In Example 3.8, (X,θ,K) is disconnected-$\check{C}\mathcal{F}$-scs. But its associative fsts (X,τ_θ,K) is connected fsts, because $\tau_\theta = \{\bar{0}_K, \bar{1}_K\}$.

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4. Feebly Connected Čech Fuzzy Soft Closure Spaces

Definition 4.1 A $\check{C}\mathcal{F}$-scs (X,θ,K) is said to be feebly disconnected-$\check{C}\mathcal{F}$-scs, if there two non-empty disjoint fuzzy soft sets λ_A and μ_B such that $\lambda_A \cup \theta(\mu_B) = \bar{1}_K$ and $\theta(\lambda_A) \cup \mu_B = \bar{1}_K$.

Definition 4.2 A $\check{C}\mathcal{F}$-scs (X,θ,K) is said to be feebly connected-$\check{C}\mathcal{F}$-scs if it is not feebly disconnected-$\check{C}\mathcal{F}$-scs.

Remark 4.3 Feebly disconnectedness in $\check{C}\mathcal{F}$-scs is not hereditary property. The following example explains that.

Example 4.4 Let $X=\{a,b,c\}$, $K=\{h_1,h_2\}$ and let $(\lambda_A)_1, (\lambda_A)_2 \in \mathcal{F}_{ss}(X,K)$ such that

$(\lambda_A)_1 = \{(h_1, a_1 \vee c_1)\}$ and

$(\lambda_A)_2 = \{(h_1,b_1),(h_2, a_1 \vee b_1 \vee c_1)\}$.

Define $\theta: \mathcal{F}_{ss}(X,K) \to \mathcal{F}_{ss}(X,K)$ as follows:

$$\theta(\lambda_A) = \begin{cases} \bar{0}_K & if\ \lambda_A = \bar{0}_K, \\ \{(h_1, a_1 \vee c_1)\} & if\ \lambda_A \subseteq (\lambda_A)_1, \\ \{(h_1,b_1),(h_2,a_1 \vee b_1 \vee c_1)\} & if\ \lambda_A \subseteq (\lambda_A)_2, \\ \bar{1}_K & otherwise. \end{cases}$$

Then (X,θ,K) is feebly disconnected-$\check{C}\mathcal{F}$-scs. Since there exist $\lambda_A = \{(h_1, a_1 \vee c_1)\}$ and $\mu_B = \{(h_1,b_1),(h_2,a_1 \vee b_1 \vee c_1)\}$ are disjoint fuzzy soft sets such that $\theta(\mu_B) \cup \lambda_A = \bar{1}_K$ and $\mu_B \cup \theta(\lambda_A) = \bar{1}_K$. Let $V = \{b\}$, then $\theta_V: \mathcal{F}_{ss}(V,K) \to \mathcal{F}_{ss}(V,K)$ defined as:

$$\theta_V(\lambda_A) = \begin{cases} \bar{0}_K & if\ \lambda_A = \bar{0}_K, \\ \bar{V}_K & otherwise. \end{cases}$$

Then (V,θ_V,K) is feebly connected-$\check{C}\mathcal{F}$-sc subspace of (X,θ,K).

Definition 4.5 An associative fsts (X,τ_θ,K) of $\check{C}\mathcal{F}$-scs (X,θ,K) is said to be feebly disconnected fsts, if there exist two non-empty disjoint fuzzy soft sets λ_A and μ_B such that $\lambda_A \cup \tau_\theta\text{-}cl(\mu_B) = \bar{1}_K$ and $\tau_\theta\text{-}cl(\lambda_A) \cup \mu_B = \bar{1}_K$.

Theorem 4.6 If (X,θ,K) is feebly disconnected -$\check{C}\mathcal{F}$-scs, then (X,τ_θ,K) is feebly disconnected fsts.

Proof. The proof follows from the definition 4.1 and Theorem 2.19. ■

Corollary 4.7 If (X,τ_θ,K) is feebly connected fsts, then (X,θ,K) is feebly connected-$\check{C}\mathcal{F}$-scs.

Proof. The proof follows by suppose (X,θ,K) is feebly disconnected-$\check{C}\mathcal{F}$-scs. From Theorem 4.6, we get (X,τ_θ,K) is feebly disconnected fsts which is a contradiction with hypothesis. Hence, the result. ■

Next we discuss the relationship between disconnectedness and feebly disconnectedness in $\check{C}\mathcal{F}$-scs's.

Remark 4.10 The concept of disconnected-$\check{C}\mathcal{F}$-scs and feebly disconnected-$\check{C}\mathcal{F}$-scs are independent. The next two examples explain our clime.

The following example shows that if (X,θ,K) is disconnected-$\check{C}\mathcal{F}$-scs, then (X,θ,K) need not to be feebly disconnected-$\check{C}\mathcal{F}$-scs.

Example 4.11 Let $X=\{a,b\}$, $K=\{h\}$. Define $\theta: \mathcal{F}_{ss}(X,K) \to \mathcal{F}_{ss}(X,K)$ as follows:

$$\theta(\lambda_A) = \begin{cases} \bar{0}_K & if\ \lambda_A = \bar{0}_K, \\ \{(h,a_1)\} & if\ \lambda_A = \{(h,a_t);\ 0 < t < 1\}, \\ \{(h,b_1)\} & if\ \lambda_A = \{(h,b_s);\ 0 < s < 1\}, \\ \bar{1}_K & other\ wise. \end{cases}$$

Then (X,θ,K) is disconnected-$\check{C}\mathcal{F}$-scs, since there exist $\lambda_A = \{(h,a_{0.5})\}$ and $\mu_B = \{(h,b_{0.3})\}$ are fuzzy soft separated sets such that $\theta(\lambda_A) \cap \theta(\mu_B) = \bar{0}_K$ and $\theta(\lambda_A) \cup \theta(\mu_B) = \bar{1}_K$. However, (X,θ,K) is not feebly disconnected-$\check{C}\mathcal{F}$-scs since for any non-empty disjoint fss's λ_A and μ_B, we have $\lambda_A \cup \theta(\mu_B) \neq \bar{1}_K$.

The next example shows that if (X,θ,K) is feebly disconnected-$\check{C}\mathcal{F}$-scs, then (X,θ,K) need not to be disconnected-$\check{C}\mathcal{F}$-scs.

Example 4.12 Let $X=\{a,b\}$, $K=\{h_1,h_2\}$. Define $\theta: \mathcal{F}_{ss}(X,K) \to \mathcal{F}_{ss}(X,K)$ as follows:

$$\theta(\lambda_A) = \begin{cases} \bar{0}_K & if\ \lambda_A = \bar{0}_K, \\ \{(h_1, a_1 \vee b_1),(h_2,b_1)\} & if\ \lambda_A \subseteq \{(h_1,b_1)\}, \\ \{(h_1,a_1),(h_2,a_1 \vee b_1)\} & if\ \lambda_A \subseteq \{(h_2,a_1)\}, \\ \bar{1}_K & other\ wise. \end{cases}$$

Then (X,θ,K) is feebly disconnected-$\check{C}\mathcal{F}$-scs. Since there are non-empty disjoint fuzzy soft sets $\lambda_A = \{(h_1,b_1)\}$ and $\mu_B = \{(h_2,a_1)\}$ such that $\theta(\lambda_A) \cup \mu_B = \bar{1}_K$ and $\lambda_A \cup \theta(\mu_B) = \bar{1}_K$.

And (X,θ,K) is connected-$\check{C}\mathcal{F}$-scs. Since for any fuzzy soft separated sets λ_A and μ_B, we have $\theta(\lambda_A) \cup \theta(\mu_B) = \bar{1}_K$ but $\theta(\lambda_A) \cap \theta(\mu_B) \neq \bar{0}_K$.

Remark 4.13 It is worth noting that the definitions of disconnected-Č$\mathcal{F}$-scs and feebly disconnected-Č$\mathcal{F}$-scs (see Definitions 3.6 and 4.1, respectively) turn to be every disconnected-Č$\mathcal{F}$-scs is feebly disconnected-Č$\mathcal{F}$-scs, if the fuzzy soft separated sets which are satisfying the conditions of disconnected-Č$\mathcal{F}$-scs are closed-fss's.

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الاتصال في فضاءات الاغلاق الضبابية الناعمة من النوع – تشيك

لينا حسين معيبد رشا ناصر مجيد
قسم الرياضيات، كلية التربية للعلوم الصرفة إبن الهيثم، جامعة بغداد، بغداد، العراق.

المستخلص :

يعتبر مفهوم فضاءات الاغلاق الضبابية الناعمة من النوع –تشيك من المفاهيم الحديثة حيث تم تعريفه ودراسة خواصه من قبل مجيد [1] . في هذا البحث قمنا بتعريف ودراسة مفهوم المجموعات الضبابية الناعمة القابلة للفصل في فضاءات الاغلاق الضبابية الناعمة من النوع – تشيك. بأستخدام المجموعات الضبابية القابلة للفصل تم تعريف ودراسة مفهوم الاتصال في كلا من فضاءات الاغلاق الضبابية الناعمة من النوع –تشيك والفضاء الضبابي الناعم المشتق منه. كذلك عرفنا مفهوم الاتصال الضعيف ودرسنا العلاقة بين مفهوم الاتصال ومفهوم الاتصال الضعيف . واخيرا، اعطينا العديد من الامثلة لتوضيح النتائج التي تم التوصل اليها في البحث.

الكلمات المفتاحية: مجموعة ضبابية ناعمة، مؤثر فضاء الاغلاق الضبابي الناعم، مجموعات ضبابية ناعمة قابلة للفصل، فضاء الاغلاق الضبابي الناعم من النوع –تشيك المتصل، فضاء الاغلاق الضبابي الناعم من النوع –تشيك المتصل الضعيف.

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On B^*c– open set and its properties

Raad Aziz Hussan Al – Abdulla **Karim Fadhil Al – Omri**

Department of Mathematics
College computer science and Information Technology
University of AL- Qadisiyah

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Abstract:

In this paper we introduced a new set is said B^*c– open set where we studied and identified its properties and find the relation with other sets and our concluded a new class of the function called B^*c– cont. function, B^*c– open function, B^*c– closed function.

Key words:

B^*c – open set, B^*c – closed set, B^*c – closure, B^*c – interior, B^*c – continuous.

Mathematics subject classification: 54xx.

1- Introduction :

The topological idea from study this set is generalization the properties and using its to prove many of the theorems. In [1]Abd El-Monsef M.E.,El.Deeb S.N. Mahmoud R.A Introduced set of class β- open, β- closed which are considered as in put to study the set of class B^*c – open, B^*c – closed and we introduced the interior and the closure as property of B^*c – open set, B^*c – closed set. In [4] Najasted O (1965) and [5] Andrijecivic D (1986) introduced a study about the set α –open, α –closed, B – open with the set β- open set and through it, we introduced proof many of proposition as the set B^*c – open set with α –closed it can lead to set β- open set. In [6] Ryszard Engelking introduced the function as concept to β- continuous, B^*c – continuous, β– open function, B^*c – open function, β– closed function, B^*c – closed function and find the relation among them.

2. On B^*c – open sets

Definition (2.1) [1]

Let X be a top. sp. Then a sub set A of X is called to be

i) a β- open set if $A \subseteq \overline{A^o}$.

ii) a β- closed set if $A \supseteq \overline{A^o}^o$

The all β- open (resp. β- closed) set sub sets of a space X will be as always symbolizes that βo(x) (resp. βc(x)).

Example (2.2):

Let X = {a, b, c, d} with topology t = { ∅ , X, {a} , { a, b}, {a, c, d}. Then the clases of β- open set and βclosed set are:

βo(X) = {∅, X, {a} ,{a, b},{a, c},{a, d}, {a, b, c},{a, b, d},{a, c, d}.

βc(X) = {∅, X, {b} ,{c},{d},{b, c}, {b, d},{c, d},{b, c, d}.

Remark (2.3):

Let X be a top. Sp. If $\overline{A}$ = X, then A is β-open set.

Remark (2.4):

If A β- open set in X, then A^C is β- closed set in X.

Proposition (2.5):

Let X be a top. Sp. Then:

i) Every open set is β- open set in X.

ii) Every closed set is β- closed set in X.

Proof :

i) Let A be open set, then $A = A^\circ$. Since $A \subseteq \overline{A}$, then $A = A^\circ \subseteq \overline{A}^o$, there for $A \subseteq \overline{A^o}$, hence A is β- open set inX.

ii) Let A be closed set, then A^C open set, then A^C β-open set in X by (i), then A β- closed set in X.

The converse of above proposition is not true in general.

Example (2.6):

Let X = {1, 2, 3}, t = {∅, X, {1}, {2,3} }.

βo(X) = {∅, X, {1}, {2}, {3}, {1,2}, {1,3}, {2,3} }.

βC (X) = βo(X).

Note that A = {3} is β- open (resp. β-closed) set, but not open (resp. closed) set.

Theorem (2.7):

Let X be atop. Sp. Then the following statement are holds:

i) The union family of β- open sets is β- open set.

ii) The intersection family of β- closed sets is β-closed set.

Proof:

i) Let $\{A_\alpha : \alpha \in \Lambda\}$ be a family of β-open set in X , then $A_\alpha \subseteq \overline{A_\alpha^o}$, then

$$\bigcup_{\alpha\in\Lambda} A_\alpha \subseteq \bigcup_{\alpha\in\Lambda} \overline{A_\alpha^o} = \overline{\bigcup_{\alpha\in\Lambda} A_\alpha^o} \subseteq \overline{[\bigcup_{\alpha\in\Lambda} A_\alpha]^o} = \overline{\bigcup_{\alpha\in\Lambda} A_\alpha^o}$$

, hence $\bigcup_{\alpha\in\Lambda} A_\alpha$ is β-open set .

ii) Let $\{A_\alpha : \alpha \in \Lambda\}$ be a family of β-closed set in X , then $\{A_\alpha^c : \alpha \in \Lambda\}$ be β-open set in X, then { $\bigcup_{\alpha\in\Lambda} A_\alpha^c : \alpha \in \Lambda$ } β - open sets .But $[\bigcap_{\alpha\in\Lambda} A_\alpha]^c = \bigcup_{\alpha\in\Lambda} A_\alpha^c$, then $[\bigcap_{\alpha\in\Lambda} A_\alpha]^c$ β - open sets in X . There for $\bigcap_{\alpha\in\Lambda} A_\alpha$ β-closed set in X.

Remark (2.8):

i) [1] the intersection of any two β- open sets is not β- open set in general.

ii) The union of any two β- closed sets is not β-closed set in general.

Example (2.9):

Let X = {1, 2, 3}, t = {∅, X, {1}, {2}, {1,2} }.

βo(X) = {∅, X, {1}, {2}, {1,2}, {1,3}, {2,3} }.

βc(X) = {∅, X, {1}, {2}, {3}, {1,3}, {2,3} }.

i) Let A = {1,3}, B = {2,3} are β- open sets, but A ∩ B = {3} not β- open in X.

ii) Let A = {1}, B = {2} are β- closed sets, but A ∪B = {1,2} not β- closed in X.

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Proposition (2.10)

Let X be atop. Sp. Then:

i) G is an open set in X iff $\overline{G \cap \bar{A}} = \overline{G \cap A}$ for each $A \subseteq X$. [2]

Proposition (2.11)

Let X be atop. Sp. Then:

i) The intersection a β- open set and open set in X is β- open set.

ii) The union a β- closed set and closed set in X is β- closed set.

Proof:

i) Let A be a β- open set, then $A \subseteq \overline{\bar{A}^{o}}$

Let B open set. Then

$A \cap B \subseteq (\overline{\bar{A}^{o}} \cap B)$.

$\subseteq \overline{(\bar{A}^{o} \cap B)}$.

$= \overline{(\bar{A}^{o} \cap B^{\circ})}$ by proposition (2.10) .

$= \overline{(\bar{A} \cap B)^{o}}$.

$\subseteq \overline{(\overline{\bar{A} \cap B})^{o}}$ by proposition (2.10)

$= \overline{\overline{A \cap B}^{o}}$ by proposition (2.10)

There fore A ∩ B is β- open set in X.

ii) Let A be a β- closed set in X, then A^{C} β- open set in X.

Let B be closed set in X, then B^{C} open set in X, then by (i) we get $A^{C} \cap B^{C}$ β- open set in X, but $(A \cup B)^{C} = (A^{C} \cap B^{C})$, then $(A \cup B)^{C}$ β- open set in X, then A ∪ B β- closed set in X.

Definition (2.12):

Let X be atop. Sp. and A ⊆ X. Then:

i) A is α – open if $A \subseteq \overline{A^{o}}^{o}$ [4].

ii) A is α – closed if $\overline{A^{o}}^{o} \subseteq A$ [4].

Definition (2.13):

Let X be atop. Sp. X and A ⊆ X. Then a β-open set A is said a B^{*}c. open set if ∀ x ∈ A ∃ F_{x} closed set ∋ x ∈ F_{x} ⊆ A. A is a B^{*}c– closed set if A^{c} is a B^{*}c – open set X.

The all B^{*}c – open (resp. B^{*}c – closed) set sub set of a space X will be as always symbolize B^{*}c O (X) (resp. B^{*}cc(X)).

Example (2.14):

In example (2.9). Note that closed set in X are:

∅, X, {2,3}, {1,3}, {3}. Then

B^{*}c O (X) = {∅, X, {2,3}, {1,3} }

Remark (2.15):

If A B^{*}c – open set in X, then A^{C} is B^{*}c – closed set in X.

Remark (2.16):

From definition (2.13). Note that:

i) Every B^{*}c – open set is β- open set.

ii) Every B^{*}c – closed set is β- closed set.

The converse of above Remark is not true in general.

Example (2.17):

Let X = {a, b, c}, t = {∅, X, {a}, {b, c}}.

β o(X) = {∅, X, {a}, {b}, {c}, {a, b}, {a, c}, {b,c} }.

βc(X) = βo (X)

B^{*}c O (X) = {∅, X, {a}, {b, c} }.

B^{*}c C(X) = B^{*}c O (X). Not that A = {c} is β- open (resp. β- closed) ser. but not B^{*}c– open (resp. B^{*}c – closed) set.

Remark (2.18):

i) The B^{*}c – open set and open set are in-dependent.

ii) The B^{*}c – closed set and closed set are in-dependent.

Example (2.19):

In example (2.9) not that B^{*}co (x) = {∅, X, {1,3}, {2,3}, B^{*}cc(x) = {∅, X, {1}, {2}}. Note that

i) A = {2,3} B^{*}c – open set, but not open and B = {1} is open, but not B^{*}c – open.

ii) A = {2} B^{*}c – closed set, but not closed and B = {3} is closed set, but not B^{*}c – closed.

Proposition (2.20):

Let X be atop. Sp. and A ⊆ X. If A α – closed. Then A β- open in X iff A B^{*}c – open.

Proof:

Suppose that A a β-open set in X, then $A \subseteq \overline{\bar{A}^{o}}$. Let $x \in A \subseteq \overline{\bar{A}^{o}}$. Since $x \in \overline{\bar{A}^{o}}$ and A α – closed set, then $\overline{\bar{A}^{o}} \subseteq A$. Thus $x \in \overline{\bar{A}^{o}} \subseteq A$, ∃ $\overline{\bar{A}^{o}}$ closed set ∋ $x \in \overline{\bar{A}^{o}} \subseteq A$. Then A B^{*}c – open set.

Conversely

Suppose that A B^{*}c – open set, then by definition (2.13), we get A β- open.

Corollary (2.21):

If A open set and α – closed, then A B^{*}c – open.

Proof:

By proposition (2.5) (i) and proposition (2.20).

Proposition (2.22):

Let X be atop. Sp. and A ⊆ X. If A α – open. Then A β- closed iff A B^{*}c – closed.

Proof:

Let A be β- closed, then A^{C} β- open. Since A α – open, then A^{C} α – closed, then by proposition (2. 20), we get A^{C} B^{*}c – open set. There fore A β-closed set.

Corollary (2.23):

If A closed set and α – open, then A B^{*}c – closed.

Proof:

By proposition (2.5) (ii) and (2.22)

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Proposition (2.24):

Let X be atop. Sp. X. Then

i) The union family of B^*c – open set is B^*c- open set.

ii) The intersection family B^*c – closed set is B^*c – closed set.

Proof:

i) Let $\{ A_\alpha : \alpha \in \Lambda \}$ be a family of B^*c – open sets, then $\{A_\alpha : \alpha \in \Lambda\}$ is β- open sets, then $\bigcup_{\alpha\in\Lambda} A_\alpha$ is β- open set by lemma (2.7) (i). Let $x \in \bigcup_{\alpha\in\Lambda} A_\alpha$, then $x \in A_\alpha$ for some $\alpha \in \Lambda$. Since A_α B^*c- open set $\forall \alpha \in \Lambda$, then ∃ F closed set in x ∋ x $\in F \subseteq A_\alpha \subseteq \bigcup_{\alpha\in\Lambda} A$. Then for $\bigcup_{\alpha\in\Lambda} A_\alpha$ is B^*c- open set.

ii) Let $\{ A_\alpha : \alpha \in \Lambda \}$ be a family of B^*c – closed sets, then $\{A^c_\alpha : \alpha \in \Lambda\}$ is a family of B^*c- open sets, then $\bigcup_{\alpha\in\Lambda} A^c_\alpha$ is B^*c- open sets by (i), them $[\bigcup_{\alpha\in\Lambda} A^c_\alpha]$ B^*c – closed. But $\bigcap_{\alpha\in\Lambda} A_\alpha = [\bigcup_{\alpha\in\Lambda} A^c_\alpha]^c$, then $\bigcap_{\alpha\in\Lambda} A_\alpha$ is B^*c – closed in X.

Remark (2.25):

i) Not every intersection of two B^*c – open set is B^*c – open set.

ii) Not every union of two B^*c – closed set is B^*c – closed set.

Example (2.26):

In example (2.9)

B^*c O (X) = {∅, X, {2,3}, {1,3} }.

B^*c C (X) = {∅, X, {1}, {2} }. Not that:

i) Let A = {1,3}, B = {2,3} are B^*c – open set, but A ∩ B = {3} not B^*c – open set in X.

ii) Let A = {1}, B = {2} are B^*c – closed set, but A ∪ B = {1,2} not B^*c – closed set in X.

Definition (2.27):

Let A subset of top. Sp. Then A is called:

i) Clopen set if A closed and open.[6]

ii) β- Clopen set if A β- closed and β- open.

iii) B^*c - Clopen set if A B^*c - closed and B^*c – open.

Proposition (2.28):[4]

Let X be atop. Sp. and A ⊆ X. Then

i) Every closed set is α – closed set.

ii) Every open set is α – open set.

Proposition (2.29):

Let X be atop. Sp. Then:

i) The union B^*c – open set and clopen set is B^*c – open.

ii) The intersection B^*c- closed set and clopen set is B^*c – closed.

Proof:

i) Let A B^*c – open set, then A^C B^*c- closed. Let B clopen, then B^C clopen, then B^C closed and open. Since B^C closed, then B^C β- closed. Since B^C open, then B^C α – open by proposition (2.28) (ii), then B^C B^*c – closed, then $A^C \cap B^C$ B^*c – closed by proposition (2.22) (ii), then $(A^C \cap B^C)^C$ B^*c – open. But $A \cup B = (A^C \cap B^C)^C$, there for A ∪ B B^*c – open set in X.

ii) Let A B^*c – closed, then A^C B^*c – open. Let B clopen, then B^C clopen, then by (i), we get $A^C \cup B^C$ B^*c – open, then $(A^C \cup B^C)^C$ B^*c – closed. But $A \cap B = (A^C \cup B^C)^C$, then A ∩ B B^*c – closed.

Proposition (2.30):

Let X be atop. Sp. Then:

i) The intersection B^*c – open set and clopen set is B^*c – open set.

ii) The union B^*c- closed set and clopen set is B^*c- closed.

Proof:

i) Let A be B^*c – open set and B clopen, then B open and closed, then A β- open set and B open, then A ∩ B is β- open set by (2.11)(i). Let x ∈ A ∩ B, then x ∈ A and x ∈ B, then ∃ F closed set in x ∋ x ∈ F ⊆ A. Since F ∩ B is closed set in x, then x ∈ F ∩ B ⊆ A ∩ B, hence A ∩ B B^*c – open.

ii) Let A B^*c – closed set, then A^C B^*c- open. Let B clopen in X, then B^C clopen, then by (i) we get $A^C \cap B^C$ B^*c- open in X, then $(A^C \cup B^C)^C$ B^*c – closed. But $A \cup B = (A^C \cap B^C)^C$, then A ∪ B B^*c – closed in X.

The following diagram shows the relation among types of open, closed sets.

(a) (b)

Fig (1)

Definition (2.31):

Let F: X → Y be a function and A ⊆ X. Then:

i) F is called continuous function [6]. If ∀A open subset of Y, then F^{-1} (A) is open subset of X.

ii) F is called β- continuous function. If ∀A open subset of Y , then F^{-1} (A) is β- open subset of X.[1]

iii) F is called B^*c-continuous function. If ∀A open subset of Y , then F^{-1} (A) is B^*c - open subset of X.

Proposition (2.32):

Let F: X → Y be a function and A ⊆ X. Then:

i) Every cont. function is a β- cont.

ii) Every B^*c -cont. function is a β- cont.

Proof:

Let F: X → Y be a function

i) Let F cont. and Let A be open in Y. Since F is cont. function , then F^{-1} (A) is open in X, then F^{-1} (A) is a β- open in X. Hence F is a β- cont.

ii) Let F B^*c -cont. and Let A be open in Y. Since F B^*c -cont. function then F^{-1} (A) B^*c – open in X, then F^{-1} (A) β- open in X, hence F is a β- cont.

The converse of above proposition is not true in general.

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Example (2.33)

Let F: X → Y be a function and let X = {1, 2, 3}

t = {∅, X, {1}, {2,3} }.

βo(X) = {∅, X, {1}, {2}, {3}, {1,2}, {1,3}, {2,3} }.

B^*c o (X) = {∅, X, {1}, {2,3} }.

Y = {a, b, c}, ŧ = {∅, Y, {a}, {b}, {a, b} }.

βo(Y) = {∅, Y, {a}, {b}, {a, b}, {a, c}, {b ,c}}.

B^*c o (Y) = {∅, Y, {a, c}, {b, c} }.

Define F(1) = a, F(2) = b, F(3) = C.

Note that F is β- cont. But

i) F not cont. Since A ={b} open in Y, but F^{-1}(A) not open in X.

ii) F not B^*c - cont. Since A = {b} open in Y, but F^{-1}(A) not B^*c- open in X.

Remark (2.34):

The continuous function and B^*c-continuous are independent in general.

Example (2.35):

Let F: X → Y be a function

Let X = {1, 2, 3}, t = {∅, X, {1}, {2}, {1,2}, {1,3}}, βo(x) = t.

B^*co (x) = {∅, X, {2}, {1,3} }.

Y = {a, b, c}, ŧ = {∅, Y, {a}, {b}, {a, b} }.

βo(Y) = {∅, Y, {a}, {b}, {a, b}, {a, c}, {b, c}}.

B^*co (Y) = {∅, Y, {a, c}, {b, c} }.

Define F(1) = a, F(2) = b, F(3) = C.

Note that F is cont. function, but not B^*c - cont. function. Since A={a} open in Y, but F^{-1}(A) not B^*c- open in X.

Example (2.36)

Let F: X → Y be a function and Let X = {1, 2, 3}.

t = {∅, X, {1}, {3}, {1,3}}, βo(X) ={∅, X, {1}, {3}, {1,2}, {1,3}, {2,3}}.

B^*c o (X) = {∅, X, {1,2}, {2,3} }.

Y = {a, b, c}, ŧ = {∅, Y, {a, b} }.

Define F(1) = a, F(2) = b, F(3) = C.

Note that F is B^*c - cont. Since A = {a, b} is open in Y, but F^{-1}(A) not open in X.

The following diagram shows the relation among type of the continuous function.

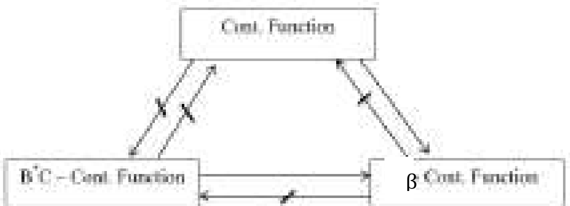

Fig (2)

3- The Closure:

Definition (3.1): [1]

The intersection of all β- closed set of atop. Sp. X which is containing A is called a β- closure of A and denoted by $\overline{A}^{\beta}$.

i. e $\overline{A}^{\beta}$ = ∩{ F: A ⊆ F, F is β- closed in X}.

Definition (3.2):

The intersection of all B^*c - closed set of atop. Sp. X which is containing A is said a B^*c - closure of A and denoted by $\overline{A}^{B^*C}$.

i. e $\overline{A}^{B^*C}$= ∩{ F: A ⊆ F, F is B^*c - closed in X}.

Lemma (3.3):

Let X be atop. Sp. and A ⊆ X. Then

i) X ∈ $\overline{A}^{\beta}$ iff ∀ β- open set G and x ∈ G ∋ G ∩ A ≠ ∅ [1].

ii) X ∈ $\overline{A}^{B^*C}$ iff ∀ B^*c - open set G and x ∈ G ∋ G ∩ A ≠ ∅.

Proof:

ii) Let x ∉ $\overline{A}^{B^*C}$, then x ∉ ∩ F ∋ F is B^*c – closed set and A ⊆ F, then x ∈ $[∩F]^C$ ∋ $[∩F]^C$ is B^*c – open containing x. Hence

$[∩F]^C$ ∩A ⊆ $[∩F]^C$ ∩ [∩F] = ∅.

Conversely

Suppose that ∃ a B^*c – open set G ∋ x ∈ G and A ∩ G = ∅., then A ⊆ G^c ∋ G^c is B^*c – closed set, hence x ∉ $\overline{A}^{B^*C}$

Remark (3.4):

Let X be a topological space and A ⊆ X. Then

i) $\overline{A}^{\beta}$ is β- closed set and $\overline{A}^{B^*C}$ is B^*c – closed set.

ii) $\overline{A}^{\beta}$ (resp. $\overline{A}^{B^*C}$) is the smallest β- closed (resp. B^*c – closed) set containing A.

iii) A ⊆ $\overline{A}^{\beta}$ also A ⊆ $\overline{A}^{B^*C}$. ∀ A ⊆ X.

Proof:

Clear.

Proposition (3.5):

Let X be a top. Sp. X and A ⊆ X. Then:

i) A β- closed set iff A = $\overline{A}^{\beta}$ [1].

ii) A B^*c – closed set iff A = $\overline{A}^{B^*C}$.

Proof:

ii) Let A be B^*c – closed set

Let X ∉ A, then X ∈ A^C, then ∃ B^*c – open set A^C ∋ A^C ∩ A = ∅, then X ∉ $\overline{A}^{B^*C}$, then $\overline{A}^{B^*C}$ ⊆ A. Since A ⊆ $\overline{A}^{B^*C}$ by Remark (3.4) (iii). Hence A = $\overline{A}^{B^*C}$.

Conversely

Let A = $\overline{A}^{B^*C}$. Since $\overline{A}^{B^*C}$ B^*c – closed set in X and A = $\overline{A}^{B^*C}$, then A B^*c – closed set.

Proposition (3.6):

Let X be a top. Sp. X and A ⊆ X. Then:

i) $\overline{\overline{A}^{\beta}}^{\beta}$ = $\overline{A}^{\beta}$ [1].

ii) If A ⊆ B, then $\overline{A}^{\beta}$ ⊆ $\overline{B}^{\beta}$.

Proof:

ii) Let A ⊆ B. Since B ⊆ $\overline{B}^{\beta}$ by Remark (3.4) (iii), then A ⊆ $\overline{B}^{\beta}$.

Since $\overline{B}^{\beta}$ is β- closed in X and $\overline{A}^{\beta}$ is the smallest β- closed set containing A. There for $\overline{A}^{\beta}$ ⊆ $\overline{B}^{\beta}$.

Proposition (3.7):

Let X be a top. Sp. X and A ⊆ X. Then

i) $\overline{\overline{A}^{B^*c}}^{B^*c} = \overline{A}^{B^*C}$.

ii) If A ⊆ B, then $\overline{A}^{B^*C} \subseteq \overline{B}^{B^*C}$.

Proof:

i) Since $\overline{A}^{B^*C}$ is B^*c – closed, then by proposition (3.5) (ii), we get the result.

ii) Let A ⊆ B. Since B ⊆ $\overline{B}^{\beta}$by Remark (3.4) (iii), then A ⊆ $\overline{B}^{B^*C}$.

Since $\overline{B}^{B^*C}$ is B^*c – closed and $\overline{A}^{B^*C}$ is the smallest B^*c – closed containing A. Then fore $\overline{A}^{B^*C} \subseteq \overline{B}^{B^*C}$.

Proposition (3.8)

Let F: X → Y be function. Then the following statements are equivalent.

i) F is β- continuous.

ii) F^{-1} (B) is β- closed in X ∀ B is closed set in Y.

iii) F ($\overline{A}^{\beta}$) ⊆ $\overline{F(A)}$ ∀ A ⊆ X.

iv) $\overline{F^{-1}(B)}^{\beta} \subseteq F^{-1}(\overline{B})$ ∀ B ⊆ Y.

Proof:

(i) ——— (ii)

Let B be closed set in Y, then Y–B is open set in Y, then F^{-1} (Y–B) is a β- open in X by (i), then X – F^{-1} (B) is a β- open in X.

Then F^{-1} (B) is a β- closed in X.

(ii) ——— (iii)

Let A ⊆ X, then F(A) ⊆ Y, then $\overline{F(A)}$ is closed set in Y, then F^{-1} ($\overline{F(A)}$) is a β- closed set in X by (ii). Since F(A) ⊆ $\overline{F(A)}$, Then A ⊆ F^{-1} ($\overline{F(A)}$), then $\overline{A}^{\beta} \subseteq F^{-1}(\overline{F(A)})$, there fore F ($\overline{A}^{\beta}$) ⊆ $\overline{F(A)}$.

(iii) ——— (iv)

Let B ⊆ Y, then F^{-1}(B) ⊆ X, then $F[\overline{F^{-1}(B)}^{\beta}] \subseteq$ F $[\overline{F^{-1}(B)}]$ by (iii), then F $[\overline{F^{-1}(B)}^{\beta}] \subseteq (\overline{B})$, then $\overline{F^{-1}(B)}^{\beta} \subseteq F^{-1}(\overline{B})$.

(iv) ——— (i)

Let B be open set in Y, then Y – B is closed set in Y. Then

$\overline{F^{-1}(Y-B)}^{\beta} \subseteq F^{-1}(\overline{Y-B}) = F^{-1}(Y-B)$. Since F^{-1} (Y – B) = X – F^{-1} (B) is a β- closed set in X, then F^{-1} (B) is a β- open set in X.

There fore F is a β- continuous.

Proposition (3.9)

Let F: X → Y be a function. Then the following statements are equivalent.

i) F is BC- continuous.

ii) F^{-1} (B) is B^*c - closed in X ∀ B is closed set in Y.

iii) F ($\overline{A}^{B^*C}$) ⊆ $\overline{F(A)}$ ∀ A ⊆ X.

iv) $\overline{F^{-1}(B)}^{B^*c} \subseteq F^{-1}(\overline{B})$ ∀ B ⊆ Y.

Proof:

(i) ——— (ii)

Let B be closed set in Y, then Y–B is open set in Y, then F^{-1} (Y–B) is a B^*c - open in X by (i), then X – F^{-1} (B) is a B^*c - open in X.

Hence F^{-1} (B) is a B^*c - closed in X.

(ii) ——— (iii)

Let A ⊆ X, then F(A) ⊆ Y, then $\overline{F(A)}$ is closed set in Y, then F^{-1} ($\overline{F(A)}$) is a BC- closed set in X by (ii). Since F(A) ⊆ $\overline{F(A)}$, Then A ⊆ F^{-1} ($\overline{F(A)}$), then $\overline{A}^{B^*C} \subseteq F^{-1}(\overline{F(A)})$, hence F ($\overline{A}^{B^*C}$) ⊆ $\overline{F(A)}$.

(iii) ——— (iv)

Let B ⊆ Y, then F^{-1}(Y) ⊆ X, then $\overline{F[F^{-1}(B)]}^{B^*c} \subseteq F[\overline{F^{-1}(B)}]$, then $\overline{F[F^{-1}(B)]}^{B^*c} \subseteq \overline{B}$, then $\overline{F^{-1}(B)}^{B^*c} \subseteq F^{-1}(\overline{B})$.

(iv) ——— (i)

Let B be open set in Y, then Y – B is closed set in Y. Then

$\overline{F^{-1}(Y-B)}^{B^*c} \subseteq F^{-1}(\overline{Y-B}) = F^{-1}(Y-B)$, then F^{-1} (Y – B) = X – F^{-1} (B) is a BC- closed set in X. F^{-1} (B) is a BC- open set in X.

There fore F is a B^*c - continuous.

Definition (3.10):

Let F: X → Y be function and A ⊆ X. Then:

i) F is called open (resp. closed) [6] . If ∀ A open (resp. closed), subset of X, then F(A) is open (resp. closed) subset of Y.

ii) F is called β- open (resp. β- closed). If ∀ A open (resp. closed), subset of X, then F(A) is β- open (resp. β- closed) subset of Y.

iii) F is called B^*c - open (resp. B^*c - closed). If ∀ A open (resp. closed), subset of X, then F(A) is B^*c - open (resp. B^*c - closed) subset of Y.

Proposition (3.11):

Let F: X → Y be a function and A ⊆ X. Then:

i) Every open function is β- open.

ii) Every closed function is β- closed.

iii) Every B^*c – open function is β- open.

iv) Every B^*c – closed function is β- closed.

Proof:

i) Let F: X → Y be a function.

Suppose that F open function and let A open in X. Since F open, then F(A) open in Y, then F(A) β- open in Y. Thus F is β- open.

ii) Similarly part (i).

iii) Suppose F is B^*c - open function and let A open in X. Since F B^*c - open, then F(A) B^*c - open in Y, then F(A) β- open in Y. Thus F is β- open.

iv) Similarly part (iii).

The Converse above proposition is not true in general.

Example (3.12):

In example (2.34)

Closed set in X are: ∅, X, {2,3},{1}.

Closed set in Y are: ∅, Y: ∅, X, {b, c}, {a,c},{c}.

βc(Y) = {∅, Y, {b, c}, {a, c}, {c},{b}, {a} }.

B^*c (Y) = {∅, Y, {b}, {a} }.

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Not that:

i) F β- open, but not open since A = {2, 3} open in X, but F(A) not open in Y.

ii) F β- closed, but not closed. Since A = {1} closed set in X, but F(A) not closed set in Y.

iii) F β- open, but not B^*c – open. Since A = {1} open in X, but F(A) not B^*c – open set in Y.

iv) F β- closed, but not B^*c – closed. Since A = {2, 3} is closed in X, but F(A) not B^*c – closed in Y.

Remark (3.13):

i) The open function and B^*c – open function are independent.

ii) The closed function and B^*c – open function are independent.

We can showing that with two the following examples.

Example (3.14):

i) Let F: X → Y be function and let X = { a, b, c}
t = {∅, X, {b}, {b, c}}, βo (x) = {∅, X, {b}, {a, b}, {b, c}}.
B^*co (X) = {∅, X}.
Y = {1, 2, 3}, ť = {∅, Y, {2}, {3}, {2, 3}, {1, 2}}.
βo(Y) = {∅, Y, {2}, {3}, {1, 2}, {2, 3} }.
B^*co (Y) = {∅, Y, {3}, {1, 2}}.

Define F(a) = 1, F(b) = 2, F(c) = 3.

ii) Let F: X → Y be a function and let X = { a, b, c}
t = {∅, X, {b, c}}, βo(x) = {∅, X, {b}, {c}, {a, b}, {a, c}, {b, c}}.
B^*co (X) = {∅, X}.
Y = {1, 2, 3}, ť = {∅, Y, {1}, {3}, {1, 3}}.
βo(Y) = {∅, Y, {1}, {3}, {1, 3}, {2, 3} }, B^*co (Y) = {∅, Y, {2, 3}}.

Define F(a) = 1, F(b) = 2, F(c) = 3.

In example (i). Note that:

1) F β- open, but not B^*c – open. Since A = {b} open in X, but F(A) not B^*c – open set in Y.

2) F b^*-closed, but not B^*c–closed. Since A= {a} closed set in X, but F(A) not B^*c–closed in Y.

In example (ii). Note that:

1) F B^*c – open, but not open. Since A = {b, c} open in X, but F(A) not open set in Y.

2) F B^*c – closed, but not closed. Since A = {a} closed in X, but F(A) not closed in Y.

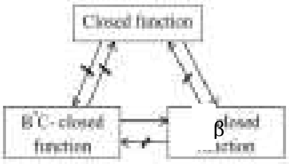

Fig (5)

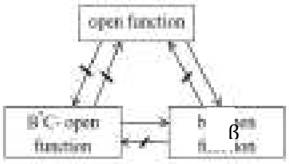

Fig (4)

4- The interior:

Definition (4.1):[1]

The union of all β - open set of atop. Sp. X contained in A is called β- interior of A and denoted $A^{o\beta}$.

i.e

$A^{o\beta} = \bigcup$ { U : U ⊆ A and U β- open set in X}.

Definition (4.2):

The union of all B^*c- open set of atop. Sp. X contained in A is called B^*c - interior of A and denoted A^{oB^*c}.

i.e

$A^{oB^*c} = \bigcup$ { U : U ⊆ A and U BC- open set in X }.

Proposition (4.3):

Let X be atop. Sp. and A ⊆ X. Then:

i) X ∈ $A^{o\beta}$ iff ∃ G β- open in X∋ x ∈G ⊆ A. [1]

ii) X ∈ A^{oB^*C} iff ∃ G B^*c-open in X∋ x ∈ G ⊆ A.

Proof:

ii) Let X ∈ A^{oB^*C}

Since A^{oB^*C} = ⋃ { G : G ⊆ A, G is B^*c - open set in X }.

Then z ∈ ⋃ {G: G ⊆ A, G is B^*c - open set in X}.

Then ∃ G B^*c – open in X ∋ X ∈ G ⊆ A.

Conversely

Let X ∈ G ⊆ A and G is B^*c – open in x ∈ G ⊆ A. Then

X ∈ U { G : G ⊆ A, G is B^*c - open set in X }.

There fore X ∈ A^{oB^*C}.

Remark (4.4):

Let x be atop. Sp. and A ⊆ X. Then:

i) $A^{o\beta}$ is β- open set and A^{oB^*C} is B^*c - open set.

ii) $A^{o\beta}$ (resp. A^{oB^*C}) is the largest β- open (resp. B^*c – open) set contained A.

iii) $A^{o\beta}$ ⊆ A also A ⊆ A^{oB^*C}.

Proof:

Clear.

Lemma (4.5): [1]

Let X be atop. Sp. and A ⊆ X. Then

i) $[A^{o\beta}]^C = \overline{A^C}^{\beta}$.

ii) $[\overline{A}^{\beta}]^C = A^{C^{o\beta}}$

Remark (4.6):

i) $[A^{oB^*C}]^C = \overline{A^C}^{B^*C}$

ii) $[\overline{A}^{B^*C}]^C = A^{C^{oB^*C}}$.

Proposition (4.7):

Let X be atop. Sp. and A ⊆ X. Then:

i) A β- open set iff A = $A^{o\beta}$.[1]

ii) i) A B^*c - open set iff A = A^{oB^*C}.

Proof:

ii) Let A B^*c – open set, then A^C B^*c – closed set, then by Remark (3.4) (iii), we have $A^C = \overline{A^C}^{B^*C}$. Since $\overline{A^C}^{B^*C} = [A^{oB^*C}]^C$ by Remark (4.6) (ii), then $A^C = [A^{oB^*C}]^C$, hence $A = A^{oB^*C}$.

Conversely

Supposedly that $A = A^{oB^*C}$.

Since A^{oB^*C} is B^*c – open set and $A = A^{oB^*C}$, then B^*c – open set.

Proposition (4.8):

Let X be atop. Sp. and A, B ⊆ X. Then

i) $[A^{o\beta}]^{o\beta} = A^{o\beta}$ [1].

ii) If A ⊆ B, then $A^{o\beta} \subseteq B^{o\beta}$.

Proof:

ii) Let A ⊆ B. Since $A^{o\beta} \subseteq A \subseteq B$, then $A^{o\beta} \subseteq B$. Since $B^{o\beta}$is the largest β- open set contained B, them $A^{o\beta} \subseteq B^{o\beta}$.

Proposition (4.9):

Let X be atop. Sp. and A, B ⊆ X. Then

i) $[A^{oB^*C}]^{oB^*C} = A^{oB^*C}$.

ii) If A ⊆ B, then $A^{oB^*C} \subseteq B^{oB^*C}$.

Proof:

i) Since A^{oB^*C} is BC– open set, then $A^{oB^*C} \subseteq B$. Since B^{oB^*C} is the largest B^*c – open set contained B, then $A^{oB^*C} \subseteq B^{oB^*C}$.

Proposition (4.10):

Let F: X → Y be function. Then the following statement are equivalent.

i) F β- open function.

ii) $F(A^\circ) \subseteq [F(A)]^{o\beta}$ ∀ A ⊆ X.

iii) $[F^{-1}(A)]^\circ \subseteq F^{-1}(A^{o\beta})$ ∀ A ⊆ Y.

Proof:

i) ------- ii)

Let A ⊆ X. Since A° open in X, then F(A°) β- open in Y by (i). Then $F(A^\circ) = [F(A^\circ)]^{o\beta} \subseteq [F(A)]^{o\beta}$. Hence $F(A^\circ) \subseteq [F(A)]^{o\beta}$.

ii) ------- (iii)

Let A ⊆ Y, then $F^{-1}(A) \subseteq X$, then $F[(F^{-1}(A^\circ))^o] \subseteq [F(F^{-1}(A))]^{o\beta}$by (ii). Then $F[(F^{-1}(A))^\circ] \subseteq A^{o\beta}$. Then $[F^{-1}(A)]^\circ \subseteq F^{-1}(A^{o\beta})$.

iii) ------ (i)

Let A open in X, then A = A°. Let F (A) ⊆ Y, then

$[F^{-1}(F(A))]^\circ \subseteq F^{-1}[(F(A))^{o\beta}]$, by (iii). Then $A = A^\circ \subseteq F^{-1}[(F(A))^{o\beta}]$, then $F(A) \subseteq [F(A)]^{o\beta}$. But $[F(A)]^{o\beta} \subseteq F(A)$, then $F(A) = [F(A)]^{o\beta}$. Hence F (A) β- open in Y, there fore F β- open function.

Proposition (4.11):

Let F: X → Y be function. Then the following statement are equivalent.

i) F B^*c – open function.

ii) $F(A^\circ) \subseteq [F(A)]^{oB^*C}$ ∀ A ⊆ X.

iii) $[F^{-1}(A)]^\circ \subseteq F^{-1}(A^{oB^*C})$ ∀ A ⊆ Y.

Proof:

i) ------- ii)

Let A ⊆ X. Since A° open in X, then F(A°) B^*c - open in Y by (i). Then $F(A^\circ) = [F(A^\circ)]^{oB^*C} \subseteq [F(A)]^{oB^*C}$. Hence $F(A^\circ) \subseteq [F(A)]^{oB^*C}$.

ii) ------- (iii)

Let A ⊆ Y, then $F^{-1}(A) \subseteq X$, then $F[(F^{-1}(A^\circ))^o] \subseteq [F(F^{-1}(A))]^{oB^*C}$ by (ii). Then $F[(F^{-1}(A))^\circ] \subseteq A^{oB^*C}$, hence $[F^{-1}(A)]^\circ \subseteq F^{-1}(A^{oB^*C})$.

iii) ------ (i)

Let A open in X, then A = A°. Let F (A) ⊆ Y, then

$[F^{-1}(F(A))]^\circ \subseteq F^{-1}[(F(A))^{oB^*C}]$, by (iii). Then $A = A^\circ \subseteq F^{-1}[(F(A))^{oB^*C}]$, then $F(A) \subseteq [F(A)]^{oB^*C}$. But $[F(A)]^{oB^*C} \subseteq F(A)$, then $F(A) = [F(A)]^{oB^*C}$. Hence F (A) BC- open in Y, there fore F Bc- open function.

Proposition (4.10):

A function F: X → Y is a β- closed iff $\overline{F(A)}^{\beta} \subseteq F(\bar{A})$ ∀ A ⊆ X.

Proof:

Suppose F is a β- closed. Let A ⊆ X, then $\bar{A}$ closed in X, then F ($\bar{A}$) is a β- closed in Y.

Then $\overline{F(A)}^{\beta} \subseteq \overline{F(\bar{A})}^{\beta} = F(\bar{A})$.

Conversely

Let A be closed in X, then $A = \bar{A}$ Since $\overline{F(A)}^{\beta} \subseteq F(\bar{A}) = F(A)$, then $\overline{F(A)}^{\beta} \subseteq F(A)$. But $F(A) \subseteq \overline{F(A)}^{\beta}$, then $F(A) = \overline{F(A)}^{\beta}$. There fore F(A) is a β- closed set in Y. Hence F is a β- closed.

Proposition (4.11):

A function F: X → Y is a B^*c- closed iff $\overline{F(A)}^{B^*c} \subseteq F(\bar{A})$ ∀ A ⊆ X.

Proof:

Suppose that F is a B^*c - closed.

Let A ⊆ X, then $\bar{A}$ closed set in X, then F ($\bar{A}$) is a B^*c - closed in Y.

Then $\overline{F(A)}^{B^*C} \subseteq \overline{F(\bar{A})}^{B^*C} = F(\bar{A})$.

Conversely

Let A be closed in X, then $A = \bar{A}$ Since $\overline{F(A)}^{B^*C} \subseteq F(\bar{A}) = F(A)$, then $\overline{F(A)}^{B^*C} \subseteq F(A)$. But $F(A) \subseteq \overline{F(A)}^{B^*C}$, then $F(A) = \overline{F(A)}^{B^*C}$. There fore F(A) is a B^*c - closed set in Y. Hence F is a B^*c - closed.

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حول خصائص المجموعة B^*c– open set

رعد عزيز العبد الله كريم فاضل العمري

قسم الرياضيات
كلية علوم الحاسوب وتكنلوجيا المعلومات
جامعة القادسية

المستخلص:

في هذا البحث قدمنا صنف جديد من المجموعات يسمى B^*c– open set تم دراسته والتعرف على خواص وايجاد العلاقات مع المجموعات الاخرى ودراسة صنف جديد من الدوال يسمى B^*c– continuous ، B^*c– open function، B^*c– closed function. حيث حصلنا على بعض النتائج التي تظهر العلاقة بين المجموعات من خلال النظريات التي تم الحصول عليها باستخدام المجموعة من النمط B^*c– open.

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Modification Of High Performance Training Algorithms for Solve Singularly Perturbed Volterra integro-differential and integral equation

Khalid. Mindeel. M. Al-Abrahemee

Department of Mathematics , College of Education , University of AL-Qadisiyha
Email: Khalid.mohammed@qu.edu.iq

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Abstract:

In this paper, we apply neural network for solve singularly perturbed Volterra integro-differential equations (SPVIDE) and singularly perturbed Volterra integral equations (SPVIE). Using Modification Of High Performance Training Algorithms such as quase-Newton, Levenberge-Marqaurdt, and Baysian Regulation. The proposed method was compared with the standard training algorithms and analytical methods. We found that the proposed method is characterized by high accuracy in the results, a lower error rate and a speed that is much convergent to standard methods.

Key words: Singularly perturbed Volterra integro-differential equations , Singularly perturbed Volterra integral equations ,Singularly perturbed problems, neural network.

Mathematical subject classification : 34K28

1. Introduction:

In this paper we consider the numerical discretization of (SPVIDE):

$$\varepsilon\mho''(x)\, f\left(\mho,\mho',\varkappa\right)\int_0^x \mathcal{M}(\varkappa,t)\mho(t)dt\;,\varkappa\in[a,b] \quad (1)$$

with boundary condition $\mho(a) = A\,,\mho(b) = B$.
And Volterra integral equations (SPVIE)

$$\varepsilon\mho(x) = f(\varkappa) + \int_0^x \mathcal{M}(\varkappa,t)\mho(t)dt \quad , \quad \varkappa\in[a,b] \quad (2)$$

with boundary condition $\mho(a) = A\,,\mho(b) = B$, where ε is a small parameter satisfying $0 \ll \varepsilon < 1$ called perturbation parameter , f and $\mathcal{M}$ are given.

An increasing interest in Voltaire's integrative and differential equations that have small parameters that stimulate this research. In the numerical solution to the problems of the value of the single hyper-limit of ordinary differential equations.

Nonlinear phenomena that appears in many applicationes in scientifec fields, such as fluids dynamics, solid state physics, plasma physics, mathematics biology and chemical, can be modeled by integral equations.. We are frequently faced with the problem of determining the solution of integral equations, one of these integral equations is SPVIE [1].

IN [2] have proposed the HPM for solving the SPVIEs, Alquran and Khair [3] solved the same problem by DTM and VIM. Finally, Dogan et al. [4] used DTM to solve the presented problem. Liao [5],[6],[7] successfully applied the HAM to solve many types of nonlinear problems.

This paper focus on building a new technique by using neural networks to arrive at an approximate solution to the integrated integrative and differential integrality equation. This structure of artificial neural networks (ANN) can calculate the corresponding production of vector inputs. The error function is now limited to the minimum in the selection points. Thus, the proposed ANN uses a training algorithm based on on quasi-Newton, Levenberg-Marquardt, and Bayesian Regulation algorithms used to modify the parameters (weights and biases) to any degree of accuracy required.

2. Modification Of High Performance Training Algorithms:

In this section we will explicate how to modify some of the training algorithms. And to avoid some disadvantage that occurs in LM algorithm, we imply singular value decompation(S V D)of Jacobian matrix ξ *and* ξ^{-1} *if* $\xi(w)$ *is* a rectanguler matrix or singular, then we use SVD of$\xi(w)$. To avoid some disadvantage that occurs in of quasi– Newton algorithm, we explicate Singular Value Decomposition of Hessian matrix H*and* H^{-1}, which is the mainly extensively use technique, and specially, it is a high-quality technique for illconditioned problems and to calculate the pseudoinverse ofH and to calculate theminimum error. To over passing the drawbackof BayesianRegulation algorithm, we imply S V D for compute ξ *and* ξ^{-1}, which is a good quality technique to calculate the pseudoinverse of ξ and to calculate the minimumerror.

3. Description of the Method

In this part we will explicate how our appear be able to use find the approximate solution of equations (1) and (2). To enter y (x) to the converter to be calculated, $\mho_\tau(\varkappa_i,\rho)$Refers to the analytical solution. In the proposed approach, the FFNF experimental solution is used and the parameter p correspond to the W_i *and* B_i of the neural architecture. We choose a model for a pilot function $\mho_\tau(\varkappa)$ to meet BC requirements. This is acheived by writing it as two parts :

$$\mho_\tau(\varkappa_i,\rho) = \mathrm{A}(\varkappa) + \mathcal{F}(\varkappa,\mathbb{N}(\varkappa,\rho,\mathcal{E}))(3)$$

where $\mathbb{N}(\varkappa,\rho,\mathcal{E})$ is the output of the neural network with one input vector $\varkappa$.

4. Illustration of the Method

To show the technique, we will consider the equations ((1) and (2)) ,where x ∈ [0, 1] and the BC Ʊ (0) = A and Ʊ (1) = B . The approximate solution can be written:

$$\mho_\tau(\varkappa_i,\rho) = A + (A+\mathcal{B})\varkappa + \varkappa(\varkappa - 1)\mathbb{N}(\varkappa,\rho,\varepsilon) \qquad (4) \ ,$$

And the error quantety to be minimzed is given by

$$E[p] = \{\varepsilon \frac{d^2\mho_\tau}{d\varkappa^2}\sum_{i=1}^{n} f\left(\mho_i, \mho_t^{'}, \varkappa_i\right) + \int_0^x \mathcal{M}(\varkappa_i,t)\mho_\tau(t)dt\}^2 \ , \qquad (5)$$

where the xi's are points in [a, b].

5. Numerical examples

In this section we will present the numerical results of some mathematical models from the various examples of some cases of numerical conditions of the proposed neural network where the structure of the network consists of three layers. . In hidden layers we used sigmoid(logsig) as an activation function that is $\sigma(\varkappa) = \frac{1}{1+e^{-\varkappa}}$. or each test problem, the analytical solution $y_a(x)$was defined in proceed. The accuracy of the approximate solutions can be tested by using the following equation $\Delta\mho(x) = |\mho_t(\varkappa) - \mho_a(x)|$.

In this section, there are three examples of different cases and each example contains four tables.

The first table shows the experimental and approximate solution of the high-level training algorithms with the analytical solution. The second table shows the error of the modified method and the third table represents the accuracy of the proposed method and the number of iterative needed to reach the target.

And the fourth table represents the initial value of W_i *and* B_is of the of the neural network. The figure illustrates the exact and approximation solutions in the training set

Example 1:

We consider the following linear SPVIE [8]

$$\varepsilon y(x) = \int_0^x [1 + t - y(t)\, dt$$ and BC: y(0)=0 ,y(1)= $y(x) = 2 - e^{-\frac{1}{\varepsilon}} - \varepsilon\left(1 - e^{-\frac{1}{\varepsilon}}\right)$

which has the exact solution $y(x) = x + 1 - e^{-\frac{x}{\varepsilon}} - \varepsilon\left(1 - e^{-\frac{x}{\varepsilon}}\right)$, $\varepsilon = 0.25$.

Example 2:

We consider the following nonlinear SPVIE [9]

$$y(x) = \frac{1}{\varepsilon}(1 - e^x) + \frac{1}{\varepsilon}\int_0^x e^{x-t}\, y^2(t)dt$$

and the boundary conditions y(0)=0 , y(1)=$\frac{2(1-e^{\frac{1}{\varepsilon}(\sqrt{1+4\varepsilon^2})})}{\varepsilon(\frac{1}{\varepsilon}(\sqrt{1+4\varepsilon^2})-1)e^{\frac{1}{\varepsilon}(\sqrt{1+4\varepsilon^2})}+\frac{1}{\varepsilon}(\sqrt{1+4\varepsilon^2})+1}$

which has the exact solution $y(x) = \frac{2(1-e^{\varphi x})}{\varepsilon(\varphi-1)e^{\varphi x}+\varphi+1}$

where the parameters φ are defined as $\varphi = \frac{1}{\varepsilon}(\sqrt{1+4\varepsilon^2})$ and ε=0.75.

Example 3:

In this problem we consider the SPVIDE [10–12]

$$\varepsilon \frac{d}{dx} y(x) = (1+\varepsilon)e^{-1} - \varepsilon - y(x) + \int_0^x (1 + \varepsilon) y(t)dt \ ,$$

with the boundary condition $y(0) = 1 + e^{-1}$, $y(1) = 1 + e^{\frac{-1}{\varepsilon(1+\varepsilon)}}$ and the analytic solution is $y(x) = e^{x-1} + e^{\frac{-x}{\varepsilon(1+\varepsilon)}}$, we get ε =2^{-10}.

6. Conclusion

This paper present new technique to solve 2nd singularly perturbed Volterra integro-differential equations (SPVIDE) and singularly perturbed Volterra integral equations (SPVIE). Using artificial neural network which have the singularly perturbed, using modification of high performance training algorethms such as quase-Newton, Levenberge-Marquard, and Baysian Regulation. The projected construction of ANN is more professional and accurate than the other approximat method. convenient outcome with a few hundred wieghts prove that the Levenberge-Marquardt (trainlm) algorethm will have the highest convergence, then trainbfg and trainbr. The performance of different algorethms can be affect by the accuracyrequired for rounding.

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Table 1: presents a comparison between the exact and approximated solutions of example 1, ε =0.25

input	Analytic solution	Solution of FFNN yt(x) for training algorithm		
x	ya(x)	Trainlm	Trainbfg	Trainbr
0.0	0	8.63846704779123e-06	6.24160951957431e-06	3.33127199026609e-06
0.1	0.347259965473271	0.347262822894421	0.347549233468808	0.347241687278199
0.2	0.613003276912084	0.613012683806799	0.612995182065023	0.613047107476904
0.3	0.824104341065849	0.823855506383785	0.824088405150215	0.824059458872570
0.4	0.998577611504008	0.998385206583588	0.998584506284787	0.998578563013621
0.5	1.14849853757254	1.14847942445457	1.14848027740797	1.14853704015194
0.6	1.28196153503294	1.28199362598582	1.28192438254163	1.28193562817729
0.7	1.40439245303109	1.40437009337511	1.40437701556679	1.40428258136672
0.8	1.51942834701623	1.51937926612973	1.51943739391913	1.51933087735538
0.9	1.62950720816453	1.62950369108927	1.62944608756291	1.62951250794293
1.0	1.73626327083345	1.73621733919038	1.73591117829189	1.73626101112071

Table 2: Accuracy of solutions of example 1

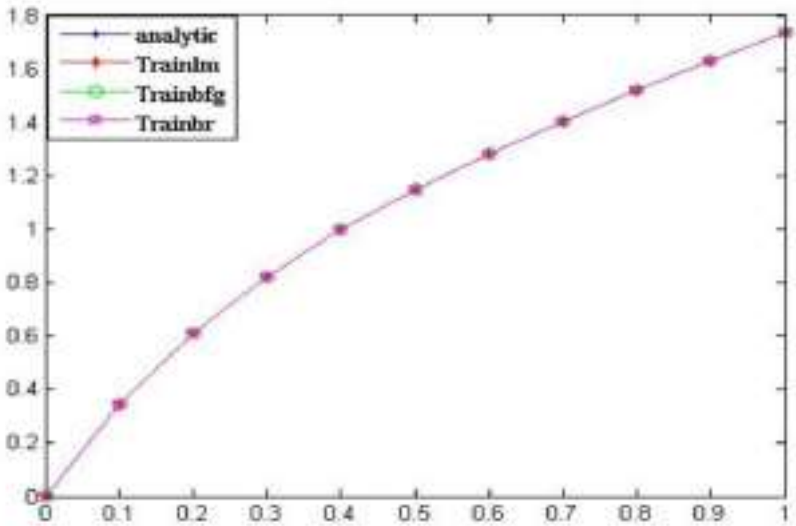
The error E(x)=|yt(x)−ya(x)| where yt(x) computed by the following training algorithm		
Trainlm	Trainbfg	Trainbr
8.63846704779123e-06	6.24160951957431e-06	3.33127199026609e-06
2.85742115052612e-06	0.000289267995537579	1.52781950711641e-05
9.40689471529144e-06	8.09484706110197e-06	4.38305648197135e-05
0.000248834682063381	1.59359156334249e-05	4.48821932786947e-05
0.000192404920420497	6.89478077875450e-06	9.51509612434620e-07
1.91131179658743e-05	1.82601645739577e-05	3.85025794009675e-05
3.20909528779279e-05	3.71524913145294e-05	2.59068556500708e-05
2.23596559758565e-05	1.54374642924449e-05	0.000109871664370598
4.90808864939130e-05	9.04890290393084e-06	9.74696608420089e-05
3.51707526458078e-06	6.11206016205568e-05	5.29977839969220e-06
4.59316430732049e-05	0.000352092541554994	2.25971274048220e-06

Table 3: The accuracy of the train of suggested FFNN.

Train function	*Performance of train*	*Epoch*	*Time*	*Msereg.*
Trainlm	3.09e-33	95	0:00:01	8.634742679417933e-09
Trainbfg	1.60e-15	442	0:00:08	1.749442398773895e-08
Trainbr	7.19e-10	301	0:00:04	2.294226818490657e-09

Table4: Initial weight and bias of FFNN

Initial weights and bias for trainlm			*Initial weights and bias for trainbfg*			*Initialweights and bias for trainbr*		
Net.IW {1,1}	Net.LW {2,1}	Net.B{1}	Net.IW {1,1}	Net.LW {2,1}	Net.B{1}	Net.IW {1,1}	Net.LW {2,1}	Net.B{1}
0.2691	0.9831	0.6981	0.7702	0.1759	0.6074	0.5523	0.0495	0.1465
0.4228	0.3015	0.6665	0.3225	0.7218	0.1917	0.6299	0.4896	0.1891
0.5479	0.7011	0.1781	0.7847	0.4735	0.7384	0.0320	0.1925	0.0427
0.9427	0.6663	0.1280	0.4714	0.1527	0.2428	0.6147	0.1231	0.6352
0.4177	0.5391	0.9991	0.0358	0.3411	0.9174	0.3624	0.2055	0.2819



Figure(1) : shows a comparison between the e×act solution and the appro×imate solution of the problem which is presented in e×ample 1 , with ε =0.25.

Table 5: : presents a comparison between the exact and approximated solutions of example 2, ε =0.75

input	Analytic solution	Solution of FFNN $y_t(x)$ for training algorithm		
x	$y_a(x)$	Trainlm	Trainbfg	Trainbr
0.0	0	-1.232966395647 85e-09	-2.29553043240571e-10	-6.76384614983760e-09
0.1	-0.129682372351998	-0.129682372416046	-0.129682331724809	-0.129682382846623
0.2	-0.279753006539272	-0.279753006437191	-0.279753106383285	-0.279753010727454
0.3	-0.448074072192466	-0.448074072225477	-0.448074029369055	-0.448074072108653
0.4	-0.630374312628864	-0.630374312671338	-0.630374165443910	-0.630374312666036
0.5	-0.820483778242515	-0.820483778253259	-0.820483734588987	-0.820483779223842
0.6	-1.01107274618158	-1.01107274623763	-1.01107281604895	-1.01107274632745
0.7	-1.19473797678062	-1.19473797682557	-1.19473800796783	-1.19473797555232
0.8	-1.36511190858948	-1.36511190848251	-1.36511196236237	-1.36511190753393
0.9	-1.51765877577841	-1.51765877584827	-1.51765912453648	-1.51765877552818
1.0	-1.64997435974891	-1.64997436156016	-1.64997435305496	-1.64997435715362

Table 6: Accuracy of solutions of example 2, ε =0.75

The error $E(x)=\|yt(x)-ya(x)\|$ where $yt(x)$ computed by the following training algorithm		
Trainlm	*Trainbfg*	*Trainbr*
1.23296639564785e-09	2.29553043240571e-10	6.76384614983760e-09
6.40480168900837e-11	4.06271892516852e-08	1.04946255297111e-08
1.02081509911756e-10	9.98440130017819e-08	4.18818218994588e-09
3.30114824365069e-11	4.28234107063830e-08	8.38128455526999e-11
4.24735802084797e-11	1.47184954446544e-07	3.71715991320798e-11
1.07445163877173e-11	4.36535279035866e-08	9.81327130666898e-10
5.60449464614976e-11	6.98673674470740e-08	1.45869538670240e-10
4.49544845793071e-11	3.11872150327019e-08	1.22829435511562e-09
1.06969100244214e-10	5.37728923610814e-08	1.05554631701921e-09
6.98625601813774e-11	3.48758070423116e-07	2.50229170717375e-10
1.81124848452896e-09	6.69394806251944e-09	2.59529575608042e-09

Table 7:The accuracy of the train of suggested FFNN, ε =0.75

Train function	*Performance of train*	*Epoch*	*Time*	*Msereg.*
Trainlm	5.03e-21	225	0:00:02	3.95987261210944 8e-19
Trainbfg	1.60e-15	442	0:00:08	1.37000413592227 6e-14
Trainbr	8.19e-18	286	0:00:03	1.504156882367969e-17

Table 8: Initial weight and bias of FFNN, ε =0.75

Initial weights and bias for trainlm			*Initial weights and bias for trainbfg*			*Initialweights and bias for trainbr*		
Net.IW{	*Net.LW{*	*Net.B{1*	*Net.IW{*	*Net.LW{*	*Net.B{1}*	*Net.IW{*	*Net.LW{*	*Net.B{1}*
0.1478	0.4674	0.4278	0.4504	0.4470	0.7462	0.5825	0.6714	0.2794
0.0198	0.6567	0.2672	0.4736	0.5876	0.4679	0.6866	0.8372	0.9462
0.9643	0.2902	0.7537	0.9497	0.8776	0.8608	0.7194	0.9715	0.9064
0.9704	0.7545	0.8984	0.0835	0.4691	0.4665	0.6500	0.0569	0.3927
0.1239	0.5581	0.7284	0.2798	0.4374	0.4981	0.7269	0.4503	0.0249

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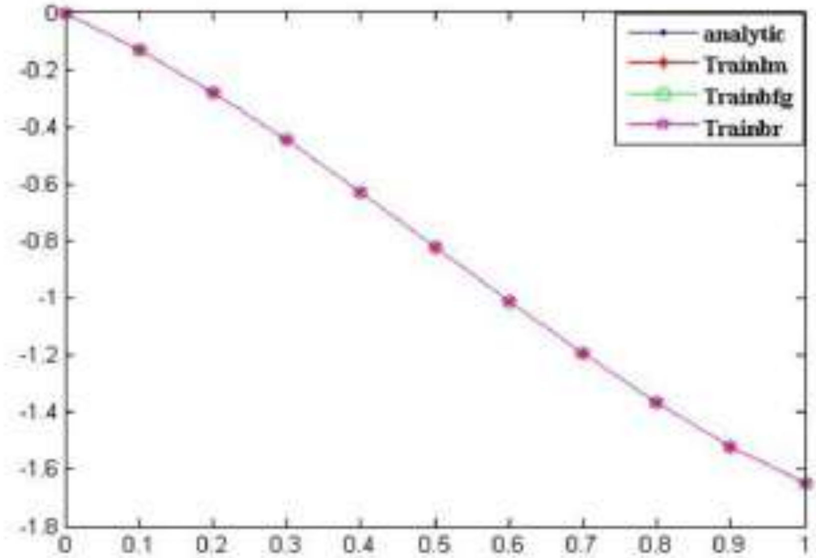

Figure2: shows a comparison between the e×act solution and the appro×imate solution of the problem which is presented in e×ample 2, with ε =0.75.

Table 9: presents a comparison between the e×act and approximated solutions of example 3, ε =10^{-4}

input	*Analytic solution*	*Solution of FFNN yt(x) for training algorithm*		
x	*ya(x)*	*Trainlm*	*Trainbfg*	*Trainbr*
0.0	1.36787944117144	1.36788008912683	1.36658579902398	1.36787348745286
0.1	0.408403141605383	0.408403158192725	0.409648772717205	0.408555958854668
0.2	0.449332325772970	0.447220714806900	0.446917028590653	0.449072090350214
0.3	0.496585309954944	0.496585884132547	0.497291215677891	0.496705926669505
0.4	0.548811636105327	0.549525400628744	0.550290995079382	0.548833632292113
0.5	0.606530659712654	0.606598111288221	0.607065207795657	0.605991033696597
0.6	0.670320046035639	0.670332186873078	0.669343251141215	0.669735307851162
0.7	0.740818220681718	0.740783737266799	0.739191349597689	0.744369056450970
0.8	0.818730753077982	0.818759261704545	0.818267940190760	0.818711070837418
0.9	0.904837418035960	0.904826537817461	0.905932626518221	0.904812403869590
1.0	1	0.999131812534284	0.998764074542181	1.00337320163045

Table 10: Accuracy of solutions of example 3, ε =10^{-4}

The error E(x)=|yt(x)−ya(x)| where yt(x) computed by the following training algorithm		
Trainlm	*Trainbfg*	*Trainbr*
6.47955391341881e-07	0.00129364214746341	5.95371858547189e-06
1.65873413893181e-08	0.00124563111182108	0.000152817249284620
0.00211161096607054	0.00241529718231664	0.000260235422756383
5.74177602963299e-07	0.000705905722946654	0.000120616714560984
0.000713764523417204	0.00147935897405427	2.19961867854446e-05
6.74515755664240e-05	0.000534548083002684	0.000539626016057282
1.21408374382792e-05	0.000976794894424593	0.000584738184477796
3.44834149189621e-05	0.00162687108402837	0.00355083576925175
2.85086265625623e-05	0.000462812887222275	1.96822405638120e-05
1.08802184987100e-05	0.00109520848226119	2.50141663692416e-05
0.000868187465716153	0.00123592545781870	0.00337320163044530

Table 11:The accuracy of the train of suggested FFNN, $\varepsilon = 10^{-6}$

Train function	*Performance of train*	*Epoch*	*Time*	*Msereg.*	*MSE of Numerical Method in* [13]
Trainlm	3.24e-10	369	0:00:06	4.687304918869537e-07	8.70 e − 003
Trainbfg	7.84e-8	209	0:00:03	1.519637055012368e-06	
Trainbr	1.20e-22	275	0:00:04	2.023134155861496e-06	

Table 12: Initial weight and bias of FFNN, $\varepsilon = 10^{-6}$

Initial weights and bias for trainlm			*Initial weights and bias for trainbfg*			*Initialweights and bias for trainbr*		
Net.IW{	*Net.LW{*	*Net.B{1*	*Net.IW{*	*Net.LW{*	*Net.B{1*	*Net.IW{*	*Net.LW{*	*Net.B{1*
0.9541	0.1820	0.6427	0.6393	0.4333	0.6240	0.6109	0.4186	0.8051
0.5428	0.0930	0.0014	0.9173	0.8842	0.3279	0.9000	0.1557	0.0672
0.5401	0.4635	0.0304	0.1616	0.3931	0.8030	0.1934	0.8190	0.9508
0.3111	0.0093	0.2085	0.7156	0.1790	0.9995	0.7544	0.6249	0.4976
0.0712	0.9150	0.4550	0.5777	0.6333	0.9810	0.3463	0.7386	0.7551

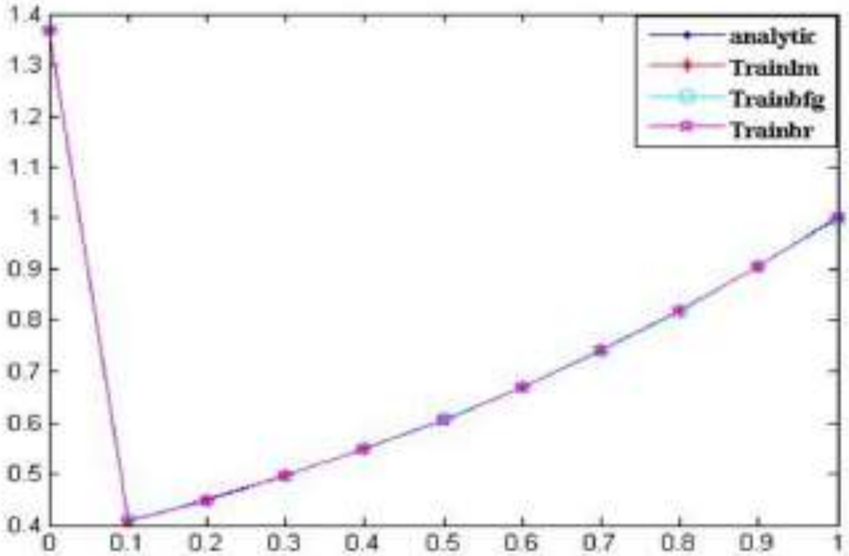

Figure 3.: shows a comparison between the e×act solution and the appro×imate solution of the problem which is presented in e×ample 3, with $\varepsilon = 2^{-6}$.

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خوارزميات التدريب ذات الاداء العالي المحسنة لحل معادلة فولتيرا التكاملية التفاضلية و التكاملية المضطربة الشاذة

خالد منديل محمد
قسم الرياضيات ، كلية التربية ، جامعة القادسية

المستخلص :

في هذا البحث، نستخدم الشبكات العصبية لحل معادلة فولتيرا التكاملية التفاضلية ذات الاضطراب المنفردة ومعادلة فولتيرا التكاملية المنفردة ذات الاضطراب . استخدمنا تحسين خوارزميات التدريب ذات الاداء العالي مثل كوازي نيوتن ، ليفنبيرك - ماركواردت وبايسن ركوليشن . تمت مقارنة الطريقة المقترحة مع خوارزميات التدريب القياسية والحلول التحليلية وجدنا أن الطريقة المقترحة تتميز بالدقة العالية في النتائج ، وانخفاض معدل الخطأ وسرعة التقارب كثيرًا مع الطرق القياسية .

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Property of oscillation of first order impulsive neutral differential equations with positive and negative coefficients

Hussain Ali Mohamad
Department of Mathematics
College of science for Women

Aqeel Falih Jaddoa
Department of Mathematics
College of Education for Pure Science
Ibn Al-Haitham

University of Baghdad, Iraq

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ABSTRACT.

In this paper, necessary and sufficient conditions for oscillation are obtained, so that every solution of the linear impulsive neutral differential equation with variable delays and variable coefficients oscillates. Most of authors who study the oscillatory criteria of impulsive neutral differential equations, investigate the case of constant delays and variable coefficients. However the points of impulsive in this paper are more general. An illustrate example is given to demonstrate our claim and explain the results.

Keywords: Oscillation, Impulses effect, Neutral differential equations.

Mathematics Subject Classification: 34K11.

1. INTRODUCTION

The investigation in the theory of impulsive differential equations is now not only wider than the theory of differential equations without impulses effect, but it describes many phenomena and processes more reality so it has a lot of applications in many natural and industrial fields to study different characters and it can used as a tool in mathematical models for instance, in medicine [1], control theory [2], population dynamics [3], in neural networks [4] and etc. In fact, many evolution processes are often developed for immediate perturbations and sudden changes in specific moments of time such as in biological system in heart beats. This period of change is very small compared to the periods of operation, therefore the situation is quite different from what it is in differential equations without impulses in many physical phenomena, and it appears as a sudden change in its state. The consideration of oscillatory solutions for impulsive neutral differential equations is a new and wide object to find the qualitative properties. There is a lot of research and monographs that deal with the conditions to guarantee the oscillation of all solutions the impulsive neutral differential equations with coefficients such as variable coefficients and constant delays see [5, 6]. There are some results of oscillation for this type of equations [7-12] and we noted that the search of oscillation with impulsive neutral differential equations is more difficult than the type without it. Shen etc. al [13-17] obtained sufficient conditions for oscillation of all solutions of first order impulsive neutral differential equation of constant delay with positive and negative coefficients are obtained. Consider neutral differential equation:

$$\left.\begin{aligned}&\big[y(t)-P(t)y(\tau(t))\big]'+Q(t)y(\sigma(t))-R(t)y(\alpha(t))=0,\\&\qquad t\neq t_k\quad k=1,2,\dots\\&y(t_k^+)+b_k y(t_k)=a_k\,y(t_k),\quad t=t_k\qquad k=1,2,\dots\end{aligned}\right\}\quad(1.1)$$

Where $P\in PC([t_0,\infty);R^+)$ and $Q,R,\in C([t_0,\infty);R^+)$, and $\tau(t),\alpha(t),\sigma(t)\in C([t_0,\infty);R)$, $\lim_{t\to\infty}\tau(t)=\infty, \lim_{t\to\infty}\alpha(t)=\infty, \lim_{t\to\infty}\sigma(t)=\infty$ where τ,α,σ are increasing functions. The functions $\tau^{-1}(t),\sigma^{-1}(t),\alpha^{-1}(t)$ are the inverse of the functions $\tau(t),\alpha(t),\sigma(t)$ respectively.

2. SOME BASIC LEMMAS

The following lemmas will be useful in the proof of our main results: Throughout the paper we assume that $\tau(t),\alpha(t),\sigma(t)<t$, for $t\in(t_k,t_{k+1}], t_1\geq t_0, k=1,2,\dots$

Lemma 2.1. [8] Suppose that $g,h\in C(R^+,R^+)$, $h(t)<t$ for $t\geq t_0$, $\lim_{t\to\infty}h(t)=\infty$ and

$$\liminf_{t\to\infty}\int_{h(t)}^{t}g(s)ds>\frac{1}{e}\qquad(2.1)$$

then the inequality $y'(t)+g(t)y(h(t))\leq 0$ has no eventually positive solutions.

Lemma 2.2. Let $y(t)$ be an eventually positive solution of equation (1.1) and there exists a continuous function $\delta(t)$ such that:

$$W(t)=y(t)-P(t)y(\tau(t))-\int_{\alpha^{-1}(\delta(t))}^{t}R(u)y(\alpha(u))du-\int_{t}^{\sigma^{-1}(\delta(t))}Q(u)y(\sigma(u))du\qquad(2.2)$$

Where $\delta(t)<t$ and $t\in(t_k,t_{k+1}]$, $0<t_0<t_1<\dots<t_k\to\infty$ as $k\to\infty$.

Also $\alpha^{-1}(\delta(t))<t$ and $\sigma^{-1}(\delta(t))>t$, in addition to the following assumptions:

H1: $Q\left(\sigma^{-1}(\delta(t))\right)(\sigma^{-1}(\delta(t)))'-R\left(\alpha^{-1}(\delta(t))\right)\left(\alpha^{-1}(\delta(t))\right)'\geq 0$, $t\in(t_k,t_{k+1}]$, $k=1,2,\dots$

H2: There exists two positive real numbers a_k and b_k such that

$0<a_k-b_k\leq 1$, $k=1,2,\dots$ And

$$\begin{cases}P(t_k^+)\geq(a_k-b_k)P(t_k)\text{ for }\tau(t_k)\neq t_i, i<k,\\ P(t_k^+)\geq\dfrac{1}{a_k-b_k}P(t_k)\text{ for }\tau(t_k)=t_i, i<k,\end{cases}$$

Where $a_k=a_i, b_k=b_i$ when $\tau(t_k)=t_i, i<k$

H3: $\limsup_{t\to\infty}\left[P(t)+\int_{\alpha^{-1}(\delta(t))}^{t}R(u)du+\int_{t}^{\sigma^{-1}(\delta(t))}Q(u)du\right]\leq 1$, $t\in(t_k,t_{k+1}]$.

Then $W(t)$ is eventually positive and nonincreasing function.

Proof. Let $y(t)$ be an eventually positive solution of equation(1.1) that is $y(t) > 0, y(\tau(t)) > 0$, $y(\sigma(t)) > 0$ and $y(\alpha(t)) > 0$, $t \geq t_0$,

Differentiate (2.2) for every interval $(t_k, t_{k+1}]$ where k = 1,2, ...

and use (1.1) we get

$$W'(t) = [y(t) - P(t)y(\tau(t))]' - R(t)y(\alpha(t)) + R(\alpha^{-1}(\delta(t)))y(\delta(t))(\alpha^{-1}(\delta(t)))' - Q(\sigma^{-1}(\delta(t)))y(\delta(t))(\sigma^{-1}(\delta(t)))' + Q(t)y(\sigma(t))$$

$$= -Q(t)y(\sigma(t)) + R(t)y(\alpha(t)) - R(t)y(\alpha(t)) + R(\alpha^{-1}(\delta(t)))y(\delta(t))(\alpha^{-1}(\delta(t)))' - Q(\sigma^{-1}(\delta(t)))y(\delta(t))(\sigma^{-1}(\delta(t)))' + Q(t)y(\sigma(t))$$

$$= -\left[Q\left(\sigma^{-1}(\delta(t))\right)(\sigma^{-1}(\delta(t)))' - R\left(\alpha^{-1}(\delta(t))\right)\left(\alpha^{-1}(\delta(t))\right)'\right]y(\delta(t)) \leq 0 \quad (2.3)$$

Hence $W(t)$ is nonincreasing function on each $t \in (t_k, t_{k+1}]$ for $k = 1,2, \ldots$

To prove that $W(t_k^+) \leq W(t_k)$ for k = 1,2, ... In view of $0 < a_k - b_k \leq 1$, $k = 1,2, \ldots$, we have $0 < a_k - b_k \leq 1$ and from (2.2) with regard to the condition $\boldsymbol{H2}$ when $\tau(t_k) = t_i$, $i < k$, then:

$$W(t_k^+) = (a_k - b_k)y(t_k) - P(t_k^+)(a_k - b_k)y(\tau(t_k)) - \int_{\alpha^{-1}(\delta(t_k))}^{t_k} R(u)y(\alpha(u))du - \int_{t_k}^{\sigma^{-1}(\delta(t_k))} Q(u)y(\sigma(u))du$$

$$\leq y(t_k) - P(t_k)y(\tau(t_k)) - \int_{\alpha^{-1}(\delta(t_k))}^{t_k} R(u)y(\alpha(u))du - \int_{t_k}^{\sigma^{-1}(\delta(t_k))} Q(u)y(\sigma(u))du = W(t_k)$$

When $\tau(t_k) \neq t_i$, $i < k$ then from (2.2) with regard to the condition $\boldsymbol{H2}$:

$$W(t_k^+) = (a_k - b_k)y(t_k) - P(t_k^+)y(\tau(t_k)) - \int_{\alpha^{-1}(\delta(t_k))}^{t_k} R(u)y(\alpha(u))du - \int_{t_k}^{\sigma^{-1}(\delta(t_k))} Q(u)y(\sigma(u))du$$

$$\leq (a_k - b_k)y(t_k) - (a_k - b_k)P(t_k)y(\tau(t_k)) - (a_k - b_k)\int_{\alpha^{-1}(\delta(t_k))}^{t_k} R(u)y(\alpha(u))du - (a_k - b_k)\int_{t_k}^{\sigma^{-1}(\delta(t_k))} Q(u)y(\sigma(u))du = (a_k - b_k)W(t_k) \leq W(t_k) \quad (2.4)$$

$W(t)$is non-increasing on $[t_0, \infty)$.Hence $-\infty \leq L < \infty$. Where $|L| = sup\{W(t_k^+), lim_{k\to\infty} W(t_k)\}$, $t \in [t_l, \infty)$for some $l \geq t_0$. We claim that $W(t_k) \geq 0$ for k=$l, l+1, \ldots$. Otherwise there exists some $m \geq l$ such that:

$W(t_m) = -\mu < 0$, implies that $W(t) \leq -\mu$ for $t \geq t_m$, since $W(t)$ is non-increasing on $[t_l, \infty)$,then for each $t \in (t_k, t_{k+1}]$, k $= l, l+1, \ldots$ we get from (2.2):

$$y(t) \leq -\mu + P(t)y(\tau(t)) + \int_{\alpha^{-1}(\delta(t))}^{t} R(u)y(\alpha(u))du + \int_{t}^{\sigma^{-1}(\delta(t))} Q(u)y(\sigma(u))du. \quad (2.5)$$

So, we have two cases to consider:

Case 1. If $y(t)$ is unbounded then there exists a sequence of points $\{s_n\}$ such that

$s_n \geq t_m, \lim_{n\to\infty} y(s_n) = \infty$ and

$y(s_n) = \max\{y(t), t_m \leq t \leq s_n\}$.

Then (2.5) reduce to:

$$y(s_n) \leq -\mu + P(s_n)y(\tau(s_n)) + \int_{\alpha^{-1}(\delta(s_n))}^{s_n} R(u)y(\alpha(u))du + \int_{s_n}^{\sigma^{-1}(\delta(s_n))} Q(u)y(\sigma(u))du.$$

$$\leq -\mu + \{P(s_n) + \int_{\alpha^{-1}(\delta(s_n))}^{s_n} R(u)du + \int_{s_n}^{\sigma^{-1}(\delta(s_n))} Q(u)du\}y(s_n) \leq -\mu + y(s_n)$$

Leads to $0 \leq -\mu$ which is a contradiction.

Case 2. If $y(t)$ is bounded that is $\limsup_{t\to\infty} y(t) = \mathrm{M} < \infty$.We can choose a sequence of points $\{s_n\}$ such that

$\lim_{n\to\infty} y(s_n) = \mathrm{M}$ and $y(\eta(s_n)) = \max\{\mathrm{y(t)}: \rho_1(s_n) \leq t \leq \rho_2(s_n)\}$.

Where $\rho_1(s_n) = min\{\tau(s_n), \sigma(s_n)\}$, $\rho_2(s_n) = max\{\tau(s_n), \alpha(s_n)\}$ it is obvious that $\lim_{n\to\infty} \eta(s_n) = \infty$ and $\limsup_{t\to\infty} y(\eta(s_n)) \leq M$.

$$y(s_n) \leq -\mu + P(s_n)y(\tau(s_n)) + \int_{\alpha^{-1}(\delta(s_n))}^{s_n} R(u)y(\alpha(u))du + \int_{s_n}^{\sigma^{-1}(\delta(s_n))} Q(u)y(\sigma(u))du$$
$$\leq -\mu + \{P(s_n) + \int_{\alpha^{-1}(\delta(s_n))}^{s_n} R(u)du + \int_{s_n}^{\sigma^{-1}(\delta(s_n))} Q(u)du\}y(\eta(s_n))$$
$$\leq -\mu + y(\eta(s_n))$$

Taking the superior limit as $n \to \infty$, we get $M \leq -\mu + M$, which is also a contradiction.

Combining the cases 1 and 2, we see that $W(t) \geq 0$ for $t \in (t_k, t_{k+1}]$, k=$l, l+1, \ldots$.

Since $W(t)$ is nonincreasing, so $W(t_k) \geq W(t) \geq 0$ for $t \in (t_k, t_{k+1}]$.

To prove $W(t) > 0$, we first prove that $W(t_k) > 0$ for $k = 1,2,\ldots$. If it is not true, then there exists some $m \geq 0$ such that $W(t_m) = 0$, integrating (2.3) on $(t_m, t_{m+1}]$ yield:

$$W(t_{m+1}) = W(t_m^+) - \int_{t_m}^{t_{m+1}} [Q(\sigma^{-1}(\delta(t)))(\sigma^{-1}(\delta(t)))' - R(\alpha^{-1}(\delta(t)))(\alpha^{-1}(\delta(t)))']y(\delta(t))\, dt$$
$$< W(t_m^+) \leq W(t_m) = 0$$

This contradiction shows that $W(t_k) > 0$ for $k = 1,2 \ldots$, as well as $W(t) \geq W(t_{k+1}) > 0$, for $t \in (t_k, t_{k+1}]$, $k = 1,2, \ldots$. Thus $W(t) > 0$ for $t \geq t_0$. The proof is complete. ∎

3. MAIN RESULTS

The next results provide sufficient conditions for the oscillation of all solutions of (1.1):

Theorem 3.1. Let $W(t)$ defined as in (2.2) and the assumptions ***H1 − H3*** hold, and there exist a continuous function $\delta(t) < t$ such that

$$\liminf_{t\to\infty} \int_{\delta(t)}^{t} [Q(\sigma^{-1}(\delta(s)))(\sigma^{-1}(\delta(s)))' - R(\alpha^{-1}(\delta(s)))(\alpha^{-1}(\delta(s)))'][1 + P(\delta(s)) + \int_{\alpha^{-1}(\delta(\delta(s)))}^{\delta(s)} R(u)du + \int_{\delta(s)}^{\sigma^{-1}(\delta(\delta(s)))} Q(u)du]\, ds > \frac{1}{e} \qquad (3.1)$$

Where $\alpha^{-1}(\delta(t)) < t$ and $\sigma^{-1}(\delta(t)) > t$, then every solution of equation (1.1) oscillates.

Proof. Suppose that $y(t)$ is eventually positive solution of (1.1) then by Lemma 2.2 it follows that $W(t)$ is positive nonincreasing function for $t \in (t_k, t_{k+1}]$, k=1,2, …, since $W(t) \leq y(t)$, hence it follows from (2.2):

$$y(t) = W(t) + P(t)y(\tau(t)) + \int_{\alpha^{-1}(\delta(t))}^{t} R(u)y(\alpha(u))du + \int_{t}^{\sigma^{-1}(\delta(t))} Q(u)y(\sigma(u))du$$
$$\geq W(t) + P(t)W(\tau(t)) + \int_{\alpha^{-1}(\delta(t))}^{t} R(u)W(\alpha(u))du + \int_{t}^{\sigma^{-1}(\delta(t))} Q(u)W(\sigma(u))du$$
$$\geq W(t) + P(t)W(\tau(t)) + W(\alpha(t))\int_{\alpha^{-1}(\delta(t))}^{t} R(u)du + W(\delta(t))\int_{t}^{\sigma^{-1}(\delta(t))} Q(u)du$$
$$\geq W(t) + P(t)W(t) + W(t)\int_{\alpha^{-1}(\delta(t))}^{t} R(u)du + W(t)\int_{t}^{\sigma^{-1}(\delta(t))} Q(u)du$$
$$= W(t)\,[1 + P(t) + \int_{\alpha^{-1}(\delta(t))}^{t} R(u)du + \int_{t}^{\sigma^{-1}(\delta(t))} Q(u)du]$$

$$y(\delta(t)) \geq W(\delta(t))[1 + P(\delta(t)) + \int_{\alpha^{-1}(\delta(\delta(t)))}^{\delta(t)} R(u)du + \int_{\delta(t)}^{\sigma^{-1}(\delta(\delta(t)))} Q(u)du] \quad (3.2)$$

Substituting the last inequality (3.2) in (2.3) we get

$$W'(t) \leq -[Q\left(\sigma^{-1}(\delta(t))\right)(\sigma^{-1}(\delta(t)))' - R\left(\alpha^{-1}(\delta(t))\right)\left(\alpha^{-1}(\delta(t))\right)'][1 + P(\delta(t)) + \int_{\alpha^{-1}(\delta(\delta(t)))}^{\delta(t)} R(u)du + \int_{\delta(t)}^{\sigma^{-1}(\delta(\delta(t)))} Q(u)du]\ W(\delta(t))$$

$$W'(t) + [Q\left(\sigma^{-1}(\delta(t))\right)(\sigma^{-1}(\delta(t)))' - R\left(\alpha^{-1}(\delta(t))\right)\left(\alpha^{-1}(\delta(t))\right)'][1 + P(\delta(t)) + \int_{\alpha^{-1}(\delta(\delta(t)))}^{\delta(t)} R(u)du + \int_{\delta(t)}^{\sigma^{-1}(\delta(\delta(t)))} Q(u)du]\ W(\delta(t)) \leq 0, \quad (3.3)$$

By Lemma 2.1, and the condition (3.1) the last inequality cannot has eventually positive solution, which is a contradiction. The proof is complete. ■

Corollary 3.2: Let $W(t)$ defined as in (2.2) and the assumptions H1-H3 hold, and there exist a continuous function $\delta(t) < t$ such that

$$[Q(\sigma^{-1}(\delta(t)))(\sigma^{-1}(\delta(t)))' - R(\alpha^{-1}(\delta(t)))(\alpha^{-1}(\delta(t)))'] \geq \frac{1}{e\ \min_{t\geq t_0}\{t-\delta(t)\}}. \quad (3.4)$$

Where $\alpha^{-1}(\delta(t)) < t$ and $\sigma^{-1}(\delta(t)) > t$, then every solution of equation (1.1) oscillates.

Proof. It is obvious that condition (3.4) implies that

$$\int_{\delta(t)}^{t} [Q(\sigma^{-1}(\delta(s)))(\sigma^{-1}(\delta(s)))' - R(\alpha^{-1}(\delta(s)))(\alpha^{-1}(\delta(s)))']ds \geq \frac{1}{e\ \min_{t\geq t_0}\{t-\delta(t)\}}(t-\delta(t)) \geq \frac{1}{e}.$$

Which leads to condition (3.1) holds. Hence according to Theorem 3.1 every solution of (1.1) oscillates.

4. EXAMPLE

In this section we give an example to illustrate the obtained results.

Example 4.1. Consider the impulsive neutral differential equation:

$$\left[y(t) - \frac{1}{9}(1 + e^{-t})y(t-2\pi)\right]' + \frac{1}{9}(8 - e^{-t})y\left(t - \frac{5\pi}{2}\right) - \frac{1}{9}e^{-t}y(t-2\pi) = 0,$$

$t \neq t_k$ and $k = 1,2,\ldots$

$$y(t_k^+) + b_k y(t_k) = a_k\, y(t_k), \quad t = t_k \text{ and } k = 1,2,\ldots \quad (4.1)$$

Where $a_k = \frac{2k}{k+1}$ and $b_k = \frac{k}{k+1}$ we can see that

$$a_k - b_k = \frac{2k}{k+1} - \frac{k}{k+1} = \frac{k}{k+1} < 1$$

$$\text{Let P(t)} = \begin{cases} \left(\frac{1}{9} + \frac{1}{9}e^{-t}\right), & t \neq t_k \\ \frac{1}{20k}, & t = t_k \end{cases}$$

Let $\delta(t) = t - \frac{9\pi}{4}$, $\sigma^{-1}(\delta(t)) = t + \frac{\pi}{4}$ and $\alpha^{-1}(\delta(t)) = t - \frac{\pi}{4}$ to see condition ***H1***

$$Q\left(\sigma^{-1}(\delta(t))\right)(\sigma^{-1}(\delta(t)))' - R\left(\alpha^{-1}(\delta(t))\right)\left(\alpha^{-1}(\delta(t))\right)' = \frac{1}{9}\left(8 - e^{-t-\frac{\pi}{4}}\right) - \frac{1}{9}e^{-t+\frac{\pi}{4}} \geq 0.5945.$$

Let $t_k = k$, $P(t_k^+) = P(k^+) = \lim_{t\to k^+} P(t) = \lim_{t\to k^+}\left(\frac{1}{9} + \frac{1}{9}e^{-t}\right) = \left(\frac{1}{9} + \frac{1}{9}e^{-k}\right) \geq \frac{1}{9}$.

$$(a_k - b_k)P(t_k) = \frac{k}{k+1}P(k) = \frac{k}{k+1}\frac{1}{20k} = \frac{1}{20(k+1)} \leq 0.025, \text{ and}$$

$$\frac{1}{(a_k - b_k)}P(t_k) = \frac{k+1}{k}P(k) = \frac{k+1}{k}\frac{1}{20k} = \frac{k+1}{20k^2} \leq 0.1,$$

so ***H2*** holds.

And the condition ***H3***leads to

$$\liminf_{t\to\infty} \int_{\delta(t)}^{t} [Q\left(\sigma^{-1}(\delta(\xi))\right)(\sigma^{-1}(\delta(\xi)))' - R\left(\alpha^{-1}(\delta(\xi))\right)\left(\alpha^{-1}(\delta(\xi))\right)'][1 + P(\delta(\xi)) + \int_{\alpha^{-1}(\delta(\delta(\xi)))}^{t} R(u)du + \int_{\delta(\xi)}^{\sigma^{-1}(\delta(\delta(\xi)))} Q(u)du]d\xi$$

$$= \frac{1}{9}\lim_{t\to\infty} \int_{t-\frac{9\pi}{4}}^{t} \left[8 - e^{-\xi-\frac{9\pi}{4}}\right][1 + \frac{1}{9} + \frac{1}{9}e^{-\xi+\frac{9\pi}{4}} + \frac{1}{9}\int_{\xi-\frac{10\pi}{4}}^{\xi-\frac{9\pi}{4}} e^{-u}du + \frac{1}{9}\int_{\xi-\frac{9\pi}{4}}^{\xi-\frac{8\pi}{4}} (8 - e^{-u})du]\, d\xi$$

$$= 11.3662 > \frac{1}{e}$$

Hence all conditions of theorem 3.1 hold, so all solutions of equation (3.1) are oscillatory. For instance the solution $y(t) = \begin{cases} sint, & t \neq t_k \\ 2 + \frac{1}{k}, & t = t_k \end{cases}$ is such a solution.

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خاصية التذبذب لمعادلات تفاضلية محايدة متسارعة من الرتبة الاولى ذات المعاملات الموجبة والسالبة

حسين علي محمد عقيل فالح جدوع
قسم الرياضيات
كلية العلوم للبنات كلية التربية للعلوم الصرفة /إبن الهيثم
جامعة بغداد

المستخلص:

في هذا البحث حصلنا على الشروط الضرورية والكافية للتذبذب، ذلك أنه كل حل لمعادلات تفاضلية محايدة متسارعة خطية ذات تباطؤات متغيرة ومعاملات متغيرة يتذبذب.إن أغلب المؤلفين والباحثين الذين درسوا ظاهرة التذبذب لمعادلات تفاضلية محايدة متسارعه كانوا قد اجروا دراسة لمعادلات ذات تباطؤات ثابتة ومعاملات ثابتة. على اية حال، إن نقاط التسارع هنا أكثر عمومية. لقد قدمنا مثال توضيحي ليبرهن صحة النتائج.

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On strongly *E*-convex sets and strongly *E*-convex cone sets

Saba Naser Majeed
Department of Mathematics, College of Education for Pure Science (Ibn-Haitham), University of Baghdad,
Baghdad, Iraq
saba.n.m@ihcoedu.uobaghdad.edu.iq

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Abstract:

E-convex sets and E-convex functions, which are considered as an important class of generalized convex sets and convex functions, have been introduced and studied by Youness [5] and other researchers. This class has recently extended, by Youness, to strongly E-convex sets and strongly E-convex functions. In these generalized classes, the definitions of the classical convex sets and convex functions are relaxed and introduced with respect to a mapping E. In this paper, new properties of strongly E-convex sets are presented. We define strongly E-convex hull, strongly E-convex cone, and strongly E-convex cone hull and we proof some of their properties. Some examples to illustrate the aforementioned concepts and to clarify the relationships between them are established.

Keywords: E-convex sets, strongly E-convex sets, strongly E-convex cone, strongly E-convex hull

Mathematics Subject Classification: 46N10, 47N10, 90C48, 90C90, 49K27.

1. Introduction and Preliminaries

Classical convex analysis takes a considerable role in pure and applied Mathematics. In particular, convex sets and convex functions are mainly employed in optimization and operation research [1]. Many researchers have extended and generalized convex sets and convex functions into other kinds of less restrictive convexity and applied them into optimization theory. For example, convex functions are extended to the class of invex functions [2] and B-vex functions [3, 4]. An important type of generalized convexity is E-convexity. Youness [5] introduced E- convex sets, E-convex functions, and E-convex programmings, defined in finite dimensional Euclidian space. In these classes, Youness relaxed the definitions of the classical convex sets and convex functions with respect to a mapping $E: \mathbb{R}^n \to \mathbb{R}^n$. The research on E-convexity is continued, improved and extended in different directions. Further study of E-convex sets are recently introduced by Sheiba and Thangavelu [6] and Majeed and Abd Al-Majeed [7]. Youness [8] studied some properties of E-convex programming and established the necessary and sufficient conditions of optimality for nonlinear E-convex programming. Recently, Megahed et al. [9,10] introduced duality in E-convex programming and studied optimality conditions for E-convex programming which has E-differentiable objective function (see also [11], for more recent results on E-convex functions and E-convex programming). The initial results of Youness inspired a great deal of subsequent work which has expanded the role of E-convexity for which an extension class of the class of E-convex sets and E-convex functions, called strongly E-convex sets and strongly E-convex functions, is established by Youness [12]. Some results related to semi strongly E-convex functions have established in [13]. The class of strongly E-convex sets and strongly E-convex functions is closely related to the class of E-convex sets and E-convex functions in the sense that the new class considers the effect of the images of any arbitrary points x and y in $\mathbb{R}^n$ with respect to a mapping $E: \mathbb{R}^n \to \mathbb{R}^n$ as well as the two arbitrary points. To the best of my knowledge, there is not much work has been obtained for the class of strongly E-convex sets and functions. This gives a motivation to study further this class and try to extract new results and notions.

Therefore, in this paper, we continue studying strongly E-convex sets by proving new properties of these sets. In addition, we define strongly (resp., E-convex hull , E-cone, E-convex cone hull) sets, and we discuss some of their properties. We show that many results of (resp., E-convex, E-cone) sets hold for the class of strongly (resp., E-convex, E-cone) sets. Some examples are given to illustrate some of these concepts and to clarify the relationships between them. In section two, we recall the definitions of E-convex and strongly E-convex sets introduced in [5,12] and some properties of strongly E-convex sets. We prove some new properties of strongly E-convex sets. For an arbitrary set, we define strongly E-convex hull. In section three, we introduce the definition of strongly E-cone and strongly E-convex cone sets, and we deduce some of their properties. We also define strongly E-convex cone hull and we show a characterization of strongly E-convex cone. Some examples to discuss the relationship between strongly (E-cone, E-convex cone, E-convex) sets are given.

Throughout this paper, we assume that $\mathbb{R}^n$ is the n-dimensional Euclidean space, all sets we consider are non-empty subsets of $\mathbb{R}^n$, and $E: \mathbb{R}^n \to \mathbb{R}^n$ is a given mapping.

2. Strongly *E*-convex Sets

A set $S \subseteq \mathbb{R}^n$ is said to be convex in the "classical sense" if the convex combinations of any two elements of S retain in S [1]. This concept has been extended by Youness [5,12] in which E-convex sets and strongly E-convex sets are, respectively, defined, and some of their basic properties are introduced. In this section, we first recall the definitions of E-convex sets and strongly E-convex sets and review some existing results of strongly E-convex sets. Then, we prove new properties of strongly E-convex sets. Note that some of these properties are satisfied for E-convex sets [5,6]. Finally, we define strongly E-convex hull and deduce a property of this set.

Definition 2.1 [5] A non-empty set S is said to be E-convex if $\forall s_1, s_2 \in S$ and for every $\lambda \in [0,1]$ we have $\lambda E(s_1) + (1-\lambda)E(s_2) \in S$.

Definition 2.2 [12] A non-empty set S is said to be strongly E-convex if and only if $\forall s_1, s_2 \in S$, for every $\lambda \in [0,1]$, and $\alpha \in [0,1]$ we have

$$\lambda(\alpha s_1 + E(s_1)) + (1-\lambda)(\alpha s_2 + E(s_2)) \in S.$$

The relation between strongly E-convex sets and E-convex (resp., convex) sets is given next.

Remark 2.3

i. Every strongly E-convex set is an E-convex (Choose $\alpha = 0$). The converse does not hold, in general [Example 2, 12].

ii. Every strongly E-convex set is convex (Choose $\alpha = 0$ and $E = I$ (the identity mapping)).

Proposition 2.4 [12] If a set S is a strongly E-convex, then $E(S) \subseteq S$.

Proposition 2.5 [12] Let S_1 and S_2 be two strongly E-convex sets, then

i. $S_1 \cap S_2$ is E-convex set.

ii. If E is a linear mapping, then $S_1 + S_2$ is strongly E-convex set.

Remark 2.6 The intersection property, in the above proposition, can be easily extended to an arbitrary family of strongly E-convex sets.

The definition of strongly E-convex sets can be generalized into the strongly E-convex combinations of any finite elements of these sets.

Definition 2.7 Let $S \subset \mathbb{R}^n$. The set of strongly E-convex combinations of p elements of S is denoted by $C(s,p)$ and is defined as

$$C(s,p) = \{s = \sum_{i=1}^{p} \lambda_i(\alpha s_i + E(s_i)) : \{s_1, \dots, s_p\} \subset S,\ \alpha \in [0,1],\ \lambda_i \geq 0 \text{ and } \sum_{i=1}^{p} \lambda_i = 1\}.$$

Next, a sufficient condition, for a set S to be strongly E-convex sets, is given in terms of the strongly E-convex combinations of its elements.

Proposition 2.8 Assume that a set $S \subset \mathbb{R}^n$ and $C(s,p)$ be the set of E-convex combinations of p elements of S defined in Definition 3 such that $C(s,p) \subset S\ \forall p \in N$. Then S is strongly E-convex set.

Proof. assume that $C(s,p) \subset S\ \forall p \in N$. In case $p = 2$, then for each $s_1, s_2 \in S$, $\alpha \in [0,1]$ and $\lambda \in [0,1]$ we have $s = \lambda(\alpha s_1 + E(s_1)) + (1-\lambda)(\alpha s_2 + E(s_2)) \in S$. Hence, S is E-convex. ∎

Proposition 2.9

i. If a set S is a strongly E-convex, then $\alpha s + E(s) \in S$ for each $s \in S$ and $\alpha \in [0,1]$.

ii. If S is a convex set and $\alpha s + E(s) \in S$ for each $s \in S$ and $\alpha \in [0,1]$, then S is strongly E-convex.

Proof. The conclusion of part (i) directly follows from the assumption, by choosing $\lambda = 1$. To show (ii), let $s_1, s_2 \in S$ and $\alpha \in [0,1]$ then from the assumption $\alpha s_1 + E(s_1) \in S$ and $\alpha s_2 + E(s_2) \in S$. Since S is convex then for each $\lambda \in [0,1]$ we have $\lambda(\alpha s_1 + E(s_1)) + (1-\lambda)(\alpha s_2 + E(s_2)) \in S$ as required to proof. ∎

Note that Proposition 2.9(ii) provides a condition under which the converse of Remark 2.3(ii) holds.

Some algebraic properties of strongly E-convex sets are given next.

Proposition 2.10

i. If S is strongly E-convex set, $a \in \mathbb{R}$ and E is linear then aS is strongly E-convex set.

ii. Assume that $E_1 : \mathbb{R}^p \to \mathbb{R}^p$ and $E_2 : \mathbb{R}^q \to \mathbb{R}^q$, and $E : \mathbb{R}^{p+q} \to \mathbb{R}^{p+q}$ are mappings such that $E(s, \bar{s}) = (E_1(s), E_2(\bar{s}))\ \ \forall s \in \mathbb{R}^p,\ \forall \bar{s} \in \mathbb{R}^q$. Let $S_1 \subseteq \mathbb{R}^p$ be a strongly E_1-convex and $S_2 \subseteq \mathbb{R}^q$ be a strongly E_2-convex. Then, $S_1 \times S_2 \subseteq \mathbb{R}^{p+q}$ is strongly E-convex set.

iii. Let S_1 and S_2 be two strongly E-convex sets, then $S_1 \times S_2$ is strongly $E \times E$-convex set.

Proof. To show (i), suppose that $as_1, as_2 \in aS$ and $\alpha, \lambda \in [0,1]$. We must show that $\lambda(\alpha a s_1 + E(as_1)) + (1-\lambda)(\alpha a s_2 + E(as_2)) \in aS$. From the linearity of E,

$$\lambda(\alpha a s_1 + E(as_1)) + (1-\lambda)(\alpha a s_2 + E(as_2)) = a[\lambda(\alpha s_1 + E(s_1)) + (1-\lambda)(\alpha s_2 + E(s_2))].$$

Since S is strongly E-convex set, the right-hand side of the above expression belongs to aS as we want to show. Let us proof (ii). Let $(s_1, s_2), (\bar{s_1}, \bar{s_2}) \in S_1 \times S_2$, thus, $s_1, \bar{s_1} \in S_1$ and $s_2, \bar{s_2} \in S_2$. Since S_1(resp., S_2) is strongly E_1-convex (resp., E_2-convex), we have $\lambda(\alpha s_1 + E_1(s_1)) + (1-\lambda)(\alpha \bar{s_1} + E_1(\bar{s_1})) \in S_1$ and $\lambda(\alpha s_2 + E_2(s_2)) + (1-\lambda)(\alpha \bar{s_2} + E_2(\bar{s_2})) \in S_2$, where $\lambda, \alpha \in [0,1]$. Thus, $(\lambda(\alpha s_1 + E_1(s_1)) + (1-\lambda)(\alpha \bar{s_1} + E_1(\bar{s_1})),\ \lambda(\alpha s_2 + E_2(s_2)) + (1-\lambda)(\alpha \bar{s_2} + E_2(\bar{s_2}))) \in S_1 \times S_2$.

In other words, $\lambda(\alpha(s_1, s_2) + (E_1(s_1), E_2(s_2))) + (1-\lambda)(\alpha(\bar{s_1}, \bar{s_2}) + (E_1(\bar{s_1}), E_2(\bar{s_2}))) \in S_1 \times S_2$.

From the definition of E, the last term can be written as

$\lambda(\alpha(s_1, s_2) + E(s_1, s_2)) + (1-\lambda)(\alpha(\bar{s_1}, \bar{s_2}) + E(\bar{s_1}, \bar{s_2})) \in S_1 \times S_2$, and this completes the proof. Part (iii) can be considered as a special case of part (ii) such that $E = E_1 = E_2$ and $p = q$. ∎

Proposition 2.11 Assume that $E_1 : \mathbb{R}^p \to \mathbb{R}^p$ and $E_2 : \mathbb{R}^q \to \mathbb{R}^q$, and $F : \mathbb{R}^p \to \mathbb{R}^q$ are mappings such that F is linear and $FoE_1 = E_2 oF$. Let $S \subseteq \mathbb{R}^p$ be a strongly E_1-convex. Then, $F(S) \subseteq \mathbb{R}^q$ is a strongly E_2-convex set.

Proof. Let $F(s_1), F(s_2) \in F(S) \subseteq \mathbb{R}^q$ and $\alpha, \lambda \in [0,1]$ then

$$\lambda\big(\alpha F(s_1) + E_2(F(s_1))\big) + (1-\lambda)\big(\alpha F(s_2) + E_2(F(s_2))\big)$$

$$= \lambda\big(\alpha F(s_1) + (E_2 o F)(s_1)\big) + (1-\lambda)\big(\alpha F(s_2) + (E_2 o F)(s_2)\big)$$

From the assumption $FoE_1 = E_2 oF$, the last expression becomes

$$= \lambda\big(\alpha F(s_1) + (FoE_1)(s_1)\big) + (1-\lambda)\big(\alpha F(s_2) + (FoE_1)(s_2)\big),$$

$$= \lambda\big(\alpha F(s_1) + F(E_1(s_1))\big) + (1-\lambda)\big(\alpha F(s_2) + F(E_1(s_2))\big).$$

Applying the linearity of F and re-arranging the last expression, we get

$$= \lambda F\big(\alpha s_1 + E_1(s_1)\big) + (1-\lambda) F\big(\alpha s_2 + E_1(s_2)\big),$$

$$= F\Big(\lambda\big(\alpha s_1 + E_1(s_1)\big) + (1-\lambda)\big(\alpha s_2 + E_1(s_2)\big)\Big) \in F(S).$$

The last conclusion is obtained since S is strongly E_1-convex set. ■

Proposition 2.12 Let $\beta \in \mathbb{R}_+, b \in \mathbb{R}^n$, and E is an idempotent and linear mapping then the upper E-half space $S = \{s \in \mathbb{R}^n : < E(s), b > \geq \beta\}$ is strongly E-convex.

Proof. Let $s_1, s_2 \in S$ and $\alpha, \lambda \in [0,1]$ we aim to prove $\lambda\big(\alpha s_1 + E(s_1)\big) + (1-\lambda)\big(\alpha s_2 + E(s_2)\big) \in S$.

i.e., we show $< E(\lambda(\alpha s_1 + E(s_1)) + (1-\lambda)(\alpha s_2 + E(s_2))), b > \geq \beta$,

where $\beta \in \mathbb{R}_+$ and $b \in \mathbb{R}^n$. Since E is an idempotent and linear mapping, then

$$< E(\lambda(\alpha s_1 + E(s_1)) + (1-\lambda)(\alpha s_2 + E(s_2))), b >$$

$$=< \lambda \alpha E(s_1) + (1-\lambda)\alpha E(s_2), b > + < \lambda E(s_1) + (1-\lambda)E(s_2), b >$$

$$= \lambda\alpha < E(s_1), b > + (1-\lambda)\alpha < E(s_2), b > + \lambda < E(s_1), b > + (1-\lambda) < E(s_2), b >$$

Since $s_1, s_2 \in S$, the last expression yield

$$\geq \lambda\alpha\beta + (1-\lambda)\alpha\beta + \lambda\beta + (1-\lambda)\beta = \alpha\beta + \beta \geq \beta.$$

Note that the right most inequality follows because $\beta \in \mathbb{R}_+$ and $\alpha \in [0,1]$. ■

Proposition 2.13 Let I be an index set and $\beta_i \in \mathbb{R}_+, b_i \in \mathbb{R}^n$ for all $i \in I$. Assume also that E is an idempotent and linear mapping then the set $S = \{s \in \mathbb{R}^n : < E(s), b_i > \geq \beta_i \quad \forall i \in I\}$ is strongly E-convex.

Proof. The conclusion follows from Proposition 2.12 and Remark 2.6 ■

Proposition 2.14 Let $S_1, S_2, \dots, S_n$ be strongly E-convex sets and E is a linear mapping. Then $S = \gamma_1 S_1 + \dots + \gamma_n S_n$ is a strongly E-convex set where $\gamma_1, \dots, \gamma_n \in \mathbb{R}$.

Proof. Let $s, \overline{s} \in S$. Then $s = \gamma_1 s_1 + \dots + \gamma_n s_n$ and $\overline{s} = \gamma_1 \overline{s_1} + \dots + \gamma_n \overline{s_n}$ such that $s_i, \overline{s_i} \in S \quad \forall i = 1, \dots, n$. For $\alpha, \lambda \in [0,1]$ we have

$$\lambda\big(\alpha s + E(s)\big) + (1-\lambda)\big(\alpha\overline{s} + E(\overline{s})\big)$$

$$= \lambda\big(\alpha(\gamma_1 s_1 + \dots + \gamma_n s_n) + E(\gamma_1 s_1 + \dots + \gamma_n s_n)\big) + (1-\lambda)\big(\alpha(\gamma_1 \overline{s_1} + \dots + \gamma_n \overline{s_n}) + E(\gamma_1 \overline{s_1} + \dots + \gamma_n \overline{s_n})\big)$$

Applying the linearity of E to the last expression and re-arranging it, we get

$$= \gamma_1\big(\lambda(\alpha s_1 + E(s_1)) + (1-\lambda)(\alpha\overline{s_1} + E(\overline{s_1}))\big) + \dots + \gamma_n\big(\lambda(\alpha s_n + E(s_n)) + (1-\lambda)(\alpha\overline{s_n} + E(\overline{s_n}))\big) \in \gamma_1 S_1 + \dots + \gamma_n S_n = S,$$

where we used the fact that $S_1, \dots, S_n$ are strongly E-convex which implies that each $\lambda(\alpha s_i + E(s_i)) + (1-\lambda)(\alpha\overline{s_i} + E(\overline{s_i})) \in S_i \quad \forall i = 1, \dots, n$. Thus,

$$\lambda\big(\alpha s + E(s)\big) + (1-\lambda)\big(\alpha\overline{s} + E(\overline{s})\big) \in S;$$

therefore, S is strongly E-convex set. ■

We pointed out in Remark 2.6 that the intersection of arbitrary strongly E-convex sets is strongly E-convex. This fact is used next to define the smallest strongly E-convex set containing a fixed set.

Definition 2.15 The strongly E-convex hull of a set $S \subset \mathbb{R}^n$, denoted by $s.E\text{-}conv(S)$ is the smallest strongly E-convex set contains S, that is,

$s.E\text{-}conv(S) = \bigcap_{N \supseteq S} N$, N are strongly E-convex sets.

Next, we provide an example of a strongly E-convex hull of a non-strongly E-convex set S.

Example 2.16 Let $S = (-2,0) \cup [1,2) \subset \mathbb{R}$ and let $E: \mathbb{R} \longrightarrow \mathbb{R}$ is given by $E(x) = -x \quad \forall x \in \mathbb{R}$. Note that, S is not strongly E-convex set. For instance, let $x = -1, y = 1, \lambda = \frac{1}{2}$ and $\alpha = \frac{1}{2}$. Then,

$$\lambda(\alpha x + E(x)) + (1-\lambda)(\alpha y + E(y)) = 0 \notin S.$$

From Definition 2.15, $s.E\text{-}conv(S) = (-2,2)$ which is strongly E-convex. i.e., s. $E\text{-}conv(S)$ is a smallest strongly E-convex set in $\mathbb{R}$ contains S. Indeed, for each $x, y \in S$ and $\alpha, \lambda \in [0,1]$, then

$$\lambda\big(\alpha x + E(x)\big) + (1-\lambda)\big(\alpha y + E(y)\big) = -(1-\alpha)(\lambda x + (1-\lambda) y \in S.$$

Remark 2.17 From the above definition, it is clear that

i. $s.E\text{-}conv(S)$ is strongly E-convex set and $S \subseteq s.E\text{-}conv(S)$.

ii. If S is strongly E-convex set then $s.E\text{-}conv(S) = S$.

Proposition 2.18 Let $S \subset \mathbb{R}^n$ and $\mathcal{L}$ be the set of all strongly E-convex combinations of elements of S. That is

$$\mathcal{L} = \bigcup_{p \in N} C(s, p),$$

where $C(s,p)$ is defined as in Definition 2.7. If $\alpha s + E(s) \subseteq \mathcal{L}\ \forall s \in S$ and $\alpha \in [0,1]$, then $s.E\text{-}conv(S) \subseteq \mathcal{L}$.

Proof. To prove $s.E\text{-}conv(S) \subseteq \mathcal{L}$, it is enough to show that $\mathcal{L}$ is a convex set. Indeed, if $\mathcal{L}$ is a convex set and $\alpha s + E(s) \subseteq \mathcal{L}\ \forall s \in S$. Then from Proposition 2.9(ii), $\mathcal{L}$ is strongly E-convex set. The last conclusion with the fact that $S \subseteq \mathcal{L}$ yield $s.E\text{-}conv(S) \subseteq \mathcal{L}$ as required. Let us show that $\mathcal{L}$ is a convex set. Take $x, y \in \mathcal{L}$, then

$x = \sum_{i=1}^{p} \lambda_i(\alpha x_i + E(x_i))$ and $y = \sum_{i=1}^{s} \gamma_i(\alpha y_i + E(y_i))$,

where $\{x_1, \dots, x_p, y_1, \dots, y_s\} \subset S$ and $\{\lambda_1, \dots, \lambda_p, \gamma_1, \dots, \gamma_s\}$ are non-negative which satisfy

$\sum_{i=1}^{p} \lambda_i = 1$ and $\sum_{i=1}^{s} \gamma_i = 1$.

Fix $\mu \in (0,1)$, then the convex combination

$$\mu x + (1-\mu)y = \mu \sum_{i=1}^{p} \lambda_i(\alpha x_i + E(x_i)) + (1-\mu) \sum_{i=1}^{s} \gamma_i(\alpha y_i + E(y_i))$$

Note that

$\mu \sum_{i=1}^{p} \lambda_i + (1-\mu) \sum_{i=1}^{s} \gamma_i = 1.$

Therefore, $\mu x + (1-\mu)y \in \mathcal{L}$. i.e., $\mathcal{L}$ is a convex set, and using the assumption $\alpha s + E(s) \subseteq \mathcal{L}\ \forall s \in S$ yield $\mathcal{L}$ is E-convex set. Because $S \subseteq \mathcal{L}$ and $S \subseteq s.E\text{-}conv(S)$.Then $s.E\text{-}conv(S) \subseteq \mathcal{L}$. ∎

3. Strongly *E*-cone and Strongly *E*-convex cone

In this section, we define strongly (E-cone, E-convex cone, E-convex cone hull) of arbitrary sets and we discuss some properties of these sets. We prove a new characterization of E-convex cone sets. Some examples, to illustrate the concepts defined in this section and to show the relationship between them, are given.

Definition 3.1 A set $C \subset \mathbb{R}^n$ is called strongly E-cone if for every $c \in C, \alpha \in [0,1]$, and $\gamma \geq 0$ we have $\gamma(\alpha c + E(c)) \in C$. If C is strongly E-cone and strongly E-convex set, it is called strongly E-convex cone.

Examples of strongly E-convex cone set, strongly E-convex set (not strongly E-cone), and strongly E-cone (not strongly E-convex set) are shown, respectively, next.

Example 3.2 Let $C \subset \mathbb{R}^2$ be defined by $C = \{ (x,y) \in \mathbb{R}^2 : x, y \geq 0 \}$, and let $E: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is given by $E(x,y) = (x,0)\quad \forall x,y \in \mathbb{R}$.

For any $(x,y) \in C, \alpha \in [0,1]$, and $\gamma \geq 0$, we have

$\gamma(\alpha(x,y) + E(x,y)) = (\gamma(\alpha+1)x, \gamma\alpha y) \in C$.

Thus, C is strongly E-cone. Also, let $(x_1,y_1), (x_2,y_2) \in C$ and $\lambda, \alpha \in [0,1]$, then

$$\lambda(\alpha(x_1,y_1) + E(x_1,y_1)) + (1-\lambda)(\alpha(x_2,y_2) + E(x_2,y_2))$$
$$= ((\alpha+1)(\lambda x_1 + (1-\lambda)x_2), \alpha(\lambda y_1 + (1-\lambda)y_2)) \in C$$

Thus, C is strongly E-convex set. Altogether, we obtain that C is a strongly E-convex cone.

Example 3.3 Let $C \subset \mathbb{R}^2$ be defined by $C = \{ (x,y) \in \mathbb{R}^2 : -1 \leq x \leq 1, -1 \leq y \leq 1 \}$, and let $E: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be given by $E(x,y) = (-x,-y)\quad \forall x,y \in \mathbb{R}$.

Note that $\alpha(x,y) + E(x,y) = ((\alpha-1)x, (\alpha-1)y) = -((1-\alpha)x, (1-\alpha)y) \in C$ and C is a convex set. From Proposition 2.9(ii), C is strongly E-convex set. To show that C is not strongly E-cone, take for example $(1,1) \in C, \alpha = \frac{1}{2}$ and $\gamma = 5$.

Then $\gamma(\alpha(x,y) + E(x,y)) = \left(\frac{-5}{2}, \frac{-5}{2}\right) \notin C$.

Example 3.4 Let $C = \{ (x,y) \in \mathbb{R}^2 : x \leq -1, -1 \leq y \leq 1 \} \cup \{ (x,y) \in \mathbb{R}^2 : x \geq 1, -1 \leq y \leq 1 \}$, and let $E: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be given by $E(x,y) = (x,0)$. For each $(x,y) \in C, \alpha \in [0,1]$, and $\gamma \geq 0$, we have $\gamma(\alpha(x,y) + E(x,y) = \gamma((\alpha+1)x, \alpha y) \in C$.

Thus, C is strongly E-cone. However, take $(-1,1), (1,1) \in C$, and $\lambda = \alpha = \frac{1}{2}$. Then

$$\lambda(\alpha(-1,1) + E(-1,1)) + (1-\lambda)(\alpha(1,1) + E(1,1)) =$$
$$\frac{1}{2}\left(-\frac{3}{2}, \frac{1}{2}\right) + \frac{1}{2}\left(\frac{3}{2}, \frac{1}{2}\right) = (0, \frac{1}{2}) \notin C$$

Thus, C is not strongly E-convex.

Remark 3.5

i. Every strongly E-cone is an E-cone. (Take $\alpha = 0$).

ii. Every strongly E-cone is a cone. (Take $E = I, \alpha = 0$).

The converse of Remark 3.5(i) does not hold as we show in the following examples.

Example 3.6 Consider C defined as in the Example 3.3. i.e., $C = \{(x,y) \in \mathbb{R}^2 : -1 \leq x \leq 1, -1 \leq y \leq 1\}$, and let $E(x,y) = (0,0)\ \forall x,y \in \mathbb{R}$. We show that C is E-cone but not strongly E-cone. For any $\gamma \geq 0$ and any $(x,y) \in C$, $\gamma E(x,y) = (0,0) \in C$, thus, C is E-cone. Now, if we take $\gamma = 5, \alpha = \frac{1}{2}$, and $(x,y) = (1,1) \in C$, then

$\gamma(\alpha(x,y) + E(x,y)) = 5(\frac{1}{2}(1,1) + (0,0)) = (\frac{5}{2}, \frac{5}{2}) \notin C$.

Thus, C is not strongly E-cone.

Example 3.7 Suppose that $E: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be defined as $E(x,y) = (x^2, y^2)\ \forall x,y \in \mathbb{R}$ and $C = \{(x,y) \in \mathbb{R}^2 : x \leq 0, y \leq 0\}$.

We show that C is a cone but not strongly E-cone. For any $\alpha \geq 0$ and for any $(x,y) \in C$, we have, $\alpha(x,y) = (\alpha x, \alpha y) \in C$. Thus, C is a cone. To show C is not strongly E-cone. Let $(x,y) = (-3,-5) \in C, \alpha = \frac{1}{2}$, and $\gamma = 3$, then $\gamma(\alpha(x,y) + E(x,y)) = 3\left(\frac{15}{2}, \frac{45}{2}\right) \notin C$ as required.

Proposition 3.8

i. If a set C is strongly E-cone, then $E(C) \subseteq C$.

ii. If C be a convex cone and $\alpha x + E(x) \in C$ for each $x \in C$ and $\alpha \in [0,1]$. Then C is strongly E-convex cone.

Proof. First, let us show (i). Let $E(x) \in E(C)$ such that $x \in C$. Since C is strongly E-cone, then $\gamma\big(\alpha x + E(x)\big) \in C \ \ \forall \gamma \geq 0$ and $\alpha \in [0,1]$. If $\gamma = 1$ and $\alpha = 0$, then $\gamma\big(\alpha x + E(x)\big) = E(x) \in C$ as required. To prove (ii), it is enough to prove that C is strongly E-cone since C is already strongly E-convex by Proposition 2.9(ii). Consider $x \in C$, then $\alpha x + E(x) \in C$. Since C is a cone, then $(\alpha x + E(x)) \in C$, for each $\gamma \geq 0$. Thus, C is strongly E-cone. ∎

Remark 3.9 The converse of Proposition 3.8(i) is not true in general (see Example 3.4).

Proposition 3.10 Let S be a strongly E_1-convex cone (resp., strongly E_2-convex cone) such that E_2 (resp., E_1) is constant, then S is a strongly $(E_1 o E_2)$-convex cone (resp., $(E_2 o E_1)$-convex cone).

Proof. Assume that $s_1, s_2 \in S$, $\alpha, \lambda \in [0,1]$, and $\gamma \geq 0$. We must show that
$\lambda\big(\alpha s_1 + (E_1 o E_2)(s_1)\big) + (1-\lambda)\big(\alpha s_2 + (E_1 o E_2)(s_2)\big) = \lambda\big(\alpha s_1 + E_1(E_2(s_1))\big) + (1-\lambda)\big(\alpha s_2 + E_1(E_2(s_2))\big) \in S$, and $\gamma\big(\alpha s_1 + (E_1 o E_2)(s_1)\big) = \gamma\big(\alpha s_1 + E_1(E_2(s_1))\big) \in S$. Now, E_2 is constant, then $E_2(s_1) = s_1 \in S$ and $E_2(s_2) = s_2 \in S$. Using the last assertion and the fact that S is strongly E_1-convex cone, $\lambda\big(\alpha s_1 + E_1(E_2(s_1))\big) + (1-\lambda)\big(\alpha s_2 + E_1(E_2(s_2))\big) \in S$ and $\gamma\big(\alpha s_1 + E_1(E_2(s_1))\big) \in S$. Similarly, one can show that S is strongly $(E_2 o E_1)$-convex cone. ∎

Proposition 3.11

i. Let $\{C_i : i \in I\}$ be a non-empty family of strongly E-cones, then $\bigcup_{i\in I} C_i$ is strongly E-cone.

ii. Let $\{C_i : i \in I\}$ be a non-empty family of strongly E-cones, then $\bigcap_{i\in I} C_i$ is strongly E-cone.

iii. If C_1 and C_2 be two strongly E-cones and let E is a linear mapping, then the set $C_1 + C_2$ is strongly E- cone.

iv. Let C be strongly E- cone, E is a linear mapping, and $\alpha \in \mathbb{R}$, then the set αC is strongly E-cone.

v. If C_1 and C_2 be two strongly E-cones, then $C_1 \times C_2$ is strongly $E \times E$- cone.

Proof. We prove part (i) and in a similar way one can show part (ii). Take an arbitrary $x \in \bigcup_{i\in I} C_i$ where C_i is strongly E-cone for each $i \in I$. Then, for $\gamma \geq 0$ and $\alpha \in [0,1]$, we have $\gamma(\alpha x + E(x)) \in C_i$ for some $i \in I$; hence $\gamma(\alpha x + E(x)) \in \bigcup_{i\in I} C_i$. Thus, $\bigcup_{i\in I} C_i$ is strongly E-cone. The proof of parts (iii)-(v) proceed in a way similar to that of Proposition 2.5, Proposition 2.10(i), and Proposition 2.10(iii), respectively. Hence, the proof of parts (iii)-(v) are omitted. ∎

Remark 2.6, Propositions 2.5, 2.10(i) and 2.10(iii) together with Proposition 3.11 yield the following result.

Proposition 3.12

i. Let $\{C_i : i \in I\}$ be a non-empty family of strongly E- convex cone sets, then $\bigcap_{i\in I} C_i$ is strongly E-convex cone set.

ii. Let C be strongly E- convex cone, E is a linear mapping, and $\alpha \in \mathbb{R}$, then the set αC is strongly E- convex cone set.

iii. If C_1 and C_2 be two strongly E- convex cones, then $C_1 \times C_2$ is strongly $E \times E$- convex cone set. Moreover, if E is a linear mapping then $C_1 + C_2$ is strongly E- convex cone set.

Proposition 3.13 Assume that $b \in \mathbb{R}^n$ and E is an idempotent and linear mapping then the upper E-half space $C = \{x \in \mathbb{R}^n : < E(x), b > \geq 0\}$ is strongly E-convex cone.

Proof. From Proposition 2.12 and by choosing $\beta = 0$, the set C is strongly E-convex. Hence, we only need to prove that C is strongly E-cone. Let $x \in C, \gamma \geq 0$, and $\alpha \in [0,1]$ we show that

$$< E(\gamma(\alpha x + E(x)), b > \geq 0.$$

Since E is an idempotent and linear mapping and $x \in C$, then

$$< E(\gamma(\alpha x + E(x)), b \geq < \gamma\alpha E(x), b > + < \gamma E(x), b \geq \gamma\alpha < E(x), b > + \gamma < E(x), b > \geq 0.$$ ∎

Proposition 3.14 Let I be an index set and $b_i \in \mathbb{R}^n$ for all $i \in I$. Assume also that E is an idempotent and linear mapping then $C = \{x \in \mathbb{R}^n : < E(x), b_i > \geq 0 \ \ \forall i \in I\}$ is strongly E-convex cone.

Proof. The required result follows from Proposition 3.12(i) and Proposition 3.13. ∎

The following proposition give an alternative characterization of strongly E-convex cone.

Proposition 3.15 A set C is a strongly E-convex cone if and only if C is a strongly E-closed (i.e., C is closed with respect to the mapping E and an arbitrary point in C) under addition and non-negative scalar multiplication.

Proof. Assume that C is a strongly E-convex cone. From the definition of strongly E-cone, we have $\gamma(\alpha x + E(x)) \in C$, for any $\gamma \geq 0, \alpha \in [0,1]$, and for any $x \in C$.

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Thus, C is strongly E-closed for non-negative scalar multiplication. Next, we show that C is strongly E-closed under addition. Fix $x, y \in C$ which is strongly E- convex set, then

$$u = \frac{1}{2}(\alpha x + E(x)) + \frac{1}{2}(\alpha y + E(y)) \in C.$$

Hence, $2u = (\alpha x + E(x)) + (\alpha y + E(y)) \in C$ as required. For proving the opposite direction, assume that C is strongly E-closed with respect to addition and non-negative scalar multiplication. Then, C is strongly E-cone automatically holds. Let $\lambda, \alpha \in [0,1]$ and $x, y \in C$ then

$\lambda(\alpha x + E(x)) \in C$ and $(1-\lambda)(\alpha y + E(y)) \in C$.

This yield $\lambda(\alpha x + E(x)) + (1-\lambda)(\alpha y + E(y)) \in C$. Hence, C is strongly E-convex cone set. ∎

Proposition 3.16 Let C be a subset of $\mathbb{R}^n$ and $K(x,p)$ is the set of strongly E-non-negative linear combinations of p elements of C. That is

$K(x,p) =$
$\{x = \sum_{i=1}^{p} \gamma_i(\alpha x_i + E(x_i)) : \{x_1, \dots, x_p\} \subset C, \gamma_i \geq 0, \alpha \in [0,1]\}$. If $K(x,p) \subset C \ \forall p \in N$ then C is strongly E-convex cone.

Proof. Assume that $K(x,p) \subset C \ \forall p \in N$. In particular, for each $x_1, x_2 \in C$, $\alpha \geq 0$, and $\gamma \in [0,1]$ we have $x = \gamma(\alpha x_1 + E(x_1)) + (1-\gamma)(\alpha x_2 + E(x_2)) \in C$ and $\gamma(\alpha x_1 + E(x_1)) \in C$. Hence, C is strongly E-convex cone.

Next, we introduce a smallest strongly E-convex cone that contains a certain set.

Definition 3.17 The strongly E-convex cone hull of a set C, denoted by $s.E\text{-}cone(C)$ is the intersection of all strongly E-convex cone sets containing C; that is, $E\text{-}cone(C) = \bigcap_{N \supseteq C} N$, N are strongly E-convex cone sets.

The following result is analogue to the one introduced in Proposition 2.18 for strongly E-convex sets.

Proposition 3.18 Let $C \subset \mathbb{R}^n$ and $\mathfrak{I}$ is the set of all strongly E-non-negative linear combinations of elements of C. That is

$$\mathfrak{I} = \bigcup_{p \in N} K(x,p),$$

where $K(x,p)$ is defined as in Proposition 3.16. If $\alpha x + E(x) \subseteq \mathfrak{I} \ \forall x \in C$ and $\alpha \in [0,1]$, then $s.E\text{-}cone(C) \subseteq \mathfrak{I}$.

Proof. First, we show that $\mathfrak{I}$ is a convex cone set. To show that $\mathfrak{I}$ is a convex set, follow similar steps that is used in Proposition 2.18 to show that $\mathcal{L}$ is a convex set. Next, we show that $\mathfrak{I}$ is a cone. Let $x \in \mathfrak{I}$, then there exists $p \in N$ such that $x = \sum_{i=1}^{p} \gamma_i(\alpha x_i + E(x_i))$ where $\{x_1, \dots, x_p\} \subset C, \alpha \in [0,1]$, and $\{\gamma_1, \dots, \gamma_p\}$ are non-negative scalars. Fix $\beta \geq 0$, then the non-negative E-linear combination

$$\beta x = \beta \sum_{i=1}^{p} \gamma_i(\alpha x_i + E(x_i))$$
$$= \sum_{i=1}^{p} \beta\gamma_i(\alpha x_i + E(x_i)) \in \mathfrak{I}$$

Thus, $\mathfrak{I}$ is a convex cone set, and since $+E(x) \subseteq \mathfrak{I}$ $\forall x \in C$, then from Proposition 3.8(ii), $\mathfrak{I}$ is strongly E-convex cone set. The last conclusion with the fact that $C \subseteq \mathfrak{I}$ yield $s.E\text{-}cone(C) \subseteq \mathfrak{I}$ as required. ∎

Conclusion

This paper proposes some strongly E-convex sets, namely, strongly E-convex hull, strongly E-convex cone, and strongly E-convex cone hull and discusses their properties with examples to illustrate the aforementioned concepts and to clarify the relationships among them. These sets are considered as extension to convex sets and convex cone sets. For possible future work, we suggest studying non-linear optimization problem in which the objective function is either convex function or strongly convex function and the constraint set is strongly closed cone. In addition, we can study the optimality criteria of this optimization problem.

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حول المجاميع المحدبة بقوة ومجاميع المخروط المحدبة بقوة من النوع E

صبا ناصر مجيد
قسم الرياضيات ، كلية التربية للعلوم الصرفة إبن الهيثم ، جامعة بغداد، بغداد، العراق.

المستخلص :

هناك نوع من التعميمات المهمة للمجموعات المحدبة ، الدوال المحدبة ، ومشاكل الأمثلية المحدبة تسمى المجموعات المحدبة بقوة، الدوال المحدبة بقوة، ومشاكل الأمثلية المحدبة بقوة من النوع E والتي عُرّفت ودُرسّت من قبل يونس وباحثين اخرين. في هذا النوع من المجموعات والدوال، قام يونس بتعريف المجموعة المحدبة بقوة والدالة المحدبة بقوة بالنسبة الى دالة تسمى E. في هذا البحث تم دراسة خواص جديدة للمجموعة المحدبة بقوة من النوع E والمخروط من نوع E ومجموعة الانغلاق المحدب بقوة من النوع E. قمنا ايضاً بتعريف مجموعات جديدة والمسماة بمجموعة الانغلاق المخروطي المحدب بقوة من نوع E وكذلك قمنا بدراسة بعض خواص هذه المجموعات. واخيراً قمنا بأعطاء بعض الأمثلة لتوضيح المفاهيم المستعرضة في البحث ولتوضيح العلاقة فيما بينها.

الكلمات المفتاحية: المجاميع المحدبة من النوع E، المجاميع المحدبة بقوة من النوع E ، مجاميع المخروط المحدبة بقوة من نوع E، مجاميع الانغلاق المحدب بقوة من النوع E .

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A new subclasses of meromorphic univalent functions associated with a differential operator

Hazha Zirar Hussain

Department of Mathematics, College of Science, University of Salahaddin, Erbil,Iraq.

E-mail : hazhazirar@yahoo.com

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Abstract

In this paper we have introduced and studied some new subclasses of meromorphic univalent functions which are defined by means of a differential operator. We have obtained numerous sharp results including coefficient conditions, extreme points, distortion bounds and convex combinations for the above classes of meromorphic univalent functions.

Keywords: Univalent Functions, Meromorphic Functions, Differential Operator, Distortion Inequality, Extreme Points.

Mathematics Subject Classification: **64S40.**

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1. Introduction

Let $\mathfrak{H}$ denote the class of functions which are analytic in the punctured disk $\mathcal{U}^* = \{z: 0 < |z| < 1\}$ of the form

$$f(z) = \frac{a_0}{z} + \sum_{n=1}^{\infty} a_n z^n, \quad a_0 > 0. \qquad (1.1)$$

Suppose that $\mathfrak{H}^*$denote the subclass of $\mathfrak{H}$ consisting of functions that are univalent in $\mathcal{U}^*$.

Further $\mathfrak{H}_m^*$ denote subclass of $\mathfrak{H}^*$ consisting of functions f of the form

$$f(z) = \frac{a_0}{z} + \sum_{n=0}^{\infty} a_{m+n} z^{m+n}, \quad a_0 > 0, a_{m+n} > 0, m \in \mathbb{N} \qquad (1.2)$$

Definition: A function $f \in \mathfrak{H}_m^*$ is said to be meromorphic starlike of order α in $\mathcal{U}^*$ if it satisfies the inequality

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > -\alpha, z \in \mathcal{U}^*, 0 \le \alpha < 1. \qquad (1.3)$$

On the other hand, a function $f \in \mathfrak{H}_m^*$ is said to be meromorphic convex of order α in $\mathcal{U}^*$ if it satisfies the inequality

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > -\alpha, z \in \mathcal{U}^*, 0 \le \alpha < 1. \qquad (1.4)$$

Various subclasses of $\mathfrak{H}$ have been introduced and studied by many authors see [1], [2], [5], [7], [8], [16],[17],[19], [20], [21] and [23] In recent years, some subclasses of meromorphic functions associated with several families of integral operators and derivative operators were introduced and investigated see [7] [8], [18] and [4],[15]. The first differential operator for meromorphic function was introduced by Fraisin and Darus [10]. Ghanim and Darus introduced a differential operator [11]:

$$I^0 f(z) = f(z),$$

$$I^1 f(z) = zf'(z) + \frac{2a_0}{z},$$

$$I^2 f(z) = z\left(I^1 f(z)\right)' + \frac{2a_0}{z},$$

$$I^k f(z) = z\left(I^{(k-1)} f(z)\right)' + \frac{2a_0}{z},$$

where $k \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}, z \in \mathcal{U}^*$.

For a function f in $\mathfrak{H}_m^*$, from definition of the differential operator $I^k f(z)$, we easily see that

$$I^k f(z) = \frac{a_0}{z} + \sum_{n=0}^{\infty} n^k a_{m+n} z^{m+n}, \quad a_0 > 0, a_{m+n} > 0, m \in \mathbb{N}, k \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}, z \in \mathcal{U}^*. \qquad (1.5)$$

By using the operator I^k, some authors have established many subclasses of meromorphic functions, for example [9], [11],[12] and [13]. With the help of the differential operator I^k, we define the following new class of meromorphic univalent functions and obtain some interesting results.

Let $\mathfrak{H}_{m,k}^*(\eta, \theta, \vartheta)$, denote the family of meromorphic univalent functions f of the form (1.2) such that

$$\left|\frac{z^2\left(I^k f(z)\right)' + a_0}{\vartheta z^2\left(I^k f(z)\right)' - a_0 + (1+\vartheta)\eta a_0}\right| < \theta, \qquad (1.6)$$

For $0 \le \eta < 1, 0 < \theta \le 1$, $0 \le \vartheta \le 1$, $k \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}$, and $z \in \mathcal{U}^*$.

For a given real number $z_0 (0 < z_0 < 1)$. Let $\mathfrak{H}_{mi} (i = 0, 1)$be a subclass of $\mathfrak{H}_m^*$satisfying the condition $z_0 f(z_0) = 1$ and $-{z_0}^2 f'(z_0) = 1$ respectively.

Let

$$\mathfrak{H}_{mi,k}^*(\eta, \theta, \vartheta, z_0) = \mathfrak{H}_{m,k}^*(\eta, \theta, \vartheta) \cap \mathfrak{H}_{mi}, (i = 0, 1). \qquad (1.7)$$

For other subclasses of meromorphic univalent functions, one may refer to the recent work of Aouf [2], Aouf and Darwish [3], Cho et al [8], Joshi et al [14], Srivastava and Owa [21] and [22]. Also we prove a necessary and sufficient condition for a subset C of the real interval $[0, 1]$ should satisfy the property $\cup_{z_r \in C} \mathfrak{H}_{m0,k}^*(\eta, \theta, \vartheta, z_r)$ and $\cup_{z_r \in C} \mathfrak{H}_{m1,k}^*(\eta, \theta, \vartheta, z_r)$ each constitute a convex family.

2. Coefficient Inequalities

In this section, we provide a necessary and sufficient condition for a function f meromorphic univalent in $\mathcal{U}^*$ to be in $\mathfrak{H}_{m,k}^*(\eta, \theta, \vartheta)$, $\mathfrak{H}_{m0,k}^*(\eta, \theta, \vartheta, z_0)$ and $\mathfrak{H}_{m1,k}^*(\eta, \theta, \vartheta, z_0)$.

Theorem 2.1: A function $f(z) \in \mathfrak{H}_m^*$ defined by equation (1.2) is in the class $\mathfrak{H}_{m,k}^*(\eta, \theta, \vartheta)$ if and only if

$$\sum_{n=0}^{\infty} n^k (m+n)(1+\vartheta\theta) a_{m+n} \le \theta a_0 (1-\eta)(1+\vartheta), \qquad (2.1)$$

where $0 \le \eta < 1, 0 < \theta \le 1$, $0 \le \vartheta \le 1$, $k \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}$, and $z \in \mathcal{U}^*$.

The result is sharp for the function given by

$$f(z) = \frac{a_0}{z} + \frac{\theta a_0(1-\eta)(1+\vartheta)}{n^k(m+n)(1+\vartheta\theta)} z^{m+n}, \quad n \geq 1 \qquad (2.2)$$

Proof: Assume that the condition (2.1) is true. We must show that $f \in \mathfrak{H}^*_{m,k}(\eta,\theta,\vartheta)$ or equivalently prove that

$$\left|\frac{z^2\left(I^k f(z)\right)' + a_0}{\vartheta z^2\left(I^k f(z)\right)' - a_0 + (1+\vartheta)\eta a_0}\right| < \theta,$$

$$\left|\frac{z^2\left(I^k f(z)\right)' + a_0}{\vartheta z^2\left(I^k f(z)\right)' - a_0 + (1+\vartheta)\eta a_0}\right| =$$

$$\left|\frac{a_0 + \left(-a_0 + \sum_{n=1}^{\infty}(m+n)n^k a_{m+n} z^{m+n+1}\right)}{\vartheta\left(-a_0 + \sum_{n=1}^{\infty}(m+n)n^k a_{m+n} z^{m+n+1}\right) - a_0 + (1+\vartheta)\eta a_0}\right| =$$

$$\left|\frac{\sum_{n=0}^{\infty}(m+n)n^k a_{m+n} z^{m+n+1}}{\vartheta\left(-a_0 + \sum_{n=0}^{\infty}(m+n)n^k a_{m+n} z^{m+n+1}\right) - a_0 + (1+\vartheta)\eta a_0}\right| \leq$$

$$\left|\frac{\sum_{n=0}^{\infty}(m+n)n^k a_{m+n}}{\vartheta\left(-a_0 + \sum_{n=0}^{\infty}(m+n)n^k a_{m+n}\right) - a_0 + (1+\vartheta)\eta a_0}\right| < \theta.$$

The last inequality is true by (2.1).
Conversely, suppose that $f \in \mathfrak{H}^*_{m,k}(\eta,\theta,\vartheta)$. We must show that the condition (2.1) holds true. We have

$$\left|\frac{z^2\left(I^k f(z)\right)' + a_0}{\vartheta z^2\left(I^k f(z)\right)' - a_0 + (1+\vartheta)\eta a_0}\right| < \theta.$$

Thus

$$\left|\frac{\sum_{n=0}^{\infty}(m+n)n^k a_{m+n} z^{m+n+1}}{\vartheta\left(-a_0 + \sum_{n=0}^{\infty}(m+n)n^k a_{m+n} z^{m+n+1}\right) - a_0 + (1+\vartheta)\eta a_0}\right| < \theta.$$

Since $Re(z) < |z|$ for all z, we have

$$Re\left\{\frac{\sum_{n=0}^{\infty}(m+n)n^k a_{m+n} z^{m+n+1}}{\vartheta\left(-a_0 + \sum_{n=0}^{\infty}(m+n)n^k a_{m+n} z^{m+n+1}\right) - a_0 + (1+\vartheta)\eta a_0}\right\} < \theta.$$

Now, choosing values of z on the real axise and allowing $z \to 1$ from the left through real values, the last inequality immediately yields the desired condition in (2.1).
Finally, it is observed that the result is sharp for the function given by

$$f(z) = \frac{a_0}{z} + \frac{\theta a_0(1-\eta)(1+\vartheta)}{n^k(m+n)(1+\vartheta\theta)} z^{m+n}, \quad n \geq 1.$$

Theorem 2.2: A function $f(z) \in \mathfrak{H}^*_m$ defined by equation (1.2) is in the class $\mathfrak{H}^*_{m0,k}(\eta,\theta,\vartheta,z_0)$ if and only if

$$\sum_{n=0}^{\infty}\left[\frac{n^k(m+n)(1+\vartheta\theta)}{\theta(1-\eta)(1+\vartheta)} + z_0{}^{m+n+1}\right] a_{m+n} \leq 1, \qquad (2.3)$$

where $0 \leq \eta < 1, 0 < \theta \leq 1, \quad 0 \leq \vartheta \leq 1, \ k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and $z \in \mathcal{U}^*$.
The result is sharp for the function given by

$$f(z) = \frac{n^k(m+n)(1+\vartheta\theta) + \theta(1-\eta)(1+\vartheta) z^{m+n+1}}{z\left[n^k(m+n)(1+\vartheta\theta) + \theta(1-\eta)(1+\vartheta) z_0{}^{m+n+1}\right]}, \quad m \in \mathbb{N}, n \geq 1 \qquad (2.4)$$

Proof: Assume that $f \in \mathfrak{H}^*_{m0,k}(\eta,\theta,\vartheta,z_0)$, then

$$f(z_0) = \frac{a_0}{z_0} + \sum_{n=0}^{\infty} a_{m+n} z_0{}^{m+n}, \quad a_0 > 0, a_{m+n} > 0, m \in \mathbb{N}$$

$$z_0 f(z_0) = a_0 + \sum_{n=0}^{\infty} a_{m+n} z_0{}^{m+n+1}, \quad a_0 > 0, a_{m+n} > 0, m \in \mathbb{N}$$

$$1 = a_0 + \sum_{n=0}^{\infty} a_{m+n} z_0{}^{m+n+1}, \quad a_0 > 0, a_{m+n} > 0, m \in \mathbb{N}$$

$$a_0 = 1 - \sum_{n=0}^{\infty} a_{m+n} z_0{}^{m+n+1}, \quad a_0 > 0, a_{m+n} > 0, m \in \mathbb{N}, \qquad (2.5)$$

Subistituting equation (2.5) in inequality (2.1), we get

$$\sum_{n=0}^{\infty} n^k(m+n)(1+\vartheta\theta) a_{m+n} \leq \theta\left(1 - \sum_{n=0}^{\infty} a_{m+n} z_0{}^{m+n+1}\right)(1-\eta)(1+\vartheta),$$

$$\sum_{n=0}^{\infty} n^k(m+n)(1+\vartheta\theta) a_{m+n} + \sum_{n=0}^{\infty} \theta(1-\eta)(1+\vartheta) a_{m+n} z_0{}^{m+n+1} \leq \theta(1-\eta)(1+\vartheta)$$

Thus,

$$\sum_{n=0}^{\infty}\left[\frac{n^k(m+n)(1+\vartheta\theta)}{\theta(1-\eta)(1+\vartheta)} + z_0{}^{m+n+1}\right] a_{m+n} \leq 1.$$

Hence the proof is complete.

Theorem 2.3: A function $f(z) \in \mathfrak{H}_m^*$ defined by equation (1.2) is in the class $\mathfrak{H}_{m1,k}^*(\eta,\theta,\vartheta,z_0)$ if and only if

$$\sum_{n=0}^{\infty}(m+n)\left[\frac{n^k(1+\vartheta\theta)}{\theta(1-\eta)(1+\vartheta)} - z_0{}^{m+n+1}\right]a_{m+n} \leq 1, \qquad (2.6)$$

where $0 \leq \eta < 1, 0 < \theta \leq 1, \quad 0 \leq \vartheta \leq 1, \; k \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}$, and $z \in \mathcal{U}^*$.

The result is sharp for the function given by

$$f(z) = \frac{n^k(m+n)(1+\vartheta\theta)+\theta(1-\eta)(1+\vartheta)z^{m+n+1}}{z(m+n)[n^k(1+\vartheta\theta)-\theta(1-\eta)(1+\vartheta)z_0{}^{m+n+1}]}, \quad m \in \mathbb{N}, n \geq 1 \qquad (2.7)$$

Proof: Assume that $f \in \mathfrak{H}_{m1,k}^*(\eta,\theta,\vartheta,z_0)$, then

$$f(z_0) = \frac{a_0}{z_0} + \sum_{n=0}^{\infty} a_{m+n}z_0{}^{m+n}, \quad a_0 > 0, a_{m+n} > 0, m \in \mathbb{N}$$

$$-z_0{}^2 f'(z_0) = a_0 + \sum_{n=0}^{\infty}(m+n)a_{m+n}z_0{}^{m+n+1}, \quad a_0 > 0, a_{m+n} > 0, m \in \mathbb{N}$$

$$1 = a_0 + \sum_{n=0}^{\infty}(m+n)a_{m+n}z_0{}^{m+n+1}, \quad a_0 > 0, a_{m+n} > 0, m \in \mathbb{N}$$

$$a_0 = 1 - \sum_{n=0}^{\infty}(m+n)a_{m+n}z_0{}^{m+n+1}, \quad a_0 > 0, a_{m+n} > 0, m \in \mathbb{N}, \qquad (2.8)$$

subistituting equation (2.8) in equation (2.1), we get

$$\sum_{n=0}^{\infty} n^k(m+n)(1+\vartheta\theta)a_{m+n}, \leq \theta\left(1 - \sum_{n=0}^{\infty}(m+n)a_{m+n}z_0{}^{m+n+1}\right)(1-\eta)(1+\vartheta)$$

and,

$$\sum_{n=0}^{\infty} n^k(m+n)(1+\vartheta\theta)a_{m+n} + \sum_{n=0}^{\infty}\theta(m+n)(1-\eta)(1+\vartheta)a_{m+n}z_0{}^{m+n+1} \leq \theta(1-)(1-\eta)(1+\vartheta)$$

Thus,

$$\sum_{n=0}^{\infty}(m+n)\left[\frac{n^k(1+\vartheta\theta)}{\theta(1-\eta)(1+\vartheta)} - z_0{}^{m+n+1}\right]a_{m+n} \leq 1.$$

Hence the proof is complete.

From Theorem 2.2 and Theorem 2.3, we have the following results:

Corollary 2.1: If a function $f(z) \in \mathfrak{H}_m^*$ defined by (1.2) is in the class $\mathfrak{H}_{m0,k}^*(\eta,\theta,\vartheta,z_0)$, then

$$a_{m+n} \leq \frac{\theta(1-\eta)(1+\vartheta)}{n^k(m+n)(1+\vartheta\theta)+\theta(1-\eta)(1+\vartheta)z_0{}^{m+n+1}}, \qquad (2.9)$$

where $0 \leq \eta < 1, 0 < \theta \leq 1 \; 0 \leq \vartheta \leq 1, \; k \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}$, and $z \in \mathcal{U}^*$.

Corollary 2.2: If a function $f(z) \in \mathfrak{H}_m^*$ defined by (1.2) is in the class $\mathfrak{H}_{m1,k}^*(\eta,\theta,\vartheta,z_0)$, then

$$a_{m+n} \leq \frac{\theta(1-\eta)(1+\vartheta)}{(m+n)[n^k(1+\vartheta\theta)-\theta(1-\eta)(1+\vartheta)z_0{}^{m+n+1}]}, \qquad (2.10)$$

where $0 \leq \eta < 1, 0 < \theta \leq 1 \; 0 \leq \vartheta \leq 1, \; k \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}$, and $z \in \mathcal{U}^*$.

3. Covering theorems

In this section, distortion theorems will be considered and covering property for functions in the classes $\mathfrak{H}_{m0,k}^*(\eta,\theta,\vartheta,z_0)$ and $\mathfrak{H}_{m1,k}^*(\eta,\theta,\vartheta,z_0)$ will also be given.

Theorem 3.1: If a function $f(z) \in \mathfrak{H}_m^*$ defined by equation (1.2) is in the class $\mathfrak{H}_{m0,k}^*(\eta,\theta,\vartheta,z_0)$, then

$$|f(z)| \geq \frac{m(1+\vartheta\theta)-\theta(1-\eta)(1+\vartheta)r^{m+1}}{r[m(1+\vartheta\theta)+\theta(1-\eta)(1+\vartheta)z_0{}^{m+1}]},$$

where $0 \leq \eta < 1, 0 < \theta \leq 1 \; 0 \leq \vartheta \leq 1, \; k \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}$, and $0 < |z| < 1$.

The result is sharp with the extremal function f given by

$$f(z) = \frac{m(1+\vartheta\theta)+\theta(1-\eta)(1+\vartheta)r^{m+1}}{r[m(1+\vartheta\theta)+\theta(1-\eta)(1+\vartheta)z_0{}^{m+1}]}.$$

Proof: Since $f \in \mathfrak{H}_{m0,k}^*(\eta,\theta,\vartheta,z_0)$, by Theorem 2.2 we have

$$m(1+\vartheta\theta)+\theta(1-\eta)(1+\vartheta)z_0{}^{m+1}\sum_{n=0}^{\infty} a_{m+n} \leq \sum_{n=0}^{\infty} n^k(m+n)(1+\vartheta\theta)+\theta(1-\eta)(1+\vartheta)z_0{}^{m+n+1}a_{m+n} \leq \theta(1-\eta)(1+\vartheta),$$

$$\sum_{n=0}^{\infty} a_{m+n} \leq \frac{\theta(1-\eta)(1+\vartheta)}{m(1+\vartheta\theta)+\theta(1-\eta)(1+\vartheta)z_0{}^{m+1}},$$

Also we have

$$a_0 = 1 - \sum_{n=0}^{\infty} a_{m+n} {z_0}^{m+n+1}, \quad a_0 > 0, a_{m+n} > 0, m \in \mathbb{N},$$

$$\geq \frac{m(1+\theta\vartheta)}{m(1+\vartheta\theta)+\theta(1-\eta)(1+\vartheta){z_0}^{m+1}},$$

Thus from the above equation we obtain

$$|f(z)| = \left|\frac{a_0}{z} + \sum_{n=0}^{\infty} a_{m+n} z^{m+n}\right|, a_0 > 0, a_{m+n} > 0, m \in \mathbb{N}$$

$$\geq \frac{a_0}{r} - r^m \sum_{n=0}^{\infty} a_{m+n}$$

$$\geq \frac{m(1+\vartheta\theta)-\theta(1-\eta)(1+\vartheta)r^{m+1}}{r[m(1+\vartheta\theta)+\theta(1-\eta)(1+\vartheta){z_0}^{m+1}]}.$$

Hence the proof is complete.

Theorem 3.2: If a function $f(z) \in \mathfrak{H}_m^*$ defined by equation (1.2) is in the class $\mathfrak{H}_{m1,k}^*(\eta,\theta,\vartheta,z_0)$, then

$$|f(z)| \leq \frac{m(1+\vartheta\theta)+\theta(1-\eta)(1+\vartheta)r^{m+1}}{r[m(1+\vartheta\theta)+\theta(1-\eta)(1+\vartheta){z_0}^{m+1}]},$$

where $0 \leq \eta < 1, 0 < \theta \leq 1$ $0 \leq \vartheta \leq 1$, $k \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}$, and $0 < |z| = r < 1$.

The result is sharp with the extremal function f given by

$$f(z) = \frac{m(1+\vartheta\theta)+\theta(1-\eta)(1+\vartheta)r^{m+1}}{rm[(1+\vartheta\theta)+\theta(1-\eta)(1+\vartheta){z_0}^{m+1}]}.$$

Proof: Since $f \in \mathfrak{H}_{m1,k}^*(\eta,\theta,\vartheta,z_0)$ by Theorem 2.3 we have

$m(1+\vartheta\theta)+\theta(1-\eta)(1+\vartheta){z_0}^{m+1}\sum_{n=0}^{\infty} a_{m+n} \leq \sum_{n=0}^{\infty} n^k(m+n)(1+\vartheta\theta)+\theta(1-\eta)(1+\vartheta){z_0}^{m+n+1}a_{m+n} \leq \theta(1-\eta)(1+\vartheta),$

$$\sum_{n=0}^{\infty} a_{m+n} \leq \frac{\theta(1-\eta)(1+\vartheta)}{m(1+\vartheta\theta)-\theta(1-\eta)(1+\vartheta){z_0}^{m+1}}.$$

Also we have

$$a_0 = 1 + \sum_{n=0}^{\infty}(m+n)a_{m+n}{z_0}^{m+n+1}, \quad a_0 > 0, a_{m+n} > 0, m \in \mathbb{N},$$

$$\leq \frac{(1+\theta\vartheta)}{(1+\vartheta\theta)+\theta(1-\eta)(1+\vartheta){z_0}^{m+1}}.$$

Thus from the above equation we obtain

$$|f(z)| = \left|\frac{a_0}{z} + \sum_{n=0}^{\infty} a_{m+n} z^{m+n}\right|, a_0 > 0, a_{m+n} > 0, m \in \mathbb{N}$$

$$\leq \frac{a_0}{r} + r^m \sum_{n=0}^{\infty} a_{m+n}$$

$$\leq \frac{m(1+\vartheta\theta)+\theta(1-\eta)(1+\vartheta)r^{m+1}}{rm[(1+\vartheta\theta)+\theta(1-\eta)(1+\vartheta){z_0}^{m+1}]}.$$

Hence the proof is complete.

Corollary 3.1: The disk $0 < |z| < 1$ is mapped onto a domain that contains the disk $|w| < \frac{m(1+\vartheta\theta)-\theta(1-\eta)(1+\vartheta)r^{m+1}}{[m(1+\vartheta\theta)+\theta(1-\eta)(1+\vartheta){z_0}^{m+1}]}$ by any function $f \in \mathfrak{H}_{m0,k}^*(\eta,\theta,\vartheta,z_0)$.

4. Extreme Points

The extreme points of the class $\mathfrak{H}_{m0,k}^*(\eta,\theta,\vartheta,z_0)$ and $\mathfrak{H}_{m1,k}^*(\eta,\theta,\vartheta,z_0)$ are given by the following theorem.

Theorem 4.1: Let $f_0(z) = \frac{1}{z}$,

and

$$f_{m+n}(z) = \frac{n^k(m+n)(1+\vartheta\theta)+\theta(1-\eta)(1+\vartheta)z^{m+n+1}}{z[n^k(m+n)(1+\vartheta\theta)+\theta(1-\eta)(1+\vartheta){z_0}^{m+n+1}]},$$

$n \geq 0$

then $f(z)$ is in the class $\mathfrak{H}_{m0,k}^*(\eta,\theta,\vartheta,z_0)$, if and only if it can be expressed in the form $f(z) = \sum_{n=0}^{\infty}\gamma_n f_{m+n}(z)$ where $\gamma_n \geq 0, \gamma_i = 0$ $(i = 1,2,\dots,m-1, m \geq 2)$ and $\sum_{n=0}^{\infty}\gamma_n = 1$.

Proof: Suppose

$$f(z) = \sum_{n=0}^{\infty}\gamma_n f_{m+n}(z)$$

$$= \frac{\gamma_0}{z} + \sum_{n=0}^{\infty}\frac{n^k(m+n)(1+\vartheta\theta)+\theta(1-\eta)(1+\vartheta)z^{m+n+1}\gamma_{m+n}}{z[n^k(m+n)(1+\vartheta\theta)+\theta(1-\eta)(1+\vartheta){z_0}^{m+n+1}]}$$

$$= \frac{1}{z}\left[\gamma_0 + \sum_{n=0}^{\infty}\frac{n^k(m+n)(1+\vartheta\theta)\gamma_{m+n}}{n^k(m+n)(1+\vartheta\theta)+\theta(1-\eta)(1+\vartheta){z_0}^{m+n+1}}\right]$$

$$+ \sum_{n=0}^{\infty}\frac{\theta(1-\eta)(1+\vartheta)\gamma_{m+n}z^{m+n+1}}{n^k(m+n)(1+\vartheta\theta)+\theta(1-\eta)(1+\vartheta){z_0}^{m+n+1}}.$$

Then, we have

$$\sum_{n=0}^{\infty}\frac{n^k(m+n)(1+\vartheta\theta)+\theta(1-\eta)(1+\vartheta){z_0}^{m+n+1}}{\theta(1-\eta)(1+\vartheta)} \times \left(\frac{\theta(1-\eta)(1+\vartheta)\gamma_{m+n}}{n^k(m+n)(1+\vartheta\theta)+\theta(1-\eta)(1+\vartheta){z_0}^{m+n+1}}\right),$$

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$$\sum_{n=0}^{\infty}\gamma_{m+n} = 1-\gamma_0 \le 1.$$

Now, we have

$$z_0 f_{m+n}(z_0) = 1.$$

Thus,

$$z_0 f(z_0) = \sum_{n=0}^{\infty}\gamma_{m+n} z_0 f_{m+n}(z_0) = \sum_{n=0}^{\infty}\gamma_{m+n} = 1.$$

This implies that $f \in \mathfrak{H}_{m0,k}$.

Therefore $f \in \mathfrak{H}^*_{m0,k}(\eta,\theta,\vartheta,z_0)$.

Conversely, suppose $f \in \mathfrak{H}^*_{m0,k}(\eta,\theta,\vartheta,z_0)$. Since

$$a_{m+n} \le \frac{\theta(1-\eta)(1+\vartheta)}{n^k(m+n)(1+\vartheta\theta)+\theta(1-\eta)(1+\vartheta){z_0}^{m+n+1}}, \quad n \ge 0.$$

Set

$$\gamma_{m+n} = \frac{[n^k(m+n)(1+\vartheta\theta)+\theta(1-\eta)(1+\vartheta){z_0}^{m+n+1}]}{\theta(1-\eta)(1+\vartheta)} a_{m+n}, n \ge 0,$$

and $\gamma_0 = 1-\sum_{n=0}^{\infty}\gamma_{m+n}$.

Then

$$f(z) = \sum_{n=0}^{\infty}\gamma_n f_n(z).$$

This completes the proof of Theorem 4.1.

Theorem 4.2: Let $f_0(z) = \frac{1}{z}$,

and

$$f_{m+n}(z) = \frac{n^k(m+n)(1+\vartheta\theta)+\theta(1-\eta)(1+\vartheta)z^{m+n+1}}{z(m+n)[n^k(1+\vartheta\theta)-\theta(1-\eta)(1+\vartheta){z_0}^{m+n+1}]}, \quad n \ge 0$$

Then $f(z)$ is in the class $\mathfrak{H}^*_{m1,k}(\eta,\theta,\vartheta,z_0)$, if and only if it can be expressed in the form $f(z) = \sum_{n=0}^{\infty}\gamma_n f_n(z)$ where $\gamma_n \ge 0, \gamma_i = 0$ $(i = 1,2,\dots,m-1, m \ge 2)$ and $\sum_{n=0}^{\infty}\gamma_n = 1$.

Corollary 4.1: The extreme points of the class $\mathfrak{H}^*_{m0,k}(\eta,\theta,\vartheta,z_0)$ are the functions $f_0(z), f_m, f_{m+1}, f_{m+2}, \dots$ in Theorem 4.1.

Corollary 4.2: The extreme points of the class $\mathfrak{H}^*_{m1,k}(\eta,\theta,\vartheta,z_0)$ are the functions $f_0(z), f_m, f_{m+1}, f_{m+2}, \dots$ in Theorem 4.2.

5. Closure Theorems

Theorem 5.1: The class $\mathfrak{H}^*_{m0,k}(\eta,\theta,\vartheta,z_0)$ is closed under convex linear combination

Proof: Suppose that the functions $f, g \in \mathfrak{H}^*_{m0,k}(\eta,\theta,\vartheta,z_0)$ defined by

$$f(z) = \frac{a_0}{z} + \sum_{n=0}^{\infty} a_{m+n}z^{m+n}, a_0 > 0, a_{m+n} > 0, z \in \mathcal{U}^*$$

and

$$g(z) = \frac{b_0}{z} + \sum_{n=0}^{\infty} b_{m+n}z^{m+n}, b_0 > 0, b_{m+n} > 0, z \in \mathcal{U}^*$$

respectively, it is sufficient to prove that the function H defined by

$$H(z) = \omega f(z) + (1-\omega)g(z), \quad (0 \le \omega \le 1)$$

is also in the class $\mathfrak{H}^*_{m0,k}(\eta,\theta,\vartheta,z_0)$.

Since

$$H(z) = \frac{\omega a_0 + (1-\omega)b_0}{z} + \sum_{n=0}^{\infty}(\omega a_{m+n} + (1-\omega)b_{m+n})z^{m+n}, a_0 > 0, a_{m+n} > 0, z \in \mathcal{U}^*$$

we observe that

$$\sum_{n=0}^{\infty}[n^k(m+n)(1+\vartheta\theta) + {z_0}^{m+n+1}](\omega a_{m+n} + (1-\omega)b_{m+n}) \le \theta(1-\eta)(1+\vartheta),$$

with the aid of theorem 2.2.

Thus $H(z) \in \mathfrak{H}^*_{m0,k}(\eta,\theta,\vartheta,z_0)$.

This completes the proof of the theorem.

In a similar manner, by using Theorem 2.3, we can prove the following theorem.

Theorem 5.2: The class $\mathfrak{H}^*_{m1,k}(\eta,\theta,\vartheta,z_0)$ is closed under convex linear combination.

Proof: The proof is similar to that of Theorem 5.1.

Theorem 5.3: Let the function $f_l(z), l = 0,1,2,\dots,q$ defined by

$$f_l(z) = \frac{a_{0,l}}{z} + \sum_{n=0}^{\infty} a_{m+n,l}z^{m+n}, a_0 > 0, a_{m+n,l} > 0, z \in \mathcal{U}^*$$

be in the class $\mathfrak{H}^*_{m0,k}(\eta,\theta,\vartheta,z_0)$. Then the function

$$\varphi(z) = \sum_{l=0}^{q} c_l f_l(z), \quad (c_l \ge 0)$$

is also in the class $\mathfrak{H}^*_{m0,k}(\eta,\theta,\vartheta,z_0)$, where $\sum_{l=0}^{q} c_l = 1$.

Proof: By Theorem 2.2 and for every $l = 0,1,2,\dots,q$ we have

$$\sum_{n=0}^{\infty}[n^k(m+n)(1+\vartheta\theta)+{z_0}^{m+n+1}]a_{m+n,l} \le \theta(1-\eta)(1+\vartheta),$$

Then,

$$\varphi(z)=\sum_{l=0}^{q}c_l\left(\frac{a_{0,l}}{z}+\sum_{n=0}^{\infty}a_{m+n,l}z^{m+n}\right),\quad (c_l\ge 0)$$

$$=\frac{c_l a_{0,l}}{z}+\sum_{n=0}^{\infty}\left(\sum_{l=0}^{q}c_l a_{m+n,l}\right)z^{m+n}.$$

Since

$$\sum_{n=0}^{\infty}[n^k(m+n)(1+\vartheta\theta)+{z_0}^{m+n+1}]\left(\sum_{l=0}^{q}c_l a_{m+n,l}\right),$$

$$=\sum_{l=0}^{q}c_l\left(\sum_{n=0}^{\infty}[n^k(m+n)(1+\vartheta\theta)+{z_0}^{m+n+1}]\,a_{m+n,l}\right),$$

$$\le\left(\sum_{l=0}^{q}c_l\right)\theta(1-\eta)(1+\vartheta),$$

$$=\theta(1-\eta)(1+\vartheta),$$

Then, $\varphi(z)\in\mathfrak{H}^*_{m0,k}(\eta,\theta,\vartheta,z_0)$.

Theorem 5.4: Let the function $f_l(z), l = 0,1,2,\dots,q$ defined by

$$f_l(z)=\frac{a_{0,l}}{z}+\sum_{n=0}^{\infty}a_{m+n,l}z^{m+n},\ a_0>0,\ a_{m+n,l}>0,\ z\in\mathcal{U}^*$$

be in the class $\mathfrak{H}^*_{m1,k}(\eta,\theta,\vartheta,z_0)$. Then the function

$$\varphi(z)=\sum_{l=0}^{q}c_l\,f_l(z),\quad (c_l\ge 0)$$

is also in the class $\mathfrak{H}^*_{m1,k}(\eta,\theta,\vartheta,z_0)$, where $\sum_{l=0}^{q}c_l=1$.

Proof: The proof is similar to that of Theorem 5.3.

6. Convex Family

Definition 6.1: The family $\mathfrak{H}^*_{m0,k}(\eta,\theta,\vartheta,C)$ is defined by

$$\mathfrak{H}^*_{m0,k}(\eta,\theta,\vartheta,C)=\cup_{z_r\in C}\,\mathfrak{H}^*_{m0,k}(\eta,\theta,\vartheta,z_r),$$

where C is a nonempty subset of the real interval $[0,1]$ and $\mathfrak{H}^*_{m0,k}(\eta,\theta,\vartheta,C)$ is defined by a convex family if the subset C consists of one element only by Theorems 5.1 and 5.3.

Now, we have the following results:

Lemma 6.1: Let $z_1,z_2\in C$ be two distinct positive numbers and $f(z)\in\mathfrak{H}^*_{m0,k}(\eta,\theta,\vartheta,z_0)\cap\mathfrak{H}^*_{m0,k}(\eta,\theta,\vartheta,z_1)$, then $f(z)=\frac{1}{z}$.

Proof: Suppose that $f(z)\in\mathfrak{H}^*_{m0,k}(\eta,\theta,\vartheta,z_1)\cap\mathfrak{H}^*_{m0,k}(\eta,\theta,\vartheta,z_2)$,

we have

$$a_0=1-\sum_{n=0}^{\infty}a_{m+n}{z_1}^{m+n+1}$$

$$=1-\sum_{n=0}^{\infty}a_{m+n}{z_2}^{m+n+1}.$$

Also

$$f(z)=\frac{a_0}{z}+\sum_{n=0}^{\infty}a_{m+n}z^{m+n},\quad a_0>0,\ a_{m+n}>0,\ m\in\mathbb{N}$$

Thus, $a_{m+n}\equiv 0,\forall\, n\ge 0$, because $a_{m+n}\ge 0, z_1>0$ and $z_2>0$, hence

$$f(z)=\frac{1}{z}.$$

This completes the proof of the Lemma.

Theorem 6.1: Suppose that $C\subset[0,1]$, then $\mathfrak{H}^*_{m0,k}(\eta,\theta,\vartheta,C)$ is a convex family if and only if C is connected.

Proof: Assume that C is connected and $z_1,z_2\in C$ with $z_1<z_2$.

$$a_0=1-\sum_{n=0}^{\infty}a_{m+n}{z_0}^{m+n+1}$$

$$=1-\sum_{n=0}^{\infty}b_{m+n}{z_1}^{m+n+1}.$$

Suppose that the functions $f\in\mathfrak{H}^*_{m0,k}(\eta,\theta,\vartheta,z_0)$ defined by

$$f(z)=\frac{a_0}{z}+\sum_{n=0}^{\infty}a_{m+n}z^{m+n},\ a_0>0,\ a_{m+n}>0,\ z\in\mathcal{U}^*$$

and $g\in\mathfrak{H}^*_{m0,k}(\eta,\theta,\vartheta,z_1)$

$$g(z)=\frac{b_0}{z}+\sum_{n=0}^{\infty}b_{m+n}z^{m+n},\ b_0>0,\ b_{m+n}>0,\ z\in\mathcal{U}^*$$

it is sufficient to prove that the function H defined by

$$H(z)=\omega f(z)+(1-\omega)g(z),\quad (0\le\omega\le 1)$$

that there exists a $z_2(z_0\le z_2\le z_1)$ is also in the class $\mathfrak{H}^*_{m0,k}(\eta,\theta,\vartheta,z_2)$.

Then

$$K(z) = zH(z)$$

$$K(z) = \omega a_0 + (1-\omega)b_0 + \sum_{n=0}^{\infty}(\omega a_{m+n} + (1-\omega)b_{m+n})z^{m+n}, a_0 > 0, a_{m+n} > 0, z \in \mathcal{U}^*$$

$$= 1 + \omega\sum_{n=0}^{\infty}(z^{m+n} - z_0{}^{m+n})a_{m+n} + (1-\omega)\sum_{n=0}^{\infty}(z^{m+n} - z_1{}^{m+n})\, b_{m+n}, a_0 > 0, a_{m+n} > 0, z \in \mathcal{U}^*$$

since z is real number, then $K(z)$ is also real number also we have

$K(z_0) \le 1$ and $K(z_1) \ge 1$, there exists $z_2 \in [z_0, z_1]$, such that $K(z_2) = 1$.

Therefore,

$$z_2H(z_2) = z_2, \quad (z_0 \le z_2 \le z_1)$$

this implies that

$$H(z) \in \mathfrak{H}^*_{m0,k}.$$

We observe that

$$\sum_{n=0}^{\infty}[n^k(m+n)(1+\vartheta\theta) + z_2{}^{m+n+1}](\omega a_{m+n} + (1-\omega)b_{m+n})$$

$$= \omega\sum_{n=0}^{\infty}[n^k(m+n)(1+\vartheta\theta) + z_0{}^{m+n+1}]\, a_{m+n} + (1-\omega)\sum_{n=0}^{\infty}[n^k(m+n)(1+\vartheta\theta) + z_1{}^{m+n+1}]\, b_{m+n}$$

$$+\theta(1-\eta)(1+\vartheta)\omega\sum_{n=0}^{\infty}(z_2{}^{m+n+1} - z_0{}^{m+n+1})\, a_{m+n} + \theta(1-\eta)(1+\vartheta)(1-\omega)\sum_{n=0}^{\infty}(z_2{}^{m+n+1} - z_1{}^{m+n+1})\, b_{m+n}$$

$$= \omega\sum_{n=0}^{\infty}[n^k(m+n)(1+\vartheta\theta) + z_0{}^{m+n+1}]\, a_{m+n} + (1-\omega)\sum_{n=0}^{\infty}[n^k(m+n)(1+\vartheta\theta) + z_1{}^{m+n+1}]\, b_{m+n}$$

$$\le \theta(1-\eta)(1+\vartheta) + (1-\omega)\,\theta(1-\eta)(1+\vartheta) = \theta(1-\eta)(1+\vartheta).$$

With the aid of theorem 2.2.

Thus, $H(z) \in \mathfrak{H}^*_{m0,k}(\eta,\theta,\vartheta,z_2)$.

Since z_1 and z_2 are arbitrary numbers, the family $\mathfrak{H}^*_{m0,k}(\eta,\theta,\vartheta,C)$ is convex.

Conversely, if the set C is not connected, then there exists z_0, z_1 and z_2 such that $z_0, z_1 \in C$ and $z_2 \notin C$ and $z_0 < z_2 < z_1$.

Now, let $f(z) \in \mathfrak{H}^*_{m0,k}(\eta,\theta,\vartheta,z_0)$, and $g(z) \in \mathfrak{H}^*_{m0,k}(\eta,\theta,\vartheta,z_1)$

Therefore,

$$K(\omega) = K(z_2,\omega)$$

$$= 1 + \omega\sum_{n=0}^{\infty}(z_2{}^{m+n+1} - z_0{}^{m+n+1})a_{m+n} + (1-\omega)\sum_{n=0}^{\infty}(z_2{}^{m+n+1} - z_1{}^{m+n+1})\, b_{m+n}, a_0 > 0, a_{m+n} > 0, z \in \mathcal{U}^*$$

for fixed z_2 and $0 \le \omega \le 1$.

Since $K(z_2,0) < 1$ and $K(z_2,1) > 1$, there exists $\omega_0; 0 < \omega_0 < 1$, such that $K(z_2,\omega_0) = 1$ or $z_2K(z_2) = 1$,

where $K(z) = \omega_0 f(z) + (1-\omega_0)g(z)$.

Therefore $K(z) \in \mathfrak{H}^*_{m0,k}(\eta,\theta,\vartheta,z_0)$

Also $K(z) \notin \mathfrak{H}^*_{m0,k}(\eta,\theta,\vartheta,C)$ using Lemma 6.1.

Since $z_2 \in C$ and $K(z) \ne z$.

Thus the family $\mathfrak{H}^*_{m0,k}(\eta,\theta,\vartheta,C)$ is not convex which is a contradiction.

This completes the proof of theorem.

Conclusion: The main impact of this paper is to is to introduce a new subclasses of meromorphic univalent functions, and study their geometrical properties , like coefficient estimate, distortion theorem, extreme points and convex family.

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حول اصناف جديدة من الدوال احادية التكافؤ الميرومورفية بمؤثر تفاضلي

هازه زرار حسين

قسم الرياضيات ، كلية العلوم، جامعة صلاح الدين ، اربيل، عراق

المستخلص :

في هذا البحث نعرف وندرس اصناف جديدة $\mathfrak{H}^{*}_{mi,k}(\eta,\theta,\vartheta,z_0)(i=1,2)$ من الدوال احادية التكافؤ الميرومورفية المعرفة بواسطة مؤثر تفاضلي ، ونحصل على العديد من النتائج المهمة مثل متباينة المعاملات، والنقاط القصوى، نظرية البعد، التركيب المحدب للاصناف من الدوال السابقة.

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Haar Wavelet Technique for Solving Fractional Differential Equations with An Application

Mohammed S. Mechee Oday I. Al-Shaher Ghassan A. Al-Juaifri
Department of Mathematics , College of Computer Science and Mathematics , Kufa university, Najaf, Iraq
mohammeds.abed@uokufa.edu.iq

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Abstract:

In this article, the approxmated solutions of ordinary differential equations of fractional order using Haar wavelet and B-spline bases are introduced. The algorithm of collection method is updated using two basis. Several initial value problems has been solved to show the applicability and efficacy of the Haar wavelet and B-spline basis. An application of Lane-Eman equation has been introduced and studied. The approximated results have clearly shown the advantage and the efficiency of the modified method in terms of accuracy and computational time.

Keywords Fractional-Order; Ordinary differential equations; Haar Wavelet; Collocation Method; B-spline; Operational matrix Approximated solution.

Mathematics Subject Classification:34-XX

1.Introduction.

Differential equations (DEs) have an important value in many applications relating to various fields such as engineering , physics, chemistry, biology and economics, however many mathematical models of physical systems are given as DE first-, second- or higher-order. Two types of DEs which depends on the domain of definition of DE we classified them as complex Des (CDEs) and real DEs (RDEs) each of them classified as follows firstly: Ordinary differential equations (ODEs) which have the branches: delay differential equations (DDEs), fractional differential equations (FrDEs), fuzzy differential equations (FDEs). Secondly: Partial differential equations (PDEs), which have the branches: DDEs, FrDEs and FDEs. Thirdly: Stochastic differential equations which have the branches: DDEs, FrDEs, FDEs. Johy[12]. The ODEs of first- or second order have many applications in branches of mechanical engineering, electrical, civil, chemical and others. The subject of PDEs has a long history with an active contemporary phase. An early phase with a separate focus on string vibrations and heat law through solid bodies. A stimulated of great importance for mathematical analysis to all manner of mathematical, physical and technical problems continues. Such stimulated is a wider concept of functions and integration and the direct relevance of PDEs.

In this paper, Haar wavelet functions used to approximate the solutions of typical ODEs or of fractional-order. Haar wavelets can be written as a family of functions constructed from transformation and dilation of a single function. Haar wavelet transform method is powerful numerical method to use it in solving DEs. Haar wavelet function and its properties are studied and used in solving of the DEs. The useful properties of Haar wavelet transform are studied in solving the DEs. The solutions of them are approximated by the summation of constant multiples of the Haar functions. The other terms of the DE usually found out using some properties of integrating and differentiating . Many researches studied Haar wavelets like Berwal et al. [3] studied the solution of DEs based on Haar operational matrix, Sahoo[23] studied the solution of DEs using Haar wavelet collocation method, Shi et al. (2007) studied the numerical solution of DEs by using Haar wavelets, Chang &Piau.[4] used Haar wavelet matrices designation in numerical solution of ODEs, Chen [5] used Haar wavelet approach to ODEs, Li & Hu [16] solved the fractional Riccati DEs using Haar wavelet while Mechee&Senu[17] studied the fractional DEs of Lane-Emden type numerically by method of collocation. Shah & Abbas [24] used Haar wavelet operational matrix method for the numerical solution of fractional order DEs, Saeed&urRehman[21] used Haar wavelet-quasi linearization technique for fractional nonlinear DEs and Lepik[14] applied Haar wavelet transform to solving IEs and DEs. Weilbeer[29] introduced efficient numerical methods for fractional DEs and their analytical background. Haar wavelet operational matrix and its application for the approximated solution of fractional Bagley Torvik equation has been used by Ray [19], while Shiralashetti et al. [26] used Haar wavelet collocation method for the numerical solution of singular initial value problems. Kilicman & Al Zhour [13] introduced Kronecker operational matrices for fractional calculus and some applications, Hosseinpour & Nazemi [10] solved fractional optimal control problems with fixed or free final states by Haar wavelet collocation method, Hsiao [11] constructed Haar wavelet direct method for solving variational problems, Hariharan et al. [8] used Haar wavelet method for solving Fisher's equation, while Aziz & Amin [2] introduced numerical solution of a class of DDEs and DPDEs via Haar wavelet. Recently, we have studied implementation of different tested problems DEs which are used as mathematical models in many physically applied science and important fields. The approximated solutions of DEs have been derived using Haar wavelet and B-spline basis which shows to be more suitable to approximate the solutions of DE.

2 Preliminary

2.1 Haar Wavelet Functions

Haar functions have been used since 1910, when they were introduced by Hungarian mathematician (Haar (1910)). The orthogonal set of Haar function is defined as square waves with value of ± 1 in some interval and zero elsewhere. Then, $h_0(x) = 1$ during the whole interval $0 \leq x \leq 1$. The second curve $h_1(x)$ is the fundamental square wave function which also spans the whole interval [0;1].

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All the other subsequent curve are generated from $h_1(x)$ with two operation translation and dilation, $h_2(x)$ is obtained from $h_1(x)$ with dilation, i.e., $h_1(x)$ is compressed from the whole interval [0;1] to half interval $[0;\frac{1}{2}]$ to generate $h_2(x)$, $h_3(x)$ is the same as $h_2(x)$ but shifted to the right direction by $\frac{1}{2}$. Similarly, $h_2(x)$ is compressed from the half interval to a quarter interval to generate $h_4(x)$. The function $h_4(x)$ is translated to the right by $\frac{1}{4},\frac{2}{4},\frac{3}{4}$, to generate $h_5(x)$, $h_6(x)$ and $h_7(x)$ respectively. In general, we have the following:

$$h_0(x) = h_j(2^j x - \frac{k}{2^j}),\text{where } n = 2^j + k, j \geq 0 \quad , \quad 0 < k \leq 2^j. \qquad (1)$$

This orthogonal basis is a reminiscent of the Walsh basis, in which each Walsh function contains many wavelets to fill the interval [0,1] completely, and to form a global basis. While each Haar function contains just one wavelet during some subinterval of time, and remains zero elsewhere the Haar set form a local basis. All the Haar wavelets are orthogonal to each other:

$$\int_0^1 h_i(x)h_j(x)dx = 2^{-j}\delta_{ij} = \begin{cases} 2^{-j}, i = j = 2^{j+k} \\ 0 \;, \qquad i \neq j \end{cases} \quad (2)$$

However, the functions give a very good transform basis. To obtain a good time resolution for high frequency transients and good frequency resolution for low frequency components, Marled (1982) first introduced the idea of wavelets as a family of functions constructed from translations and dilations of a single function called mother wavelet and defined by

$$H_{a,b}(t) = \frac{1}{\sqrt{|a|}} H\left(\frac{t-b}{a}\right) \;, a \neq 0 \;, \qquad a \in R \quad (3)$$

where a is scaling parameter measures degree of compression and b is the translation parameter determines time location of wavelet.

Definition 2.1. (Haar functions)[13]

The Haar wavelet functions defined as follows on $[0,X]$.

$$h_0(x) = \frac{1}{\sqrt{M}} \quad , \; 0 \leq x \leq X \qquad (4)$$

$$h_1(x) = \frac{1}{\sqrt{M}} \begin{cases} 1, & 0 \leq x \leq \frac{X}{2} \\ -1, & \frac{X}{2} \leq x \leq X \\ 0 & , o.w \end{cases} \qquad (5)$$

$$h_j(x) = \frac{1}{\sqrt{M}} \begin{cases} \sqrt{2^j}, & \frac{k-1}{2^j} \leq x \leq \frac{k-\frac{1}{2}}{2^j} X \\ -\sqrt{2^j}, & \frac{k-\frac{1}{2}}{2^j} X \leq x \leq \frac{k}{2^j} X \\ 0 & , o.w \end{cases} \qquad (6)$$

For $i = 1,2,3, \ldots, m-1, M = 2^j$ and $i = 2^j + k - 1$.

We say that $h_1(x)$ is mother function and

$$h_i(x) = 2^{\frac{j}{2}} h_1(2^j x - k) \qquad (7)$$

For $i = 1,2,3, \ldots, m-1$

Note that:

$$(h_p(x), h_q(x)) = \int_0^x h_p(x), h_q(x)dx \qquad (8)$$

$$= \begin{cases} \frac{x}{m} \;, & p \neq q \\ 0 \;, & p = q \end{cases} \qquad (9)$$

To approximate the function $f(x)$ using Haar functions consider

$$f(x) = \sum_{i=0}^{m-1} a_i h_i(x)$$

$$\int_0^x f(x)h_j(x)dx$$

$$= \sum_{i=0}^{m-1} a_i \int_0^x h_i(x)\, h_j(x)dx, \qquad (10)$$

$$= a_j \int_0^x h_j^2(x)dx, \qquad (11)$$

Where

$$a_j = \frac{\int_0^x f(x)\, h_j(x)dx}{\int_0^x h_j^2(x)dx}$$

$$= \frac{m}{X} \int_0^x f(x)h_j(x)dx$$

2.2 Spline Functions

The spline functions are used in applications of numerical analysis due to they have a wide class of smoothness. One of these applications is data interpolation. The data structure may be either one-dimensional or multi-dimensional. In the interpolation, spline functions are normally determined as the minimizers of suitable measures of roughness subject to the interpolation constraints. Smoothness splines may be show as generalizations of interpolation splines where the functions are determined to minimize a weighted linear combination of the average squared approximation error over observed data.

The spline functions are constructed to be finite dimensional in the applications. Here, we have focus on one-dimensional, polynomial B-splines and use the term B-spline in this restricted sense. The base $\Phi(x) = \{\Phi_1(x), \Phi_2(x), \dots, \Phi_n(x)\}$ is called B- spline base of order n if the basis functions satisfy $\Phi_i(x) \in C^{n-1}(\infty, -\infty) for\ i = 1,2, \dots, n$

First of all, we will partition [0,1] by choosing a positive integer n and defining $h = \frac{1}{n+1}$.This produces the equally-spaced nodes $x_i = ih$, for each $i = 1,2, \dots, n + 1$. Then, we have defined the basis functions$\{\Phi_i(x)\}_{i=0}^{n+1}$on the interval [0,1].

2.2.1 Linear Spline

The simplest spline is a piecewise polynomial function, with each polynomial having a single variable .The spline S takes values from an interval [a,b] and maps them to $\mathfrak{R}$ where $S : [a, b] \to \mathfrak{R}$ Since S is piecewise defined, choose k subintervals to partition [a,b]. The simplest choice of spline functions basis involves piecewise-linear polynomials. The first step is to form a partition of [0,1] by choosing points $x_0, x_1, \dots, x_n$. Let $h_i = x_{i+1} - x_i$,for each $i = 1,2, \dots, n$.We have defined the basis functions $\{\Phi_1(x), \Phi_2(x), \dots, \Phi_n(x)\}$. Linear spline is linear polynomial S(x) which satisfy S(x)∈ $C_0(\infty, -\infty)$To construct linear spline base in which satisfy the boundary conditions $\Phi_i(0) = \Phi_i(1)$for i $1,2, \dots, n$We have constructed the following component linear spline functions:

$$\Phi_i(x) = \begin{cases} 0\ , & 0 \le x \le x_{i-1} \\ \frac{1}{h_{i-1}}(x - x_{i-1})\ , & x_{i-1} < x \le x_i \\ \frac{1}{h_i}(x_{i+1} - x)\ , & x_i < x \le x_{i+1} \\ 0\ , & x_{i+1} < x \le 1. \end{cases} \quad (12)$$

for each $i = 1,2, \dots, n$.We can prove that the functions are orthogonal because $\Phi_i(x)\ and\ \Phi_i'(x)$ are nonzero only on (x_{i-1}, x_{i+1}) such that $\Phi_i(x)\Phi_j(x) = 0$ and $\Phi_i'(x)\Phi_j'(x)$=0 if $i \ne j, i - 1, j - 1$.Consequence $\Phi_i(x) \in C(\infty, -\infty)$.

2.2.2 Quadratic B-Spline

Quadratic B-spline base is quadratic B-Spline polynomials S(x) which satisfy S(x) ∈ $C_0^1(\infty, -\infty)$To construct quadratic spline base in which satisfy the boundary conditions $\Phi_i(0) = \Phi_i(1)$ for i=1,2,… n We have constructed the following component quadratic spline functions:

$$\emptyset_m(x) = \begin{cases} T + 3(x_m - x)^2\ ; & [x_{m-1}; x_m] \\ T; & [x_m; x_{m+1}] \\ (x_{m+2} - x)^2; & [x_{m+1}; x_{m+3}] \\ 0 & o.w. \end{cases} \quad (13)$$

Where $T = (x_{m+2} - x)^2 - 3(x_{m+1} - x)^2$

2.2.3 Cubic B-Spline

Many researchers used B-cubic spline in numerical analysis. We have defined B cubic spline base as follows:

$$s(x) = \frac{1}{4}\begin{cases} 0, & x < -2 \\ (2 + x)^3, & -2 \le x \le -1 \\ (2 + x)^3 - 4(1 + x)^3, & -1 \le x \le 0 \\ (2 - x)^3 - 4(1 - x)^3, & 0 \le x \le 1 \\ (2 - x)^3, & 1 \le x \le 2 \\ 0, & x > 2 \end{cases} \quad (14)$$

Consequence S(x) ∈ $C_0^2(-\infty, \infty)$. To construct cubic spline base in which satisfy the boundary conditions$\emptyset_i(0) = \emptyset_i(1)$ for i = 1,2, … , n. We have constructed the following component cubic spline functions on the interval $[x_{i-2}, x_{i+2}]$ as follows [1]:

$$\emptyset_i(x) = \frac{1}{4}\begin{cases} s\left(\frac{x}{h}\right) - 4s\left(\frac{x+h}{h}\right), & i = 0 \\ s\left(\frac{x-h}{h}\right) - s\left(\frac{x+h}{h}\right), & i = 1 \\ s\left(\frac{x-ih}{h}\right), & 2 \le i \le n \\ s\left(\frac{x-nh}{h}\right) - s\left(\frac{x-(n+2)h}{h}\right), & i = n \\ s\left(\frac{x-(n+1)h}{h}\right) - 4s\left(\frac{x-(n+2+h}{h}\right), & i = n+1 \end{cases} \quad (15)$$

2.3 Fractional Derivatives

The fractional integrals have been defined by many researches as follows:

The left hand Riemann-Liouville fractional derivatives of order $\alpha > 0$; $n \in N$ (N is natural numbers set), is given by:

$$D_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_0^x \frac{f(t)}{(x-t)^{\alpha-n+1}}dt \quad (16)$$

2.3.1 Operational Matrix of the Fractional-Order Integration of the Haar Wavelet

Shiralashetti & Deshi [26] had introduced the Haar wavelet operational matrix FH_α of integration of the fractional order α is given by

$$FH_{\alpha,i} = \begin{cases} f_1, x \in [\frac{k}{m}, \frac{k+0.5}{m}) \\ f_2, x \in [\frac{k+0.5}{m}, \frac{k+1}{m}) \\ f_3, x \in [\frac{k+1}{m}, 0) \\ 0, \quad o.w \end{cases} \quad (17)$$

Where

$$f_1(x) = \frac{1}{\Gamma(n-\alpha)}(x - \frac{k}{m})^\alpha$$

$$f_2(x) = \frac{1}{\Gamma(n-\alpha)}\left(\left(x - \frac{k}{m}\right)^\alpha - 2\left(x - \frac{k+0.5}{m}\right)^\alpha\right)$$

$$f_3(x) = \frac{1}{\Gamma(n-\alpha)}(\left(x - \frac{k}{m}\right)^\alpha - 2\left(x - \frac{k+0.5}{m}\right)^\alpha + \left(x - \frac{k+1}{m}\right)^\alpha)$$

2.4 Operational Matrix of the Fractional-Order Integration of the B-Spline Bas

2.4.1 Linear Spline

We have introduced the linear B-spline operational matrix FSa of integration of the fractional order as follows:

$$J_{x\,i}^{\alpha}(x) = \frac{1}{\Gamma(n+2)} \begin{cases} 0, \quad 0 \le x \le x_{i-1} \\ \frac{1}{h_{i-1}}(x - x_{i-1})^{\alpha+1}, x_{i-1} \le x \le x_i \\ \frac{1}{h_{i-1}}(x - x_{i-1})^{\alpha+1} + \frac{1}{h_i}(h_i\alpha(x - x_i)^\alpha - (x - x_i)^{\alpha+1}, x_i \le x \le x_{i+1} \\ 0, \quad x_{i+1} \le x \le 1 \end{cases} \quad (18)$$

2.4.2 Quadratic B-Spline

We have introduced the quadratic B-spline operational matrix FS_a of integration of the fractionalorder as follows:

$$J_x^\alpha(x) = \frac{1}{\Gamma(n+3)} \begin{cases} 2(x-1)^{\alpha+2}, & [1,2] \\ 2(x-1)^{\alpha+2} - 6(x-2)^{\alpha+2}, & [2,3] \\ 2(x-1)^{\alpha+2} - 6(x-2)^{\alpha+2} + 6(x-3)^{\alpha+2}, & [2,3] \\ 0, & o.w \end{cases} \quad (19)$$

2.4.3 Cubic B-Spline

We have introduced the cubic B-spline operational matrix FSa of integration of the fractional order as follows:

$$J_x^\alpha(x) = \frac{1}{\Gamma(n+3)} \begin{cases} 0, & x < -2 \\ \frac{3}{2}x^{\alpha+3} & -2 \le x \le -1 \\ \frac{3}{2}x^{\alpha+3} - 6(x-1)^{\alpha+3}, & -1 \le x \le 0 \\ \frac{3}{2}x^{\alpha+3} - 6(x-1)^{\alpha+3} + 9(x-2)^{\alpha+3}, & 0 \le x \le 1 \\ \frac{3}{2}x^{\alpha+3} - 6(x-1)^{\alpha+3} + 9(x-2)^{\alpha+3} - 6(x-3)^{\alpha+3}, & 1 \le x \le 2 \\ 0, & x > 2 \end{cases} \quad (20)$$

3 Analysis of Collection Method[1]

Define the collocation points $x_i = a + ih$for $i = 0,1,2,\dots,n$ discretize the functions:

$$\phi(x) = \{\phi_1(x), \phi_2(x), \phi_3(x), \dots, \phi_n(x)\}$$

Suppose

$$y(x) = \sum_{i=1}^{n} c_i \phi_i(x)$$

Put the approximation of $y(x)$ at the point $x_j$in the DE, we get the function coefficient matrix $\Phi_{i,j}(x)=\Phi_i(x_j)$ and $\Phi'_{i,j}(x) = \Phi'_i(x_j)$ The matrix of coefficients has the dimension $n \times n$. Any function $y(x)$ which is square integrable in the interval (0,1) can be expressed as an infinite sum of Haar wavelet. The above series terminates at finite terms if $y(x)$ is piecewise constant or can be approximated as piecewise constant during each subinterval [1].

3.1 The Quadratic B-Spline Base

Consider the quadratic B-spline Base

$$s(x) = \{s_1(x), s_2(x), s_3(x), \dots, s_n(x)\}$$

Suppose $y(x) = \sum_{i=1}^{n} c_i s_i(x)$ The general ODE of first-order has the following form:

$$a_0(t)y'(t) + a_1(t)y(t) = f(t), \quad 0 \le t \le 1, (21)$$

subject to the initial condition is $y(0) = \alpha$.

Problem 3.1

$$y'(t) = y(t) = \sin(t) + \cos(t), \quad 0 \le t \le 1.$$

subject to the initial condition is $y(0) = 0$, The coefficients area$_0(t) = a_1(t) = 1$ and $f(t) = \sin(t) + \cos(t)$ Consider the quadratic B-spline base, Then, The matrix of coefficients has the following formula:

$$\boldsymbol{A_{ij} = s_{i'}(t_j) + s_i(t_j)}$$

And $b_i = \sin t_i + \cos t_i$ for $i, j = 1,2,\dots,n$. By solving the system of coefficients $Ac = b$ we will obtain the coefficients of approximation where c=[.1,.2,-.1,.001,4].

3.2 The Haar Wavelet Base

We introduce the Haar wavelet technique for solving general linear first-order ODEs.

3.2.1 First-Order Linear First-Order ODEs

Consider the following general linear first-order ODE:

$$y'(t) + f(t)y(t) = g(t), 0 \le t \le a, f(t) \neq 0, \quad (23)$$

$$y(0) = \beta. \quad (24)$$

Substituting $t = ax$ in Equation (23) which reduces to

$$y'(x) + af(ax)y(ax) = ag(ax), 0 \le x \le a, f(x) \neq 0, \quad (25)$$

$$y(0) = \beta. \quad (26)$$

We assume that $y'(x) = \sum_{i=1}^{k} c_i h_i(x)$ (27)

where $c_i s$ are Haar coefficients to be determined. Integrating Equation (27) with respect to x, we get the following

$$y(x) = \beta + \sum_{i=1}^{k} c_i p_{1,i}(x) \quad (28)$$

Substituting Equations (27) and (28) in Equation (25), we get the following system of equation:

$$\sum_{i=1}^{k} c_i h_i(x) + af(x)(\beta + \sum_{i=1}^{k} c_i p_{1,i}(x)) = ag(x) \quad (29)$$

Put $x = t_j$ for $j = 1,2,\dots n$. in Equation (29), we get linear system in which the matrix of coefficients has the following formula:

$$A_{ij} = (1 + af(t_i))h_i(t_i) \quad \text{and} \quad b_i = ag(t_i)$$

for $j, i = 1,2,\dots n$ By solving the linear system of coefficients $Ac = b$ we obtain the coefficients of approximated solution.

3.3 Fractional Differential equations with Haar Base

We will introduce the Haar wavelet technique for solving FrDEs

Problem 3.2. Consider the general fractional-order linear DE

$$y^{\alpha}(t) + A(t) + B(t)y(t) = C(t)$$
$$0 \le t \le a;\ n-1 < \alpha < n \quad (30)$$

subject to initial conditions $y_j(0) = a_j$ for $j = 0,1,2,\dots,n$-1 where $A(t)$,$B(t)$ and $C(t)$ are given functions , $a'_j s$ are arbitrary constants and α is a parameter describing the order of the fractional derivative. The general response expression contains a parameter describing the order of the fractional derivative that can be varied to obtain various responses. Substituting $t = ax$ in Equation (27) which reduces to

$$y^{\alpha}(ax) + aA(ax) + aB(ax)y(ax) = aC(ax)$$
$$0 \le x \le 1;\ n-1 < \alpha < n \quad (31)$$
$$y_j(0) = a_j$$

We assume that $y'^{\alpha}(x) = \sum_{i=1}^{k} c_i h_i(x)$ (32)

If $\alpha = \frac{1}{2}$,integrating Equation (28) once, we get

$$y(x) = a_0 + \sum_{i=1}^{k} c_i FH_{\frac{1}{2},i}(x) \quad (33)$$

Substituting Equations (32) and (33) in Equation (31), we get

$$\sum_{i=1}^{k} c_i h_i(x) - aA(x) - aB(x)\left(a_0 + \sum_{i=1}^{k} c_i FH_{\frac{1}{2},i}(x)\right) = aC(x)$$

If $\alpha = \frac{3}{2}$,integrating Equation (28) once, we get

$$y^{\frac{1}{2}}(x) = a_1 + \sum_{i=1}^{k} c_i FH_{\frac{1}{2},i}(x) \quad (34)$$

And

$$y(x) = a_0 + a_1 x + \sum_{i=1}^{k} c_i FH_{\frac{3}{2},i}(x) \quad (35)$$

Substituting Equations (32) and (35) in Equation (31), we get

$$\sum_{i=1}^{k} c_i h_i(x) - aA(x) - aB(x)\left(a_0 + a_1 x + \sum_{i=1}^{k} c_i FH_{\frac{3}{2},i}(x)\right) = aC(x) \quad (36)$$

Put $x = t_j$ for $j = 1,2,\dots,n$. in Equation (35) in case $\alpha = \frac{1}{2}$,or in Equation (35) in case $\alpha = \frac{3}{2}$,we get the linear system in which the matrix of coefficients has the following formula:

$$\boldsymbol{A_{ij} = h'_i(t_j) + aB(t_j)}FH_{\alpha,i}(\boldsymbol{t_j})$$

and

$$b_i = c(\boldsymbol{t_j}) + aA(\boldsymbol{t_j}) - aa_0 B(\boldsymbol{t_j})$$

for $i, j = 1,2,\dots,n$. By solving the linear system of coefficients, we obtain the coefficients of approximated solution $y(t)$of Equation (31).

3.4 Fractional Differential equations with B-Spline Base

We will introduce the B-spline technique for solving FrDE (31). Consider the quadratic B-spline base

$$s(x) = \{s_1(x), s_2(x), s_3(x), \dots\dots, s_n(x)\}$$

Suppose $y(x) = \sum_{i=1}^{n} c_i s_i(x)$

We assume that $y'^{\alpha}(x) = \sum_{i=1}^{k} c_i s_i(x)$ (37)

If $\alpha = \frac{1}{2}$,integrating Equation (37) once, we get

$$y(x) = a_0 + \sum_{i=1}^{k} c_i FS_{\frac{1}{2},i}(x) \quad (38)$$

Substituting Equations (37) and (38) in Equation (31), we get

$$\sum_{i=1}^{k} c_i S_i(x) - aA(x) - aB(x)\left(a_0 + \sum_{i=1}^{k} c_i FS_{\frac{1}{2},i}(x)\right) = aC(x)$$

If $\alpha = \frac{3}{2}$,integrating Equation (31) once, we get

$$y(x) = a_0 + a_1 x + \sum_{i=1}^{k} c_i FS_{\frac{3}{2},i}(x) \quad (39)$$

Substituting Equations (32) and (39) in Equation (31), we get

$$\sum_{i=1}^{k} c_i S_i(x) - aA(x) - aB(x)\left(a_0 + a_1 x + \sum_{i=1}^{k} c_i FS_{\frac{3}{2},i}(x)\right) = aC(x) \quad (36)$$

Put $x = t_j$ for $j = 1,2,\dots,n.$ in Equation (39) in case $\alpha = \frac{1}{2}$,or in Equation (35) in case $\alpha = \frac{3}{2}$,we get the linear system in which the matrix of coefficients has the following formula:

$$\boldsymbol{A_{ij}} = \boldsymbol{h'}_i(\boldsymbol{t_j}) + \boldsymbol{aB}(\boldsymbol{t_j}) FS_{\alpha,i}(\boldsymbol{t_j})$$

and

$$b_i = c(\boldsymbol{t_j}) + aA(\boldsymbol{t_j}) - aa_0 B(\boldsymbol{t_j})$$

for $i, j = 1,2,\dots,n$. By solving the linear system of coefficients, we obtain the coefficients of approximated solution $y(t)$of Equation (31).

4 Lane-Emden Fractional Differential Equation

We generalize the definition of Lane- Emden equations up to fractional order as following:

$$D^{\alpha} y(t) + \frac{k}{t^{\alpha-\beta}} D^{\beta} y(t) + f(t, y) = g(t)$$

$0 < t \leq 1,\ k > 0.$ (40)

with the initial condition $y(0) = A; y_0(0) = B$ where $1 < \alpha \leq 2; 0 < \beta \leq 1$ and $A;B$ are constants

and $f(t; y)$ is a continuous real-valued function and $g(t; y) \in [0,1]$: The theory of singular boundary value problems has become an important area of investigation in the past three decades. One of the equations describing this type is the Lane-Emden equation. Lane-Emden type equations, first published by Homer Lane (1870), and further explored in detail by Emden [6], represents such phenomena and having significant applications, is a second-order ODE with an arbitrary index, known as the polytropic index, involved in one of its terms.

The Lane-Emden equation describes a variety of phenomena in physics and astrophysics. Mechee&Senu[18] imposed the Lane-Emden DE of fractional order and the approximate solution is obtained by employing the method of power series and a numerical solution is established by the least squares method for these equations. Mechee&Senu[17] approximate the solution of DE by employing the method of power series and the numerical solution is established by collection method.

5 Analysis of the Method of Solution Lane-Emden of Fractional Order

Berwal et al. [3] studied the solution of DEs based on Haar operational matrix, Sahoo[23] studied the solution of DEs using Haar wavelet collocation method, Shi et al. [25] studied the 10numerical solution of DEs by using Haar wavelets, Chen [5] used Haar wavelet approach toODEs, Li & Hu [16] solved the fractional Riccati DEs using Haar wavelet while Saeedi et al. [22] introduced an operational Haar wavelet method for solving fractional Volterra integral equations, Lepik[15] solved fractional integral equations by the Haar wavelet method, Saeed &Rehman[20] used Haar wavelet-quasi linearization technique for fractional nonlinear DEs,Lepik[15] solved the fractional integral equations by the Haar wavelet method, Wang et al. [28] used Haar wavelet method for solving fractional PDEs numerically. In Equation (40), consider α >b, $f(t, y) = \frac{1}{t^{\alpha+2}}\, y(t)$ and $g(t) = 0$ However, $D^{\alpha} W(t) = ah(t) = \sum_{i=1}^{m} c_i h_i(x)$ and

$$, D^{\alpha} W(t) = (I^{\alpha-\beta} D^{\alpha}) W(t) + W^{\beta}(0) = aP^{\alpha-\beta} h(t) + W^{\beta}(0)$$

$$W(t) = (I^{\alpha} D^{\alpha}) W(t) + W(0) = aP^{\alpha} h(t) + A$$

Hence,

$$ah(t) + \frac{k}{t^{\alpha-\beta}} ap^{\alpha-\beta} h(t) + W^{\beta}(0) + ap^{\alpha} h(t) + A = ch(t)$$

If we consider $\alpha = \frac{3}{2}$and $\beta = \frac{1}{2}$we solve the system of equations to obtainthe coefficients $(c_0, c_1, c_2, \dots, c_m)$.

6 Comparison Study Using Numerical Collection Method

Collocation method for solving DEs is one of the most powerful approximated methods. This method has its basis upon approximate the solution of FrDEs by a series of complete sequence of functions, a sequence of linearly independent functions which has no non-zero function perpendicular to this sequence of functions. In general, $y(t)$ is approximated by Mechee&Senu [17]

$$y(x) = \sum_{i=1}^{n} a_i \theta_i(x) \qquad (41)$$

where a_i for $i = 0,1,2,\dots,n$ are an arbitrary constants to be evaluated and θ_i for $i = 0,1,2,\dots,n$ aregiven set of functions. Therefore, the problem in Equation (40) of evaluating $y(t)$ is approximated by (42) then, is reduced to the problem of evaluating the coefficients for . $i = 0,1,2,\dots,n$ Let $\{t_1, t_2, \dots, t_n\}$is a partition to interval [0,1] and $t_j = jh$and$h = \frac{1}{n}n$ and $j = 0,1,2,\dots,n$ See the comparison of absolute errors of the problem using numerical collection method with polynomial basis and Haar wavelet basis.

7 Discussion and Conclusion

The numerical solutions of ordinary differential equations of fractional order using Haar wavelet and B-spline bases have been studied. Haar wavelet technique is used to approximate the solution of the differential equations. The algorithm of collection method is updated for using the two basis. An application of Lane-Emden has been studied numerically. The numerical results have clearly shown the advantage and the efficiency of the modified method in terms of accuracy and computational time.

8 Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Table 1: Absolute Errors of Example1 Using Numerical Collection Method with (a) Polynomial Basis (b) Haar wavelet Basis

$n \backslash t$	0	0.25	0.5	0.75	1
5	0	1.3345e-3	0.0015	5.0673e-3	3.6339e-3
	0	1.311e-3	0.0005	5.0683e-3	3.6229e-3
10	0	1.3232e-5	2.0342e-5	1.5634e-6	4.1443e-5
	0	1.3211e-5	2.1212e-5	1.2341e-6	4.0101e-5
50	0	2.3416e-7	1.0611e-7	5.1126e-7	2.1233e-7
	0	2.1414e-7	1.2211e-7	5.2233e-7	2.1266e-7
100	0	4.9383e-8	3.4455e-8	5.0347e-8	6.4332e-7
	0	4.9121e-8	3.4564e-8	5.0111e-8	6.4222e-7

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مويجات هار لحل المعادلات التفاضلية الكسورية مع تطبيقات

محمد صاحب مجي عدي عمران عيسى الشاهر غسان عبد الأمير الجعيفري
قسم الرياضيات ، كلية علوم الحاسوب والرياضيات ، جامعة الكوفة

المستخلص :

قدمت في هذا البحث الحلول العددية للمعادلات التفاضلية الاعتيادية ذات الرتب الكسرية باستخدام أساس مويجات هار وأساس بي سبلاين. طورت طريقة الحشد باستخدام الأساسين. تم حل بعض مسائل القيم الأبتدائية لتبين كفاءة مويجات هار وأساس بي سبلاين. قدمت معادلة لان امان ودرس حلها. أظهرت النتائج العددية وبوضوح كفاءة ومزايا الطريقة المحورة بالدقة والوقت الأحتسابي.

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On the class of multivalent analytic functions defined by differential operator for derivative of first order

Zainab Odeh A.Mohammed
Department of Mathematics
College of Education
University of AL-Qadisiyah
AL-Qadisiya –Iraq
hsyn47421@gmail.com

Kassim A. Jassim
Department of Mathematics
College of Science
University of Baghdad
Baghdad-Iraq
Kasimmathphd@gmail.com

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Abstract

In the submitted search ,by making use of Differential operator ,we drive coefficient bounds and some important properties of the subclass $T_j(n,p,q,\alpha,\lambda)$ $(p,j \in N = \{1,2,\dots\}; q,n \in N_0 = N \cup \{0\}; 0 \le \alpha < p-q)$ of analytic and multivalent function with negative coefficients .Distortion property for functions in the class $T_j(n,p,q,\alpha,\lambda)$ are investigated once by using the composition involving an integral operator and certain fractional calculus operator and other once by using the composition involving an integral operator and certain fractional calculus inverse operator .

Keywords. Multivalent function, Coefficient bounds ,Distortion inequality , δ- neighbourhood , Differential operator, integral and fractional operators .

Mathematics Subject Classification: 30C45.

1-Introduction

Let $T(j,p)$ be the class of analytic and multivalent functions $f(z)$ in the open unit disk $U=\{z: z\in C: |z|<1\}$ that defined by

$$f(z)=z^p-\sum_{k=j+p}^{\infty}a_k z^k \quad (a_k\geq 0; j,p\in N=\{1,2,\dots.\}\ . \quad \textbf{(1)}$$

Let $T(j,p)$ the class consists of function of the form

$$f^q(z)=\frac{p!}{(p-q)!}z^{p-q}-\sum_{k=j+p}^{\infty}\frac{k!}{(k-q)!}a_k z^{k-q} \quad (a_k\geq 0\,;\ q\in N_0=N\cup\{0\}\,;\ p>q) \quad \textbf{(2)}$$

The Differential operator for a function in $T(j,p)$ is define by

$$D_p^n(f^q(z))=\frac{p!}{(p-q)!}z^{p-q}-\sum_{k=j+p}^{\infty}\frac{k!}{(k-q)!}\left(\frac{k-q}{p-q}\right)^n a_k z^{k-q} \quad (p,j\in N=\{1,2,\dots\}\,; q,n\in N_0\,;\ p>q) \quad \textbf{(3)}$$

The operator D_p^n was studied by M.K. Aouf [5] and Altintas et al. [7] , earlier by Owa [13] ,Yamakawa[9] ,Owa [12] ,Srivastava Owa[4] .

It is easy to see that

$$D_p^{n+1}f^q(z)=\frac{z}{(p-q)}(D_p^n f^q(z))' \quad \textbf{(4)}$$

By using the operator $D_p^n f^q(z)$ Given by **(3)** ,a function $f(z)$ belonging to $T_j(n,p,q,\alpha,\lambda)$ if and only if

$$R\left(\frac{D_p^{n+1}(f^q(z))}{(1-\lambda)D_p^n(f^q(z))+\lambda D_p^{n+1}(f^q(z))}\right)>\alpha \quad (p\in N\,; q,n\in N_0=N\cup\{0\}\,;\ p>q)\,, \quad \textbf{(5)}$$

for some α $(0\leq\alpha<p)$ and for all $z\in U$.

Next the following earlier investigations by Osman Altintas,Huseyin Irmak and H.M.Srivastava [6] ,

when $f(z)\in T(j,p)$ we define the δ - neighborhood by

$$N_\delta(f)=\{g: g\in T(j,p), g(z)=z^p-\sum_{k=j+p}^{\infty}b_k z^k \ and\ \sum_{k=j+p}^{\infty}k|a_k-b_k|\leq\delta\}. \quad \textbf{(6)}$$

So that ,obviously ,

$$N_\delta(h)=\{g: g\in T(j,p), g(z)=z^p-\sum_{k=j+p}^{\infty}b_k z^k$$

And $\sum_{k=j+p}^{\infty}|b_k|\leq\delta\}$ **(7)**

Where, and in what follows,

$$h(z)=z^p \quad (k\geq j+p; n,p\in N; q\in N_0=N\cup\{0\}). \quad \textbf{(8)}$$

we using the familiar operator $J_{c,p}$ defined by Bernardi [10], Libera[8] and Srivastave et al. [2] as follows

$$(J_{c,p}f)(z)=\frac{c+p}{z^c}\int_0^z t^{c-1}f(t)dt \quad (f(z)\in T(j,p)\,; c>-p\,; p\in N), \quad \textbf{(9)}$$

and fractional calculus operator D_z^μ Srivastave [11] , Srivastava et al.[3] that known as the form

$$D_z^\mu(z^p)=\frac{\Gamma(p+1)}{\Gamma(p-\mu+1)}z^{p-\mu} \quad (p>-1; \mu\epsilon R) \quad \textbf{(10)}$$

2-Coefficient Inequalities

We drive sufficient condition for $f(z)$ that defined by using differential operator .

Theorem 1. Assume that $f(z)\in T(j,p)$. Then $f(z)\in T_j(n,p,q,\alpha,\lambda)$ if and only if

$$\sum_{k=j+p}^{\infty}\left(\frac{k-q}{p-q}\right)^n\left[\left(\frac{k-q}{p-q}\right)-\alpha\left(1+\lambda\left(\frac{k-q}{p-q}-1\right)\right)\right]\delta(k,q)a_k<(1-\alpha)\delta(p,q)$$

$$(0\leq\alpha<p-q\,;\ p,j\in N\,;\ q\,,n\in N_0\,; p>q\,) \quad \textbf{(11)}$$

Where

$$\delta(p,q)=\frac{p!}{(p-q)!}=\begin{cases}p(p-1)\dots(p-q+1) & q\neq 0\\ 1 & q=0\end{cases} \quad \textbf{(12)}$$

Proof. If $f(z)\in T_j(n,p,q,\alpha,\lambda)$, then

$$R\left(\frac{D_p^{n+1}(f^q(z))}{(1-\lambda)D_p^n(f^q(z))+\lambda D_p^{n+1}(f^q(z))}\right)=$$

$$R\left(\frac{\frac{p!}{(p-q)!}-\sum_{k=j+p}^{\infty}\frac{k!}{(k-q)!}(\frac{k-q}{p-q})^{n+1}a_k z^{k-p}}{\frac{p!}{(p-q)!}-\sum_{k=j+p}^{\infty}\frac{k!}{(k-q)!}(\frac{k-q}{p-q})^n[1-\lambda+\lambda(\frac{k-q}{p-q})]a_k z^{k-p}}\right)$$

$$\left\{\frac{\frac{p!}{(p-q)!}-\sum_{k=j+p}^{\infty}\frac{k!}{(k-q)!}(\frac{k-q}{p-q})^{n+1}a_k z^{k-p}}{\frac{p!}{(p-q)!}+\sum_{k=j+p}^{\infty}\frac{k!}{(k-q)!}(\frac{k-q}{p-q})^n[\lambda-1-\lambda(\frac{k-q}{p-q})]a_k z^{k-p}}\right\}>\alpha$$

$$(1-\alpha)\frac{p!}{(p-q)!}>\sum_{k=j+p}^{\infty}(\frac{k-q}{p-q})^n\left(\left(\frac{k-q}{p-q}\right)+\lambda\alpha-\alpha-\lambda\alpha\left(\frac{k-q}{p-q}\right)\right)\frac{k!}{(k-q)!}a_k z^{k-p}\,,$$

Since $z\to 1^-$,we have $\sum_{k=j+p}^{\infty}(\frac{k-q}{p-q})^n\left(\left(\frac{k-q}{p-q}\right)-\alpha\left(1+\lambda\left(\left(\frac{k-q}{p-q}\right)-1\right)\right)\right)\delta(k,q)a_k z^{k-p}<(1-\alpha)\delta(p,q)$

Conversely ,assume that inequality **(11)** holds true ,since

$R(w)>\alpha$ if and only if $\left|\frac{w-1}{w+(1-2\alpha)}\right|<1$

Since

$$\left|\frac{\frac{D_p^{n+1}(f^q(z))}{(1-\lambda)D_p^n(f^q(z))+\lambda D_p^{n+1}(f^q(z))}-1}{\frac{D_p^{n+1}(f^q(z))}{(1-\lambda)D_p^n(f^q(z))+\lambda D_p^{n+1}(f^q(z))}+(1-2\alpha)}\right|$$

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$$=\left|\frac{(1-\lambda)D_p^{n+1}(f^q(z))-(1-\lambda)\,D_p^n(f^q(z))}{(1+\lambda-2\alpha\lambda)D_p^{n+1}(f^q(z))+(1-2\alpha)-\lambda(1-2\alpha)\,D_p^n(f^q(z))}\right|$$

$$=\left|\frac{-\sum_{k=j+p}^{\infty}\frac{k!}{(k-q)!}\left(\frac{k-q}{p-q}\right)^n\left[\frac{k-q}{p-q}-\lambda\left(\frac{k-q}{p-q}\right)-1+\lambda\right]a_kz^{k-p}}{2\,\frac{p!}{(p-q)!}(1-\alpha)+\sum_{k=j+p}^{\infty}\frac{k!}{(k-q)!}\left(\frac{k-q}{p-q}\right)^n\left[2\alpha+\lambda-2\alpha\lambda-1-\frac{k-q}{p-q}-\lambda\left(\frac{k-q}{p-q}\right)+2\alpha\lambda\left(\frac{k-q}{p-q}\right)\right]a_kz^{k-p}}\right|$$

$$\le\frac{\sum_{k=j+p}^{\infty}\left(\frac{k-q}{p-q}\right)^n\left[\left(\frac{k-q}{p-q}\right)-\lambda\left(\frac{k-q}{p-q}\right)-1+\lambda\right]\frac{k!}{(k-q)!}a_kz^{k-p}}{2\,(1-\alpha)\frac{p!}{(p-q)!}+\sum_{k=j+p}^{\infty}\left(\frac{k-q}{p-q}\right)^n\left[2\alpha+\lambda-2\alpha\lambda-1-\left(\frac{k-q}{p-q}\right)\{1+\lambda-2\alpha\lambda\}\right]\frac{k!}{(k-q)!}a_kz^{k-p}}$$
$$\le 1\,.$$

Putting $j=1, n=1, q=0$ and $\lambda=0$ in Theorem 1,we have the following corollary :

Corollary 1 . Let the function $f(z)\in T(j,p)$. Then $f(z)\in C(p,\alpha)$ if and only if
$\sum_{k=1+p}^{\infty}\frac{k}{p}\left[\frac{k}{p}-\alpha\right]a_k<(1-\alpha)$ $\quad(0\le\alpha<p\,;p\in N)$.
Not that this result obtained by Salagean et al[1] .

Corollary 2. Assume that the function $f(z)$ defined by **(2)** be in the class $T_j(n,p,q,\alpha,\lambda)$ .Then
$$\sum_{k=j+p}^{\infty}\delta(k,q)a_k<\frac{(1-\alpha)\delta(p,q)}{\left(\frac{j}{p-q}+1\right)^n\left(\left(\frac{j}{p-q}(1-\alpha\lambda)\right)+(1-\alpha)\right)}$$
(13)
$(k\ge j+p\ ;p\,,j\in N\,;q\,,n\in N_0\,;\ p>q\,)$.
The result is sharp for the function $f(z)$ given by

$$f(z)=z^p-\left(\frac{(1-\alpha)\delta(p,q)}{\left(\frac{k-q}{p-q}\right)^n\left(\left(\frac{k-q}{p-q}\right)-\alpha\left(1+\lambda\left(\left(\frac{k-q}{p-q}\right)-1\right)\right)\right)\delta(k,q)}\right)z^k\ \ (k\ge j+p\,;\ p,j\in N\,;q\,,n\in N_0\,;p>q\,)\,.\qquad\textbf{(14)}$$

3-Extreme points

Theorem 2. Let $f_p(z)=z^p$ and $f_k(z)=z^p-\left(\frac{(1-\alpha)\delta(p,q)}{\left(\frac{k-q}{p-q}\right)^n\left(\left(\frac{k-q}{p-q}\right)-\alpha\left(1+\lambda\left(\left(\frac{k-q}{p-q}\right)-1\right)\right)\right)\delta(k,q)}\right)z^k$, for $k\ge j+p$ and $p>q$. Then $f(z)\in T_j(n,p,q,\alpha,\lambda)$ if and only if it is of the form
$f(z)=\sum_{k=p}^{\infty}\omega_kf_k(z)$,where $\omega_k\ge 0$ for all $k\ge j+p$ and $\sum_{k=p}^{\infty}\omega_k=1$.

Proof . Suppose that $f(z)=\sum_{k=p}^{\infty}\omega_kf_k(z)=\omega_pf_p(z)+\sum_{k=j+p}^{\infty}\omega_kf_k(z)$
$=z^p+\sum_{(k=j+p)}^{\infty}$ 〖ω_k (((1−α)δ(p,q))/(((k−q)/(p−q))^n (((k−q)/(p−q))−α(1+λ(((k−q)/(p−q))−1)))δ(k,q))) z^k〗 .

Then $\sum_{k=j+p}^{\infty}\left(\frac{k-q}{p-q}\right)^n\left(\left(\frac{k-q}{p-q}\right)-\alpha\left(1+\lambda\left(\left(\frac{k-q}{p-q}\right)-1\right)\right)\right)\delta(k,q)\omega_k\left[\frac{(1-\alpha)\delta(p,q)}{\left(\frac{k-q}{p-q}\right)^n\left(\left(\frac{k-q}{p-q}\right)-\alpha\left(1+\lambda\left(\left(\frac{k-q}{p-q}\right)-1\right)\right)\right)\delta(k,q)}\right]$
$=\sum_{k=j+p}^{\infty}\omega_k(1-\alpha)\delta(p,q)$
$=(1-\omega_p)(1-\alpha)\delta(p,q)$
$\le(1-\alpha)\delta(p,q)$.
Thus ,it follows from Theorem **(1)** that $f(z)\in T_j(n,p,q,\alpha,\lambda)$.
Conversely ,suppose that $f(z)\in T_j(n,p,q,\alpha,\lambda)$,since

$$a_k<\frac{(1-\alpha)\delta(p,q)}{\left(\frac{j}{p-q}+1\right)^n\left[\frac{j}{p-q}(1-\alpha\lambda)+(1-\alpha)\right]\delta(j+p,q)}\ ,k\ge j+p\,.$$
We define

$$\omega_k=\frac{\left(\frac{k-q}{p-q}\right)^n\left(\left(\frac{k-q}{p-q}\right)-\alpha\left(1+\lambda\left(\left(\frac{k-q}{p-q}\right)-1\right)\right)\right)\delta(k,q)}{(1-\alpha)\delta(p,q)}a_k$$
$(k\ge j+p)$.
And $\omega_p=1-\sum_{k=j+p}^{\infty}\omega_k$ by simple calculation ,we get
$f(z)=z^p-\sum_{k=j+p}^{\infty}a_kz^k$
$$=z^p-\sum_{k=j+p}^{\infty}\left(\frac{(1-\alpha)\delta(p,q)}{\left(\frac{k-q}{p-q}\right)^n\left(\left(\frac{k-q}{p-q}\right)-\alpha\left(1+\lambda\left(\left(\frac{k-q}{p-q}\right)-1\right)\right)\right)\delta(k,q)}\right)\omega_kz^k$$
$=\omega_pz^p-\sum_{k=j+p}^{\infty}\omega_kf(z)$.
Thus we get the result .

4-Neighbouhoods for the function class $T_j(n,p,q,\alpha,\lambda)$

In this section ,we conclude the neighborhood properties for each of the following slightly mutated function in the class $T_j(n,p,q,\alpha,\lambda,\gamma)$ .Our first implication relation including the δ- neighbourhood $N_\delta(h)$ is given below in the following Theorem .

Theorem 3. If $f(z)$ belonging to $T_j(n,p,q,\alpha,\lambda)$,then
$T_j(n,p,q,\alpha,\lambda)\subset N_\delta(h)$. **(15)**
Where $h(z)$ is defined as **(8)** and

$$\delta:=\frac{(1-\alpha)\delta(p,q)}{\left(\frac{j}{p-q}+1\right)^n\left(\left(\frac{j}{p-q}(1-\alpha\lambda)\right)+(1-\alpha)\right)}$$

Proof . For $f(z) \in T(j,p)$,Theorem **(1)** ,immediately yields .

$$\left(\frac{j}{p-q}+1\right)^n\left(\left(\frac{j}{p-q}(1-\alpha\lambda)\right)+(1-\alpha)\right)\delta(j+p,q)\sum_{k=j+p}^{\infty}a_k \le (1-\alpha)\delta(p,q) ,$$

so that $\delta(j+p,q)\sum_{k=j+p}^{\infty}ka_k \le \left(\frac{(1-\alpha)\delta(p,q)}{\left(\frac{j}{p-q}+1\right)^n\left(\left(\frac{j}{p-q}(1-\alpha\lambda)\right)+(1-\alpha)\right)}\right)$.

Thus ,we have

$$\sum_{k=j+p}^{\infty}ka_k \le \frac{(1-\alpha)\delta(p,q)}{\left(\frac{j}{p-q}+1\right)^n\left(\left(\frac{j}{p-q}(1-\alpha\lambda)\right)+(1-\alpha)\right)\delta(j+p,q)} := \delta .$$

This complete the proof .

A function $f(z) \in T(j,p)$ is said to be in the class $T_j(n,p,q,\alpha,\lambda,\gamma)$ if there exists another function $g(z) \in T_j(n,p,q,\alpha,\lambda,\gamma)$ such that

$$\left|\frac{f(z)}{g(z)}-1\right| < p-\gamma \quad (z \in U : 0 \le \gamma < p).$$

Theorem 4. Let $g(z) \in T_j(n,p,q,\alpha,\lambda,\gamma)$ .Suppose also that

$$\gamma = p - \frac{\delta}{j+p}\left[\frac{(j+p)!\left(\frac{j}{p-q}+1\right)^n\left(\left(\frac{j}{p-q}(1-\alpha\lambda)\right)+(1-\alpha)\right)}{\left(\frac{j}{p-q}+1\right)^n\left(\left(\frac{j}{p-q}(1-\alpha\lambda)\right)+(1-\alpha)\right)(j+p)!-(1-\alpha)\delta(p,q)(j+p-q)!}\right] \quad \textbf{(16)}$$

Then

$$N_\delta(g) \subset T_j(n,p,q,\alpha,\lambda,\gamma)$$

Proof . Suppose that $f(z) \in N_\delta(g)$,we then find from **(6)** that

$$\sum_{k=j+p}^{\infty}k|a_k-b_k| \le \delta$$

Which readily implies the following coefficient inequality

$$\sum_{k=j+p}^{\infty}|a_k-b_k| \le \frac{\delta}{(j+p)} \quad , (j,p \in N; p > q)$$

Next ,since $g(z) \in T_j(n,p,q,\alpha,\lambda)$,we have

$$\sum_{k=j+p}^{\infty}b_k \le \frac{(1-\alpha)\delta(p,q)((j+p-q)!)}{\left(\frac{j}{p-q}+1\right)^n\left(\left(\frac{j}{p-q}(1-\alpha\lambda)\right)+(1-\alpha)\right)(j+p)!}$$

So that

$$\left|\frac{f(z)}{g(z)}-1\right| < \frac{\sum_{k=j+p}^{\infty}|a_k-b_k|}{1-\sum_{k=j+p}^{\infty}b_k}$$

$$\le \frac{\delta}{(j+p)}\left(\frac{1}{1-\frac{(1-\alpha)\delta(p,q)(j+p-q)!}{\left(\frac{j}{p-q}+1\right)^n\left(\left(\frac{j}{p-q}(1-\alpha\lambda)\right)+(1-\alpha)\right)(j+p)!}}\right)$$

$$= \frac{\delta\ (j+p)!\left(\frac{j}{p-q}+1\right)^n\left[\frac{j}{p-q}(1-\alpha\lambda)+(1-\alpha)\right]}{\left((j+p)\left(\frac{j}{p-q}+1\right)^n\left(\left(\frac{j}{p-q}(1-\alpha\lambda)\right)+(1-\alpha)\right)\right)(j+p)!-\left((1-\alpha)\delta(p,q)\right)(j+p-q)!}$$

$$= p-\gamma$$

Provided that γ is given properly by **(16)** .Thus we have $f(z) \in T_j(n,p,q,\alpha,\lambda,\gamma)$ for every γ given by **(16)**.This obviously completes the proof of Theorem **(4)**.

5- Properties involving the operator $J_{c,p}$ and D_z^μ

Lemma 1.[6] let the function $f(z) \in T(j,p)$,then

$$D_z^\mu\left(J_{c,p}f(z)\right) = \frac{\Gamma(p+1)}{\Gamma(p-\mu+1)}z^{p-\mu} - \sum_{k=j+p}^{\infty}\left(\frac{(c+p)\Gamma(k+1)}{(c+k)\Gamma(k-\mu+1)}\right)a_k z^{k-\mu}$$

$$(\mu \in R; c > -p; j,p \in N) \quad \textbf{(17)}$$

And

$$J_{c,p}\left(D_z^\mu f(z)\right) = \frac{(c+p)\Gamma(p+1)}{(p-\mu+c)\Gamma(p-\mu+1)}z^{p-\mu} - \sum_{k=j+p}^{\infty}\left(\frac{(c+p)\Gamma(k+1)}{(k-\mu+c)\Gamma(k-\mu+1)}\right)a_k z^{k-\mu}$$

$$(\mu \in R\ ;\ c > -p\ ;\ j\ ,p \in N\ ) \quad \textbf{(18)}$$

Provided that there are no zeros appear in the denominators in (17) and (18) .This in general ,the operators $J_{c,p}$ and D_z^μ are non-commutative .

So as to give growth and distortion properties for functions in the class $T_j(n,p,q,\alpha,\lambda)$ including the operators $J_{c,p}$ and D_z^μ ,we find it to be convenient to use the order operation exhibited by **(18)** and **(19)** as we shown in the following Theorems .

Theorem 5 .If $f(z)$ is in the class $T_j(n,p,q,\alpha,\lambda)$, then

$$\left\{\frac{\Gamma(p+1)}{\Gamma(p-\mu+1)} - \left(\frac{(c+p)\Gamma(j+p+1)(1-\alpha)\delta(p,q)}{(j+p+c)\Gamma(j+p+\mu+1)\left(\frac{j}{p-q}+1\right)^n\left(\left(\frac{j}{p-q}(1-\alpha\lambda)\right)+(1-\alpha)\right)}\right)|z|^j\right\}|z|^{p+\mu} \le \left|D_z^{-\mu}\left(J_{c,p}f(z)\right)\right|$$

$$\le \left\{\frac{\Gamma(p+1)}{\Gamma(p-\mu+1)} + \left(\frac{(c+p)\Gamma(j+p+1)(1-\alpha)\delta(p,q)}{(j+p+c)\Gamma(j+p+\mu+1)\left(\frac{j}{p-q}+1\right)^n\left(\left(\frac{j}{p-q}(1-\alpha\lambda)\right)+(1-\alpha)\right)}\right)|z|^j\right\}|z|^{p+\mu}$$

$(z \in U\,;\ 0 \le \alpha < p-q\,;\ \mu > 0\,;\ n\,,q \in N_0\,;\ j\,,p \in N, c > -p\,;\ p > q\,)$
(19)

The result is sharp for the function give by

$$J_{c,p}(f(z)) = z^p - \left(\frac{(c+p)(1-\alpha)\delta(p,q)}{(j+p+c)\left(\frac{j}{p-q}+1\right)^n\left(\frac{j}{p-q}(1-\alpha\lambda)+(1-\alpha)\right)}\right) z^{j+p}$$
(20)

Proof . It follows from Theorem **(1)** that

$$\left(\frac{j}{p-q}+1\right)^n\left(\left(\frac{j}{p-q}(1-\alpha\lambda)\right)+(1-\alpha)\right)\delta(j+p,q)\sum_{k=j+p}^{\infty} a_k \le$$

$$\sum_{k=j+p}^{\infty}\left[\frac{k-q}{p-q}\right]^n\left(\left(\frac{k-q}{p-q}\right)-\alpha\left(1+\lambda\left(\left(\frac{k-q}{p-q}\right)-1\right)\right)\right)\delta(k,q)a_k$$

$$\le (1-\alpha)\delta(p,q)\ .$$

Which readily yields

$$\sum_{k=j+p}^{\infty} a_k \le \frac{(1-\alpha)\delta(p,q)}{\left(\frac{j}{p-q}+1\right)^n\left(\left(\frac{j}{p-q}(1-\alpha\lambda)\right)+(1-\alpha)\right)\delta(j+p,q)}\ .$$
(21)

Assumed that the function defined in U by

$$F(z) = \left(\frac{\Gamma(p+\mu+1)}{\Gamma(p+1)}\right) z^{-\mu} D_z^{-\mu}\left(J_{c,p}f(z)\right)$$
$$= z^p - \sum_{k=j+p}^{\infty}\left(\frac{(p+c)\Gamma(p+\mu+1)\Gamma(k+1)}{(k+c)\Gamma(p+1)\Gamma(k+\mu+1)}\right) a_k z^k$$
$$= \delta(p,q)z^{p-q} - \sum_{k=j+p}^{\infty}\theta(k)a_k z^{k-q} \qquad (z \in U)\ ,$$
(22)

if we set $\theta(k) = \frac{(p+c)\Gamma(p+\mu+1)\Gamma(k+1)}{(k+c)\Gamma(p+1)\Gamma(k+\mu+1)}$ $(k \ge j+p; j,p \in N)$. **(23)**

Then it is easily seen that $\theta(k)$ is decreasing function of k when $\mu > 0$,and hence

$$0 < \theta(k) \le \theta(j+p) = \frac{(p+c)\Gamma(p+\mu+1)\Gamma(j+p+1)}{(j+p+c)\Gamma(p+1)\Gamma(j+p+\mu+1)} \qquad (c > -p : \mu > 0;\ j,p \in N))\ .$$
(24)

Where

$$D_z^{-\mu}\left(J_{c,p}f(z)\right) = z^{p+\mu} - \left(\frac{(p+c)\Gamma(p+\mu+1)\Gamma(j+p+1)(1-\alpha)\delta(p,q)}{(j+p+c)\Gamma(p+1)\Gamma(j+p+\mu+1)\left(\frac{j}{p-q}+1\right)^n\left(\left(\frac{j}{p-q}(1-\alpha\lambda)\right)+(1-\alpha)\right)\delta(j+p,q)}\right) a_k z^{j+p+\mu}$$

By using **(21)** and **(24)**, we deduce that

$$|z|^p - \theta(j+p)|z|^{j+p}\sum_{k=j+p}^{\infty} a_k \le |F(z)| \le$$

$$|z|^p + \theta(j+p)|z|^{j+p}\sum_{k=j+p}^{\infty} a_k$$

That is

$$|z|^p - \left(\frac{(p+c)\Gamma(p+\mu+1)\Gamma(j+p+1)(1-\alpha)\delta(p,q)}{(j+p+c)\Gamma(p+1)\Gamma(j+p+\mu+1)\left(\frac{j}{p-q}+1\right)^n\left(\left(\frac{j}{p-q}(1-\alpha\lambda)\right)+(1-\alpha)\right)\delta(j+p,q)}\right)|z|^{j+p} \le |F(z)| \le$$

$$|z|^p + \left(\frac{(p+c)\Gamma(p+\mu+1)\Gamma(j+p+1)(1-\alpha)\delta(p,q)}{(j+p+c)\Gamma(p+1)\Gamma(j+p+\mu+1)\left(\frac{j}{p-q}+1\right)^n\left(\left(\frac{j}{p-q}(1-\alpha\lambda)\right)+(1-\alpha)\right)\delta(j+p,q)}\right)|z|^{j+p}$$

Which yields inequality **(19)**

Theorem 6 . If $f(z)$ is in $T_j(n,p,q,\alpha,\lambda)$, then

$$\left\{\frac{\Gamma(p+1)}{\Gamma(p-\mu+1)} - \left(\frac{(c+p)\Gamma(j+p+1)(1-\alpha)\delta(p,q)}{(j+p+c)\Gamma(j+p-\mu+1)\left(\frac{j}{p-q}+1\right)^n\left(\left(\frac{j}{p-q}(1-\alpha\lambda)\right)+(1-\alpha)\right)}\right)|z|^j\right\}|z|^{p-\mu} \le \left|D_z^{\mu}\left(J_{c,p}f^q(z)\right)\right|$$

$$\le \left\{\frac{\Gamma(p+1)}{\Gamma(p-\mu+1)} + \left(\frac{(c+p)\Gamma(j+p+1)(1-\alpha)\delta(p,q)}{(j+p+c)\Gamma(j+p+\mu+1)\left(\frac{j}{p-q}+1\right)^n\left(\left(\frac{j}{p-q}(1-\alpha\lambda)\right)+(1-\alpha)\right)}\right)|z|^j\right\}|z|^{p-\mu}$$

$(z \in U; 0 \le \alpha < p-q; 0 \le \mu \le 1; n,q \in N_0; j,p \in N, c > -p; p > q\,)$. **(25)**

The result is sharp for the function give by **(20)** .

Proof . It follows from Theorem **(1)** that

$$\sum_{k=j+p}^{\infty} k a_k \le \frac{(j+p)(1-\alpha)\delta(p,q)}{\left(\frac{j}{p-q}+1\right)^n\left(\left(\frac{j}{p-q}(1-\alpha\lambda)\right)+(1-\alpha)\right)\delta(j+p,q)} \qquad (0 \le \alpha < p: 0 \le \mu \le 1; j,p \in N)\ ,$$
(26)

suppose that the function defined in U as follows

$$G(z) = \frac{\Gamma(p-\mu+1)}{\Gamma(p+1)} z^{\mu} D_z^{\mu}\left(J_{c,p}f(z)\right)$$

$$= z^p - \sum_{k=j+p}^{\infty}\left(\frac{(p+c)\Gamma(p-\mu+1)\Gamma(k)}{(k+c)\Gamma(p+1)\Gamma(k-\mu+1)}\right) k a_k z^k$$

$$= z^p - \sum_{k=j+p}^{\infty}\vartheta(k) k a_k z^k \qquad (z \in U)\ ,$$
(27)

if we set $\vartheta(k) = \frac{(p+c)\Gamma(p-\mu+1)\Gamma(k)}{(k+c)\Gamma(p+1)\Gamma(k-\mu+1)}$ $(k \ge j+p; 0 \le \mu < 1; j,p \in N)$. **(28)**

Then it is easily seen that $\vartheta(k)$ is decreasing function of k when $\mu < 1$,and hence

$0 < \vartheta(k) \leq \vartheta(j+p) = \frac{(p+c)\Gamma(p-\mu+1)\Gamma(j+p)}{(j+p+c)\Gamma(p+1)\Gamma(j+p-\mu+1)}$
$(c > -p\,;\ p\,,j \in N\,;0 \leq \mu < 1)$. **(29)**

By using **(26)** and **(29)**, we deduce that

$$|z|^p - \vartheta(j+p)|z|^{j+p}\sum_{k=j+p}^{\infty} ka_k \leq |F(z)| \leq |z|^p + \vartheta(j+p)|z|^{j+p}\sum_{k=j+p}^{\infty} ka_k$$

That is

$$|z|^p - \left(\frac{(c+p)\Gamma(p-\mu+1)\Gamma(j+p+1)(1-\alpha)\delta(p,q)}{(j+p+c)\Gamma(p+1)\Gamma(j+p-\mu+1)\left(\frac{j}{p-q}+1\right)^n\left(\left(\frac{j}{p-q}(1-\alpha\lambda)\right)+(1-\alpha)\right)}\right)|z|^{j+p} \leq |F(z)| \leq$$

$$|z|^p + \left(\frac{(c+p)\Gamma(p-\mu+1)\Gamma(j+p+1)(1-\alpha)\delta(p,q)}{(j+p+c)\Gamma(p+1)\Gamma(j+p-\mu+1)\left(\frac{j}{p-q}+1\right)^n\left(\left(\frac{j}{p-q}(1-\alpha\lambda)\right)+(1-\alpha)\right)}\right)|z|^{j+p}$$

.

Which yields inequality **(25)** .

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حول اصناف الدوال التحليلية المتعددة التكافوء المعرفة بواسطة المؤثر التفاضلي للمشتقة من الرتبة الاولى

زينب عودة اثبينة
قسم الرياضيات
كلية التربية
جامعة القادسية

قاسم عبدالحميد جاسم
قسم الرياضيات
كلية العلوم
جامعة بغداد

المستخلص :

في البحث المقدم .وباستخدام العامل التفاضلي ، وجدنا حدود المعامل ، بعض الخواص المهمة لدوال التحليلية المتعددة التكافوء لمعاملات سالبه للصنف الجزئي $T_j(n,p,q,\alpha,\lambda)$ حيث $(p,j \in N = \{1,2,\ldots\}; q,n \in N_0 = N \cup \{0\}; 0 \leq \alpha < p-q)$ لدوال .قدمنا خاصية النشوء لتلك الدوال باستخدام التركيب المتضمن العامل التفاضلي والعامل الكسري الحسابي *مرة ومرة اخرى استخدمنا* التركيب المتضمن العامل التفاضلي والعامل الكسري الحسابي المعكوس .

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Some Properties of the Prolongation Limit Random Sets in Random Dynamical Systems

Sundus Talib Mohsin Ihsan Jabbar Kadhim

College of Computer Science and Mathematics , University of Al-Qadiysih
sundustalibm@gmail.com Ihsan.kadhim@qu.edu.iq

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Abstract:
The aim of this paper is to study the omega limit set with new concepts of the prolongation limit random sets in random dynamical systems, where some properties are proved and introduced such as the relation among the orbit closure, orbit and omega limit random set. Also we prove that the first prolongation of a closed random set containing this set, the first prolongation is closed and invariant. In addition, it is connected whenever it is compact provided that the phase space of the random dynamical systems is locally compact. Then, we study the prolongational limit random set and examined some essential properties of this set. Finally, the relation among the first prolongation, the prolongational limit random set and the positive trajectory of a random set is given and proved.

Keywords: random dynamical system, trajectories,Omega-Limi set, prolongations and prolongational limit of random dynamical system.

Mathematics Subject Classification: 37HXX.

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1. Introduction.

Random dynamical systems arise in the modeling of many phenomena in physics, biology, economics, climatology, etc. , and the random effects often reflect intrinsic properties of these phenomena rather than just to compensate for the defects in deterministic models. The history of study of random dynamical systems goes back to Ulam and von Neumann in 1945 [1] and it has flourished since the 1980s due to the discovery that the solutions of stochastic ordinary differential equations yield a cocycle over a metric dynamical system which models randomness, i.e. a random dynamical system. Arnold and I.D. Chueshov (1998) [2] presented the universal view of an order-preserving random dynamical system, offered several examples and studied the chattels of their random equilibria and attractor. Son (2009)[3] studied the Lyapunov exponents for random dynamical systems. Yingchao (2010)[4] used the theory of random dynamical systems and stochastic analysis to research the existence of random attractors and also stochastic bifurcation behavior for stochastic Duffing-van der Pol equation with jumps under some assumptions. Kadhim and A.H. Khalil(2016)[5] they define the random dynamical system and random sets in uniform space are and proved some necessary properties of these two concepts. Also they study the expansivity of uniform random operator.

The structure of this paper is as follows: In Section 2 we recall same basic definition and facts about random dynamical. In Section 3 we study the definition of trajectories in random dynamical system. In Section 4 we recall some basic fact about omega-limit random set in random dynamical system. In Section 5 will be devoted to the concept of prolongations and prolongational limit random sets under a random dynamical system. We define the first prolongations and prolongational limit random sets of random dynamical system (Definition 5.1,5.5) .If $M(\omega)$ is invariant. We have first prolongations and prolongational limit sets of random dynamical system so invariant (Theorem5.3, 5.7). the first prolongation and the prolongational limit random set are closed sets (Theorem5.2 ,5.6) .If X is locally compact. We have first prolongations and prolongational limit sets of random dynamical system are connected (Theorem5.4, 4.13).

2. Notation and basic definitions

In this Section we recall some basic definition and facts about random dynamical system and notation .

2.1. Notations

(1) $\mathbb{G}$ =locally compact group.

(2) $\mathbb{X}$=metric space with metric d.

(3) $(\Omega, \mathbb{F}, \mathbb{P})$ is a probability space.

(4) $\mathbb{X}_{\mathcal{B}}^{\Omega}$ = the set of all measurable functions from Ω to $\mathbb{X}$.

(5) $S[A,r]$ the set$\{y: d(y,A) \leq r\}$.

(6)$H(A,r)$ the set$\{y: d(y,A) = r\}$.

2.2. Basic definitions

Definition 2.2.1 [6-7]: The metric dynamical system (MDS) is the 5-tuple $(\mathbb{G}, \Omega, \mathbb{F}, \mathbb{P}, \theta)$ where $(\Omega, \mathbb{F}, \mathbb{P})$ is a probability space and $\theta: G \times \Omega \to \Omega$ is $(\beta(\mathbb{G}) \otimes \mathbb{F}, \mathbb{F})$ −measurable, with

(i) $\theta(e,\omega) = Id_{\Omega}$,

(ii) $\theta(g * h, \omega) = \theta(g, \theta(h,\omega))$ and

(iii) $\mathbb{P}(\theta_g F) = \mathbb{P}(F)$, $\forall F \in \mathbb{F}\ \forall \omega \in G$.

Definition2.2.2[6]: The MDS $(\mathbb{G}, \Omega, \mathbb{F}, \mathbb{P}, \theta)$ is said to be topological metric dynamical system (TMDS) if Ω is topological space and $\theta: G \times \Omega \to \Omega$ is continuous.

Definition2.2.3 [6-8]: The mapping $\varphi: \mathbb{G} \times \Omega \times \mathbb{X} \to \mathbb{X}$ is said to be measurable random dynamical system on the measurable space $(\mathbb{X}, \beta(\mathbb{X}))$ over an MDS $(\mathbb{G}, \Omega, \mathbb{F}, \mathbb{P}, \theta)$ with if it has the following properties:

(i) φ is $\beta(\mathbb{G})\otimes\mathbb{F}\otimes\beta(\mathbb{X}), \beta(\mathbb{X})$ − measurable.

(ii) The mappings $\varphi(t,\omega) := \varphi(g,\omega,\cdot): \mathbb{X} \to \mathbb{X}$ form a cocycle over $\theta(\cdot)$, that is, $\forall g,h \in \mathbb{G}, \omega \in \Omega$ they satisfy

$\varphi(e,\omega) = id_X\ \forall\omega \in \Omega,$ (2.2.1)

$\varphi(g * h, \omega) = \varphi(g, \theta_h\omega) \circ \varphi(h,\omega)$ (2.2.2)

The RDS $(\mathbb{G}, \Omega, \mathbb{X}, \theta, \varphi)$ shall denote by (θ, φ).

If the function $\varphi(\cdot, \omega, \cdot): \mathbb{T} \times \mathbb{X} \to \mathbb{X}$, $(t,x) \mapsto \varphi(t,\omega,x)$, is continuous for every $\omega \in \Omega$ then the measurable dynamical system is called continuous or topological R

Definition 2.2.4 [9]: Let (θ, φ) be a measurable RDS and $C \subset \Omega \times X$ a set.

(i) C is called forward invariant if for $t > 0$

$C(\omega) \subset \varphi(t,\omega)^{-1} C(\theta(t,\omega))\mathbb{P}$ −a.s.

equivalently

$\varphi(t,\omega)C(\omega) \subset C(\theta(t,\omega))\mathbb{P}$ −a.s..

(ii) C is called invariant if for all $t \in \mathbb{T}$

$C(\omega) = \varphi(t,\omega)^{-1} C(\theta(t,\omega))\mathbb{P}$ −a.s.,

for two-sided time equivalent to

$\varphi(t,\omega)C(\omega) = C(\theta(t,\omega))\mathbb{P}$ −a.

Definition 2.2.5 [9-10]: Let $(\Omega, \mathcal{F})$ be a measurable space and (X,d) be a metric space which is considered a measurable space with Borel σ − algebra $\mathcal{B}(\mathbb{X})$. The set-valued function $A: \Omega \to \mathcal{B}(\mathbb{X}), \omega \longmapsto A(\omega)$, is said to be random set if for each $x \in \mathrm{X}$ the function $\omega \longmapsto d(x, A(\omega))$is measurable. If $A(\omega)$is closed (connected) (compact) for all $\omega \in \Omega$, it is called a random closed (connected) (compact) set.

Definition 2.2.6 [10]:
An RDS (θ, φ) is said to be asymptotically compact in the universe $\mathcal{D}$, if there exists an attracting random compact set $\{B_0(\omega)\}$, i.e., foe any $D \in \mathcal{D}$ and for any $\omega \in \Omega$ we have

$$\lim_{t\to\infty} d_X\{\varphi(t,\theta(-t)\omega)D(\theta(-t)\omega/B_0(\omega)\} = 0, \quad (2.2.3)$$

where $d_X\{A/B\} = \sup_{x\in A} \operatorname{dist}(x, B)$.

3. Definitions and characterizations

In this section we study the trajectories in random dynamical system. First we shall state the definition of trajectories in random dynamical system and We describe some measurable properties of the trajectory of random dynamical system.

Definition 3.1: Let $D: \omega \mapsto D(\omega)$ be a multifunction. We call the multifunction

$$\omega \mapsto \gamma_D^t(\omega) := \bigcup_{\tau\geq t} \varphi(\tau, \theta_{-\tau}\omega)D(\theta_{-\tau}\omega)$$

the tail (from the moment t) of the pull back trajectories emanating from D. If $D(\omega) = \{v(\omega)\}$ is a single valued function, then $\omega \mapsto \gamma_v(\omega) = \gamma_D^0(\omega)$ is said to be the (pull back) trajectory (or orbit) emanating from v. That is $\omega \mapsto \gamma_v(\omega) := \bigcup_{\tau\geq 0} \varphi(\tau, \theta_{-\tau}\omega)v(\theta_{-\tau}\omega)$

Definition 3.2: Let $v \in \mathbb{X}_{\mathcal{B}}^{\Omega}$ and γ_v, γ_v^+ and γ_v^- be the mappings form X in to 2^X defined as follows

(1) $\gamma_v(\omega) = \{\varphi(t, \theta_{-t}\omega)v(\theta_{-t}\omega) : t \in \mathbb{R}\}$
(2) $\gamma_v^+(\omega) = \{\varphi(t, \theta_{-t}\omega)v(\theta_{-t}\omega) : t \in \mathbb{R}^+\}$
(3) $\gamma_v^-(\omega) = \{\varphi(t, \theta_{-t}\omega)v(\theta_{-t}\omega) : t \in \mathbb{R}^-\}$

For every $v \in X_{\mathcal{B}}^{\Omega}$, the sets γ_v, γ_v^+, and γ_v^- are respectively called the trajectory, the forward semi-trajectory and backward semi-trajectory.

Definition 3.3: Let $x \in \mathbb{X}$. and γ_x, γ_x^+ and γ_x^- be the mappings form $\mathbb{X}$. in to $2^{\mathbb{X}.}$ defined as follows

(1) $\gamma_x(\omega) = \{\varphi(t, \omega)x : t \in \mathbb{R}\}$
(2) $\gamma_x^+(\omega) = \{\varphi(t, \omega)x : t \in \mathbb{R}^+\}$
(3) $\gamma_x^-(\omega) = \{\varphi(t, \omega)x : t \in \mathbb{R}^-\}$

For every $x \in X$, the sets γ_x, γ_x^+, and γ_x^- are respectively called the trajectory, the forward semi-trajectory and backward semi-trajectory.

Proposition 3.4: For and $v \in \mathbb{X}_{\mathcal{B}}^{\Omega}$, the sets γ_v, γ_v^+, and γ_v^- are invariant random sets.

Proof. Let $v \in \mathbb{X}_{\mathcal{B}}^{\Omega}$. To show that γ_v is an invariant. Let $x \in \gamma_v(\omega)$ and $t \in \mathbb{R}$. Then there exists $s \in \mathbb{R}$ such that $x = \varphi(s, \theta_{-s}\omega)v(\theta_{-s}\omega)$. Now

$\mathbb{P}\{\omega : \varphi(t,\omega)x \in \gamma_v(\theta_t\omega)\} = \mathbb{P}\{\omega : x \in \varphi(-t, \theta_t\omega)\gamma_v(\theta_t\omega)\}$

$= \mathbb{P}\{\omega : \varphi(s, \theta_{-s}\omega)v(\theta_{-s}\omega) \in \varphi(-t, \theta_t\omega)\gamma_v(\theta_t\omega)\}$

$= \mathbb{P}\{\omega : v(\theta_{-s}\omega) \in \varphi(-s, \omega) \circ \varphi(-t, \theta_t\omega)\gamma_v(\theta_t\omega)\}$

$= \mathbb{P}\{\omega : v(\theta_{-s}\omega) \in \varphi(-s, \omega) \circ \varphi(-t, \theta_t\omega)\gamma_v(\theta_t\omega)\}$

$= \mathbb{P}\{\omega : v(\theta_{-s}\omega) \in \varphi(-s, \theta_{-t}\omega') \circ \varphi(-t, \omega')\gamma_v(\omega')\}$

where $\omega' = \theta_t\omega$.

$= \mathbb{P}\{\omega : v(\theta_{-s}\omega) \in \varphi(-s-t, \omega')\gamma_v(\omega')\}$

$= \mathbb{P}\{\theta_{-t}\omega' : v(\theta_{-s}\theta_{-t}\omega') \in \varphi(-s-t, \omega')\gamma_v(\omega')\}$

$= \mathbb{P}\{\omega' : v(\theta_r\omega') \in \varphi(r, \omega')\gamma_v(\omega')\}$, $r = -s - t$.

$= \mathbb{P}\{\omega' : \varphi(-r, \theta_r\omega')v(\theta_r\omega') \in \gamma_v(\omega')\} = 1.$

Thus for every $x \in \gamma_v(\omega)$ and $t \in \mathbb{R}$, we have

$\mathbb{P}\{\omega : \varphi(t,\omega)x \in \gamma_v(\theta_t\omega)\} = 1.$

This means that the set $\gamma_v(\omega)$ is an invariant. In a similar way we can show that γ_v^+, and γ_v^- are invariant random sets.

4. Omega-limit set in random dynamical system

In this section, we state the definition of omega-limit set in random dynamical system is due to [10-11].Thus, we give some basic properties of omega-limit set in random dynamical system.

Definition 4.1: The multifunctions $\omega \mapsto \Gamma_M^+(\omega) := \{y \in \mathbb{X}. :$ there is a sequences $\{t_n\}$ in $\mathbb{R}$ and $\{x_n\}$in $M(\theta_{-t_n}\omega)$ with $t_n \longrightarrow +\infty$ and $\varphi(t_n, \theta_{-t_n}\omega)x_n \longrightarrow y$ for all $\omega\}$

$\omega \mapsto \Gamma_M^-(\omega) := \{y \in \mathbb{X}. :$ there is a sequence $\{t_n\}$ in $\mathbb{R}$ with $\{x_n\}$in $M(\theta_{-t_n}\omega)$ $t_n \longrightarrow -\infty$ and $\varphi(t_n, \theta_{-t_n}(\omega))x_n \longrightarrow y$ for all $\omega\}$

are said to be the omega (alpha) -limit set of the trajectories emanating from x respectively.

If $M = \{x\}$, the we have

⟦ $\omega \mapsto \Gamma$⟧ _x^ + $(\omega) := \{y \in \mathbb{X}.$: there is a sequences $\{t\_n\}$ in $\mathbb{R}$ with $t\_n \longrightarrow +\infty$ and $\varphi(t\_n, \theta\_($⟦ $-t$⟧ _n) $\omega)x \longrightarrow y$ for all $\omega\}$

$\omega \mapsto \Gamma_x^-(\omega) := \{y \in \mathbb{X}. :$ there is a sequence $\{t_n\}$ in $\mathbb{R}$ with $t_n \longrightarrow -\infty$ and $\varphi(t_n, \theta_{-t_n}(\omega))x \longrightarrow y$ for all $\omega\}$.

The following assertion gives another description of omega-limit sets.

Theorem 4.2: Let $\Gamma_M^+(\omega)$ be the omega-limit set of the trajectories emanating from M. Then $\Gamma_M(\omega) = \bigcap_{t>0} \overline{\gamma_D^t(\omega)} = \bigcap_{t>0} \overline{\bigcup_{\tau\geq t} \varphi(\tau, \theta_{-\tau}\omega)M(\theta_{-\tau}\omega)}$

Proof. Suppose that $y \in \Gamma_M(\omega)$, the for any $t > 0$ there exists $\{t_n\}$ in $\mathbb{R}$ and $\{x_n\}$ in $M(\theta_{-t_n}\omega)$ such that $\varphi(t_n, \theta_{-t_n}\omega)x_n \longrightarrow y$. Hence $x_n \in \bigcup_{\tau\geq t} M(\theta_{-\tau}\omega)$. Thus

$$\varphi(t_n, \theta_{-t_n}\omega)x_n \in \bigcup_{\tau\geq t} \varphi(\tau, \theta_{-\tau}\omega)M(\theta_{-\tau}\omega) \subset \overline{\bigcup_{\tau\geq t} \varphi(\tau, \theta_{-\tau}\omega)M(\theta_{-\tau}\omega)}.$$

Therefore

$$y \in \overline{\bigcup_{\tau\geq t} \varphi(\tau, \theta_{-\tau}\omega)M(\theta_{-\tau}\omega)}, \text{ for all } t > 0.$$

Thus $y \in \bigcap_{t>0} \overline{\bigcup_{\tau\geq t} \varphi(\tau, \theta_{-\tau}\omega)M(\theta_{-\tau}\omega)}$.

To prove the converse inclusion, let

$y \in \bigcap_{t>0} \overline{\bigcup_{\tau\geq t} \varphi(\tau, \theta_{-\tau}\omega)M(\theta_{-\tau}\omega)}$

then $y \in \overline{\bigcup_{\tau\geq t} \varphi(\tau, \theta_{-\tau}\omega)M(\theta_{-\tau}\omega)}$ for all $t > 0$.

In particular,

$y \in \overline{\bigcup_{\tau\geq n} \varphi(\tau, \theta_{-\tau}\omega)M(\theta_{-\tau}\omega)}$ for all $n = 1,2, \ldots$.

Therefore there exists a sequence $\{y_n\}$ in $\bigcup_{\tau\geq n}\varphi(\tau,\theta_{-\tau}\omega)M(\theta_{-\tau}\omega)$ such that $y_n \longrightarrow y$. Thus $y_n \in \bigcup_{\tau\geq n}\varphi(\tau,\theta_{-\tau}\omega)M(\theta_{-\tau}\omega)$ and $d(y,y_n) < 1/n$, $n = 1,2,\ldots$. It follows that there exists $t_n \geq n$ and $x_n \in M(\theta_{-\tau}\omega)$ such that $y_n = \varphi(t_n,\theta_{t_n}\omega)x_n$. That is $\varphi(t_n,\theta_{t_n}\omega)x_n \longrightarrow y$. Consequently, $y \in \Gamma_M(\omega)$.

$\Gamma_M(\omega) = \bigcap_{t>0}\overline{\gamma_D^t(\omega)} = \bigcap_{t>0}\overline{\bigcup_{\tau\geq t}\varphi(\tau,\theta_{-\tau}\omega)M(\theta_{-\tau}\omega)}$.

$\Gamma_M(\omega) = \bigcap_{t>0}\overline{\gamma_D^t(\omega)} = \bigcap_{t>0}\overline{\bigcup_{\tau\geq t}\varphi(\tau,\theta_{-\tau}\omega)M(\theta_{-\tau}\omega)}$.Since $\overline{\gamma_D^t(\omega)}$ is closed an invariant, then so is $\Gamma_M(\omega) = \bigcap_{t>0}\overline{\gamma_D^t(\omega)}$.

Theorem 4.3: Let $\Gamma_M^+(\omega)$ is a random closed set, then the proof is devided in two parts:

1: Indirect Proof. By above theorem we have $\Gamma_M(\omega) = \bigcap_{t>0}\overline{\gamma_D^t(\omega)} = \bigcap_{t>0}\overline{\bigcup_{\tau\geq t}\varphi(\tau,\theta_{-\tau}\omega)M(\theta_{-\tau}\omega)}$.
Since $\overline{\gamma_D^t(\omega)}$ is closed an invariant, then so is $\Gamma_M(\omega) = \bigcap_{t>0}\overline{\gamma_D^t(\omega)}$.

2: Direct proof. Let $y \in \overline{\Gamma_M^+(\omega)}$. Then there exists $\{y_n\}$ in be a sequence in $\Gamma_M^+(\omega)$ such that $y_n \longrightarrow y$. We wish to show that $y \in \Gamma_M^+(\omega)$. Indeed for each positive integer k, there is a sequence $\{t_n^k\}$ in $\mathbb{R}$ and $\{x_n^k\}$ in $M(\theta_{-t_n^k}\omega)$ with $t_n^k \longrightarrow +\infty$ and $\varphi(t_n^k,\theta_{-t_n^k}(\omega))x_n^k \longrightarrow y_k$. We assume without loss of generality that $d\left(y_k,\varphi\left(t_n^k,\theta_{-t_n^k}\omega\right)x_n^k\right) < 1/k$ and $t_n^k \geq k$ for $n \geq k$. Consider now the sequence $\{t_n\}$ in $\mathbb{R}$ with $t_n = t_n^n$ and a sequence $\{x_n\}$ in $M(\theta_{-t_n}\omega)$ with $x_n = x_n^n$. Then $t_n \longrightarrow +\infty$ and we claim that $\varphi\left(t_n^k,\theta_{-t_n^k}\omega\right)x_n \longrightarrow y$.To see this observe that

$d(\varphi(t_n,\theta_{-t_n}\omega)x_n,y) \leq d(\varphi(t_n,\theta_{-t_n}\omega)x_n,y_n) + d(y_n,y)$

$$< 1/n + d(y_n,y).$$

Since $1/n$ and $d(y_n,y)$ tend to zero we conclude that

$$d(\varphi(t_n,\theta_{-t_n}\omega)x_n,y) \longrightarrow 0$$

Consequently $\varphi(t_n,\theta_{-t_n}\omega)x_n \longrightarrow y$ and $y \in \Gamma_M^+(\omega)$. Thus $\Gamma_M^+(\omega) = \overline{\Gamma_M^+(\omega)}$, i.e., $\Gamma_M^+(\omega)$ is closed.

Theorem 4.4: Let $\mathbb{X}.$ be any metric space and $x \in \mathbb{X}..$ Then

$$\Gamma_x^+(\theta_t w) = \varphi(t,\omega)\Gamma_x^+(\omega) \text{ for every } t \in \mathbb{R}.$$

Proof. To prove $\Gamma_x^+(\theta_t w) = \varphi(t,\omega)\Gamma_x^+(\omega)$ .Let $z \in \Gamma_x^+(\theta_t w)$.Then there is a sequences $\{t_n\}$ in $\mathbb{R}$ with $t_n \longrightarrow +\infty$ and $\varphi(t_n,\theta_{-t_n}\theta_t w)x \longrightarrow z$

$$\varphi(t_n + t - t,\theta_{-t_n}\theta_t w)x \longrightarrow z$$
$$\varphi(t,\omega)\,\varphi(t_n - t,\theta_{-t_n}\theta_t w)x \longrightarrow z$$
$$\varphi(t_n - t,\theta_{t-t_n} w)x \longrightarrow \varphi(t,\omega)^{-1}z$$

,where $t_n - t \longrightarrow +\infty$. Thus we have $\varphi(t,\omega)^{-1}z \in \Gamma_x^+(\omega)$. Then $z \in \varphi(t,\omega)\Gamma_x^+(\omega)$

Then $\Gamma_x^+(\theta_t w) \subseteq \varphi(t,\omega)\Gamma_x^+(\omega)$ .Now let $z \in \varphi(t,\omega)\Gamma_x^+(\omega)$. Then there is $y \in \Gamma_x^+(\omega)$ suchthat $z = \varphi(t,\omega)y$ .Then there is a sequences $\{t_n\}$ in $\mathbb{R}$ with $t_n \longrightarrow +\infty$ and $\varphi(t_n,\theta_{-t_n})x \longrightarrow y$. By continuity $\varphi(t,\omega)$, $\varphi(t,\omega)\varphi(t_n,\theta_{-t_n}w)x \longrightarrow \varphi(t,\omega)y$, $\varphi(t + t_n,\theta_{-t_n}w)x \longrightarrow z$

$\varphi(t_n + t,\theta_{-t-t_n}\theta_t w)x \longrightarrow z$ .Thus we have $z \in \Gamma_x^+(\theta_t w)$ .Then $\varphi(t,\omega)\Gamma_x^+(\omega) \subseteq \Gamma_x^+(\theta_t w)$ .Then $\Gamma_x^+(\theta_t w) = \varphi(t,\omega)\Gamma_x^+(\omega)$.

Theorem 4.5. If $(t,x) \longmapsto \varphi(t,\theta_{-t}\omega)x$ is continuous, then

$$\overline{\gamma_M^t(\omega)} = \gamma_M^t(\omega) \cup \Gamma_M^+(\omega).$$

Proof. First, note that $\gamma_M^t(\omega) \subset \overline{\gamma_M^t(\omega)}$. By Theorem (4.2), we have $\Gamma_M^+(\omega) \subset \overline{\gamma_M^t(\omega)}$. Therefore $\overline{\gamma_M^t(\omega)} \supset \gamma_M^t(\omega) \cup \Gamma_M^+(\omega)$. To prove the converse inclusion, let $y \in \overline{\gamma_M^t(\omega)}$. then there exists a sequence $\{y_n\}$ in $\gamma_M^t(\omega)$ such that $y_n \longrightarrow y$. Now $y_n \in \bigcup_{\tau\geq t}\varphi(\tau,\theta_{-\tau}\omega)M(\theta_{-\tau}\omega)$, then there exists a sequence $\{\tau_n\}$ with $\tau_n \geq t$ for every n and $\{x_n\}$ in $M(\theta_{-\tau_n}\omega)$such that $y_n = \varphi(\tau_n,\theta_{-\tau_n}\omega)x_n$. We have two cases:

Case I: The sequence $\{\tau_n\}$ has the property that $\tau_n \longrightarrow +\infty$, in which case $y \in \Gamma_M^+(\omega)$.

Case II: There is a subsequence $\{\tau_{n_k}\}$ in $\mathbb{R}^+$ such that $\tau_{n_k} \longrightarrow \tau \in \mathbb{R}^+$ (as $\mathbb{R}^+$ is closed). But then $\varphi\left(\tau_{n_k},\theta_{-\tau_{n_k}}\omega\right)x \longrightarrow \varphi(\tau,\theta_{-\tau}\omega)x \in \gamma_M^t(\omega)$ (since $(t,x) \longmapsto \varphi(t,\theta_{-t}\omega)$ (since $(t,x) \longmapsto \varphi(t,\theta_{-t}\omega)x$ is continuous). Since $\left(\tau_{n_k},\theta_{-\tau_{n_k}}\omega\right)x \longrightarrow y$, then from the uniqueness of the limit we have $\varphi(\tau,\theta_{-\tau}\omega)x = y \in \gamma_M^t(\omega)$. From Case I and Case II, we have $y \in \gamma_M^t(\omega) \cup \Gamma_M^+(\omega)$. Hence

$$\overline{\gamma_M^t(\omega)} \subset \gamma_M^t(\omega) \cup \Gamma_M^+(\omega).$$

Therefore $\overline{\gamma_M^t(\omega)} = \gamma_M^t(\omega) \cup \Gamma_M^+(\omega)$

Corollary 4.6: For any $x \in \mathbb{X}..\overline{\gamma_x^+(\omega)} = \gamma_x^+(\omega) \cup \Gamma_x^+(\omega)$ and $\overline{\gamma_x^-(\omega)} = \gamma_x^-(\omega) \cup \Gamma_x^-(\omega)$.

Proof. By the definition we have $\gamma_x^+(\omega) \cup \Gamma_x^+(\omega) \subseteq \overline{\gamma_x^+(\omega)}$.To show that $\overline{\gamma_x^+(\omega)} \subseteq \gamma_x^+(\omega) \cup \Gamma_x^+(\omega)$,let $y \in \overline{\gamma_x^+(\omega)}$.Then there is a sequence $\{y_n\}$ in$\gamma_x^+(\omega)$ such that $y_n \rightarrow y$. Sincey_n in $\gamma_x^+(\omega)$ .Then $y_n = \varphi(\tau_n,\theta_{-\tau_n}\omega)x$ for a τ_n in R^+.Either the sequence $\{\tau_n\}$ has the property that $\tau_n \rightarrow +\infty$,in which case $y \in \Gamma_x^+(\omega)$, or there is a subsequence $\tau_{nk} \rightarrow t \in R^+$(as R^+ is closed).But then $\varphi(\tau_{nk},\theta_{-\tau_n}\omega)x \rightarrow \varphi(\tau,\theta_{-\tau}\omega)x \in \gamma_x^+(\omega)$, and since also $\varphi(\tau_{nk},\theta_{-\tau_n}\omega)x \rightarrow y$ we have$\varphi(\tau,\theta_{-\tau}\omega)x = y \in \gamma_x^+(\omega)$.Thus$\overline{\gamma_x^+(\omega)} \subseteq \gamma_x^+(\omega) \cup \Gamma_x^+(\omega)$.Thus $\overline{\gamma_x^+(\omega)} = \gamma_x^+(\omega) \cup \Gamma_x^+(\omega)$.■

5. Some Properties of the Limit Random Sets in Random Dynamical Systems.

the concepts of prolongations and prolongational limit sets are played an essential role. In the deterministic dynamical system the formal definition of prolongation is due to Ura [12] and the concept of prolongational limit set is due to Bhatia, Szegö [13]. By following this line of investigation, the present paper introduces the notions of prolongations and prolongational limit random sets of random dynamical systems. We simplify several concepts and effects of reclusiveness and depressiveness from Bhatia and Szegö [2]. We consider (θ,φ) random dynamical system then we define the first prolongations and prolongational limit random set of M .we prove some new properties of the studying of prolongations and prolongational limit random sets.

Definition 5.1: Let $M:\omega\longmapsto M(\omega)$ be multifunction. The multifunction $\omega\longmapsto D_M^+(\omega)$, where
$D_M^+(\omega)\coloneqq\{y\in\mathbb{X}.$
: there is a sequences $\{t_n\}$ in $\mathbb{R}^+$ $and\ \{x_n\} in\ M(\theta_{-t_n}\omega\ )$ with $,x_n\longrightarrow x\in\bigcap_{n=1}^{\infty}M(\theta_{-t_n}\omega\ )$ and $\varphi(t_n,\theta_{-t_n}\omega)x_n\longrightarrow y$for all $\omega\}$, is said to be to be first positive prolongation of M. If the set $\mathbb{R}^+$ replaced by $\mathbb{R}^-$ in above we get the notation of first negative prolongation of M and shall denoted by $D_M^-(\omega)$.
If $M=\{x\}$, the we have
$D_x^+(\omega)\coloneqq\{y\in\mathbb{X}.:$
there is a sequences $\{t_n\}$ in $\mathbb{R}^+$ $and\ \{x_n\} in\ X\ with\ ,x_n\longrightarrow x$ and $\varphi(t_n,\theta_{-t_n}\omega)x_n\longrightarrow y$ for all $\omega\}$

Theorem 5.2: $D_M^+(\omega)$ is closed.

Proof. To show that $D_M^+(\omega)$ is closed. Let $y\in\overline{\mathrm{D}_M^+(\omega)}$, then there exists sequence $\{y_n\}$ in $\mathrm{D}_M^+(\omega)$ such that $y_n\longrightarrow y$. Since $y_n\in\mathrm{D}_M^+(\omega)$ for every n. Then by definition of $\mathrm{D}_M^+(\omega)$ there exists sequences $\{t_n^k\}\in\mathbb{R}^+$ and $\{x_n^k\}\in M(\theta_{-t_n^k}\omega\ )$ such that $x_n^k\longrightarrow x\in\bigcap_{n=1}^{\infty}M(\theta_{-t_n^k}\omega\ )$ and $\varphi(t_n^k,\theta_{-t_n^k}\omega)x_n^k\longrightarrow y_k$. We assume by taking subsequences if necessarily that $t_n^k>k$, $d(x_n^k,x)\leq 1/k$ and $d(\varphi\left(t_n^k,\theta_{-t_n^k}\omega\right)x_n^k,y_k)\leq 1/k$ for $n\geq k$. Now consider the sequences $\{x_n^n\}$, $\{t_n^n\}$. Clearly $x_n^n\longrightarrow x\in\bigcap_{n=1}^{\infty}M(\theta_{-t_n^n}\omega\ )$ and $\{t_n^n\}\in\mathbb{R}^+$. Note that

$$d\big(\varphi(t_n^n,\theta_{-t_n^n}\omega)x_n^n,y\big)\leq d\big(\varphi(t_n^n,\theta_{-t_n^n}\omega)x_n^n,y_n\big)+d(y_n,y)$$
$$\leq 1/n+d(y_n,y).$$

Since $\{1/n\}$ and $d(y_n,y)$ tend to zero, then $\varphi\big(t_n^n,\theta_{-t_n^n}\omega\big)x_n^n\longrightarrow y$, then $y\in D_M^+(\omega)$. This means $\overline{\mathrm{D}_M^+(\omega)}=\mathrm{D}_M^+(\omega)$ and so $\mathrm{D}_M^+(\omega)$ is closed.

Theorem 4.3: If M is invariant, then so is $\mathrm{D}_M^+(\omega)$.

Proof. We need to show that $\varphi(t,\omega)\mathrm{D}_M^+(\omega)=\mathrm{D}_M^+(\theta_t\omega\ )$.

Let $z\in\varphi(t,\omega)\mathrm{D}_M^+(\omega)$, then there exists $y\in\mathrm{D}_M^+(\omega)$ such that
$z=\varphi(t,\omega)y.$
To show that $z\in\mathrm{D}_M^+(\theta_t\omega\ )$.Since $y\in\mathrm{D}_M^+(\omega)$, there exist sequences $\{t_n\}$ in $\mathbb{R}^+$ and $\{x_n\}$ in $M(\theta_{-t_n}\theta_t\omega\ )$ with $x_n\longrightarrow x\in\bigcap_{n=1}^{\infty}M(\theta_{-t_n}\omega\ )$ and
$\varphi(t_n,\theta_{-t_n}\omega)x_n\longrightarrow y$. Since $\varphi(t,\omega)$ is continuous, then
$\varphi(t,\omega)\circ\ \varphi(t_n,\theta_{-t_n}\omega)x_n\longrightarrow\varphi(t,\omega)\ y$, then
$\varphi(t+t_n,\theta_{-t-t_n}\circ\theta_t\omega)x_n\longrightarrow\varphi(t,\omega)\ y$
for and $y\in\mathrm{D}_M^+(\omega)$ by Definition. According to Def. $z=\varphi(t,\omega)y\in\mathrm{D}_M^+(\theta_t\omega)$. then $\varphi(t,\omega)\mathrm{D}_M^+(\omega)\subset\mathrm{D}_M^+(\theta_t\omega)$. To prove the converse inclusion, let $z\in\mathrm{D}_M^+(\theta_t\omega)$. by Def. there exist sequences $\{t_n\}$ in $\mathbb{R}^+$ and $\{y_n\}$ in $M\big(\theta_{-t_n}\theta_t\omega\ \big)$ with
$y_n\longrightarrow y\in\bigcap_{n=1}^{\infty}M(\theta_{-t_n}\theta_t\omega\ )$ and so
$y\in M(\theta_{-t_n}\theta_t\omega\ )$ for all n. Since M is an invariant $y_n\in M(\theta_{-t_n}\theta_t\omega\ )$, then
$y_n\in\varphi(t,\theta_{-t_n}\omega)M(\theta_{-t_n}\omega\ )$, then there exists $x_n\in M(\theta_{-t_n}\omega\ )$ such that $y_n=\varphi(t,\theta_{-t_n}\omega)x_n$, then $x_n=\varphi(-t,\theta_t\theta_{-t_n}\omega)y_n$. Now,
$$y\in\bigcap_{n=1}^{\infty}\varphi\big(t,\theta_{-t_n}\omega\big)M\big(\theta_{-t_n}\omega\ \big).$$
Then $y\in\varphi\big(t,\theta_{-t_n}\omega\big)M\big(\theta_{-t_n}\omega\ \big)$ for all n.
Then there exists $x\in\bigcap_{n=1}^{\infty}M(\theta_{-t_n}\omega\ )$ such that $y=\varphi(t,\theta_{-t_n}\omega)x$ for all n. Since $y_n\longrightarrow y$, i.e. $\varphi(t_n,\theta_{-t_n}\omega)x_n\longrightarrow\varphi(t,\theta_{-t_n}\omega)x$
$\varphi(t_n,\theta_{-t_n}\theta_t\omega)y_n\longrightarrow z.$
$\varphi(t,\omega)\ \circ\varphi(t_n-t,\theta_{-t_n+t}\omega)x_n\longrightarrow z$
$\Longrightarrow\varphi(t,\omega)\ z_n\longrightarrow z$, with
$z_n\coloneqq\varphi(t_n-t,\theta_{-t_n+t}\omega)x_n.$ (5.1)
From (2.2.3) we have that $z_n\longrightarrow B_0(\omega)$ as $n\longrightarrow\infty$. Since $B_0(\omega)$is compact, there exist $\{n_k\}$ and $b\in B_0(\omega)$ such that $z_{n_k}\longrightarrow b$ as $k\longrightarrow\infty$. Moreover by Def. $b\in\mathrm{D}_M^+(\omega)$. From (5.1) we obtain that $z=\varphi(t,\omega)b$. Therefore $\mathrm{D}_M^+(\theta_t\omega)\subset\varphi(t,\omega)\mathrm{D}_M^+(\omega)$ for all $t>0$ and $\omega\in\Omega$. Thus $\mathrm{D}_M^+(\omega)$ is invariant.
We now discuss about the connectedness of the First Prolongation .

Theorem 4.4: Let $\mathbb{X}$. be locally compact. Then $\mathrm{D}_M^+(\omega)$ is connected whenever it is compact.

Proof. Let $\mathrm{D}_M^+(\omega)$ be compact but disconnected. Then there are two compact non- empty sets$P$ and Q such that $P\cup\ Q=\mathrm{D}_M^+(\omega)$ and $P\cap\ Q=\emptyset$ .Since P and Q are compact $d(P,Q)>0$.Thus there is $r>0$ such that$S[P,r]$, $S[Q,r]$are compact and disjoint .Now $x\in$ P or $x\in$ Q .Let $x\in$ P .Then there is a sequence $\{x_n\}$ in X and a sequence $\{t_n\}$ in $\mathbb{R}^+$such that$x_n\ \longrightarrow x$,and $\varphi(t_n,\theta_{-t_n}\omega)x_n\longrightarrow y\in Q$ .We may assume $x_n\in S[P,r]$ and $\varphi(t_n,\theta_{-t_n}\omega)x_n\ \in S[Q,r]$ .Then the trajectory segments $\varphi(s_n,\theta_{-s_n}\omega)x_n$, $0\leq s_n\leq t_n$ intersect $H(P,r)$,and therefor is a sequence $\{\tau_n\}$, $,0\leq\tau_n\leq t_n$ such that $\varphi(\tau_n,\theta_{\tau_n}\omega)x_n\ \in\ H(P,r)$ .Since $H(P,r)$ is compact we may assume that $\varphi(\tau_n, \theta_{\tau_n}\omega)x_n \longrightarrow z \in H(P,r)$.Then $z \in \mathrm{D}_M^+(\omega)$,but $z \notin \mathrm{P} \cup Q$ as $z \in H(P,r)$ .Thus contradiction shows that $\mathrm{D}_M^+(\omega)$ is connected.

Definition 5.5: Let $M: \omega \longmapsto M(\omega)$ be multifunction.The multifunction $\omega \longmapsto J_M^+(\omega)$, where $J_M^+(\omega) \coloneqq \{y \in \mathbb{X}$.
: there is a sequences $\{t_n\}$ in $\mathbb{R}^+$ *and* $\{x_n\}$*in* $M(\theta_{-t_n}\omega)$ with $t_n \longrightarrow +\infty$, $x_n \longrightarrow M(\omega)$and $\varphi(t_n, \theta_{-t_n}\omega)x_n \longrightarrow y$for all $\omega\}$, is said to be to be first positive prolongational limit set of M. If the set $\mathbb{R}^+$ replaced by $\mathbb{R}^-$ in above we get the notation of first negative prolongational limit set of M and shall denoted by $J_M^-(\omega)$.

If $M = \{x\}$, then the definition of $J_M^+(\omega)$ becomes

$J_x^+(\omega) \coloneqq \{y \in \mathbb{X}$.
: there is a sequences $\{t_n\}$ in $\mathbb{R}^+$ *and* $\{x_n\}$*in*
X with $t_n \longrightarrow +\infty$,
$x_n \longrightarrow x$ and $\varphi(t_n, \theta_{-t_n}\omega)x_n \longrightarrow y$ for all $\omega\}$.

The following result show that the prolongational limit set is closed and invariant.

Theorem 5.6: $J_M^+(\omega)$ is closed.

Proof. To show that $J_M^+(\omega)$ is closed. Let $y \in \overline{J_M^+(\omega)}$, then there exists sequence $\{y_n\}$ in $J_M^+(\omega)$ such that $y_n \longrightarrow y$. Since $y_n \in J_M^+(\omega)$ for every n. Then by definition of $J_M^+(\omega)$ there exists sequences $\{t_n^k\} \in \mathbb{R}^+$ and $\{x_n^k\} \in M(\theta_{-t_n^k}\omega)$ such that $x_n^k \longrightarrow M(\omega)$, $t_n^k \longrightarrow +\infty$ and $\varphi(t_n^k, \theta_{-t_n^k}\omega)x_n^k \longrightarrow y_k$. We assume by taking subsequences if necessarily that $t_n^k > k$, $d(x_n^k, x) \leq 1/k$ and $d(\varphi\left(t_n^k, \theta_{-t_n^k}\omega\right)x_n^k, y_k) \leq 1/k$ for $n \geq k$. Now consider the sequences $\{x_n^n\}$, $\{t_n^n\}$. Clearly $x_n^n \longrightarrow M(\omega)$ and $t_n^k \longrightarrow +\infty$. Note that

$$d\left(\varphi\left(t_n^n, \theta_{-t_n^n}\omega\right)x_n^n, y\right) \leq d\left(\varphi\left(t_n^n, \theta_{-t_n^n}\omega\right)x_n^n, y_n\right) + d(y_n, y)$$
$$\leq 1/n + d(y_n, y).$$

Since $\{1/n\}$ and $d(y_n, y)$ tend to zero, then $\varphi\left(t_n^n, \theta_{-t_n^n}\omega\right)x_n^n \longrightarrow y$, then $y \in J_M^+(\omega)$. This means $\overline{J_M^+(\omega)} = J_M^+(\omega)$ and so $J_M^+(\omega)$ is closed.

Theorem 4.7: If M is invariant, then so is $J_M^+(\omega)$.

Proof. We need to show that $\varphi(t,\omega)J_M^+(\omega) = J_M^+(\theta_t\omega)$.

Let $z \in \varphi(t,\omega)J_M^+(\omega)$, then there exists $y \in J_M^+(\omega)$ such that $z = \varphi(t,\omega)y$.

To show that $z \in J_M^+(\theta_t\omega)$.Since $y \in J_M^+(\omega)$, there exist sequences $\{t_n\}$ in $\mathbb{R}^+$ and $\{x_n\}$ in $M(\theta_{-t_n}\theta_t\omega)$ with $x_n \longrightarrow M(\omega)$, $t_n \longrightarrow +\infty$ and $\varphi(t_n, \theta_{-t_n}\omega)x_n \longrightarrow y$.
Since $\varphi(t,\omega)$ is continuous, then

$$\varphi(t,\omega) \circ \varphi(t_n, \theta_{-t_n}\omega)x_n \longrightarrow \varphi(t,\omega)\, y.$$

By the cocycle property, we have

$$\varphi(t + t_n, \theta_{-t-t_n} \circ \theta_t\omega)x_n \longrightarrow \varphi(t,\omega)\, y$$

for and $y \in J_M^+(\omega)$ by definition. According to definition. $z = \varphi(t,\omega)\, y \in J_M^+(\theta_t\omega)$. then $J_M^+(\omega) \subset J_M^+(\theta_t\omega)$. To prove the converse inclusion, let $z \in J_M^+(\theta_t\omega)$.By definition there exist sequences $\{t_n\}$ in $\mathbb{R}^+$ and $\{x_n\}$ in $M(\theta_{-t_n}\theta_t\omega)$ with $x_n \longrightarrow M(\omega)$, $t_n \longrightarrow +\infty$ and $\varphi(t_n, \theta_{-t_n}\theta_t\omega)x_n \longrightarrow z$. By the cocycle property we have

$$\varphi(t,\omega) \circ \varphi(t_n - t, \theta_{-t_n+t}\omega)x_n \longrightarrow z$$
$$\Longrightarrow \varphi(t,\omega)\, z_n \longrightarrow z, \text{ with}$$

$$z_n \coloneqq \varphi(t_n - t, \theta_{-t_n+t}\omega)x_n. \qquad (5.2)$$

From (2.2.3) we have that $z_n \longrightarrow B_0(\omega)$ as $n \longrightarrow \infty$. Since $B_0(\omega)$is compact, there exist $\{n_k\}$ and $b \in B_0(\omega)$ such that $z_{n_k} \longrightarrow b$ as $k \longrightarrow \infty$. Moreover by Def. $b \in J_M^+(\omega)$. From (5.2) we obtain that $z = \varphi(t,\omega)b$. Therefore $J_M^+(\theta_t\omega) \subset \varphi(t,\omega)J_M^+(\omega)$ for all $t > 0$ and $\omega \in \Omega$. Thus $J_M^+(\omega)$ is invariant.

Theorem 5.8: $\mathrm{D}_M^+(\omega) = \gamma_M^+(\omega) \cup J_M^+(\omega)$.

Proof. $\gamma_M^+(\omega) \cup J_M^+(\omega) \subset \mathrm{D}_M^+(\omega)$. To prove the converse inclusion. Let $y \in \mathrm{D}_M^+(\omega)$ by Def. there exist sequences $\{t_n\}$ in $\mathbb{R}^+$ and $\{x_n\}$ in $M(\theta_{-t_n}\omega)$ with $x_n \longrightarrow M(\omega)$ and $\varphi(t_n, \theta_{-t_n}\omega)x_n \longrightarrow y$. We may assume that either $t_n \longrightarrow t \in \mathbb{R}^+$ or $t_n \longrightarrow +\infty$, if necessarily by taking subsequences. In the first case $\varphi(t_n, \theta_{-t_n}\omega)x_n \longrightarrow \varphi(t, \theta_{-t}\omega)x$ (since $\varphi(\cdot,\omega,\cdot): \mathbb{R} \times X \longrightarrow X$ is continuous for every $\omega \in \Omega$). By uniqueness of the limit we have $\varphi(t, \theta_{-t}\omega)x = y \in \gamma_M^{+t}(\omega)$. In the second case $y \in J_M^+(\omega)$ by Def. of $J_M^+(\omega)$. Thus $y \in \gamma_M^{+t}(\omega) \cup J_M^+(\omega)$. Hence $\mathrm{D}_M^+(\omega) = \gamma_M^+(\omega) \cup J_M^+(\omega)$.

Corollary 5.9: $\mathrm{D}_x^+(\omega) = \gamma_x^+(\omega) \cup J_x^+(\omega)$.

Proof. By definitions $\gamma_x^+(\omega) \cup J_x^+(\omega) \subset \mathrm{D}_x^+(\omega)$. . To prove the converse inclusion. Let $y \in \mathrm{D}_x^+(\omega)$ by Def. there exist a sequences $\{t_n\}$ in $\mathbb{R}^+$ and a sequences $\{x_n\}$ with $x_n \longrightarrow x$ such that $\varphi(t_n, \theta_{-t_n}\omega)x_n \longrightarrow y$. We may assume that either $t_n \longrightarrow t \in \mathbb{R}^+$ or $t_n \longrightarrow +\infty$, if necessarily by taking subsequences. In the first case $\varphi(t_n, \theta_{-t_n}\omega)x_n \longrightarrow \varphi(t, \theta_{-t}\omega)x$ (since $\varphi(\cdot,\omega,\cdot): \mathbb{R} \times X \longrightarrow X$ is continuous for every $\omega \in \Omega$). By uniqueness of the limit we have $\varphi(t, \theta_{-t}\omega)x = y \in \gamma_x^+(\omega)$. In the second case $y \in J_x^+(\omega)$ by Def. of $J_x^+(\omega)$. Thus $y \in \gamma_x^+(\omega) \cup J_x^+(\omega)$. Hence
$\mathrm{D}_x^+(\omega) = \gamma_x^+(\omega) \cup J_x^+(\omega)$.

Theorem5.10: Let

$x, y \in X$ with the property that

$x = \varphi(t, \theta_{-t}\omega)y$, for every t in $\mathbb{R}$ and

$\omega \in \Omega$. Then $y \in J_x^+(\omega)$ if and only if

$x \in J_y^-(\omega)$.

Proof. Suppose that $y \in J_x^+(\omega)$. Then there exist sequences $\{t_n\}$ in $\mathbb{R}^+$ and $\{x_n\}$ in X with $x_n \longrightarrow x$, $t_n \longrightarrow +\infty$ and $\varphi(t_n, \theta_{-t_n}\omega)x_n \longrightarrow y$. Set $\tau_n \coloneqq -t_n$ and $y_n \coloneqq \varphi(t_n, \theta_{-t_n}\omega)x_n$. Then $\{\tau_n\}$ is a sequence in $\mathbb{R}^-$ with $\tau_n \longrightarrow -\infty$ and $\{y_n\}$ is a sequence in X and $y_n \longrightarrow y$.

Finlay we need to show that $\varphi(\tau_n,\theta_{-\tau_n}\omega)y_n \longrightarrow x$.

$$d\big(\varphi(\tau_n,\theta_{-\tau_n}\omega)y_n,x\big)=$$
$$d\big(\varphi(\tau_n,\theta_{-\tau_n}\omega)\circ\varphi(t_n,\theta_{-t_n}\omega)x_n,x\big)$$
$$= d\big(\varphi(\tau_n,\theta_{-\tau_n}\omega)\circ\varphi(t_n,\theta_{-t_n}\omega)x_n,\varphi(\tau_n,\theta_{-\tau_n}\omega)y\big),$$
$$= d\big(\varphi(t_n,\theta_{-t_n}\omega)x_n,y\big)\longrightarrow 0$$

Then we have $\varphi(\tau_n,\theta_{-\tau_n}\omega)y_n \longrightarrow x$ .Thus $x\in J_y^-(\omega)$. Similarly we can prove the converse.

Theorem 5.11: $J_x^+(\theta_t\omega) = \varphi(t,\omega)J_x^+(\omega)$

Proof. To prove $J_x^+(\theta_t\omega) = \varphi(t,\omega)J_x^+(\omega)$ .Let $z\in J_x^+(\theta_t\omega)$. Then there is a sequence $\{t_n\}$ in $\mathbb{R}^+$ with $t_n \longrightarrow +\infty$ and a sequence $\{x_n\}$ in X with $x_n \longrightarrow x$ such that $\varphi(t_n,\theta_{-t_n}\theta_t\omega)x_n \longrightarrow z$,

$$\varphi(t_n-t+t,\theta_{-t_n}\theta_t\omega)x_n \longrightarrow z$$
$$\varphi(t,\omega)\varphi(t_n-t,\theta_{t-t_n}\omega)x_n \longrightarrow z$$

$\varphi(t_n-t,\theta_{t-t_n}\omega)x_n \longrightarrow \varphi(t,\omega)^{-1}z$,where $t_n-t \longrightarrow +\infty$.

Thus we have $\varphi(t,\omega)^{-1}z\in J_x^+(\omega)$. Then $z\in \varphi(t,\omega)J_x^+(\omega)$,

, then $J_x^+(\theta_t\omega) \subseteq \varphi(t,\omega)J_x^+(\omega)$. To prove the converse inclusion

Let $z\in\varphi(t,\omega)J_x^+(\omega)$. Then there is $y\in J_x^+(\omega)$ with $z=\varphi(t,\omega)y$ and a sequence $\{t_n\}$ in $\mathbb{R}^+$ with $t_n \longrightarrow +\infty$ and a sequence $\{x_n\}$ in X with $x_n \longrightarrow x$ such that $\varphi(t_n,\theta_{-t_n}\omega)x_n \longrightarrow y$. By the continuity of $\varphi(t,\omega)$,

$\varphi(t,\omega)\circ\varphi(t_n,\theta_{-t_n}\omega)x_n \longrightarrow \varphi(t,\omega)y$

$\varphi(t+t_n,\theta_{-t_n}\omega)x_n \longrightarrow z$

$\varphi(t+t_n,\theta_{-t-t_n}\theta_t\omega)x_n \longrightarrow z$, where $t_n+t \longrightarrow +\infty$, $x_n \longrightarrow x$.

Thus $z\in J_x^+(\theta_t\omega)$**,**we have

$\varphi(t,\omega)J_x^+(\omega)\subseteq J_x^+(\theta_t\omega)$.

Then $J_x^+(\theta_t\omega) = \varphi(t,\omega)J_x^+(\omega)$.

Theorem 5.12: If $\mathbb{X}$. is locally compact.Then $\Gamma_M^+(\omega) \neq \emptyset$ whenever $J_M^+(\omega)$ is non-empty and compact.

Proof. If possible let $\Gamma_M^+(\omega)=\emptyset$ .Then we claim that $\gamma_M^+(\omega)$ is closed and disjoint with$J_M^+(\omega)$.That$\gamma_M^+(\omega)$ is closed follows from $\overline{\gamma_M^+(\omega)}=\gamma_M^+(\omega)\cup\Gamma_M^+(\omega)$=$\gamma_M^+(\omega)$ as $\Gamma_M^+(\omega)=\emptyset$,That $\Gamma_M^+(\omega)\cap J_M^+(\omega)=\emptyset$ follows from the fact that if $\Gamma_M^+(\omega)\cap J_M^+(\omega)\neq\emptyset$,then by invariance of $J_M^+(\omega)$, $\Gamma_M^+(\omega)\subseteq J_M^+(\omega)$.Since $J_M^+(\omega)$ is compact ,we will have $\Gamma_M^+(\omega)\neq\emptyset$ and compact(remember that any sequence$\{y_n\}$in a compact set Q has a convergent subsequence). This again contradicts the assumption $\Gamma_M^+(\omega)$=$\emptyset$ .Thus$\gamma_M^+(\omega)$ is closed and $\Gamma_M^+(\omega)\cap J_M^+(\omega)=\emptyset$ .Since $J_M^+(\omega)$ is non-empty and compact we have $d(\gamma_M^+(\omega),J_M^+(\omega))\geq 0$. thus there is a $r>0$ such that $S[J_M^+(\omega),r]$ is compact and disjoint with $\gamma_M^+(\omega)$. Now choose any of $y\in J_M^+(\omega)$. There is a sequence $\{x_n\}$ in $M(\theta_{-t_n}\omega)$ and a sequence $\{t_n\}$ in R^+ such that $x_n\to x\in\cap M(\theta_{-t_n}\omega)$ and $t_n\to+\infty$ and, $\varphi(t_n,\theta_{-t_n}\omega)x_n \longrightarrow y$ .We may assume that $x\notin\gamma_M^+(\omega)$, $\varphi(t_n,\theta_{-t_n}\omega)x_n\in S[J_M^+(\omega),r]$ for all n .Then the trajectory segments $\varphi(s_n,\theta_{-s_n}\omega)x_n$ with $0\leq s_n\leq t_n$, intersect $H(J_M^+(\omega),r)$ and therefor there is a sequence $\{\tau_n\}$, $0\leq\tau_n\leq t_n$,such that $\varphi(\tau_n,\theta_{-\tau_n}\omega)x_n\in H(J_x^+(\omega),r)$ .Since $H(J_x^+(\omega),r)$ is compacte we may assume that $\varphi(\tau_n,\theta_{-\tau_n}\omega)x_n \longrightarrow z\in H(J_x^+(\omega),r)$ .By taking subsequences we may assume that either $\tau_n\to t\in R^+$ or $\tau_n\to+\infty$. If $\tau_n\to t$, then by the continuity axiom $\varphi(\tau_n,\theta_{-\tau_n}\omega)x_n \longrightarrow \varphi(t,\omega)x=z$,i.e, $z\in\gamma_M^+(\omega)$ which contradicts $\gamma_M^+(\omega)\cap S[J_M^+(\omega),r]=\emptyset$ .If $\tau_n\to+\infty$,then $z\in J_M^+(\omega)$,but this contradicts $z\in H(J_x^+(\omega),r)$ as $J_M^+(\omega)\cap H(J_M^+(\omega),r)$ =$\emptyset$.

Theorem 5.13. Let $\mathbb{X}$. be locally compact. Then $J_M^+(\omega)$ is non –empty

And compact if and only if $D_M^+(w)$ is compact.

Proof. Let $J_M^+(\omega)$ be non –empty and compact .Then $\Gamma_M^+(\omega)$ is non empty and compact .But then $\overline{\gamma_M^+(\omega)}$ is compact($\overline{\gamma_M^+(\omega)}$ is closed with $\mathbb{X}$ be locally compact).Hence $D_M^+(w)=\gamma_M^+(\omega)\cup J_M^+(\omega)$

$=\overline{\gamma_M^+(\omega)}\cup J_M^+(\omega)$is compact .Now $D_M^+(w)$ is compact . Since $J_M^+(\omega)\subseteq D_M^+(w)$ .Then $J_M^+(\omega)$ is compact.

Theorem 5.14: If $\mathbb{X}$ is locally compact. Then $J_M^+(\omega)$ is connected.

Proof: Let $J_M^+(\omega)$ be compact . If $J_M^+(\omega)$ =$\emptyset$ there is nothing to prove. So let $J_M^+(\omega)\neq\emptyset$ .If $J_M^+(\omega)$ is disconnected ,then there are non-empty compact sets P,Q such that $J_M^+(\omega)=P\cup Q$ and $=P\cap Q=\emptyset$ .Since $\Gamma_M^+(\omega)$ is non- empty and compact ,hence connected ,we have $\Gamma_M^+(\omega)\subset P$ or $\Gamma_M^+(\omega)\subset Q$ .Let $\Gamma_M^+(\omega)\subset P$ .Since$\gamma_M^+(\omega)\cup P = \overline{\gamma_M^+(\omega)}\cup P$ as $\Gamma_M^+(\omega)\subset P$ and $\overline{\gamma_M^+(\omega)}$ is compact. Then $\gamma_M^+(\omega)\cup P$ is compact. Now let $Q\cap(\gamma_M^+(\omega)\cup P)\neq\emptyset$ $(Q\cap\gamma_M^+)\cup(Q\cap P)\neq\emptyset$,then $Q\cap\gamma_M^+\neq\emptyset$.But Q must be invariant .Thus will show that $\Gamma_M^+(\omega)\subset Q$,aconradiction .Then $\gamma_M^+(\omega)\cup P$ is compact and disjoint from Q , $D_M^+(w)$= $\gamma_M^+\cup J_M^+(\omega)=(\gamma_M^+(\omega)\cup P)\cup Q$ .since $\gamma_M^+(\omega)\cup P$ and Q are disjoint compact sets we have $D_M^+(w)$ is disconnected. Thus is a contradiction . Then $J_M^+(\omega)$ is connected.

6. Conclusion

This paper has been studied the concept of Prolongation Limit Random Sets in Random Dynamical Systems. we prove that the First Prolongation of a closed random set containing this set, the First Prolongation is closed and invariant, also it is connected whenever it is compact provided that the phase space of the RDS is locally compact. Then we study the Prolongational Limit Set for RDS and proved some essential properties of this set.

Where we prove that the Prolongational Limit Set for RDS is closed and invariant. Also the relation among the the First Prolongation, the Prolongational Limit Set and the positive trajectory of a random set is given and proved. Also if the phase space of RDS is locally compact then the following statements are true : if the Prolongational Limit Set for RDS is nonempty and compact, then the omega-limit set is non-empty; the Prolongational Limit Set for RDS is nonempty and compact if and only if the the First Prolongation is compact. Finally the Prolongational Limit Set for RDS is connected.

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بعض الخصائص مجموعات الغايه المستطيله العشوائيه في النظام الديناميكي العشوائي

سندس طالب محسن احسان جبار كاظم
قسم الرياضيات ، كلية علوم الحاسوب والرياضيات ، جامعة القادسية

المستخلص :

الهدف من هذا البحث هو دراسه مجموعه الغايه من نمط اوميكا مع مفاهيم جديده للمجموعات الغايه المستطيله العشوائيه في الانظمه الديناميكيه العشوائيه ،حيث تم برهان بعض الخواص الجديده مثل العلاقه بين اغلاق المسار والمسار ومجموعه الغايه من النمط اوميكا بالنسبه للنظم الديناميكيه العشوائيه ،وكذلك برهنا بان الاستطاله الأولى لمجموعه عشوائيه مغلقه تحتوي تلك المجموعه وان الاستطاله الأولى لمجموعه تكون مغلقه وغير متغايره بشرط ان تلك المجموعه غير متغايره ، وكذلك تكون محموعه مترابطه عندما تكون متراصه بشرط ان فضاء الطور للنظام الدينامكي العشوائي متراص محليا ومن ثم درسنا مجموعه الغايه المستطيله للنظم الديناامكيه العشوائيه وبرهنا بعض الخواص الاساسيه،وأخيرا برهنا العلاقه بين الاستطاله الأولى ومجموعه الغايه المستطيله والمسار الموجب لمجموعه عشوائيه.

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On Differential Sandwich Results For Analytic Functions

Waggas Galib Atshan **Sarah Abd Al-Hmeed Jawad**
Department of Mathematics
College of Computer Science and Information Technology,
University of Qadisiayh,Diwaniyah,
Iraq.
waggas.galib@qu.edu.iq **Sarahabdalhmeed94@gmail.com**

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Abstract: In this paper , we obtain some subordination and superordination results involving the integral operator F_c^δ.Also,we get Differential sandwich results for classes of univalent functions in the unit disk.

Keywords: Analytic function, univalent function, differential subordination , superordination.

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1-Introduction :

Let H=H(U) be the class of analytic functions in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. For n a positive number additionally $a \in \mathbb{C}$. Let H[a, n] be the subclass of H entailing of functions of the form:

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots \quad (a \in \mathbb{C}). \quad (1.1)$$

Too, let A be the subclass of H entailing of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k . \quad (1.2)$$

Let $f, g \in H$. The function f is said to be subordinate to g, or g is said to be subordinate to f, if there exists a Schwarz function w analytic in U with $w(0) = 0$ and $|w(z)| < 1$ ($z \in U$), to such an extent that $f(z) = g(w(z))$, In such a case we compose $f \prec g$ or $f(z) \prec g(z) (z \in U)$. If g is univalent function in U, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.

Let $p, h \in H$ and $\psi(r, \delta, t, z): \mathbb{C}^3 \times U \to \mathbb{C}$. If p and $\psi(p(z), zp'(z), z^2p''(z); z)$ are univalent functions in U and if p fulfills the second-order differential superordination.

$$h(z) \prec \psi(p(z), zp'(z), z^2p''(z); z), \quad (1.3)$$

then p is called a result of the differential superordination (1.3). (If f is subordinate to g, then g is superordinate to f) . An analytic function q is called a subordinant of (1.3) , if $q \prec p$ for very the functions p filling (1.3).

An univalent subordinant $\tilde{q}$ that fulfills $q \prec \tilde{q}$ for all the subordinants q of (1.3) is called the best subordinant. Miller and Mocanu [5] have gotten conditions on the functions h, q and ψ for which the accompanying ramifications holds:

$$h(z) \prec \psi(p(z), zp'(z), z^2p''(z); z) \to q(z) \prec p(z). \quad (1.4)$$

For $f \in A$,Al-shaqsi [2] defined the following integral operator:

$$F_c^{\delta} f(z) = (1+c)^{\delta} \emptyset_{\delta}(c; z) * f(z) = \frac{(1-c)^{\delta}}{\Gamma(\delta)} \int_0^1 t^{c-1} (\log \tfrac{1}{t})^{\delta-1} f(tz)\, dt, (c > 0, \delta > 1 \text{ and } z \in U). \quad (1.5)$$

We also note that the operator $F_c^{\delta} f(z)$ characterized by (1.5) can be communicated by the arrangement development as pursues:

$$F_c^{\delta} f(z) = z + \sum_{k=2}^{\infty} (\tfrac{1+c}{k+c})^{\delta} a_k z^k. \quad (1.6)$$

In addition, from (1.6), it pursues that

$$z(F_c^{\delta+1} f(z))' = (c+1) F_c^{\delta} f(z) - c F_c^{\delta+1} f(z). \quad (1.7)$$

Ali et al.[1] gotten adequate conditions for certain standardized scientific capacities to satisfy

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z),$$

where q_1 and q_2 are given univalent functions in U with $q_1(0) = q_2(0) = 1$. Additionally, Tuneski [9] acquired adequate conditions for starlikeness of f in relations of the amount $\frac{f''(z) f(z)}{(f'(z))^2}$ .Recently, Shanmugam et al.[7,8], Goyal et al .[4] also gotten sandwich consequences for certain classes of analytic functions.

The principle question of the present paper is to discover adequate conditions for certain standardized systematic capacities f to fulfill:

$$q_1(z) \prec (\frac{F_c^{\delta+1} f(z)}{z})^{\lambda} \prec q_2(z),$$

and

$$q_1(z) \prec (\frac{t\, F_c^{\delta+1} f(z) + (1-t)\, F_c^{\delta} f(z)}{z})^{\lambda} \prec q_2(z),$$

wherever q_1 and q_2 are known univalent functions in U with $q_1(0)= q_2(0)= 1$.

2-Preliminaries :

With the end goal to demonstrate our subordination and superordination result , we require the accompanying definition and lemmas.

Definition 2.1 [5] : Denote by Q the set of all functions f that are analytic and injective on $\bar{U} \setminus E(f)$, where

$$E(f) = \{\xi \in \partial U : \lim_{z \to \xi} f(z) = \infty\} \quad (2.1)$$

and are such that $f'(\xi) \neq 0$ for $\xi \in \partial U \setminus E(f)$.

Lemma 2.1 [5] : Let q be univalent in the unit disk U and let θ and $\emptyset$ be analytic in a domain D containing q(U) with $\emptyset(w) \neq 0$ when $w \in q(U)$. Set $Q(z) = zq'(z)\emptyset(q(z))$ and $h(z) = \theta(q(z)) + Q(z)$.

Suppose that

(i) Q(z) is starlike univalent in U,

(ii) $\mathrm{Re}\{\frac{zh'(z)}{Q(z)}\} > 0$ for $z \in U$.

If p is analytic in U with $p(0) = q(0), p(U) \subset D$ and

$$\theta(p(z)) + zp'(z)\emptyset(p(z)) \prec \theta(q(z)) + zq'(z)\emptyset(q(z)) \quad (2.2)$$

then $p \prec q$ and q is the best dominant of (2.2).

Lemma 2.2 [6]: Let q be convex univalent in function in U and let $\alpha \in \mathbb{C}, \beta \in \mathbb{C} / \{0\}$ with

$$\mathrm{Re}\,(1 + \frac{zq''(z)}{q'(z)}) > \max(0, -\mathrm{Re}\left(\frac{\alpha}{\beta}\right)).$$

If p is analytic in U, and

$$\alpha p(z) + \beta z p'(z) \prec \alpha q(z) + \beta z q'(z), \quad (2.3)$$

then $p \prec q$ and q is the best dominant of (2.3).

Lemma 2.3 [6]: Let q be convex univalent in U and let $\beta \in \mathbb{C}$, further assume that $\mathrm{Re}\ (\beta) > 0$. If $P \in H[q(0)] \cap Q$ and $p(z) + \beta z p'(z)$ is univalent in U, then

$q(z) + \beta z q'(z) \prec p(z) + \beta z p'(z)$,
(2.4)

which implies that $q \prec p$ and q is the best subordinant of (2.4).

Lemma 2.4 [3]: Let q be convex univalent in the unit disk U and let θ and $\emptyset$ be analytic in domain D containing q (U) . Suppose that

(i) $\mathrm{Re}\left\{\frac{\theta'(q(z))}{\emptyset(q(z))}\right\} > o$ for $z \in U$,

(ii) $Q(z) = zq'(z)\emptyset(q(z))$ is starlike univalent in U.

If $p \in H\,[q(0),1] \cap Q$, with $p(\cup) \subset D$, $\theta(p(z)) + zp'(z)\,\emptyset p(z)$ is univalent in U and

$\theta(q(z)) + zq'(z)\emptyset(q(z)) \prec \theta(p(z)) + zp'(z)\emptyset(p(z))$, (2.5)

then $q \prec p$ and q is the best subordination of (2.5).

3- Subordination Consequences :

Theorem 3.1 : Let q be convex univalent function in U with $q(0) = 1$, $0 \neq \Psi \in \mathbb{C}, \lambda > 0$ also, assume that q satisfies:

$$\mathrm{Re}\left(1 + \frac{zq''(z)}{q'(z)}\right) > \max\left(0, -\mathrm{Re}\left(\frac{\lambda}{\Psi}\right)\right). \qquad (3.1)$$

If $f \in A$ satisfies the subordination

$$\left(1 - \Psi(c+1)\right)\left(\frac{F_c^{\delta+1}f(z)}{z}\right)^{\lambda} + \Psi(c+1)\left(\frac{F_c^{\delta+1}f(z)}{z}\right)^{\lambda}\left(\frac{F_c^{\delta}f(z)}{F_c^{\delta+1}f(z)}\right) \prec q(z) + \frac{\Psi}{\lambda}zq'(z), \qquad (3.2)$$

then

$$\left(\frac{F_c^{\delta+1}f(z)}{z}\right)^{\lambda} \prec q(z), \qquad (3.3)$$

and q is the best dominant of (3.2).

Proof : Characterize the capacity p by

$$p(z) = \left(\frac{F_c^{\delta+1}f(z)}{z}\right)^{\lambda}. \qquad (3.4)$$

Differentiating (3.4) with admiration to z logarithmically, we get

$$\frac{zp'(z)}{p(z)} = \lambda\left(\frac{z(F_C^{\delta+1}f(z))'}{F_C^{\delta+1}f(z)} - 1\right). \qquad (3.5)$$

Presently , in perspective of (1.7), we get the accompanying subordination

$$\frac{zp'(z)}{p(z)} = \lambda\left(c\left(\frac{F_C^{\delta}f(z)}{F_C^{\delta+1}f(z)} - 1\right) + \left(\frac{F_C^{\delta}f(z)}{F_C^{\delta+1}f(z)} - 1\right)\right),$$

therefore $\frac{zp'(z)}{\lambda} = \left(\frac{F_C^{\delta+1}f(z)}{z}\right)^{\lambda}\left(c\left(\frac{F_C^{\delta}f(z)}{F_C^{\delta+1}f(z)} - 1\right) + \left(\frac{F_C^{\delta}f(z)}{F_C^{\delta+1}f(z)} - 1\right)\right)$.

The subordination (3.2) from the speculation moves toward becoming

$$p(z) + \frac{\Psi}{\lambda}zp'(z) \prec q(z) + \frac{\Psi}{\lambda}zq'(z).$$

An request of Lemma 2.2 with $\beta = \frac{\Psi}{\lambda}$ and $\alpha = 1$, we get (3,3)

Putting $q(z) = \left(\frac{1+z}{1-z}\right)$ in Theorem 3.1 ,we get the following

Corollary 3.1 : Let $0 \neq \Psi \in \mathbb{C}, \lambda > 0$ also $\mathrm{Re}\left\{1 + \frac{2z}{1-z}\right\} > \max\{0, -\mathrm{Re}(\frac{\lambda}{\Psi})\}$.

If $f \in A$ satisfies the subordination

$$\left(1 - \Psi(c+1)\right)\left(\frac{F_c^{\delta+1}f(z)}{z}\right)^{\lambda} + \Psi(c+1)\left(\frac{F_c^{\delta+1}f(z)}{z}\right)^{\lambda}\left(\frac{F_c^{\delta}f(z)}{F_c^{\delta+1}f(z)}\right) \prec \left(\frac{1 - z^2 + 2\frac{\Psi}{\lambda}z}{(1-z)^2}\right),$$

then

$$\left(\frac{F_c^{\delta+1}f(z)}{z}\right)^{\lambda} \prec \left(\frac{1+z}{1-z}\right),$$

and $q(z) = \left(\frac{1+z}{1-z}\right)$ is the best dominant.

Theorem 3.2 : Let q be convex univalent function in U with $q(0) = 1, q(z) \neq 0 (z \in U)$ furthermore, accept that q fulfills

$$\mathrm{Re}\left(1 - \frac{\lambda}{\Psi} + \frac{zq''(z)}{q'(z)}\right) > 0, \qquad (3.6)$$

where $\Psi \in \mathbb{C}/\{0\}, \lambda > 0$ and $z \in U$.

Supposing that $-\Psi zq'(z)$ is starlike univalent function in U, if $f \in A$ fulfills:

$$\emptyset(\lambda, \delta, c, \Psi; z) \prec \lambda q(z) - \Psi zq'(z), \qquad (3.7)$$

where $\emptyset(\lambda, \delta, c, \Psi; z) =$

$$\lambda\left(\frac{tF_c^{\delta+1}f(z) + (1-t)F_c^{\delta}f(z)}{z}\right)^{\lambda} - \lambda\Psi\left(\frac{tF_c^{\delta+1}f(z) + (1-t)F_c^{\delta}f(z)}{z}\right)^{\lambda}\left(\frac{tF_c^{\delta}f(z) + (1-t)F_c^{\delta-1}f(z)}{tF_c^{\delta+1}f(z) + (1-t)F_c^{\delta}f(z)} - 1\right), \qquad (3.8)$$

then

$$\left(\frac{tF_c^{\delta+1}f(z) + (1-t)F_c^{\delta}f(z)}{z}\right)^{\lambda} \prec q(z), \qquad (3.9)$$

and q(z)is the best dominant of (3.7).

Proof: Express the function p by

$$p(z) = \left(\frac{tF_c^{\delta+1}f(z) + (1-t)F_c^{\delta}f(z)}{z}\right)^{\lambda}, \qquad (3.10)$$

by setting :

$\theta(w) = \lambda w$ and $\emptyset(w) = -\Psi, w \neq 0$.

We see that $\theta(w)$ is analytic in $\mathbb{C}$, $\emptyset(w)$ is analytic in $\mathbb{C}/\{0\}$ and so on $\emptyset(w) \neq 0, w \in \mathbb{C}^*$.
Too, we get

$$Q(z) = zq'(z)\emptyset q(z) = -\Psi zq'(z),$$

and

$$h(z) = \theta q(z) + Q(z) = \lambda q(z) - \Psi zq'(z).$$

It is clear that Q(z)is starlike univalent in U ,

$$\operatorname{Re}\left\{\frac{zh'(z)}{Q(z)}\right\} = \operatorname{Re}\left\{1 - \frac{\lambda}{\Psi} + \frac{zq''(z)}{q'(z)}\right\} > 0.$$

By a straightforword computation , we obtain$\lambda p(z) - \Psi zp'(z) = \emptyset(\lambda, \delta, c, \Psi; z)$, (3.11)
where $\emptyset(\lambda, \delta, c, \Psi; z)$ is given by (3.8).
From (3.7) and (3.11), we have
$\lambda p(z) - \Psi z\, p'(z) \prec \lambda q(z) - \Psi z\, q'(z)$.
(3.12)
So , by Lemma 2.1, we become $p(z) \prec q(z)$. By using (3.10) , we get the result .
Putting $q(z) = \frac{1+Az}{1+Bz}$ ($-1 \leq B < A \leq 1$) in Theorem 3.2 , we obtain the next corollary :
Corollary3.2 : Let $-1 \leq B < A \leq 1$ while

$$\operatorname{Re}\left\{1 - \frac{\lambda}{\Psi} + \frac{z2B}{(1+Bz)}\right\} > 0,$$

where $\Psi \in \mathbb{C}/\{0\}$ and $z \in U$, if $f \in A$ contents
$\emptyset(\lambda, \delta, c, \Psi; z) \prec \left(\lambda\left(\frac{1+Az}{1+Bz}\right) - \Psi z\frac{A-B}{(1+Bz)^2}\right)$,
and $\emptyset(\lambda, \delta, c, \Psi; z)$ is given by (3.8),

$$\left(\frac{tF_c^{\delta+1}f(z) + (1-t)F_c^{\delta}f(z)}{z}\right)^{\lambda} \prec \frac{1+Az}{1+Bz}$$

while $q(z) = \frac{1+Az}{1+Bz}$ is the best dominant.

4-Superordination Consequences :

Theorem 4.1: Let q be convex univalent function in U with $q(0) = 1, \lambda > 0$ and $\operatorname{Re}\{\Psi\} > 0$. Let $f \in A$ satisfies

$$\left(\frac{F_c^{\delta+1}f(z)}{z}\right)^{\lambda} \in H[q(0), 1] \cap Q,$$

and

$$\left(1 - \Psi(c+1)\right)\left(\frac{F_c^{\delta+1}f(z)}{z}\right)^{\lambda} + \Psi(c+1)\left(\frac{F_c^{\delta+1}f(z)}{z}\right)^{\lambda}\left(\frac{F_c^{\delta}f(z)}{F_c^{\delta+1}f(z)}\right),$$

exist univalent in U . If

$$q(z) + \frac{\Psi}{\lambda}zq'(z) \prec \left(1 - \Psi(c+1)\right)\left(\frac{F_c^{\delta+1}f(z)}{z}\right)^{\lambda} + \Psi(c+1)\left(\frac{F_c^{\delta+1}f(z)}{z}\right)^{\lambda}\left(\frac{F_c^{\delta}f(z)}{F_c^{\delta+1}f(z)}\right), \qquad (4.1)$$

then

$$q(z) \prec \left(\frac{F_c^{\delta+1}f(z)}{z}\right)^{\lambda}, \qquad (4.2)$$

and q is the best subordinant of (4.1).
Proof: Express the function p by

$$p(z) = \left(\frac{F_c^{\delta+1}f(z)}{z}\right)^{\lambda}. \qquad (4.3)$$

Differentiating (4.3) with respect to z logarithmically , we get

$$\frac{zp'(z)}{p(z)} = \lambda\left(\frac{z(F_c^{\delta+1}f(z))'}{F_c^{\delta+1}f(z)} - 1\right).$$

(4.4)
After some computations and using (1.7) , from (4.4), we obtain

$$\left(1 - \Psi(c+1)\right)\left(\frac{F_c^{\delta+1}f(z)}{z}\right)^{\lambda} + \Psi(c+1)\left(\frac{F_c^{\delta+1}f(z)}{z}\right)^{\lambda}\left(\frac{F_c^{\delta}f(z)}{F_c^{\delta+1}f(z)}\right)$$

$= p(z) + \frac{\Psi}{\lambda}zp'(z)$,
and now , by using Lemma 2.3, we get the desired result .
Putting $q(z) = \frac{1+z}{1-z}$ in Theorem 4.1 , we acquire the accompanying corollary :
Corollary 4.1: Let $\lambda > 0$ and $\operatorname{Re}\{\Psi\} > 0$. If $f \in A$ satisfies:

$$\left(\frac{F_c^{\delta+1}f(z)}{z}\right)^{\lambda} \in H[q(0), 1] \cap Q,$$

and

$$\left(1 - \Psi(c+1)\right)\left(\frac{F_c^{\delta+1}f(z)}{z}\right)^{\lambda} + \Psi(c+1)\left(\frac{F_c^{\delta+1}f(z)}{z}\right)^{\lambda}\left(\frac{F_c^{\delta}f(z)}{F_c^{\delta+1}f(z)}\right),$$

be univalent in U . If

$$\left(\frac{1-z^2+2\frac{\Psi}{\lambda}z}{(1-z)^2}\right) \prec \left(1 - \Psi(c+1)\right)\left(\frac{F_c^{\delta+1}f(z)}{z}\right)^{\lambda} + \Psi(c+1)\left(\frac{F_c^{\delta+1}f(z)}{z}\right)^{\lambda}\left(\frac{F_c^{\delta}f(z)}{F_c^{\delta+1}f(z)}\right),$$

then

$$\left(\frac{1+z}{1-z}\right) \prec \left(\frac{F_c^{\delta+1}f(z)}{z}\right)^{\lambda},$$

and $q(z) = \frac{1+z}{1-z}$ is the best subordinant.
Theorem 4.2: Let q be convex univalent function in U with $q(0) = 1$, also, accept that q fulfills

$$\operatorname{Re}\left\{\frac{-\lambda q'(z)}{\Psi}\right\} > 0, \qquad (4.5)$$

where $\eta \in \mathbb{C}/\{0\}$ and $z \in U$.
Assume that $-\Psi zq'(z)$ is starlike univalent function in U , let $f \in A$ satisfies

$$\left(\frac{tF_c^{\delta+1}f(z) + (1-t)F_c^{\delta}f(z)}{z}\right) \in H[q(0), 1] \cap Q,$$

and $\emptyset(\lambda, \delta, c, \Psi; z)$ is univalent function in U , where $\emptyset(\lambda, \delta, c, \Psi; z)$ is given by (3.8). If

$$\lambda q(z) - \Psi zq'(z) \prec \emptyset(\lambda, \delta, c, \Psi; z), \qquad (4.6)$$

then

$$q(z) \prec \left(\frac{tF_c^{\delta+1}f(z) + (1-t)F_c^{\delta}f(z)}{z}\right)^{\lambda}. \qquad (4.7)$$

and q is the best subordinant of (4.6).

Proof: Express the function p by

$$p(z) = \left(\frac{tF_c^{\delta+1}f(z)+(1-t)F_c^{\delta}f(z)}{z}\right)^{\lambda}, \quad (4.8)$$

by setting

$\theta(w) = \lambda w$ and $\emptyset(w) = -\Psi$, $w \neq 0$,

we see that $\theta(w)$ is analytic in $\mathbb{C}, \emptyset(w)$ is analytic in $\mathbb{C}^*$ and that $\emptyset(w) \neq 0$, $w \in \mathbb{C}^*$. Too , we get

$Q(z) = zq'(z)\emptyset q(z) = -\Psi zq'(z)$.

It is clear that $Q(z)$is starlike univalent function in U ,

$$\operatorname{Re}\left\{\frac{\theta'(q(z))}{\emptyset(q(z))}\right\} = \operatorname{Re}\left\{\frac{-\lambda q'(z)}{\Psi}\right\} > 0 .$$

By a straightforword computation ,we obtain

$$\emptyset(\lambda, \delta, c, \Psi; z) = \lambda\, p(z) - \Psi zp'(z), \quad (4.9)$$

where $\emptyset(\lambda, \delta, c, \Psi; z)$is given by (3.8).

From (4.6) and (4.9) , we have

$$\lambda q(z) - \Psi zq'(z) \prec \lambda p(z) - \Psi p'(z) \quad . \quad (4.10)$$

So , by Lemma 2.4, we become $q(z) \prec p(z)$. By using (4.8), we get the outcome.

5-Sandwich Consequences :

Concluding the consequences of differential subordination and superordination we arrive at the next "sandwich consequence".

Theorem 5.1 : Let q_1 be convex univalent function in U with $q_1(0)$=1,Re $\{\Psi\}> 0$ and let q_2 be univalent in U ,$q_2(0)$=1 and fulfills (3,1), let $f \in A$ satisfies :

$$\left(\frac{F_c^{\delta+1}f(z)}{z}\right)^{\lambda} \in H\,[1,1] \cap Q ,$$

and

$$(1-\Psi(c+1))\left(\frac{F_c^{\delta+1}f(z)}{z}\right)^{\lambda} + \Psi(c+1)\left(\frac{F_c^{\delta+1}f(z)}{z}\right)^{\lambda}\left(\frac{F_c^{\delta}f(z)}{F_c^{\delta+1}f(z)}\right),$$

be univalent in U . If

$$q_1(z) + \frac{\Psi}{\lambda} zq'_1(z) \prec (1-\Psi(c+1))\left(\frac{F_c^{\delta+1}f(z)}{z}\right)^{\lambda} + \Psi(c+1)\left(\frac{F_c^{\delta+1}f(z)}{z}\right)^{\lambda}\left(\frac{F_c^{\delta}f(z)}{F_c^{\delta+1}f(z)}\right) \prec q_2(z) + \frac{\Psi}{\lambda} z\, q'_2(z),$$ then

$$q_1(z) \prec \left(\frac{F_c^{\delta+1}f(z)}{z}\right)^{\lambda} \prec q_2(z),$$

and $q_1$and q_2 are correspondingly , the best subordinant and the best dominant .

Theorem 5.2: Let q_1 be convex univalent function in U with $q_1(0)$=1, and fulfills (4.5), let q_2 be univalent function in U $q_2(0)$=1, satisfies (3.6), let $f \in A$ satisfies

$$\left(\frac{tF_c^{\delta+1}f(z)+(1-t)F_c^{\delta}f(z)}{z}\right)^{\lambda} \in H\,[1,1] \cap Q.$$

And $\emptyset(\lambda, \delta, c, \Psi; z)$ is univalent in U . Where $\emptyset(\lambda, \delta, c, \Psi; z)$is given by (3.8) . If $\lambda q_1(z) - \Psi zq'_1(z) \prec \emptyset(\lambda, \delta, c, \Psi; z) \prec \lambda q_2(z) - \Psi zq'_2(z)$ then

$$q_1(z) \prec \left(\frac{tF_c^{\delta+1}f(z)+(1-t)F_c^{\delta}f(z)}{z}\right)^{\lambda} \prec q_2(z).$$

In addition $q_1$and q_2 are correspondingly , the best subordinant and the best dominant .

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نتائج الساندوج التفاضلية للدوال التحليلية

وقاص غالب عطشان سارة عبدالحميد جواد

قسم الرياضيات ، كلية علوم الحاسوب وتكنولوجيا المعلومات ، جامعة القادسية - العراق

الملخص:

في هذا البحث، نحصل على بعض نتائج التبعية والتبعية العليا باستخدام المشغل التكاملي F_c^{δ}.ايضا، وحصلنا على نتائج الساندوج التفاضلية لصنف من الدوال احادية التكافؤ في قرص الوحدة .

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Properties of The Space GFB(V, U)

Jehad R. Kider **Manar N. Gheeab**

Department of Mathematics and Computer Applications , School of Applied Sciences , University of Technology, Baghdad, Iraq.
jehadkider@gmail.com **manar.naji33@gmail.com**

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Abstract:

Our goal in the present paper is to recall the concept of general fuzzy normed space and its basic properties to define the general fuzzy bounded operator as a background to introduce the notion general fuzzy norm of a general fuzzy bounded linear operator. After that we proved any operator from a general fuzzy normed space into a general complete general fuzzy normed space has an extension. Also we prove that a general fuzzy bounded operator on a general fuzzy normed space is equivalent to a general fuzzy continuous. Finally different types of fuzzy approaches of operators is introduced in order to prove that the general fuzzy normed space GFB(V,U) is general complete when U is general complete.

KeyWords: The general fuzzy normed space GFB(V,U), General Fuzzy continuous operator, General Fuzzy bounded operator, General Fuzzy normed space.

Mathematics Subject Classification: 30C45, 30C50.

1.Introduction. Zadeh in 1965[1] was the first one who introduced the theory of fuzzy set. When Katsaras in 1984 [2] studying the notion of fuzzy topological vector spaces he was the first researcher who studied the notion of fuzzy norm on a linear vector space. A fuzzy metric space was also studied by Kaleva and Seikkala in 1984 [3]. The fuzzy norm on a vector space have been studied by Felbin in 1992 [4] where Kaleva and Seikkala introduce this type of fuzzy metric. Another type of fuzzy metric spaces was given by Kramosil and Michalek in [5].

Certain type of fuzzy norm on a linear space was given by Cheng and Mordeson in 1994 [6] where Kramosil and Michalek present this type of fuzzy metric. A finite dimensional fuzzy normed space was studied by Bag and Samanta in 2003 [7]. Saadati and Vaezpour in 2005 [8] where studied complete fuzzy normed spaces and proved some results. Also Bag and Samanta in 2005 [9] were studied fuzzy bounded linear operators on a fuzzy normed space.

Again Bag and Samanta in 2006 and 2007 [10], [11] used the fuzzy normed spaces that introduced by Cheng and Mordeson to prove the fixed point theorems. The fuzzy topological structure that introduced by Cheng and Mordeson of the fuzzy normed space was studied by Sadeqi and Kia in 2009 [12]. Kider introduced a new fuzzy normed space in 2011 [13]. Also he proved this new fuzzy normed space has a completion in [14]. The properties of fuzzy continuous mapping which was defined on a fuzzy normed spaces by Cheng and Mordeson was studied by Nadaban in 2015 [15]. The concepts of fuzzy norm is developed by a large number of researches with different authors have been published for reference one may see [18, 19, 20, 19,22, 23, 24, 25].

In this paper first the definition of general fuzzy normrd space is recalled and also its basic properties in order to define the general fuzzy norm of a general fuzzy bounded linear operator from a general fuzzy normed space V into another general fuzzy normed space U.

2.Basic Properties of General Fuzzy Norm

Definition 2.1:[10]

A binary operation $\odot : [0,1]\times[0,1]\to[0,1]$ satisfying

(1) $a\odot b = b\odot a$

(2) $b\odot 1 = b$

(3) $a\odot[b\odot t] = [a\odot b]\odot t$

(4) if $b\leq a$ and $t\leq s$ then $b\odot t \leq a \odot s$.

for all a ,b ,s ,t ∈[0,1] is called a continuous **triangular norm [or t-norm].**

Example 2.2:[11]

(1)Let $m\otimes n=m.n$ for all n, m ∈ [0,1] where m .n is multiplication in [0,1]. Then ⊗ is continuous t-norm.

(2)Let $m\otimes n= m \wedge n$ for all n, m∈[0,1] then ⊗ is continuous t-norm.

Remark 2.3:[24]

(1)for all n > m there is k with $n\otimes k \geq m$ where n, m, k ∈[0,1].

(2)there is q with $q\otimes q \geq n$ where n, q ∈[0,1].

First we need the following definition

Definition 2.4:[26]

Let ℝ be a vector a space of real numbers over filed ℝ and ⊙,⊗ be continuous t-norm. A fuzzy set $L_{\mathbb{R}}$:ℝ×[0,∞) is called **fuzzy absolute value on** ℝ if it satisfies

(A1) $0\leq L_{\mathbb{R}}(n,a) < 1$ for all a>0.

(A2) $L_{\mathbb{R}}(n,a) = 1 \Leftrightarrow n= 0$ for all a>0.

(A3) $L_{\mathbb{R}}(n+m, a+b) \geq L_{\mathbb{R}}(n,a)\odot L_{\mathbb{R}}(m,b)$.

(A4) $L_{\mathbb{R}}(nm, ab) \geq L_{\mathbb{R}}(n,a) \otimes L_{\mathbb{R}}(m,b)$.

(A5) $L_{\mathbb{R}}(n,.):[0,\infty)\to[0,1]$ is continuous function of t.

(A6) $\lim_{a\to\infty} L_{\mathbb{R}}(n,a) = 1$.

For all m , n ∈ℝ and for all a, b ∈[0,1]. Then (ℝ, $L_{\mathbb{R}}$, ⊙,⊗) is called **a fuzzy absolute value space.**

Example 2.5:[26]

Define $L_{\mathbb{R}}(a, t) = \frac{t}{t+|a|}$ for all a ∈ ℝ then $L_{\mathbb{R}}$ is a fuzzy absolute value on ℝ where t ⊙ s = t · s and t ⊗ s = t · s for all t, s ∈ **[0, 1]** where t · s is the ordinary multiplication of t and s.

Example 2.6 :[28]
Define L_d: ℝ× [0 ,∞) →[0 ,1] by

$$L_d(u, a)=\begin{cases}0 & \text{if } a \le |u| \\ 1 & \text{if } a > |u|\end{cases}$$

then L_d is a fuzzy absolute value on ℝ . L_d is called the **discrete fuzzy absolute value on ℝ.**

Definition 2.7:[28]
Let V be a vector space over the filed ℝ and ⊙,⊗ be a continuous t-norms. A fuzzy set G_V:V×[0,∞) is called a **general fuzzy norm on V** if it satisfies the following conditions for all u, v∈V and for all α ∈ ℝ, s, t ∈ [0,∞):

(G1) 0≤ G_V(u, s)<1 for all s>0.

(G2) G_V(u, s)=1 ⇔ u= 0 for all s>0.

(G3) $G_V(\alpha$u, st) ≥ $L_{\mathbb{R}}(\alpha$, s) ⊗ G_V(u, t) for all $\alpha \neq 0$ ∈ℝ.

(G4) G_V(u+ v, s +t) ≥ G_V(u, s) ⊙ G_V(v, t).

(G5) G_V(u, .) : [0,∞) → [0,1] is continuous function of t.

(G6) $\lim_{t\to\infty} G_V$(u ,t) = 1

Then (V, G_V,⊙,⊗) is called **a general fuzzy normed space.**

Example 2.8:[28]
Define $G_{|.|}(u,a)=\frac{a}{a+|u|}$ for all u∈ℝ .Then (ℝ ,$G_{|.|}$,⊙,⊗) is a general fuzzy normed space with s⊙ t = s.t and t⊗ s = t.s for all s ,t ∈ [0,1]. Then $G_{|.|}$ is called the **standard general fuzzy norm induced by the absolute value** |.|.

Example 2.9 :[28]
If (V, ||.||) is normed space and $G_{\|.\|}$: V × [0,∞) → [0,1] is defined by :
$G_{\|.\|}(u,a) = \frac{a}{a+\|u\|}$ then (V,$G_{\|.\|}$,⊙ ,⊗) is general fuzzy normed space where s⊙t = s ∧ t and t⊗ s= t. s for all t ,s ∈ [0,1]. Then $G_{\|.\|}$ is called the **standard general fuzzy norm induced by the norm ||.||**.

Example 2.10 :[28]
Let (V,|| .||) be vector space over ℝ, define

$$G_d(u,t)=\begin{cases}1 & if \ \ \|u\| < t \\ 0 & if \ \ \|u\| \ge t\end{cases}$$

Where u ⊙ v = u ⊗ v = u∧ v for all u , v∈ [0,1] and u⊗ v= u . v for all u, v∈ [0,1]. Then G_d is called the **discrete general fuzzy norm on V.**

Proposition 2.11:[28]
Suppose that (V, ||.||) is a normed space define $G_V(u, s) = \frac{s}{s+\|u\|}$ for all u ∈ V and 0< s. Then (V, G_V,⊙,⊗) is general fuzzy normed space where a ⊙ b = a ⊗ b = a . b for all a, b ∈ [0,1].

Lemma 2.12:[28]
G_V(u, .) is a nondecreasing function of t in the general fuzzy normed space (V, G_V,⊙,⊗) for all u ∈V this means when 0< t <s implies G_V(u, t) < G_V(u, s) .

Remark 2.13:[28]
Assume the general fuzzy normed space (V,G_V,⊙,⊗). Then for any u ∈ V, s>0 ,0<n<1 .

1-If G_V(u, s)≥(1−n) we can find 0< t< s with G_V(u ,t)> (1−n).

2- If G_V(u, s)≥(1−n) we can find 0< s< t with G_V(u ,t)> (1−n).

Definition 2.14:[28]
If (V,G_V,⊙,⊗) is a general fuzzy normed space. Then GFB(u ,n ,s) = {
m ∈V: G_V(u−m, s)>(1−n) } is called a **general fuzzy open ball** with center u∈V radius n and s>0 and GFB[u, n ,s] = {m ∈V : G_V(u−m ,s) ≥ (1−n)} is called a **general fuzzy closed ball** with center u∈V radius n and s>0**.**

Definition 2.15:[28]
Suppose that (V, G_V ,⊙,⊗) is a general fuzzy normed space and M⊆ V. Then M is called a **general fuzzy open** if for any u ∈ M we can find 0< n<1, s>0 with FB(u, n, t) ⊆ M . A subset W ⊆ V is called a **general fuzzy closed** set if W^C is a general fuzzy open.

Definition 2.16:[28]
Suppose that (V,G_V,⊙,⊗) is a general fuzzy normed space. A sequence (u_n) in V is said to be **general fuzzy approaches** to u if every 0 < ε < 1 and 0< s there is N ∈ ℕ such that G_V(u_n – u , s) > (1 $-\varepsilon$) for every n≥ N. If (u_n) is general fuzzy approaches to the fuzzy limit u we write $\lim_{n\to\infty} u_n$= u or u_n → u. Also $\lim_{n\to\infty} G_V(u_n$ − u, s) = 1 if and only if (u_n) is general fuzzy approaches to u.

Definition 2.17:[28]
Suppose that (V, G_V,⊙,⊗) is a general fuzzy normed space. A sequence (v_n) in V is called a **general Cauchy sequence** if for each $0 < r < 1$, $t > 0$ there exists a positive number N ∈ ℕ such that $G_V[v_m - v_n, t] > (1 - r)$ for all m , n ≥ N.

Definition 2.18:[28]
Let (V,G_V,⊙,⊗) be a general fuzzy normed space and let M⊆ V. Then the **general closure** of M is denote by $\overline{M^G}$ or GCL(M) is smallest general fuzzy closed set contains M .

Definition 2.19:[28]

Suppose that $(V, G_V, \odot, \otimes)$ is a general fuzzy normed space and let $M \subseteq V$. Then M is said to be **general fuzzy dense** in V if $\overline{M^G} = V$

Lemma 2.20:[28]

If $(V, G_V, \odot, \otimes)$ is a general fuzzy normed space and let $M \subseteq V$. Then $m \in \overline{M^G}$ if and only if we can find (m_n) in M such that $m_n \to m$.

Definition 2.21:[28]

Suppose that $(V, G_V, \odot, \otimes)$ is a general fuzzy normed space A sequence (v_n) in V is called a **general Cauchy sequence** if for each $0 < r < 1$, $t > 0$ we can find $N \in \mathbb{N}$ with $G_V[v_j - v_k, t] > (1 - r)$ for all j, k≥N.

Definition 2.22:[28]

Let $(V, G_V, \odot, \otimes)$ be a general fuzzy normed space. A sequence (u_n) is said to be **general fuzzy bounded** if there exists 0< q <1 such that $G_V(u_n, s) > (1-q)$ for all s>0 and $n \in \mathbb{N}$.

Definition 2.23 :[28]

Let $(V, G_V, \odot, \otimes)$ and $(U, G_U, \odot, \otimes)$ be two general fuzzy normed spaces the operator $S: V \to U$ is called **general fuzzy continuous at** $v_0 \in V$ for every s>0 and every $0 < \gamma < 1$ there exist t and there exists δ such that for all $v \in V$ with $G_V[v - v_0, s] > (1 - \delta)$ we have $G_U[S(v) - S(v_0), t] > (1 - \gamma)$ if S is fuzzy continuous at each point $v \in V$ then S is said to be **general fuzzy continuous .**

Theorem 2.24:[28]

Suppose that $(V, G_V, \odot . \otimes)$ and $(U, G_U, \odot, \otimes)$ are general fuzzy normed spaces. Then $S: V \to U$ is a general fuzzy continuous at $u \in V$ if and only if $u_n \to u$ in V implies $S(u_n) \to S(u)$ in U.

Definition 2.25:[28]

Suppose that $(V, G_V, \odot, \otimes)$and $(U, G_U, \odot, \otimes)$ are general fuzzy normed spaces. Let $T: V \to U$ then T is called **uniformly general fuzzy continuous** if for $t > 0$ and for every $0 < \alpha < 1$ there is β and there is $s > 0$ with $G_U[T(v) - T(u), t] > (1 - \alpha)$ whenever $G_V[v - u, s] > (1 - \beta)$ for all $v, u \in V$.

Theorem 2.26:[28]

Suppose that $(V, G_V, \odot, \otimes)$ and $(U, G_U, \odot, \otimes)$ are two general fuzzy normed spaces. Let $T: V \to U$ be uniformly general fuzzy continuous operator. If (u_n) is a general Cauchy sequence in V then $(T(u_n))$ is a general Cauchy sequence in U.

Definition 2.27:[28]

Suppose that $(V, G_V, \odot, \otimes)$ is a general fuzzy normed space. Then V is called **a general complete** if every general Cauchy sequence in V is general fuzzy approaches to a vector in V.

3.General Fuzzy Bounded Linear Operator

Definition 3.1 :

Suppose that $(V, G_V, \odot, \otimes)$ and $(U, G_U, \odot, \otimes)$ are two general fuzzy normed spaces. The operator $S: D(S) \to U$ is called **general fuzzy bounded** if we can find α, $0 < \alpha < 1$ with

$$G_U(Sv, t) \geq (1 - \alpha) \quad \ldots . (3.1)$$

for each $v \in D(S)$ and $t > 0$

Notation :

Suppose that $(V, G_V, \odot, \otimes)$and $(U, G_U, \odot, \otimes)$are a general fuzzy normed spaces. Put GFB(V,U) = $\{S: V \to U: G_U(Sv, t) \geq (1 - \alpha)\}$ with $0 < \alpha <$ **1**

Proposition 3.2:

If $(V, G_V, \odot, \otimes)$ is a general fuzzy normed space. Then the sum of any two general fuzzy bounded subset of V is again general fuzzy bounded also the scalar multiple of any general fuzzy bounded subset of V by a real number is again a general fuzzy bounded.

Proof :

Suppose that $A \subseteq V$, $B \subseteq V$ are general fuzzy bounded we will prove that A + B and αA are general fuzzy bounded for every $\alpha \neq 0$. By our assumption A and B are general fuzzy bounded so there is p, $0 < p < 1$ and q, $0 < q < 1$ such that $G_V(a, t) \geq (1 - p)$ for all $a \in A$ and $t > 0$ also $G_V(b, s) \geq (1 - q)$ for all $b \in B$ and $s > 0$. Now

$$G_V(a + b, t + s) \geq G_V(a, t) \odot G_V(b, s)$$
$$\geq (1 - p) \odot (1 - q)$$

Put $(1 - p) \odot (1 - q) \geq (1 - r)$ for some r, $0 < r < 1$

Hence $G_V(a + b, t + s) \geq (1 - r)$ so A + B is general fuzzy bounded. Similarly $G_V(\alpha a, ts) = L_{\mathbb{R}}(\alpha, t) \otimes G_V(a, s))$ put $L_{\mathbb{R}}(\alpha, t) = (1-\beta)$. Now choose $0 < \delta < 1$ with that $(1-\beta) \otimes (1 - p) \geq (1-\delta)$. Thus $G_V(\alpha a, ts) \geq (1-\delta)$. Hence αA is general fuzzy bounded

Lemma 3.3 :

Suppose that $(V, G_V, \odot, \otimes)$ and $(U, G_U, \odot, \otimes)$ are two general fuzzy normed spaces then $T_1 + T_2 \in$ GFB(V, U) and $\alpha T \in$ GFB(V, U) for all $T_1, T_2 \in$ GFB(V, U) and $0 \neq \alpha \in F$.

Proof :

Let T_1 and T_2 be general fuzzy bounded linear operator then there is $0< r_1 < 1$ and $0 < r_2 < 1$ such that $G_U(T_1(v),t) \geq (1-r_1)$ and $G_U(T_2(v),t) \geq (1-r_2)$ for any $v \in D(T_1) \cap D(T_2)$ and any s , t > 0 . Now

$G_U(T_1 + T_2)(v), t+s) = G_U(T_1(v) + T_2(v)), t+s)$

$\geq G_U(T_1(v), t) \odot G_U(T_2(v), s)$

$\geq (1-r_1) \odot (1-r_2)$

Choose r, 0 < r < 1 such that

$(1-r_1) \odot (1-r_2) \geq (1-r)$

Hence $G_U[(T_1 + T_2)(v), t+s] > (1-r)$

Thus $T_1 + T_2$ is general fuzzy bounded operator. Also

$G_U(\alpha T, ts) = L_{\mathbb{R}}(\alpha, t) \otimes G_V(T, s))$ let $L_{\mathbb{R}}(\alpha, t) = (1-\beta)$ and $G_U(T, s) \geq$

$(1-p)$. Now choose $0 < \delta < 1$ with that $(1-\beta) \otimes (1-p) \geq (1-\delta)$.

Hence $G_U(\alpha T, ts) \geq (1-\delta)$.Thus αT is general fuzzy bounded

Theorem 3.4:

Suppose that $(V, G_V, \odot, \otimes)$ and $(U, G_U, \odot, \otimes)$ are two general fuzzy normed spaces. Put $\mathbf{G(T, t)} = \mathbf{inf}_{v \in D(T)} G_U(\mathbf{Tv}, \mathbf{t})$ for all $T \in GFB(V, U)$, $t > 0$. Then $[GFB(V, U), G, \odot, \otimes]$ is general fuzzy normed space.

Proof :

(G1)Since $0 \leq G_U(\mathbf{Tv}, \mathbf{t}) < 1$ with all $v \in D(T)$ and $t > 0$ so $0 \leq G(\mathbf{T}, \mathbf{t}) < 1$ for all t>0

(G2) For all $t > 0$, $G(T, t) = 1 \Leftrightarrow \inf_{v \in D(T)} G_U(Tv, t) = 1 \Leftrightarrow G_U(Tv, t) = 1$

$\Leftrightarrow T(v) = 0$ for all $v \in D(T)$ $\Leftrightarrow T = 0$

(G3) For all $0 \neq \alpha \in F$ we have

$G(\alpha T, ts) = \inf_{v \in D(T)} G_U(\alpha T, ts)$

$\geq \inf_{v \in D(T)} L_{\mathbb{R}}(\alpha, t) \otimes G_U(T, s)$

$= L_{\mathbb{R}}(\alpha, t) \otimes \inf_{v \in D(T)} G_U(T, s)$

$= L_{\mathbb{R}}(\alpha, t) \otimes G(T, s)$

(G4) $G(T_1 + T_2, t+s) =$

$\inf G_{U_{v \in D(T_1) \cap D((T_2)}}((T_1 + T_2)(v), t+s)$

$= \inf G_{U_{v \in D(T_1) \cap D((T_2)}}(T_1(v) + T_2(v), t+s)$

$\geq \inf G_{U_{v \in D(T_1)}}(T_1 v, t) \odot \inf G_{U_{v \in D(T_2)}}(T_2 v, s)$

$= G(T_1, t) \odot G(T_2, s)$

(G5) Let (t_n) be a sequence in $[0, \infty)$ with $t_n \to t \in [0, \infty)$ then

$G_U(Tv, t_n) \to G_U(Tv, t)$ so $G(T, t_n) \to G(T, t)$ that is $(T, \bullet)$ is a continuous.

(G6) $\lim_{t \to \infty} G(T, t) = \lim_{t \to \infty} \inf G_U(Tv, t) = \inf \lim_{t \to \infty} G_U(Tv, t) = 1$

Hence $(GFB(V, U), G, \odot, \otimes)$ is general fuzzy normed space.

Note 3.5 :

We can rewrite 3.1 by :

$G_U(Sv, t) \geq G(S, t)$.................(3.2)

Example 3.6 :

Let V be that vector space of all polynomials on C[0,1] with

$\|v\| = \max|v(x)|$, $x \in [0,1]$. Let

$$G_V(v, t) = \begin{cases} 1 & \text{if } \|v\| < t \\ 0 & \text{if } \|v\| \geq t \end{cases}$$

Then by example 2.10 $(V, G_V, \odot, \otimes)$ is a general fuzzy normed space.

Let $T: V \to V$ be defined by

$T[v(x)] = \acute{v}(x)$ then T is linear. Let $v_n(x) = x^n$ indeed $\|v_n\| = 1$ so

$$G_V[T(v_n), t] = \begin{cases} 1 & n < t \\ 0 & t \leq n \end{cases}$$

Hence there is no $c, 0 < c < 1$ satisfies the inquality

$G_V(T(v), t) \geq (1-c)$. Therefore T is not general fuzzy bounded.

Theorem 3.7 :

Suppose that $(V, G_V, \odot, \otimes)$ and $(U, G_U, \odot, \otimes)$ are general fuzzy normed spaces with U is a general complete. Assume that $T: D(T) \to U$ be a linear operator and a general fuzzy bounded. Then T has an extension $S: \overline{D(T)} \to U$ with S is linear and general fuzzy bounded such that $G(T, t) = G(S, t)$ for all $t > 0$.

Proof:

Suppose that $v \in \overline{D(T)}^G$ then by Lemma 2.22 there is (v_n) in D(T) such that

$v_n \to v$. But T is linear and general fuzzy bounded we have $G_U(T(v), t) \geq (1-r)$

for all $v \in D(T)$ and $t > 0$ where r, $0 < r < 1$. Now

$G_U[T v_n - T v_m, t] = G_U[T(v_n - v_m), t]$

$\geq (1-r)$

Thus $(T(v_n))$ is general Cauchy sequence in U but by our assumption U is general complete so that $(T(v_n))$ fuzzy approaches to $u \in U$. Define S(v)=u. Let $v_n \to v$ and $w_n \to v$ then $y_m \to v$ where $(y_m) = (v_1, w_1, v_2, w_2, \dots\dots)$. Hence (Ty_m) fuzzy approaches and (Tv_n) and (Tw_n) the two subsequences of (Ty_m) will has equal limit. Hence S is well defined for any $v \in \overline{D(T)}^G$. S linear is clear also S(d)=T(d) for every $d \in D(T)$ thus S is an extension of T.

Now we have $G_U[T v_n, t] \geq G(T,t)$ let $n \to \infty$ then $Tv_n \to S(v) = u$ thus we obtain $G_U[Sv, t] \geq G(T,t)$. Hence S is general fuzzy bounded and $G[S,t] \geq G[T,t]$ but $G[S,t] \leq G[T,t]$ by the definition of general fuzzy. Together we have $G[S,t] = G[T,t]$

Theorem 3.8 :

If $(V, G_V, \odot, \otimes)$ and $(U, G_U, \odot, \otimes)$ are general fuzzy normed spaces and let $T: D(T) \to U$ be a linear operator where $D(T) \subseteq V$. Then T is general fuzzy continuous if and only if T is general fuzzy bounded.

Proof :

Let T be general fuzzy bounded and let $\varepsilon, 0 < \varepsilon < 1$ be given and $t > 0$ then for every $z \in D(T)$ we have $G_U[Tz, t] \geq (1-\varepsilon)$. Now let $y \in D(T)$ then for any choice of $0 < r < 1$ with $G_V[x-y, s] \geq (1-r)$ which implies that

$G_U[Tx - Ty, t] = G_U[T(x-y), t] \geq (1-\varepsilon)$. Thus T is general fuzzy continuous at x. Hence T is general fuzzy continuous.

For the Converse let T be a general fuzzy continuous at any point $x \in D(T)$. Then given $\varepsilon, 0 < \varepsilon < 1$ and $t > 0$ there is r, $0 < r < 1$ and $s > 0$ with $G_U[Tx - Ty, t] > (1-\varepsilon)$ for all $y \in D(T)$ satisfying

$G_V[y-x, s] > (1-r)$. Take any $z \neq 0 \in V$ and set $y = x + z$, hence for all $t > 0$

$G_U(Tz, t) = G_U[T(y-x), t]$

$= G_U[Ty - Tx, t] > (1-\varepsilon)$.

Thus T is general fuzzy bounded .

Corollary 3.9 :

Suppose that $(V, G_V, \odot, \otimes)$ and $(U, G_U, \odot, \otimes)$ are two general fuzzy normed spaces with $T: D(T) \to U$ is a linear operator where $D(T) \subseteq V$. If T is a general fuzzy continuous at an arbitrary vector $v \in D(T)$ then T is general fuzzy continuous.

Proof :-

Assume that T is fuzzy continuous at $v \in D(T)$ then by Theorem 3.8, T is general fuzzy bounded which implies that T is a general fuzzy continuous.

Theorem 3.10:

Suppose that $(V, G_V, \odot, \otimes)$ and $(U, G_U, \odot, \otimes)$ are two general fuzzy normed spaces and assume that $T: V \to U$ is a general fuzzy bounded operator. Then

1) $v_n \to v$ [where $v_n, v \in D(T)$] implies

$Tv_n \to Tv$

2) The kernel of T N(T) is general closed .

Proof :

Since T is general fuzzy bounded then there is $r, 0 < r < 1$ such that

$G_U[T(v), t] \geq (1-r)$ for each $v \in D(T)$ and $t > 0$. Now

$G_U[Tv_n - Tv, t] = G_U[T(v_n - v), t] \geq (1-r)$.

Therefore $Tv_n \to Tv$.

2) Let $v \in \overline{N(T)}^G$ then we can find (v_n) in N(T) with $v_n \to v$.

Hence $Tv_n \to Tv$ by part (1) . Also T(v)=0 since $T(v_n) = 0$ so that $v \in N(T)$. Since $v \in \overline{N(T)}^G$ was arbitrary so N(T) is general closed.

Definition 3.11 :

Suppose that $(V, G_V, \odot, \otimes)$ is a general fuzzy normed space and $(\mathbb{R}, L_{\mathbb{R}}, \odot, \otimes)$ is fuzzy absolute value space $(\mathbb{R}, L_{\mathbb{R}}, \odot, \otimes)$. Then a linear function $f: V \to \mathbb{R}$ is called **general fuzzy bounded** if there exists $\sigma \in (0, 1)$ with $L_{\mathbb{R}}[f(v), t] \geq (1-\sigma)$ for any $v \in D(f)$, $t > 0$. Furthermore, the fuzzy norm of f is

$L(f,t) = \inf L_{\mathbb{R}}(f(v), t)$

and $L_{\mathbb{R}}(f(v), t) \geq L(f, t)$.

The proof of the next results follows directly from Theorem 3.8

Corollary 3.12 :

Suppose that $(V, G_V, \odot, \otimes)$ is a general fuzzy normed space and $(\mathbb{R}, L_{\mathbb{R}}, \odot, \otimes)$ is fuzzy absolute value space. Then a linear function $f: V \to \mathbb{R}$ with $D(f) \subseteq V$ is general fuzzy bounded if and only f is general fuzzy continuous.

Definition 3.13 :

Suppose that $(V, G_V, \odot, \otimes)$ is a general fuzzy normed space. Then GFB(V, $\mathbb{R}$) = { $f: V \to \mathbb{R}$: f is general fuzzy bounded linear } forms a general fuzzy normed space with general fuzzy norm defined by $L(f,t) = \inf L_{\mathbb{R}}(f(v), t)$ which is said to be the general fuzzy dual space of V.

Definition 3.14:

Suppose that $(V, G_V, \odot, \otimes)$ is general fuzzy normed space. A sequence (v_n) in V is **general fuzzy weakly approaches** if we can find $v \in V$ with every $h \in GFB(V, \mathbb{R})$ $\lim_{n\to\infty} h(v_n) = h(v)$. This is written $v_n \to^w v$ the element v is said to be the weak limit to (v_n) and (v_n) is said to be general fuzzy approaches weakly to v.

Theorem 3.15:
Suppose that $(V, G_V, \odot, \otimes)$ is a general fuzzy normed space and (v_n) is in V.
1. If $v_n \to v$ then $v_n \to^w v$.
2. $v_n \to^w v$ implies $v_n \to v$ when dimension of V is finite.

Proof:
1.Since $v_n \to v$ so for given $t > 0$, $\sigma \in (0, 1)$ there is N ∈ with
$G_V[v_n - v, t] > (1 - \sigma)$ for all $n \geq N$. Now for every $f \in GFB(V, R)$
$L_{\mathbb{R}}[f(v_n) - f(v), t] = L_{\mathbb{R}}[f(v_n - v), t] \geq L[f, t]$. Put $L[f, t] = (1 - \varepsilon)$
Hence $L_{\mathbb{R}}[f(v_n) - f(v), t] > (1 - \varepsilon)$. This shows that $v_n \to^w v$.
2.Suppose that $v_n \to^w v$ and dimV=m let $\{e_1, e_2, \ldots\ldots e_m\}$ be a basis for V so $v_n = \alpha_1^{(n)} e_1 + \alpha_2^{(n)} e_2 + \cdots \ldots \ldots + \alpha_m^{(n)} e_m$ and
$v = \alpha_1 e_1 + \alpha_2 e_2 + \cdots \ldots + \alpha_m e_m$. But $f(v_n) \to f(v)$ for every $f \in GFB(V,\mathbb{R})$ put $f_1, f_2, \ldots, f_m$ by: $f_j(e_j) = 1$and $f_j(e_k) = 0$ when $k \neq j$.
Then $f_j(v_n) = \alpha_j^{(n)}$ and $f_j(v) = \alpha_j$ hence $f_j(v_n) \to f_j(v)$ implies $\alpha_j^{(n)} \to \alpha_j$.
Now for $n \geq N$

$$G_V[v_n - v, ts] = G_V[\sum_{j=1}^{m} \left(\alpha_j^{(n)} - \alpha_j\right) e_j, st]$$
$$\geq L_{\mathbb{R}}[\alpha_1^{(n)} - \alpha_1, s] \otimes G_V\left[e_1, \frac{t}{m}\right] \odot L_{\mathbb{R}}[\alpha_2^{(n)} - \alpha_2, s] \otimes G_V\left[e_2, \frac{t}{m}\right] \odot \ldots \ldots \odot L_{\mathbb{R}}[\alpha_n^{(n)} - \alpha_n, s] \otimes G_V\left[e_n, \frac{t}{m}\right].$$

Put $L_{\mathbb{R}}[\alpha_j^{(n)} - \alpha_j, s] = (1 - r_j)$ and $G_V\left[e_j, \frac{t}{m}\right] = (1 - q_j)$.Choose r ,0 < r <1 with
$(1 - r_1) \otimes (1 - q_1) \odot (1 - r_2) \otimes (1 - q_2) \odot \ldots \odot (1 - r_n) \otimes (1 - q_n) > (1 - r)$
Hence $G_V[v_n - v, t] > (1 - r)$ for all $n \geq N$.
Therefore $v_n \to v$

Definition 3.16:
Suppose that $(V, G_V, \odot, \otimes)$ and $(U, G_U, \odot, \otimes)$ are two general fuzzy normed spaces. A sequence (T_n) of operators $T_n \in GFB(V, U)$ is said to be:
1.**Uniformly operator general fuzzy approaches** if there is T∈ GFB(V, U)
$G[T_n - T, t] \to 1$ as $n \to \infty$.
2.**Strong operator general fuzzy approaches** if $(T_n v)$ general fuzzy approaches in U for every $v \in V$.
3.**Weakly operator general fuzzy approaches** if $(T_n v)$ general fuzzy approaches weakly in U for every $v \in V$.

Definition 3.17:
Suppose that $(V, G_V, \odot, \otimes)$ is a general fuzzy normed space. A sequence (h_n) with $h_n \in GFB(V, \mathbb{R})$ is called
1)**Strong general fuzzy approaches** in the general fuzzy norm on GFB(V, ℝ) that is $h \in GFB(V, \mathbb{R})$ with $G[h_n - h, t] \to 1$ for all $t > 0$ this written $h_n \to h$
2)**Weak general fuzzy approaches** in the fuzzy absolute value on ℝ that is $h \in GFB(V, \mathbb{R})$ with $h_n(v) \to h(v)$ for every $v \in V$ written by $\lim_{n\to\infty} h_n(v) = h(v)$.

Theorem 3.18 :
Suppose that $(V, G_V, \odot, \otimes)$ and $(U, G_U, \odot, \otimes)$ are two general fuzzy normed spaces. Then GFB(V, U) is general complete when U is general complete.

Proof :
Let (T_n) be a general Cauchy sequence in GFB(V, U) . Hence for every $\varepsilon, 0 < \varepsilon < 1$, $t > 0$ there is a number N with $G_U[T_n - T_m, t] \geq (1 - \varepsilon)$ for all $m, n \geq N$.
Now for $v \in V$ and $m, n \geq N$ we have by Remark 3.5
$G_U[T_n v - T_m v, t] \geq G_U[(T_n - T_m)(v), t] > (1 - \varepsilon)$……(3.3)
Now for any fixed v and given ε_v, $0 < \varepsilon_v < 1$ and we have from (3.3)
$G_U[T_n v - T_m v, t] > (1 - \varepsilon_v)$ so that $(T_n v)$ is a general Cauchy sequence in U but U is general complete hence $(T_n v)$ fuzzy approaches to $u \in U$ that is $T_n v \to u$. The vector u depends on $v \in V$ this defines an operator $T: V \to U$ defined by T(v)=u. The operator T is linear since

$$T[\alpha x + \beta z] = \lim_{n\to\infty} T_n[\alpha x + \beta z]$$
$$= \alpha \lim_{n\to\infty} T_n x + \beta \lim_{n\to\infty} T_n z$$
$$= \alpha T(x) + \beta T(z)$$

We will prove that T is general fuzzy bounded and $T_n \to T$ since (3.3) is satisfied for all $m \geq N$ and $T_m v \to Tv$ we may let $m \to \infty$ we have from (3.3) for every $n \geq N$ and $t > 0$ where for all $v \in V$ we obtain

$$G_U[(T_n - T)(v), t] = G_U[T_n v - \lim_{m\to\infty} T_m v, t] = \lim_{m\to\infty} G_U[(T_n - T_m)(v), t] > (1 - \varepsilon) \ldots\ldots(3.4)$$

Thus $(T_n - T)$ with $n \geq N$ is general fuzzy bounded linear but T_n is general fuzzy bounded

so $T = T_n - (T_n - T)$ is general fuzzy bounded that is $T \in GFB(V, U)$ also from (2.4) we obtain by taking the infimum for all v
$G(T_n - T, t) \geq (1 - \varepsilon)$. for all $n \geq N$ and $t > 0$ that is $T_n \rightarrow T$.
The proof of the next result follows immediately from Theorem 3.18

Corollary 3.19 :

Suppose that $(V, G_V, \odot, \otimes)$ is a general fuzzy normed space and $(\mathbb{R}, L_{\mathbb{R}}, \odot, \otimes)$ is fuzzy absolute value space. Then $GFB(V, \mathbb{R})$ is general complete if $(\mathbb{R}, L_{\mathbb{R}}, \odot, \otimes)$ is general complete.

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خواص الفضاء GFB(V, U)

جهاد رمضان خضر منار ناجي غايب
الجامعة التكنولوجية ، قسم العلوم التطبيقية ، فرع الرياضيات وتطبيقات الحاسوب

المستخلص

هدفنا في هذا البحث هو استخدام مفاهيم الفضاء القياسي الضبابي العام وخواصه الاساسية لتعريف التقيد الضبابي العام للمؤثرات كمقدمة لتقديم مفهوم القياس الضبابي العام لمؤثر خطي مقيد ضبابيا عاما بعد ذلك برهنا ان اي مؤثر من فضاء القياس الضبايي العام الى فضاء القياس الضبابي كامل عام يمتلك توسيع. كذلك برهنا المؤثر المقيد ضبابيا العام المعرف على فضاء القياس الضبابي العام يكافيء الاستمرارية العامة. واخيرانواع مختلفة من التقارب الضبابي للمؤثرات تم تقديمها لغرض برهان ان فضاء القياس الضبابي العام GFB(V,U) يكون كامل عام متى ماكان U كامل عام.

الكلمات المفتاحية:فضاء القياس الضبابي العام GFB(V,U)، المؤثرات المستمرة الضبابية العامة، المؤثرات المقيدة ضبابيا العامة، فضاء القياس الضبابي العام.

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A Class of Meromorphic p-valent Functions

Waggas Galib Atshan **Sarah Abd Al-Hmeed Jawad**

Department of Mathematics , College of Computer Science and Information Technology , University of Qadisiayh,Diwaniyah,
waggas.galib@qu.edu.iq Sarahabdalhmeed94@gmail.com

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Abstract:

In this paper, we define a class of meromorphic p-valent functions and study some properties as coefficient inequality, closure theorem , growth and distortion bounds , arithmetic mean**,** radius of convexity, Convex linear combination and partial sums .

Keywords: Meromorphic p-valent function,Convex function, Integral operator .

Mathematics Subject Classification: 30C45.

1-Introduction: Let A_p^* denote the class of functions f of the form

$$f(z) = z^{-p} + \sum_{n=p}^{\infty} a_n\, z^n, (a_n \geq 0, n \geq p, p \in N), \quad (1.1)$$

which are analytic and p-valent in the punctured unit disk U^*={z∈ ℂ: $0 < |z| < 1$}. Jum-Kim Srivastara [2] defined an integral operator $I_p^\sigma f(z)$ for $f \in A_p^*$ as follows

$$I_p^\sigma f(z) = \frac{1}{z^{p+1}\Gamma(\sigma)} \int_0^z (\log \frac{z}{t})^{\sigma-1} t^p f(z)dt, (n \in N). \quad (1.2)$$

If $f(z)$ is of the form (1.1), then

$$I_p^\sigma f(z) = z^{-p} + \sum_{n=p}^{\infty} (\frac{1}{n+p+1})^\sigma\ a_n\, z^n (n \geq p, p \in N). \quad (1.3)$$

In particular, when p=1 we have:

$$I_p^\sigma f(z) = z^{-1} + \sum_{n=1}^{\infty} (\frac{1}{n+2})^\sigma\ a_n\, z^n (n \geq p, p \in N).$$

Let f and g be analytic in unit disk U, then g is said to be subordinate of f, written as $g \prec f$ or g(z)$\prec f$ (z), if there exists a schwartz function ω which is analytic in U with ω (0)=0 and $|\omega(z)| < 1 (z \in U)$ such that g(z)= f (ω(z)).

In particular, if the function f is univalent in U, we have the following equivalence ([3],[4]).

$$g(z) \prec f(z) (z \in U) \Leftrightarrow g(0) = f(0) and\ g(U) \subseteq f(U).$$

Definition(1.1): A function $f \in A_p^*$ is said to be in the class $A_p^*(\sigma, b, x, y)$ of functions of the form (1.1), which satisfies the condition

$$p - \frac{1}{b}\left\{1 + \frac{z^2 (I_p^\sigma f(z))''}{z(I_p^\sigma f(z))'} + p\right\} \prec p\frac{1+xz}{1+yz}, \quad (1.4)$$

where

$-1 \leq y \leq x \leq 1, p \in N, \sigma < 0, b$ *non zero complex number*.

We can re-write the condition (1.4) as

$$\left|\frac{z(I_p^\sigma f(z))'' + (1+p)(I_p^\sigma f(z))'}{yz(I_p^\sigma f(z))'' + [y(1+p(1-b)) + xbp](I_p^\sigma f(z))'}\right| < 1 . (1.5)$$

2.Coefficient inequality:

In the following theorem, we give a sufficient and necessary condition to be the function in the class $A_p^*(\sigma, b, x, y)$.

Theorem (2.1): Let $f \in A_p^*$ be given by (1.1). Then $f \in A_p^*(\sigma, b, x, y)$ if and only if

$$\sum_{n=p}^{\infty} [n(n+p)(1-y) - np|b|\left(\frac{x}{-y}\right)](\frac{1}{n+p+1})^\sigma a_n \leq p^2|b|(x-y). \quad (2.1)$$

The results is sharp for the function f given by

$$f(z) = z^{-p} + \left(\frac{p^2|b|(x-y)}{[n(n+p)(1-y) - np|b|(x-y)]}\right)(n+p+1)^\sigma z^n, (n \geq p, n \in N). \quad (2.2)$$

Proof: Assuming that the inequality (2.1) holds true and $|z| = 1$. Then ,we have

$$|z^2 (I_p^\sigma f(z)'' + (1+p)z(I_p^\sigma f(z))'| - |yz^2 (I_p^\sigma f(z))'' + [y(1+p(1-b)) + xbp]z(I_p^\sigma f(z))'|$$

$$= \left|\sum_{n=p}^{\infty} n(n+p)(\frac{1}{n+p+1})^\sigma a_n z^n\right| - \left|p^2|b|(x-y) + \sum_{n=p}^{\infty} [yn(n+p) + n|b|p(x-y)](\frac{1}{n+p+1})^\sigma a_n z^n\right|$$

$$\leq \sum_{n=p}^{\infty} n(n+p)(\frac{1}{n+p+1})^\sigma a_n |z|^n - p^2|b|(x-y) - \sum_{n=p}^{\infty} [yn(n+p) + n|b|p(x-y)](\frac{1}{n+p+1})^\sigma a_n |z|^n$$

$$= \sum_{n=p}^{\infty} n(n+p)(\frac{1}{n+p+1})^\sigma a_n - p^2|b|(x-y) - \sum_{n=p}^{\infty} [yn(n+p) + n|b|p(x-y)](\frac{1}{n+p+1})^\sigma a_n \leq 0,$$

by hypothesis.

Hence, by the Maximum Modulus Theorem, we have $f(z) \in A_p^*(\sigma, b, x, y)$.

Conversely, suppose that $f(z) \in A_p^*(\sigma, b, x, y)$. Then from (1.5) ,we have

$$\left|\frac{z^2 (I_p^\sigma f(z)'' + (1+p)z(I_p^\sigma f(z))'}{yz^2 (I_p^\sigma f(z))'' + [y(1+p(1-b)) + xbp]\, z(I_p^\sigma f(z))'}\right|$$

$$= \left|\frac{\sum_{n=p}^{\infty} n(n+p)(\frac{1}{n+p+1})^\sigma a_n\ z^n}{p^2|b|(x-y) + \sum_{n=p}^{\infty} [yn(n+p) + n|b|p(x-y)](\frac{1}{n+p+1})^\sigma a_n z^n}\right| < 1.$$

Since Re(z) $\leq |z|$ for all z(z∈ U) , *we have*

$$Re\left(\frac{\sum_{n=p}^{\infty} n(n+p)(\frac{1}{n+p+1})^\sigma a_n z^n}{p^2|b|(x-y) + \sum_{n=p}^{\infty} [yn(n+p) + n|b|p(x-y)](\frac{1}{n+p+1})^\sigma a_n z^n}\right) \leq 1.$$

We choose the value of z on the real and $z \to 1^-$, we get

$$\sum_{n=p}^{\infty}[n(n+p)(1-y)-np|b|\binom{x}{-y}](\frac{1}{n+p+1})^{\sigma}a_n \leq p^2|b|(x-y),$$

which give (2.1).Sharpness of the result follows by setting

$f(z)=z^{-p}+\left(\frac{p^2|b|(x-y)}{[n(n+p)(1-y)-np|b|(x-y)]}\right)(n+p+1)^{\sigma}z^n,\ (n\geq p, n\in N).$

Corollary (2.1) : Let $f(z)\in A_p^*(\sigma,b,x,y)$ .Then $a_n \leq \frac{p^2|b|(x-y)}{[n(n+p)(1-y)-np|b|(x-y)](\frac{1}{n+p+1})^{\sigma}}$, $(n\geq p, n\in N)$.

3. Growth and the Distortion Bounds:

In the following theorems, we obtain the growth and the distortion theorems for the function in the class $A_p^*(\sigma,b,x,y)$.

Theorem (3.1): If the function $f(z)$defined by (1.1) is in the class $A_p^*(\sigma,b,x,y)$, then for $0<|z|=r<1$, we have:

$$r^{-p}-\left(\frac{|b|(x-y)(2p+1)^{\sigma}}{2(1-y)-|b|(x-y)}\right)r^p \leq |f(z)| \leq r^{-p}+\left(\frac{|b|(x-y)(2p+1)^{\sigma}}{2(1-y)-|b|(x-y)}\right)r^p, \quad (3.1)$$

where equality holds true for the function

$$f(z)=z^{-p}+\left(\frac{|b|(x-y)(2p+1)^{\sigma}}{2(1-y)-|b|(x-y)}\right)z^p. \quad (3.2)$$

Proof: Since $f(z)\in A_p^*(\sigma,b,x,y)$ .Then from (2.1)

$2p^2(1-y)-p^2|b|(x-y)(\frac{1}{2p+1})^{\sigma}\sum_{n=p}^{\infty}|a_n| \leq \sum_{n=p}^{\infty}[n(n+p)(1-y)-p|b|(x-y)](\frac{1}{n+p+1})^{\sigma}a_n \leq p^2|b|(x-y)$,

we conclude that

$$\sum_{n=p}^{\infty}|a_n| \leq \frac{|b|(x-y)(2p+1)^{\sigma}}{2(1-y)-|b|(x-y)} \quad (3.3)$$

Thus for $0<|z|=r<1$,

$$|f(z)| \leq |z|^{-p}+\sum_{n=p}^{\infty}a_n|z|^n \leq r^{-p}+r^p\sum_{n=p}^{\infty}a_n, \quad (3.4)$$

or

$$|f(z)| \leq r^{-p}-\left(\frac{|b|(x-y)(2p+1)^{\sigma}}{2(1-y)-|b|(x-y)}\right)r^p, \quad (3.5)$$

and

$$|f(z)| \geq |z|^{-p}-\sum_{n=p}^{\infty}a_n|z|^n \geq r^{-p}-r^p\sum_{n=p}^{\infty}a_n,$$

or

$$|f(z)| \geq r^{-p}-\left(\frac{|b|(x-y)(2p+1)^{\sigma}}{2(1-y)-|b|(x-y)}\right)r^p.$$

On using (3.4) and (3.5) inequality (3.1) follows.

Theorem (3.2): If $f\in A_p^*(\sigma,b.x,y)$ then

$$r^{-(p+1)}-\left(\frac{p|b|(x-y)(2p+1)^{\sigma}}{2(1-y)-p^2|b|(x-y)}\right)r^{p-1} \leq |f(z)'| \leq r^{-(p+1)}+\left(\frac{p|b|(x-y)(2p+1)^{\sigma}}{2(1-y)-p^2|b|(x-y)}\right)r^{p-1}.$$

The result is sharp for the function f is given by (1.3)

Proof: The proof is similar to that of Theorem (3.1).

4. Extreme Points

In the next theorems, we obtain extreme points for the class $A_p^*(\sigma,b,x,y)$.

Theorem (4.1): Let $f_{p-1}(z)=z^{-p}$ and $f_n(z)=z^{-p}+\left(\frac{p^2|b|(x-y)(n+p+1)^{\sigma}}{[n(n+p)(1-y)-np|b|(x-y)]}\right)z^n, \quad (4.1)$

for $n\geq p$. Then $f(z)\in A_p^*(\sigma,b,x,y)$ if and only if it can be expressed in the form

$$f(z)=\sum_{n=p-1}^{\infty}\mu_n f_n(z), \text{ where } \mu_n \geq 0 \text{ and } \sum_{n=p-1}^{\infty}\mu_n=1. \quad (4.2)$$

Proof: Let

$$f(z)=\sum_{n=p-1}^{\infty}\mu_n f_n(z)=z^{-p}+\sum_{n=p}^{\infty}\left(\frac{p^2|b|(x-y)(n+p+1)^{\sigma}\mu_n}{[n(n+p)(1-y)-np|b|(x-y)]}\right)z^n.$$

Then

$$\sum_{n=p}^{\infty}\frac{[n(n+p)(1-y)-np|b|(x-y)](\frac{1}{n+p+1})^{\sigma}}{p^2|b|(x-y)}\cdot\frac{p^2|b|(x-y)}{[n(n+p)(1-y)-np|b|(x-y)](\frac{1}{n+p+1})^{\sigma}}$$

$$=\sum_{n=p}^{\infty}\mu_n=1-\mu_{p-1}\leq 1.$$

Using Theorem (2.1),we easily get $f(z)\in A_p^*(\sigma,b,x,y)$.

Conversely, let $(z)\in A_p^*(\sigma,b,x,y)$.

From the Theorem (2.1), we have

$$a_n \leq \frac{p^2|b|(x-y)(n+p+1)^\sigma}{[n(n+p)(1-y)-np|b|(x-y)]} \text{ for } n \geq p.$$

Setting

$$\mu_n = \frac{n(n+p)(1-y)-np|b|(x-y)}{p^2|b|(x-y)} . (\frac{1}{n+p+1})^\sigma \text{ for } n \geq p,$$

$$and\ \mu_{p-1} = 1 - \sum_{n=p}^{\infty} \mu_n .$$

Then

$f(z) = z^{-p} + \sum_{n=p}^{\infty} a_n z^n =$
$z^{-p} + \sum_{n=p}^{\infty} \left(\frac{p^2|b|(x-y)(n+p+1)^\sigma \mu_n}{[n(n+p)(1-y)-np|b|(x-y)]}\right) z^n =$
$\mu_{p-1} z^{-p} + \sum_{n=p}^{\infty} \mu_n f_n(z) = \sum_{n=p-1}^{\infty} \mu_n f_n(z)$

This completes the proof.

5. Radius of convexity

In the following theorem, we obtain the radius of convexity for the function in the class $A_p^*(\sigma,b,x,y)$.

Theorem (5.1): Let f the function f (z) defined by (1.1) is in the class $A_p^*(\sigma,b,x,y)$. Then f is meromorphically p-valent convex of order λ$(0 \leq \lambda < p)$ in the disk $|z| < r_2$, where $r_2 = r_2(p,\sigma,b,x,y) =$

$$\inf_{n\geq p}\left[\frac{(p-\lambda)[n(n+p)(1-y)-p|b|(x-y)](\frac{1}{n+p+1})^\sigma}{(n+2p-\lambda)p|b|(x-y)}\right]^{\frac{1}{n+p}} \quad (5.1)$$

The result is sharp for the function f given by (3.4).

Proof: A function f meromorphic p-valent convex of order λ $(0 \leq \lambda < p)$ if

$$-Re\{1 + \frac{f''(z)}{f'(z)}\} > \lambda .$$

We must show that

$$\left|\frac{zf''(z)}{f'(z)} + (1+p)\right| < p - \lambda , \quad \text{for} \quad |z| < r_2 .$$

(5.2)

We have $\left|\frac{zf''(z)}{f'(z)} + (1+p)\right| = \left|\frac{zf''(z)+(1+p)f'(z)}{f'(z)}\right| = \left|\frac{\sum_{n=p}^{\infty} n(n+p)a_n z^{n+p}}{-p+\sum_{n=p}^{\infty} n\, a_n z^{n+1}}\right| \leq \frac{\sum_{n=p}^{\infty} n(n+p)a_n |z|^{n+p}}{p-\sum_{n=p}^{\infty} n\, a_n |z|^{n+p}}$.

Thus ,(5.2) will be satisfied if

$$\sum_{n=p}^{\infty} \frac{n(n+2p-\lambda)}{p(p-\lambda)} a_n |z|^{n+p} \leq 1 .$$

(5.3)

Since $f \in A_p^*(\sigma,b,x,y)$, we have

$$\sum_{n=p}^{\infty} \frac{[n(n+p)(1-y)-np|b|(x-y)](\frac{1}{n+p+1})^\sigma}{p^2|b|(x-y)} a_n \leq 1.$$

Hence,(5.3) will be true if

$$\frac{n(n+2p-\lambda)}{p(p-\lambda)}|z|^{n+p} \leq \left[\frac{n(n+p)(1-y)-np|b|(x-y)(\frac{1}{n+p+1})^\sigma}{p^2|b|(x-y)}\right],$$

or equivalently

$$|z| \leq \left(\frac{(p-\lambda)(n+p)(1-y)-np|b|(x-y)(\frac{1}{n+p+1})^\sigma}{(n+2p-\lambda)p^2|b|(x-y)}\right)^{\frac{1}{n+p}}, n \geq p$$

which follows the result.

6. Convex linear combination:

Theorem (6.1): The class $A_p^*(\sigma,b,x,y)$ is closed under convex linear combinations.

Proof: Let f_1 and f_2 be the chance elements of $A_p^*(\sigma,b,x,y)$. Then for each t $(0 < t < 1)$ plus $(a_n, b_n \geq 0)$. we show that $(1-t)f_1 + tf_2 \in A_p^*(\sigma,b,x,y)$. Thus we have

$(1-t)f_1 + tf_2 = z^{-p} + \sum_{n=p}^{\infty}[(1-t)\, a_n + tb_n]z^n$.

Hence

$$\sum_{n=p}^{\infty}[n(n+p)(1-y) - np|b|(x-y)](\frac{1}{n+p+1})^\sigma[(1-t)\, a_n + tb_n].$$

$$= (1-t)\sum_{n=p}^{\infty}[n(n+p)(1-y) - np|b|\left(\frac{x}{-y}\right)](\frac{1}{n+p+1})^\sigma a_n + t\sum_{n=p}^{\infty} \frac{n(n+p)(1-y)}{-np|b|(x-y)}(\frac{1}{n+p+1})^\sigma b_n$$

$$\leq (1-t)p^2|b|(x-y) + tp^2|b|(x-y)$$

$$= p^2|b|(x-y).$$

This completes the proof.

7. The anthmetic mean:

Theorem (7.1): Let the functions $f_k$sharp by $f_k(z) = z^{-p} + \sum_{n=p}^{\infty} a_{n,k}$, $(a_{n,k} \geq 0, n \in N, k = 1,2,\dots l)$,

be in the class $A_p^*(\sigma,b,x,y)$ for each k = (1,2,3,… l), then the function h sharp by

$$h(z) = z^{-p} + \sum_{n=p}^{\infty} e_n z^n , (e_n \geq 0, n \in N)$$

also belong to the class $A_p^*(\sigma,b,x,y)$, where $e_n = \frac{1}{l}\sum_{n=p}^{\infty} a_{n,k}$, $(n \geq p, p \in N)$.

proof: As $f_k \in A_p^*(\sigma,b,x,y)$, it follows the Theorem (2.1) that

$$\sum_{n=p}^{\infty}[n(n+p)(1-y)-np|b|(x-y)](\frac{1}{n+p+1})^{\sigma}a_{n,k}\le p^2|b|(x-y),$$

for each k = 1,2,3,…. l Hence

$$\sum_{n=p}^{\infty}[n(n+p)(1-y)-np|b|(x-y)](\frac{1}{n+p+1})^{\sigma}e_n$$
$$=\sum_{n=p}^{\infty}[n(n+p)(1-y)-np|b|(x-y)](\frac{1}{n+p+1})^{\sigma}\left(\frac{1}{l}\sum_{k=1}^{l}a_{n,k}\right)$$
$$=\frac{1}{l}\sum_{k=1}^{l}(\sum_{n=p}^{\infty}[n(n+p)(1-y)-np\,|b|(x-y)\,]\left(\frac{1}{n+p+1}\right)a_{n,k})$$
$$\le\frac{1}{l}\sum_{k=1}^{l}p^2\,|b|\,(x-y)$$
$$=p^2|b|\,(x-y).$$

Then $h\in A_p^*\,(\sigma,b,x,y)$.

8. Partial sums

Theorem(8.1): Let $f\in A_p^*(\sigma,b,x,y)$ be assumed by(1.1)and $g\in A_p^*(\sigma,b,x,y)$ be assumed by

$$g(z)=z^{-p}+\sum_{n=p}^{\infty}b_nz^n.$$

We define the partial sums $S_1(z)\,and\,S_k(z)$ as follows :

$$S_1(z)=z^{-1}and\;\;S_k(z)=z^{-p}+\sum_{n=p}^{k-1}a_nz^n,(k\in N|\{1\}). \qquad (8.1)$$

Also suppose that

$$\sum_{n=p}^{\infty}c_na_n\le 1\,,c_n=\frac{[n(n+p)(1-y)-np\,|b|(x-y)](\frac{1}{n+p+1})^{\sigma}}{p^2|b|\,(x-y)}. \qquad (8.2)$$

Then, we hav $Re\left\{\frac{f(z)}{S_k(z)}\right\}>1-\frac{1}{c_k}\;,(z\in U\,,k\in N),$ (8.3)

and $Re\left\{\frac{S_k(z)}{f(z)}\right\}>\frac{c_k}{1+c_k}\;,(z\in U\,,k\in N).$ (8.4)

Each of the bounds in (8.3) and (8.4) is the best possible for $k\in N$.

Proof: We can see from (8.2) that $c_{n+1}>c_n>1\,,n=p,p+1,p+2,p+3,\ldots$

Therefore, we have:

$$\sum_{n=p}^{k=1}a_n+c_k\sum_{n=k}^{\infty}a_n\le\sum_{n=p}^{\infty}c_na_n\le 1. \quad (8.5)$$

By setting

$$g_1(z)=c_k\left[\frac{f(z)}{S_k(z)}-\left(1-\frac{1}{c_k}\right)\right]=1+\frac{c_k\sum_{n=k}^{\infty}a_nz^{n+1}}{1+\sum_{n=p}^{k-1}a_nz^{n+1}}, \qquad (8.6)$$

and applying (8.5) we find that

$$\left|\frac{g_1(z)-1}{g_1(z)+1}\right|\le\frac{c_k\sum_{n=k}^{\infty}a_n}{2-2\sum_{n=p}^{k-1}a_n-c_k\sum_{n=k}^{\infty}\;a_n}, \qquad (8.7)$$

which readily yields the assertion (8.3) if ,we take

$$f(z)=z^{-p}-\frac{z^k}{c_k}. \qquad (8.8)$$

Then

$\frac{f(z)}{S_k(z)}=1-\frac{z^k}{c_k}\to 1-\frac{1}{c_k}\;(z\to 1^-)$, which shows that the bound in (8.3) is the best possible for $k\in N$.

Similarly, if we put

$$g_2(z)=(1+c_k)\left[\frac{S_k(z)}{f(z)}-\frac{c_k}{1+c_k}\right]=1-\frac{(1+c_k)\sum_{n=k}^{\infty}\;a_nz^{n+1}}{1+\sum_{n=p}^{k-1}a_nz^{k+1}}, \qquad (8.9)$$

and make use of (8.9), we have

$$\left|\frac{g_2(z)-1}{g_2(z)+1}\right|\le\frac{(1+c)\sum_{n=k}^{\infty}\;a_n}{2-2\sum_{n=p}^{k-1}a_n+(1-c_k)\sum_{n=k}^{\infty}\;a_n}, \quad (8.10)$$

which leads us to the assertion (8.4). The bound (8.5) is sharp for each $k\in N$ with the function given by (6.7). The proof of the theorem is complete.

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الصنف من الدوال متعددة التكافؤ الميرومورفية

وقاص غالب عطشان سارة عبدالحميد جواد

قسم الرياضيات ، كلية علوم الحاسوب وتكنلوجيا المعلومات
جامعة القادسية ، الديوانية - العراق

المستخلص :

في هذا البحث ،عرفنا الصنف من الدوال متعددة التكافؤ الميرومورفية و درسنا بعض الخصائص مثل حدود المعامل ، مبرهنة الإنغلاق ، حدود التشوية والنمو، النقاط المتطرفة ، نصف قطر التحدب ، التركيب الخطي المحدب والمجاميع الجزئية .

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On Differential Sandwich Theorems of Multivalent Functions Defined by a Linear operar

Waggas Galib Atshan **Salwa Kalf Kazim**
Department of Mathematics , College of Computer Science and Information Technology , University of AL-Qadisiyah , Diwaniyah-Iraq
waggas.galib@qu.edu.iq , waggashnd@gmail.com

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Abstract:
The main object of the present paper is to derive some results for multivalent analytic functions defined by linear operator by using differential subordination and superordination

Keywords: Analytic functions, multivalent functions, Hadamard product, subordination, linear operators.

Mathematics Subject Classification: 30C45.

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1. Introduction

Let A_p denote the class of functions f of the form:

$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n}$, $(p \in \mathbb{N} = \{1,2,\ldots\};\ z\ U)$, (1.1)

which are analytic in the open unit disk $U = \{z \in \mathbb{C}: |z| < 1\}$.

For two functions f and g are analytic in U, we say that the function f is subordinate to g in U, written $f \prec g$, if there exists Schwarz function w, analytic in U with $w(0) = 0$ *and* $|w(z)| < 1$ *in* U *such that* $f(z) = g(w(z)), z \in U$ If g is univalent and $g(0) = f(0)$, then $f(u) \subset g(u)$.

If $f \in A_p$ is given by (1.1) and $g \in A_p$ given by

$$g(z) = z^p + \sum_{n=1}^{\infty} b_{p+n}\, z^{p+n}.$$

Then Hadamard product (or convolution) is defined by

$$(f * g)(z) = z^p + \sum_{n=1}^{\infty} a_{p+n}\, b_{p+n} z^{p+n} \quad .$$

The linear operator $J_{\mu,v}^{\lambda,p}(a,c): A_p \to A_p$ defined by

$J_{\mu,v}^{\lambda,p}(a,c)f(z) = \emptyset_{\mu,v}^{\lambda,p}(a,c;z) * f(z), \quad (f \in A_p, z \in U),$ (1.2)

where

$\emptyset_{\mu,v}^{\lambda,p}(a,c;z) = z^p + \sum_{n=1}^{\infty} \frac{(a)_n (p+1)_n (p+1-\mu+v)_n}{(c)_n (p+1-\mu)_n} z^{p+n}$ (1.3)

and

$$d_n = \begin{cases} 1 & n = 0 \\ d(d+1)(d+2)\ldots(d+n-1) & n \in N \end{cases} .$$

For $a \in R, c \in R \setminus z_\circ^-$, *where* $z_\circ^- = \{0,-1,-2,\ldots\}, 0 \le \lambda < 1, \mu, v \in R$ *and* $\mu - v - p < 1$ *and* $f \in A_p$ .Then linear operator $I_{\mu,v}^{\lambda,p,\alpha}(a,c): A_p \longrightarrow A_p$ (*see*[9]) is defined by

$I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z) \coloneqq \psi_{\mu,v}^{\lambda,p,\alpha}(a,c;z) * f(z)$, (1.4)

where $\psi_{\mu,v}^{\lambda,p,\alpha}(a,c;z)$ is the function defined in terms of the Hadamard product by the following condition:

$\emptyset_{\mu,v}^{\lambda,p}(a,c;z) * \psi_{\mu,v}^{\lambda,p,\alpha}(a,c;z) = \frac{z^p}{(1-z)^{\alpha+p}} \quad (\alpha > -p).$ (1.5)

We can easily find from (1.3) - (1.5) that

$I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z) = z^p + \sum_{n=1}^{\infty} \frac{(c)_n (p+1-\lambda+v)_n (\alpha+p)_n (p+1-\mu)_n}{(a)_n (p+1)_n (p+1-\mu+v)_n n!} a_{p+n} z^{p+n}$ (1.6)

It is easily verified from (1.6) that

$\boldsymbol{z(I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z))' = (\alpha+p)I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z) - \alpha I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z).}$ ***(1.7)***

Note that the linear operator $I_{\mu,v}^{\lambda,p,\alpha}(a,c)$ unifies many other operators considered earlier. In particular

1) $I_{0,v}^{0,p,\alpha}(a,c) \equiv J_p^{\alpha}(a,c)$ (*see Cho al.*[5]).
2) $I_{0,v}^{0,p,\alpha}(a,a) \equiv D^{\alpha+p-1}$ (*see Goel and Sohi*[6]).
3) $I_{0,v}^{0,p,1}(p+1-\lambda,1) \equiv \Omega_Z^{(\lambda,P)}$ (*see Srivastava and Aouf*[16]).
4) $I_{0,v}^{0,p,\alpha-1}(a,c) \equiv J_p^{a,\alpha}$(*see Hohlov*[8]).
5) $I_{0,v}^{0,1-\alpha,\alpha}(a,c) \equiv L_P(a,c)$ (*see Saition*[13]).
6) $I_{0,v}^{0,p,1}(p+\alpha,1) \equiv J_{\alpha,P,\alpha} \in z, \alpha > -p$ (*see Liu an Noor*[10]).

The main object of this idea is to find sufficient conditions for certain normalized analytic functions $\boldsymbol{f}$ to satisfy:

$$\boldsymbol{q_1(z) \prec \left(\frac{t_1 I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z) + t_2 I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{(t_1+t_2)z^p}\right)^{\delta} \prec q_2(z),}$$

and

$$q_1(z) \prec \left(\frac{I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{z^p}\right)^{\delta} \prec q_2(z) \ ,$$

where $q_1(z)$ and $q_2(z)$ are given univalent functions in U with $q_1(0)$and $q_2(0) = 1$.

2- Preliminaries

In order to prove our subordinations and superordinations results, we need the following definition and lemmas .

Definition 2.1. [11]: Denote by Q the set of all functions q that are analytic and injective on $\bar{U} \setminus E(q)$, where
$\bar{U} = U \cup \{z \in \partial U\}$, and
$E(q) = \{\zeta \in \partial U: \lim_{z\to\zeta} q(z) = \infty\}$ (2.1)
and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial U / E(q)$.
Further ,let the subclass of Q for which $q(0) = a$ be denoted by $Q(a), Q(0) \equiv Q_0$ and $Q(1) \equiv Q_1$.

Lemma 2.1.[1]: Let $q(z)$ be convex univalent function in U, let $\alpha \in \mathbb{C}. \beta \in \mathbb{C} \setminus \{0\}$ and suppose that

$Re(1 + \frac{zq''(z)}{q'(z)}) > \max\{0, -Re(\frac{\alpha}{\beta})\}$.

If $p(z)$ is analytic in U and
$\alpha p(z) + \beta zp'(z) \prec \alpha q(z) + \beta zq'(z)$,
then $p(z) \prec q(z)$ and q is the best dominant.

Lemma 2.2. [3]: Let $\boldsymbol{q}$ *be univalent in* $\boldsymbol{U}$*and let* $\emptyset$ *and* $\boldsymbol{\theta}$*be analytic in the domain* $\boldsymbol{D}$*containing* $\boldsymbol{q(U)}$ *with* $\emptyset(\boldsymbol{w}) \neq \boldsymbol{0}$, *when* $\boldsymbol{w} \in \boldsymbol{q(U)}$.
Set $Q(z) = zq'(z)\emptyset(q(z))$ and $h(z) = \theta(q(z)) + Q(z)$, suppose that

1- Q is starlike univalent in U,
2- $\mathrm{Re}\left(\frac{zh'(z)}{Q(z)}\right) > 0,\ z \in U$.

If p is analytic in U with $p(0) = q(0), p(U) \subseteq D$ and
$\emptyset(p(z)) + zp'(z)\emptyset(p(z)) \prec \emptyset(q(z)) + zq'(z)\emptyset(q(z))$,
then $p \prec q$, and q is the best dominant .

Lemma 2.3.[12]: Let $q(z)$ be convex univalent in the unit disk U and let θ and $\emptyset$ be analytic in a domain D containing $q(U)$. Suppose that
$1 - Re\{\frac{\theta'(q(z))}{\emptyset(q(z))}\} > 0$ for $z \in U$,
$2 - zq'(z)\emptyset(q(z))$ is starlike univalent in $z \in U$.
If $p \in \mathcal{H}[q(0),1] \cap Q$, with $p(U) \subseteq D$, and $\theta(p(z)) + zp'(z)\emptyset(p(z))$ is univalent in U , and
$\theta(q(z)) + zq'(z)\emptyset(q(z)) \prec \theta(p(z)) + zp'(z)\emptyset(p(z))$, (2.2)
then $q \prec p$, and q is the best subordinant.

Lemma 2.4.[12]: Let $q(z)$ be convex univalent in U and $q(0) = 1$. Let $\beta \in \mathbb{C}$, that $\mathrm{Re}(\beta) > 0$. If $p(z) \in \mathcal{H}[q(0),1] \cap Q$ and $p(z) + \beta zp'(z)$ is univalent in U , then
$q(z) + \beta zq'(z) \prec p(z) + \beta zp'(z)$,
which implies that $q(z) \prec p(z)$ and $q(z)$ is the best subordinant.

3-Subordination Results

Theorem 3.1. Let $q(z)$ be convex univalent in U with $q(0) = 1, \eta, \delta \in \mathbb{C} \setminus \{0\}$. Suppose that

$$Re\left(1 + \frac{zq''(z)}{q'(z)}\right) > \max\left\{0, -Re\left(\frac{\delta}{\eta}\right)\right\}. \quad (3.1)$$

If $f \in W$ is satisfies the subordination

$$G(z) \prec q(z) + \frac{\eta}{\delta} zq'(z), \quad (3.2)$$

where

$$G(z) = \left(\frac{t_1 I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z) + t_2 I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{(t_1+t_2)z^p}\right)^{\delta} \times \left(1 + \eta\left(\frac{(pt_2 - t_2\alpha) I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)(z) + (t_2 - t_1\alpha + t_2 p - pt_1)}{t_1 I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z) + t_2 I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)} \frac{I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z) + (t_1\alpha - t_1 p) I_{\mu,v}^{\lambda,p,\alpha+2}(a,c)f(z)}{t_1 I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z) + t_2 I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}\right)\right), \quad (3.3)$$

then

$$\left(\frac{t_1 I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z) + t_2 I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{(t_1+t_2)z^p}\right)^{\delta} \prec q(z), \quad (3.4)$$

and $q(z)$ is the best dominant.

Proof: Define a function $k(z)$ by

$$k(z) = \left(\frac{t_1 I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z) + t_2 I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{(t_1+t_2)z^p}\right)^{\delta}, \quad (3.5)$$

then the function $k(z)$ is analytic in $U$and $q(0) = 1$, therefore,differentiating (3.5) logarithmically with respect to z and using the identity (1.7) in the resulting equation,

$$G(z) = \left(\frac{t_1 I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z) + t_2 I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{(t_1+t_2)z^p}\right)^{\delta} \times \left(1 + \eta\left(\frac{(pt_2 - t_2\alpha) I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)(z) + (t_2 - t_1\alpha + t_2 p - pt_1)}{t_1 I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z) + t_2 I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)} \frac{I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z) + (t_1\alpha - t_1 p) I_{\mu,v}^{\lambda,p,\alpha+2}(a,c)f(z)}{t_1 I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z) + t_2 I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}\right)\right)$$

Thus the subordination (3.2) is equivalent to

$$k(z) + \frac{\eta}{\delta} zk'(z) \prec q(z) + \frac{\eta}{\delta} zq'(z).$$

An application of Lemma (2.1) with $\beta = \frac{\eta}{\delta}$ and $\alpha = 1$, we obtain (3.4).
Taking $q(z) = \frac{1+Az}{1+Bz}$,$(-1 \leq B < A \leq 1)$, in Theorem (3.1), we obtain the following Corollary.

Corollary 3.1. Let $\eta, \delta \in \mathbb{C} \setminus \{0\}$*and* $(-1 \leq B < A \leq 1)$.Suppose that

$$Re\left(\frac{1-Bz}{1+Bz}\right) > \max\left\{0, -Re\left(\frac{\delta}{\eta}\right)\right\}.$$

If $f \in W$ is satisfy the following subordination condition:

$$G(z) \prec \frac{1+Az}{1+Bz} + \frac{\eta}{\delta}\frac{(A-B)z}{(1+Bz)^2},$$

where $G(z)$ given by (3.3) , then

$$\left(\frac{t_1 I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z) + t_2 I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{(t_1+t_2)z^p}\right)^{\delta} \prec \frac{1+Az}{1+Bz},$$

and $\frac{1+Az}{1+Bz}$ is the best dominant .
Taking $A = 1$ *and* $B = -1$ in Corollary (3.1), we get following result.

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Corollary 3.2. Let $\eta,\delta \in \mathbb{C}\setminus\{0\}$ and suppose that

$Re\left(\frac{1+z}{1-z}\right) > \max\{0, -Re\left(\frac{\delta}{\eta}\right)\}$.

If $f \in W$ is satisfy the following subordination

$$G(z) \prec \frac{1+z}{1-z} + \frac{\eta}{\delta}\frac{2z}{(1-z)^2},$$

where

$G(z)$ given by (3.3), then

$$\left(\frac{t_1 I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z) + t_2 I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{(t_1+t_2)z^p}\right)^{\delta} \prec \frac{1+z}{1-z},$$

and $\frac{1+z}{1-z}$ is the best dominant.

Theorem 3.2. Let $q(z)$ be convex univalent in unit disk U with $q(0)=1$, let $\eta,\delta \in \mathbb{C}\{0\}, \gamma, t, \psi, \tau \in \mathbb{C}, f \in W$, and suppose that f and q satisfy the following conditions:

$$Re\left\{\frac{\psi}{s}q(z) + \frac{2\tau\gamma}{s}q^2(z) + 1 + z\frac{q''(z)}{q'(z)} - z\frac{q'(z)}{q(z)}\right\} > 0, \quad (3.6)$$

and

$$\frac{I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{z^p} \neq 0 \,. \quad (3.7)$$

If $r(z) \prec t + \psi q(z) + \tau\gamma q^2(z) + s\frac{zq'(z)}{q(z)}$, (3.8)

where

$$r(z) = \left(\frac{I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{z^p}\right)^{\delta}\left(\psi + t\gamma\left(\frac{I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{z^p}\right) + t + s_\delta(\alpha+p)\left(\frac{I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z)}{I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)} - 1\right)\right),$$

(3.9)

then

$\left(\frac{I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{z^p}\right)^{\delta} \prec q(z)$, and $q(z)$ is best dominant .

Proof : Define analytic function $k(z)$ by

$$k(z) = \left(\frac{I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{z^p}\right)^{\delta}. \quad (3.10)$$

Then the function $k(z)$ is analytic in U and $g(0)=1$,

differentiating (3.10) logarithmically with respect to z, we get

$$\frac{zk'(z)}{k(z)} = \delta(\alpha+p)\left(\frac{I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z)}{I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)} - 1\right). \quad (3.11)$$

By setting $\theta(w) = t + \psi w + \tau\gamma w^2$ and $\phi(w) = \frac{s}{w}$, it can be easily observed that $\theta(w)$ is analytic in $\mathbb{C}$, $\phi(w)$ is analytic in $\mathbb{C}\setminus\{0\}$ and that $\phi(w) \neq 0, w \in \mathbb{C}\setminus\{0\}$.

Also , if we let

$$\phi(z) = zq'(z)\phi(q(z)) = s\frac{zq'(z)}{q(z)},$$

and

$$h(z) = \theta(q(z)) + Q(z) = t + \psi q(z) + \tau\gamma q^2(z) + s\frac{zq'(z)}{q(z)},$$

we find $Q(z)$ is starlike univalent in U , we have

$$h'(z) = \psi q'(z) + 2\tau\gamma q(z)q'(z) + s\frac{q'(z)}{q(z)} + sz\frac{q''(z)}{q(z)} - sz\left(\frac{q'(z)}{q(z)}\right)^2,$$

and

$$\frac{zh'(z)}{Q(z)} = \frac{\psi}{s}q(z) + \frac{2\tau\gamma}{s}q^2(z) + 1 + z\frac{q''(z)}{q'(z)} - z\frac{q'(z)}{q(z)},$$

hence that

$$Re\left(\frac{zh'(z)}{Q(z)}\right) = Re\left(\frac{\psi}{s}q(z) + \frac{2\tau\gamma}{s}q^2(z) + 1 + z\frac{q''(z)}{q'(z)} - z\frac{q'(z)}{q(z)}\right) > 0 \,.$$

By using (3.11), we obtain

$$\psi k(z) + \tau\gamma k^2(z) + s\frac{zk'(z)}{k(z)} = \left(\frac{I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{z^p}\right)^{\delta}\left(\psi + \tau\gamma\gamma\left(\frac{I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{z^p}\right)^{\delta}\right) + t + \left(s_\delta(\alpha+p)\left(\frac{I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z)}{I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)} - 1\right)\right).$$

By using (3.8), we have

$$\psi k(z) + \tau\gamma k^2(z) + s\frac{zk'(z)}{k(z)} \prec \psi q(z) + \tau\gamma q^2(z) + s\frac{zq'(z)}{q(z)} \quad (3.8)$$

and by using Lemma (2.2), we deduce that subordination (3.8) implies that $k(z) \prec q(z)$ and the function $q(z)$ is the best dominant .

Taking the function $q(z) = \frac{1+Az}{1+Bz}$ $(-1 \le B < A \le 1)$, in Theorem (3.2) , the condition (3.6) becomes.

$$Re\left(\frac{\psi}{s}\frac{1+Az}{1+Bz} + \frac{2\tau\gamma}{s}\left(\frac{1+Az}{1+Bz}\right)^2 + 1 + \frac{(A-B)z}{(1+Bz)(1+Az)} - \frac{2Bz}{1+Bz}\right) > 0, \quad (3.12)$$

hence, we have the following Corollary.

Corollary 3.3. Let $(-1 \le B < A \le 1), s, \delta \in \mathbb{C}\setminus\{0\}, \gamma, t, \tau, \psi \in \mathbb{C}$. Assume that (3.12) holds.

If $f \in W$ and

$$r(z) \prec t + \psi\frac{1+Az}{1+Bz} + \tau\gamma\left(\frac{1+Az}{1+Bz}\right)^2 + s\frac{(A-B)z}{(1+Bz)(1+Az)},$$

where $r(z)$ is defined in (3.9), then

$\left(\frac{I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{z^p}\right)^{\delta} \prec \frac{1+Az}{1+Bz}$, and $\frac{1+Az}{1+Bz}$ is best dominant .

Taking the function $q(z) = \left(\frac{1+z}{1-z}\right)^{\rho}$ $(0 < p \le 1)$, in Theorem (3.2), the condition (3.6) becomes

$$Re\left\{\frac{\psi}{s}\left(\frac{1+z}{1-z}\right)^{\rho} + \frac{2\tau\gamma}{s}\left(\frac{1+z}{1-z}\right)^{2\rho} + \frac{2z^2}{1-z^2}\right\}0, (s \in \mathbb{C}\setminus\{0\}), \quad (3.13)$$

hence ,we have the following Corollary .

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Corollary3.4. Let $0<\rho\le 1$,$S,\delta\in\mathbb{C}\setminus\{0\}$ $\gamma,t,\tau,\psi\in\mathbb{C}$. Assume that (3.13) holds. If $f\in W$ and

$$r(z)\prec t+\psi\left(\frac{1+z}{1-z}\right)^{\rho}+\tau\gamma\left(\frac{1+z}{1-z}\right)^{2\rho}+s\frac{2\rho z}{1-z^2},$$

where $r(z)$ is defined in (3.9), then

$$\left(\frac{I_{M,v}^{\lambda,p,\alpha}(a,c)f(z)}{z^p}\right)^{\delta}\prec\left(\frac{1+z}{1-z}\right)^{\rho},\ and\ \left(\frac{1+z}{1-z}\right)^{\rho}$$ is the best dominant.

4-Superordination Results

Theorem 4.1. Let $q(z)$ be convex univalent $U$with $q(0)=1,\delta\in\mathbb{C}\setminus\{0\},Re\{\eta\}>0,$ if $f\in W$,such that

$$\frac{t_1I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z)+t_2I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{(t_1+t_2)z^p}\neq 0$$

and

$$\left(\frac{t_1I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z)+t_2I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{(t_1+t_2)z^p}\right)^{\delta}\mathcal{H}[q(0),1]\cap Q\,. \qquad (4.1)$$

If the function $G(z)$ defined by (3.3) is univalent and the following superordination condition:

$$q(z)+\frac{\eta}{\delta}zq'(z)\prec G(z), \qquad (4.2)$$

holds , then

$$q(z)\prec\left(\frac{t_1I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z)+t_2I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{(t_1+t_2)z^p}\right)^{\delta} \qquad (4.3)$$

and $q(z)$ is the best subordinant.

Proof: Define a function $k(z)$ by

$$k(z)=\left(\frac{t_1I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z)+t_2I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{(t_1+t_2)z^p}\right)^{\delta}. \qquad (4.4)$$

Differentiating (4.4) with respect to z logarithmically, we get.

$$\frac{z\acute{k}(z)}{k(z)}=\delta\left(\frac{t_1\left(z\left(I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z)\right)'\right)+t_2\left(z\left(I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)\right)'\right)-}{t_1\left(I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z)\right)+t_2\left(I_{\mu,V}^{\lambda,P,\alpha}(a,c)f(z)\right)}\right.$$
$$\left.\frac{pt_1I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z)+pt_2I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{t_1\left(I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z)\right)+t_2\left(I_{\mu,V}^{\lambda,P,\alpha}(a,c)f(z)\right)}\right) \qquad (4.5)$$

A simple computation and using (1.7) from (4.5), we get

$$\left(\frac{t_1I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z)+t_2I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{(t_1+t_2)z^p}\right)^{\delta}\times$$
$$\left(1+\eta\left(\frac{(pt_2-\alpha t_2)I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)+(t_2-\alpha t_1+pt_2-pt_1)}{t_1I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z)+t_2I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}\right.\right.$$
$$\left.\left.\frac{I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z)+(\alpha t_1+pt_1)I_{\mu,v}^{\lambda,p,\alpha+2}(a,c)f(z)}{t_1I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z)+t_2I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}\right)\right)$$
$$=k(z)+\frac{\eta}{\delta}zk'(z),$$

now , by using Lemma(2.4), we get the desired result .

Taking$q(z)=\frac{1+Az}{1+Bz}(-1\le B<A\le 1)$, in Theorem (4.1),we get the following Corollary.

Corollary 4.2. *Let* $Re\{\eta\}>0,\delta\in\mathbb{C}\setminus\{0\}$ and $-1\le B<A\le 1,$ such that

$$\left(\frac{t_1I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z)+t_2I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{(t_1+t_2)z^p}\right)^{\delta}\in\mathcal{H}[q(0),1]\cap Q.$$

.

If the function $G(z)$ given by (3.3) is univalent in U and $f\in W$ satisfies the following superordination condition:

$$\frac{1+Az}{1+Bz}+\frac{\eta}{\delta}\frac{(A-B)Z}{(1+BZ)^2}\prec G(z),$$

then

$$\frac{1+Az}{1+Bz}\prec\left(\frac{t_1I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z)+t_2I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{(t_1+t_2)z^p}\right)^{\delta},$$

and the function $\frac{1+Az}{1+Bz}$ is the best subordinant .

Theorem 4.2. Let $q(z)$ be convex univalent in unit disk U, Let $\delta,s\in\mathbb{C}\setminus\{0\}.\gamma,t,\psi,\tau\in\mathbb{C},q(z)\neq 0,and\ f\in W$. Suppose that

$$Re\left\{\frac{q(z)}{s}(2\tau\gamma q(z)+\psi)\right\}q'(z)>0,$$

and satisfies the next conditions

$$\left(\frac{I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{z^p}\right)^{\delta}\in\mathcal{H}[q(0),1]\cap Q, \qquad (4.6)$$

and

$$\frac{I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{z^p}\neq 0\,.$$

If the function $r(z)$ is given by (3.9) is univalent in U,

$$t+\psi q(z)+\tau\gamma q^2(z)+s\frac{zq'(z)}{q(z)}\prec \mathrm{r}(z) \qquad (4.7)$$

implies

$$q(z)\prec\left(\frac{I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{z^p}\right)^{\delta}$$,*and* $q(z)$ is the best subordinant.

Proof: Let the function $k(z)$ defined on U by (3.14).

Then a computation show that

$$\frac{zk'(z)}{k(z)}=\delta(\alpha+p)\left(\frac{I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z)}{I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}-1\right), \qquad (4.8)$$

by setting $\theta(w)=t+\psi\omega+\tau\gamma\omega^2$ *and* $\emptyset(w)=\frac{s}{\omega}$,it can be easily observed that$\theta(w)$is analytic in $\mathbb{C}$,$\phi(w)$is analytic in $\mathbb{C}\setminus\{0\}$ *and that* $\emptyset(w)\neq 0$ $(W\in\mathbb{C}\setminus\{0\})$.

Also , we get $Q(z)=zq'(z)\emptyset(q(z))=s\frac{zq'(z)}{q(z)}$, it observed that $Q(z)$ is starlike univalent in U .
Since $q(z)$ is convex , it follows that

$Re\left(\frac{z\theta'(q(z))}{\phi(q(z))}\right) = Re\{\frac{q(z)}{s}(2\tau\gamma q(z)) + \psi\}\acute{q}(z) > 0.$

By making use of (4.8) the hypothesis (4.7) can be equivalently written as

$$\theta\left(q(z) + zq'(z)\phi(q(z))\right) = \theta\left(k(z) + zk'(z)\phi(k(z))\right) \quad ,$$

thus , by applying Lemma (2.3), the proof is completed.

5.Sandwich Results

Combining Theorem (3.1) with Theorem (4.1), we obtain the following sandwich Theorem.

Theorem 5.1. Let q_1 and q_2 be convex univalent in U with $q_1(0) = q_2(0) = 1$ and q_2 satisfies (3.1). Suppose that $Re\{\eta\} > 0, \eta, \delta \in \mathbb{C} \setminus \{0\}$.
If $f \in W, such\ that$

$$\left(\frac{t_1 I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z) + t_2 I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{(t_1+t_2)z^p}\right)^\delta \in \mathcal{H}[q(0),1] \cap Q,$$

and the function $G(z)$ defined by (3.3) is univalent and satisfies

$$q_1(z) + \frac{\eta}{\delta}zq_1'(z) \prec G(z) \prec q_2(z) + \frac{\eta}{\delta}zq_2'(z), \quad (5.1)$$

then

$$q_1(z) \prec \left(\frac{t_1 I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z) + t_2 I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{(t_1+t_2)z^p}\right)^\delta \prec q_2(z),$$

where q_1 and q_2 are respectively , the subordinant and the best dominant of (5.1).
Combining Theorem (3.2) with Theorem (4.2), we obtain the following sandwich Theorem.

Theorem 5.2. Let q_i be two convex univalent functions in U , such that $q_i(0) = 1$, $q_i(0) \neq 0$ (i=1,2).Suppose that q_1 and $q_2$satisfies (3.8) and (4.8), respectively.
If $f \in W$ and suppose that f satisfies the next conditions:

$$\left(\frac{I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{z^p}\right)^\delta \in \mathcal{H}[Q(0),1] \cap Q,$$

and

$$\frac{I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{z^p} \neq 0\,,$$

and $r(z)$ is univalent in U , then

$$t + \psi q_1(z) + \tau\gamma q_1^2(z) + s\frac{zq_1'(z)}{q_1(z)} \prec t + \psi q_1(z) + \tau\gamma q_1^2(z) + s\frac{zq_1'(z)}{q_1(z)},$$

implies

$$q_1(z) \prec \left(\frac{I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{z^p}\right)^\delta \prec q_2(z)\,,$$

and $q_1$and $q_2$are the best subordinant and the best dominant respectively and $r(z)$ is given by (3.9).

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على نظريات الساندويتش التفاضلية من وظائف متعددة التكافؤ المحددة من قبل المشغل الخطي

وقاص غالب عطشان سلوى كلف كاظم
قسم الرياضيات ، كلية علوم الحاسوب وتكنلوجيا المعلومات ، جامعة القادسية ، الديوانية-العراق

المستخلص :

ا لهدف الرئيسي من هذا البحث هو استخلاص بعض النتائج للوظائف التحليلية متعددة التكافؤ التي يحددها المشغل الخطي باستخدام التبعية التفاضلية والإخضاع .

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Cuneiform symbols recognition by support vector machine (SVM)

Ali Adel Saeid Abdul Monem S. Rahma Abdul Mohssen .j. Abdul Hossen

Department of Computer Sciences , University of Technology

zeenali736@gmail.com monem.rahma@yahoo.com Abdulmoohsen53@yahoo.com

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Abstract

Cuneiform character recognition represents a complex problem in pattern recognition as result of problems that related to style of this type of writing and the diversity of its features according to distortion and shadows problems. This research proves that polygon approximation method is an optimal feature extra action method , which has been adopted for recognition task compeer with elliptic Fourier descriptor, according to the achieved high accuracy recognition results after applying multiple classes of support vector machine classifier along with depending on its discriminate functions .This work is applied by using two Data set , the first one contains 320 images of cuneiform symbols patterns for evaluate the optimal feature extraction method. The second contains 240 images of cuneiform characters to evaluate the recognition system, agents training dataset consists of 2D four triangular patterns.

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1. Introduction

Cuneiform writing is one of the oldest language systems which emerged in the third millennium BC, where the first character of writing was invented in Urk city in south of Iraq. Writing system is subjected to many stages of development to facilitate its characteristics about the shape of symbols and numbers that represent a development state of old Sumerian scrip language to Babylonian and Assyrian cuneiform language . At the beginning of the 19th century thousands of cuneiform tablets were discovered In Iraq and Iran, Which represent various Assyrian Babylonian and Persian civilizations. Today these tablets reside in many museums and the process of interpretation requires experience and time .However, the need for information technology was required to solve this problem about recognition task, therefore this research presents a recognizing way for the cuneiform symbols by applying OCR principle, through applying its chine sequential principals steps (preprocessing , segmentation , feature extraction and classification). The aim of research is to review a comparison state on features extraction methods, particularly between elliptic Fourier descriptor and polygon approximation methods to evaluate which one of them is adopted to design a recognition system. Where the adopted classifier is a support vector machine (SVM) according its discriminant functions. Finally, it must be noted in this research that the cuneiform character will be treated via collecting of cuneiform symbols, each of them will be classified according to two factors: the first one is to determine the cuneiform patterns and the second is related to its direction.

2-Literature Review

In 2017 Rahma.A. proposed a new method for recognized cuneiform symbols by adopting a polygon approximation as features extraction method. Where K-nearest neighbor classifier is adopted and the recognition raita that was achieved is 91% [1] .In 2014 Mostofi f. proposed intelligent recognition system for Ancient Persian Cuneiform Characters that is based on supervised back propagation neural network model, (classification model). The training data set is created by subjecting the original training set to Gaussian Filter with different values of stander devotion. The otsu's binirized model was adopted for computing global threshold value. The recognition achieved rate was 89-100 %,[2].In 2013 Naktal M. proposed a method for recognizing the cuneiform symbols depending on statistical and structure features derived by projection histogram, center of gravity and connected component features. However to separate each distinguish feature according to each class of symbols, the k-mean clustering was used. Multilayer Neural Network (MLP) was applied for classifying a task where the recognition rate of accuracy level was different according to each class from 83.3% to 95.1%[3].In 2001 Al-Aany proposed recognition approach for cuneiform symbols depending on extract recognition features that generated from binary cuneiform image symbol by depending on suggested seven transform forms applied on each pixel's with their neighbors. However, each cuneiform symbol will have distinguish features related to directions used for recognition task .The classification process is implemented by indexing process that was distributed on tree structure [4].

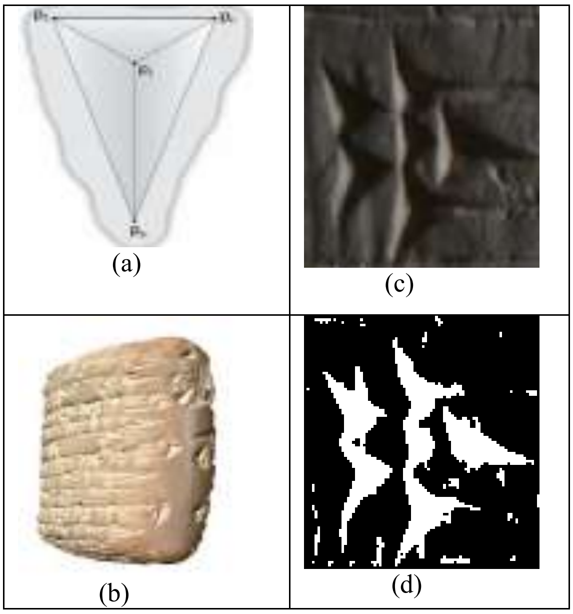
3-Cuneiform writing

The Assyrian cuneiform language represents one of the stages of the development of cuneiform writing in Mesopotamia, which continued from the beginning of the first millennium to 600 BC. It relies on drilling cuneiform symbols on clay tablets or tablets of stone from left to right to form cuneiform groups which reflect the basic language meanings. This language consists of a set of letters. Each one consists of one or set of cuneiform symbols, these symbols or wedges are organized in different directions either horizontal, vertical, Oblique or diagonal, therefore, these letters with their symbols vary from one character to another according to the (number of symbols, their direction and their location)[5] figure (1)

Figure (1) cuneiforms character images from Iraqi museum

The problems related with recognition tasks have more than one face. The First problem relates to the nature of the writing medium, whether it is stone or clay(2.c), with a three-dimensional writing form. The second relates to the nature of geometry of the cuneiform symbol, which takes the three-dimensional form (three surfaces) figure(2.a),The third problem is the cuneiform writing style, it does not depend on writing on one face but may take writing on all the surfaces of the tablet [5], figure(2.b) .

The last problem that interfaces the cuneiform recognition relates with undesirable results like spots after being subjected to segmentation process, figure (2.d), where their features differ from one image to another which affect negatively on the recognition task [1].



Figure(2) cuneiform writing style . a) the 3-D geometry shape of cuneiform symbol , b) the style of cuneiform writing take more than one surface. ,c) cuneiform image character ,d) spots problem

4. Image enhancement:

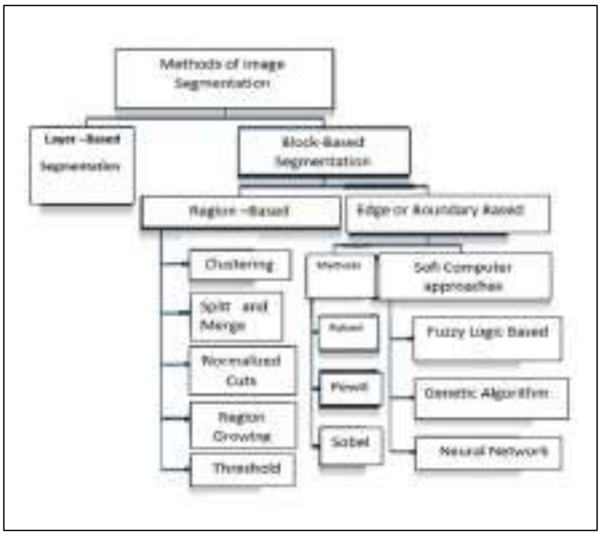
Image enhancement is one of the most important image processing techniques, which leads to reconstruct the image's features to suit the nature of application's requirements . The primary objective is to treat all the associated problems related with blurring ,contrast and noise. The process of enhancement task takes two directions: the first one submits to human vision as criterion for the evaluation, and the second is moving towards supporting and improving image qualities used to support the identity process by machine vision. Enhancement techniques can be classified into two categories[6]:

1- Frequency Domain.
2- Spatial Domain.

In this research the frequency domain is adopted for applying the enhancement process as the ideal low pass filter.

5.Image segmentation .

It is an image processing technique that leads to segment the image's pixel to segments of regions where each of them has distinguished labels. This simplification process is used to simplify images features to easier or meaningful feature form to be used to support the advanced analysis's or recognition stages. Image segmentation techniques are categorized into two branches: its block and layer based segmentation as seen in following figure (3) [7].



Figure(3):image segmentation techniques

Thresholding is a popular image segmentation technique that is adopted by large number of binriztion methods. It leads to separate the image into two sets group of regions based on selected threshold value (T). If pixel intensity color value is larger than the threshold, it will represent foreground region in the opposite case. It is considered as background, as mathematical formula below.

$$G(x,y)=\begin{cases}1\,, if\ f(x,y) > T \\ 0\,, if\ \ f(x,y) < T\end{cases} \quad \ldots(1)$$

Therefore,to apply image segmentation by thresholding, two formals are adopted to apply this task, these are Niblack and Sauvola's method and choosing one of them depends on statistical Skewness metric [1].

6-Image labeling by Extraction of Connected Components

To reach to labeled image, (Multi-scan strategy) will be applied (which is represented by **Extraction of Connected Components**) on binary image, ***A*** as it contains foreground pixels with labeled value equals (1) and background, their pixel labeled value is (0). This process is implemented iteratively with restricted condition depending on dilation concepts.

Initial step starts by locating the first foreground pixel (p) which it represents seed point for reconstructed matrix $\mathbf{X_K}$ with the structure element ***B,*** scan the image ***A*** for computing the following form[8],[9] .

$$X_k = (X_{k-1} \oplus B) \cap A, \quad k = 1, 2, 3, \ldots \qquad \ldots(2)$$

let X_0=p where K=0,1,2…n

This iterative process would be terminated after the terminated condition was satisfied as $X_K=X_{K-1}$. For applying image labeling concept, each regenerated connected component will be assigned by **distinguished label** for each connected component element

7. Elliptical Fourier Descriptor:

Elliptical Fourier Descriptor (EFD) represents boundary shape descriptor where the classification feature is generated from Fourier coefficients [ai,bi,ci,di] about each **N** harmonic to recognize closed contour of **k** elements , Where harmonic coefficients are defined as follows[10]:

The closed coordinate points on N harmonic can be calculated as follows :

$$a_n=\frac{T}{2n^2\pi^2}\sum_{i=1}^{k}\frac{dx_i}{dt_i}\left[\cos\frac{2n\pi t_i}{T}-\cos\frac{2n\pi t_{i-1}}{T}\right] \quad ...(3)$$

$$b_n=\frac{T}{2n^2\pi^2}\sum_{i=1}^{k}\frac{dx_i}{dt_i}\left[\sin\frac{2n\pi t_i}{T}-\sin\frac{2n\pi t_{i-1}}{T}\right] \quad ...(4)$$

$$c_n=\frac{T}{2n^2\pi^2}\sum_{i=1}^{k}\frac{dy_i}{dt_i}\left[\cos\frac{2n\pi t_i}{T}-\cos\frac{2n\pi t_{i-1}}{T}\right] \quad ...(5)$$

$$d_n=\frac{T}{2n^2\pi^2}\sum_{i=1}^{k}\frac{dy_i}{dt_i}\left[\sin\frac{2n\pi t_i}{T}-\sin\frac{2n\pi t_{i-1}}{T}\right] \quad ...(6)$$

$$Xi=X_{c+}\sum_{n=1}^{N} a_n \cos\frac{2n\pi t_i}{T} + b_n \sin\frac{2n\pi t_i}{T} \quad ...(7)$$

$$Yi=Y_{c+}\sum_{n=1}^{N} a_n \cos\frac{2n\pi t_i}{T} + b_n \sin\frac{2n\pi t_i}{T} \quad ...(8)$$

Xc and Yc is centroid's coordinate. Defined by below equations.

$$X_{c=}\frac{1}{T}\sum_{I=1}^{K}\frac{dx_i}{2d_i}(t_i^2 - t_{i-1}^2) + \beta_i(t_i^2 - t_{i-1}^2) \quad ...(9)$$

$$y_{c=}\frac{1}{T}\sum_{I=1}^{K}\frac{dy_i}{2d_i}(t_i^2 - t_{i-1}^2) + \alpha_i(t_i^2 - t_{i-1}^2) \quad ...(10)$$

T= The length of chain code.

$$dt_i=1+\left(\frac{\sqrt{2}-1}{2}\right)(1-(-1)^{ui}) \quad ...(11)$$

$$t_n=\sum_{i=1}^{n} dt_i \quad ...(12)$$

8. Polygon approximation with Dominant point approaches

In 2007 Asif Masood [11] proposed a new approach for polygon approximation, it is defined as **revers polygonization** principles which provide a good representation about boundary of 2D shapes with high accuracy for data reduction , shape matching and pattern recognition [11]. Basically, this method implementation depends on dominant points (DP) concept , assigned for endpoints of approximated line segments ,where the initial set of (DP) starting with detected break points(BP) is extracted from shape boundary. However, (BP) is derived after applying **freeman's chain code** on shape boundary points. Therefore, any boundary point is defined as break point (BP) that its chain code (Ci) value is not equal the chain code of prior point [12][11], However to apply polygon approximation technique, there are two optimization approaches being used to solve this approximation problem[14][13] .

1-Min-ϵ problems: let polygon **P** vertices to be approximated by another polygon **Q** with predefined number of segments **M** along with minimizing approximation error.

2-Min-# problems: let polygon **P** vertices to be approximated by another polygon **Q** with minimum numbers of segments M, where the error dose not skip the predefined tolerance value .

9.Support Vector Machine (SVM)

In 1995, Vapnik proposed a binary classification model for the supervised learning, it is a linear classifier in a feature space. Generally, it is a classification task with two classes once the training parameter is determined, however this model is based on custom functions for classification .It is defined as kernel functions. Therefore the SVM algorithm with initial form is like a decision boundary that separates between two classes, but this model can be improved to increase the separation process in new space ,Which results in creating non liner decision boundary (Kernel function) . The Kernel functions(K) in table(1) was used by SVM to map input data feature space to high feature space when the training data are not linearly separable**[15],[16]**.

table 1: kernel functions models

Core	**Formula**
Linear	K(x,y) = x.y
Sigmoid	K(x,y) = tanh(ax.y +b)
Radial Basis Function (RBF)	K(x,y) = exp(-‖x-y‖2/σ2)
Polynomial	$K(x,y) = (ax.y +b)^d$

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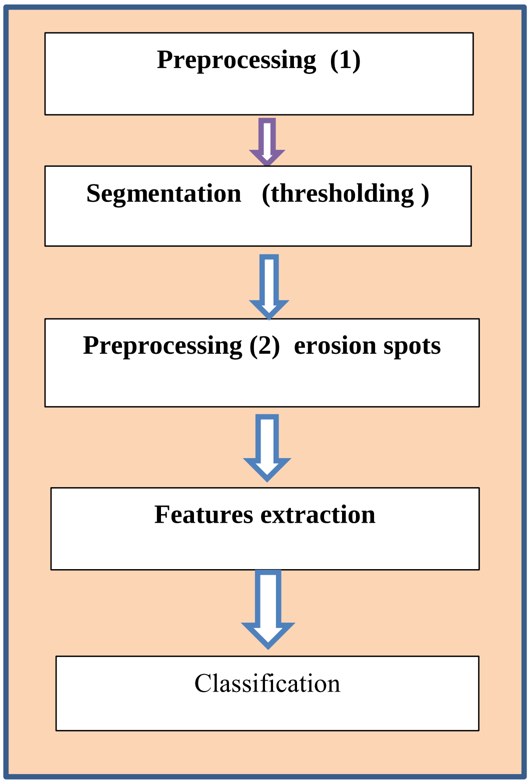
10.Proposed recognition system

This section presents the main diagram figure(4) for cuneiform recognition system starting with preprocessing stage (1)to image segmentation , preprocessing stage (2),features extra action and classification , in addition to review the proposed algorithms according each stage.



Figure(4):proposed recognition system

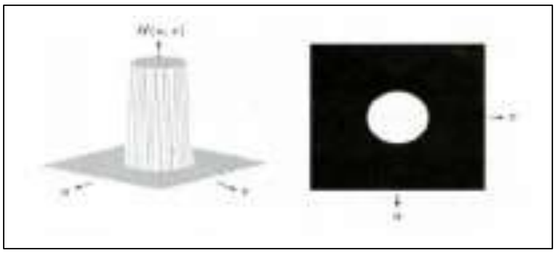
10.1Preprocessing stage (1)

Image enhancement. In this research the frequency domain is adopted for applying the enhancement process. The ideal low pass filter is dependent ,where as ideal low pass filters can be defined as follows:-

:-

$$H(u,v) = \begin{cases} 1 & if\ D(u,v) \le D0 \\ 0 & if\ D(u,v) > D0 \end{cases} \quad .... (13)$$

Where D0 is non negative value which represents the radios of cutoff frequency and $D(u,v)$ is distance value starting from point to center of frequency figure (5).



Figure(5): ideal low pass filter.

10.2. Image segmentation (Thresholding)

Therefore for applying image segmentation by thresholding, two formals that are adopted to apply this task are Niblack and Sauvola methods, these are defined respectively in following forms, choosing one of them depends on statistical Skewness metric [1].

$$T=M+k\sigma…(14)$$

$$T=m(1-k(1-\frac{\sigma}{r}))…(15)$$

Where k is constant, (m, σ) represents the mean and standard deviation respectively.

10.3 Preprocessing stage (2)Spot removing

For eliminate the unwanted elements segments like (spots) ,figure(6.b) that resulted from subjecting the gray cuneiform image ,figure (6.a) to thrsholding process. However the target of this process is to create uniform features about cuneiform image symbols to be clear from their elements figure (6.c). Therefore to satisfy this principle, this research adopts the Image connected-component labeling (CCL) concepts to erase the spots according to bellow algorithm .This process get satisfied after applying bellow Algorithm several times to extra each connected elements and assign distinguish label for each one and apply the erosion process for each segments which has small ratio to all image's pixels.

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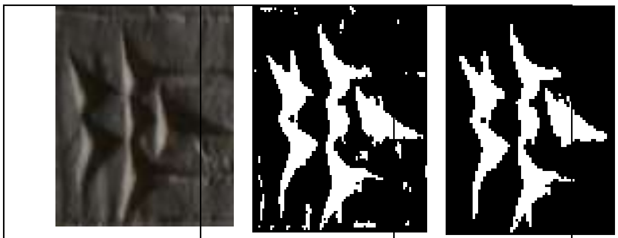
Algorithm (1) : Connected Components Extraction. **Input :Binary image.** **Output : Connected Components image**
Step1: read input binary image IB Step2: locate the first foreground pixel p and it's location p(x,y). Step3:initlize the stricture element B . Where B={1…9}=1; Step4:k=0; Step4:intilize the connection component matrix $X_K(0,0)$. Step5 :set $X_K(x,y)$ =p(x,y). Step6: **repeat** Y=Xk Applied the dilation process on X_k and interest the result with original Image ***IB*** as following formula. X_{k+1}= dilation(B,X_K)∩ ***IB*** . K=K+1. **Untiled** (Y==X_{K+1}) Step7: set CC_MATRIX=Y. Step8:return (CC_MATRIX).



(a) (b) (c)

Figure (6) spots erosion a) cuneiform image ,b) binary image with spots ,c) spots off image

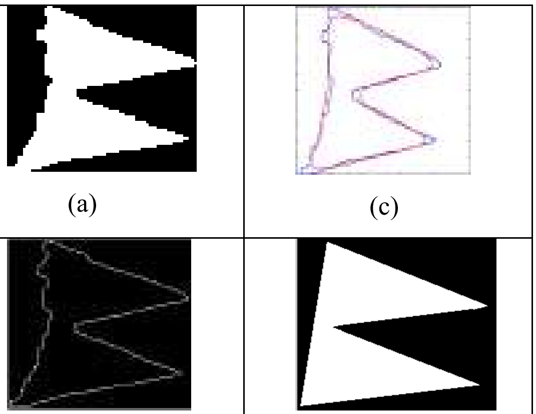
10.3.Feature extraction

This section presents two features extraction methods depending on boundary pixels to apply the comparison state between them

10.3.1Feature extraction by Elliptical Fourier Descriptor (EFD).

The features vector generated by EFD is based on quadruple Fourier coefficient (***ai,bi.ci,di***) values that are defined by equations (3-6) ,where the predefined number of harmonic ***N***, determines the size of features vector. That means that each one of these four parts corresponds to the quadruple coefficient sequentially and its length agrees with degree of harmonic. However, for extracting the boundary coefficient to generate quadruple Fourier coefficient, each cuneiform image's symbol is subjected to extract its boundary by (canny operators) as closed contour to generate freeman chain code that leads to determine the length of chain (***T***)as seen in figure (7). The Elliptical Fourier algorithm works follows:

Algorithm 2: quadruple Fourier oefficient **Input**: cuneiform binary image symbol **Output:**. Fourier coefficient (***ai,bi.ci,di***)
Step1: read binary image symbol $\mathbf{I_s}$. step2: extract boundary pixels of cuneiform symbol by edge detection method and save the results which represent a boundary image I_B. Strp3: apply the thinning operation on boundary pixel . Step4: compute edge encoding process to generate freeman direction code $\mathbf{U_i}$. where $U_{i=\{0,1,2,3,4,5,6,7\}}$. Step5: compute the length of previous direction code dt_i Where $dt_i = 1+\left(\frac{\sqrt{2}-1}{2}\right) * (1-(-1)^{ui})$. Step6: compute the harmonic coefficient (***ai,bi.ci,di***.)



(a) (c) (b) (d)

Figure (7) Elliptical Fourier Descriptors ,a) binary cuneiform symbols, b) boundary image . c)approximated boundary figure ,d) matching class .

10.3.2 Feature extraction by polygon approximation with dominate point

The proposed algorithm to generate features vectors about each cuneiform symbols depends on combining the approximation principals (**Min-ϵ problems, Min-# problems**) in approximation approaches as illustrated in (7). That means briefly with first approximation principles (**Min-ϵ**) the number of segments determined previously is compatible with same number of cuneiform classis segments as seen in figure(11) and the tolerance value will be pre-defined where it decreases gradually in each iteration until the termination condition gets satisfied,[1]. The termination condition is satisfied if the number of segments equals the number of one the cuneiform patterns segments (3,5,7,9). the proposed algorithm is as follows:-

Algorithm 3:Polygon approximation
Input: binary cuneiform image.

Output: approximated points

Step1: read binary image symbol $\mathbf{I_s}$
Step2: apply edge detection method with suitable filter,
Step3: apply thinning technique ;
Step4: compute Freeman's chain code for boundary .
Step5: find break points DP_b.
Repeat
Step6: compute AVE for all DP
Step7: repeat

Step8: determine DP that minimum value DP_{min}
Step9: remove DP_{min} from dominat table
Step10: recalculate AVE for DP_{min}'s adjacent neighbor.
Step11: compute max $_{error}$
step12: until (max $_{error}$<th)
step13.z= Remaining points about DP's is approximate polygon points
step14: deleted all DP'S which construct with its neighbors a straight angle
step15: th=th-eps; < eps epsilon value ex=0.009 >
step16 : until ((z==3)or(z==5)or(z==7)or(z==9))
step 17: end <where z= number of head >
step 18:return approximate points vector .

the recognition process depends on generating a **features vector** for the training set and testing symbol, as illustrated in figure (8), where each feature vector consists of Cartesian coordinate (approximate points) for each class .

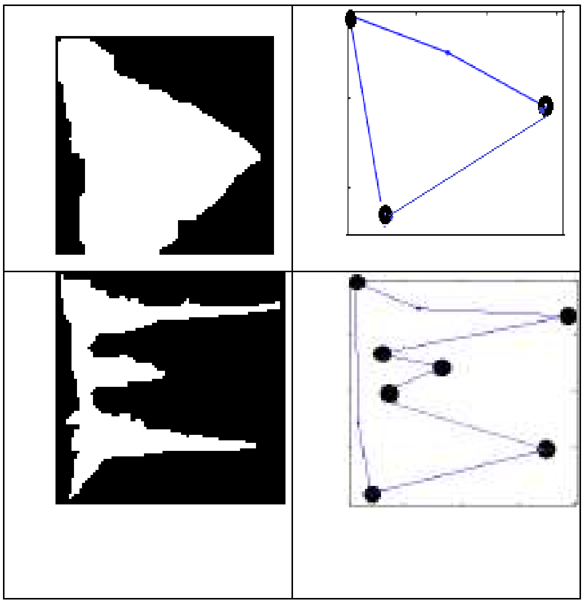
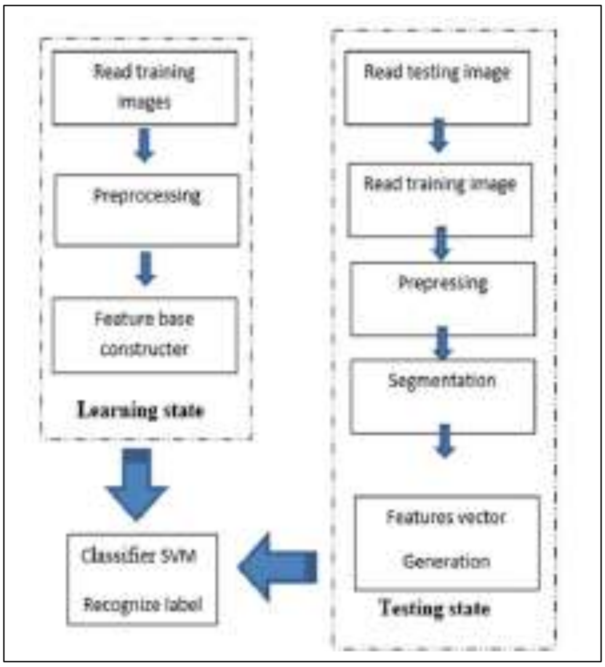


Figure (8): approximated points.

10.4 Classification

The classification task is applied with each cuneiform symbols by SVM classifier as a generated feature vector is adopted **for each features extraction method**, to apply the comparison state. However, the classification stage concerns with two steps. The First one is a training state to generate features base with assigning a distinguished labile for each class (triangle class) .The other step applies the classification task with test feature vector. Below is the classification diagram figure (9) .

.



Figure(9): classification diagram steps.

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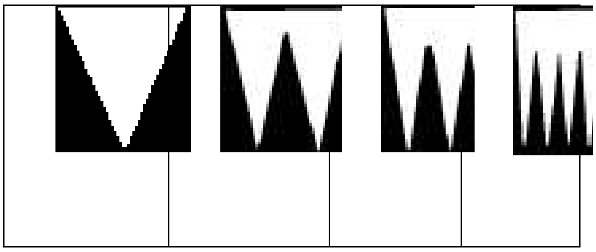
However, after applying the multiple classification task for each symbol the constructed recognition code consists of three parts , which represent number of cuneiform symbol distributed to numbers of vertical, horizontal and diagonal. For example as seen in figure (10) below the cuneiform character, the recognition code (6#2#1#3) which means the total cuneiform symbols equals 6 , which are distributed to 2 vertical ,1 horizontal and 3 diagonal.

Figure(10):cuneiform character .

11.Proposed training dataset

The design of the dataset is one of the most important aspects that play an important role in the classification process. Therefore this proposed training dataset is a virtual dataset consists of 2D four triangular patterns with forms compatible with three-dimensional geometric shape form of the cuneiform symbols, figure (11) . Covering all the possibilities and situations is taken by the cuneiform character's symbols with different directions (horizontal, vertical or diagonal), and compatible with patterns of cuneiform symbols to solve the problem of shadows. These patterns are distributed in 16 classes compatible with cuneiform directions.



Figure(11) :cuneiform patterns

11.Results and discussion:

This section presents the evaluation state according to features extraction metrics (EFD and polygon approximation) **.The first testing state** consists of 320 symbols related to probabilities of conform symbols to determine the recognition accuracy. Therefore, each binary cuneiform symbol is subjected to extra boundary pixels and thinning process to generate features vectors according to each method of comparison state. Therefore, **the second experiment state** which deals with cuneiform character dataset consists of 240 of cuneiform image characters, after evaluating the comparison state. However after applying the first testing process the recognition accuracy with average **processing time** is illustrated in the following tables (table 2) according to each feature extraction method.

Table 2: Comparison of recognition results about each features extraction method with processing time.

Experiment/n	**Classification approaches**	Accuracy results with (EFD)	Processing time	Accuracy results with (Polygon approximation)	Processing time
1	SVM with RBF kernel	0.6093	0.332	0.940	0.452
2	SVM with **polyn-omial** kernel	0.6845	0.322	0.925	0.421
3	SVM with linear kernel	0.6594	0.320	0.613	0.441

As seen in previous table, the higher recognition result is achieved when polygon approximation method is adopted to construct features vector especially with RBF discriminate function (Experiment 1). Therefore, to evaluate the reliability about the proposed features extraction method the previous testing set is subjected to resizing process according to the different sizes. Bellow the accuracy results about each size is illustrated in table (3)

Table 3: Comparison of recognition results according to different image sizes, where the SVM classifier is adopted with RBF function.

Image size	**Accuracy**
256x256	0.940
128x128	0.865
64x64	0.853

Along with applying Gaussian filter on testing set many times according each standard deviation values **δ** as seen in figure (12) to evaluate the recognition reliability.

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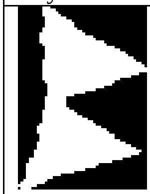
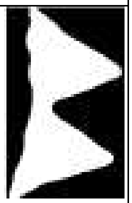
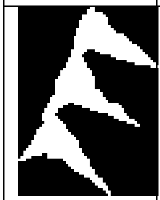
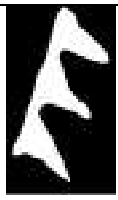
Binary cuneiform symbols	δ =1	δ =3	δ=7
			
			

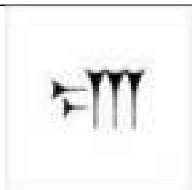
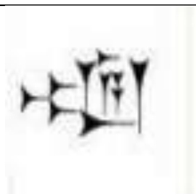
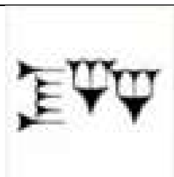
Figure (12): sample of conform symbol deformation. applied Gaussian filter on each cuneiform symbols (in first dataset) with different values of each standard deviation .

Therefore, the first dataset is subjected to deformation process by Gaussian filter several times according to each experiment depending on different standard deviation value, the accuracy results are illustrated in table (4) .

Table4 : Comparison of recognition accuracy results according to different standard deviation values **δ.**

standard deviation value	Accuracy
1	**0.860**
3	**0.867**
4	**0.870**
7	**0.860**

However, now after adopting the polygon approximation with SVM classifier, according to its discriminant function (RBF) the recognition result about the **second dataset** is satisfied against each cuneiform character image which can be seen in the following figure(13),**where the recognition state for each character results from a cumulative recognition process**.

Cuneiform character image	Recognition text image
	
	
	
	
	

Figure(13): cuneiform character image sample recognized by proposed system.

When adopting spatial low pass filter domain for enhancement process (according to the second dataset) the output of accuracy character recognition results are illustrated in following table (Table 4):-

Table 4:Comparison of results of recognition accuracy ratio after applying different LPF with different sizes

Filter size	3X3	5X5	7X7
Medina	0.496	0.512	0.496
Gaussian	0.536	0.440	0.512
Average	0.472	0.544	0.504

Table 5: Comparison of recognition accuracy results

According to cut of frequency values

Experiment/no	Cutoff frequency values	accuracy
1	0.2	0.51
2	0.4	0.61
3	0.6	0.94
4	0.8	0.62
5	0.9	0.67

As seen in the previous table (5), the higher accuracy character recognition is archived in experiment (3) ,where the value of cut off frequency equals (0.4).

Now the evaluated recognition state is applied among the discriminant functions about SVM with (**second dataset**), where the value of cutoff frequency equals (0.4) .bellow are accuracy character recognition values and **processing time** in table (6).

Experiment/no	Discriminant function	Accuracy	Average Processing Time(s)
1	RBF	0.92	4.028
2	liner	0.77	4.136
3	polynomial	0.75	4.418

As it can be seen in the previous table, the highest accuracy character recognition with low processing time when adopting the RBF is compared with other discriminant functions LPF..

Conclusion

This research presents a comparison state between two feature extracting methods ,the elliptic Fourier descriptor (EFD) and polygon approximation methods, where the support vector machine is adopted as a classifier model with its discriminant kernel functions .Consequently, the evaluated recognition results indicate that the polygon approximation by dominate points is more accurate to be adopted in classification model as the achieved accuracy result is 94% with RBF kernel function compared with (EFD) method . To achieve more reliable decision about the polygon approximation method, the testing set is subjected to deformation state by Gaussian filter with different values of stander divisions, accordingly, the accuracy result is maintained to high quality .The testing state is applied by two testing data set. The frequency domain with low pass enhancement filter is more accurate than the spatial domain according to the accuracy of the achieved results that were compared between them..

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تميز الحروف المسمارية باستخدام مصنف الدعم الأتجاهي

علي عادل سعيد عبد المنعم صالح رحمه عبد المحسن جابر عبد الحسين

قسم علوم الحاسوب ، الجامعة التكنولوجية ، بغداد العراق

المستخلص :

يهدف هذا البحث الى اجراء مقارنه بين طريقتي بناء متجه الصفات بين طريقة تقارب المنحنى وطريقة الواصف الأهليجي الدوري مع بيان اي الطريقتين يمكن اعتمادها في انشاء متجه الصفات ،وذلك اعتمادا على النتائج المتحققه المتمثله بعاملي الوقت والدقة. اظهرت النتائج المتحققه الى تفوق الطريقه الأولى في بناء متجه الصفات على الطريق الثانية . تم ومن خلال البحث اجراء الأختبارات المطلوبه على مجموعتي اختبار ،الأولى مكونه من 320 ثنائية اللون لأختبار اي الطريقتين افضل في بناء متجه الصفات . اما مجموعة البيانات الأختبارية الثانيه فهي مكونه 240 صوره للحروف المسماريه الأشوريه يتم اختبارها بعد استكمال المرحله الأولى .تم اعتماد المصنف الدعم الأتجاهي بدواله المميزه الخطيه ،متعدده الحدود.وداله توزيع كاوس. حيث اثبتت النتائج ا لأختباره على افضليه اعتماد طريقة تقارب المنحنى في بناء متجه الصفات حينما تكون الداله المميزه كاوس معتمده.

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Segmentation Of Tumor Brain Based On The Colour

Awfa Hasan Dakheel

Collage of Basic Education , University of Babylon , Iraq
Awfa_d@yahoo.com

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Abstract :

The objective of this study is to present a method that aids in the diagnosis of diseases in brain due to tumors from MRI brain image whereby segmenting the brain tumor is done by using a novel algorithm that depends on the colour of the 2D image. Therefore, this work consists of three main stages, the first one is loading image into memory, and then the segmentation algorithm is applied. Finally, in order to obliterate the noise object, the 2D median algorithm is conducted. After applying the method the results which are obtained show better output to determine the tumor, simultaneously the diameter of the tumor can be calculated.

Keywords: Noise removal, Segmentation algorithm, 2D median algorithm, Tumor.

1.Introduction

One of the main causes of the increasing rate of mortality among children and adults is brain tumor, where a tumor is defined as any mass that is produced by abnormal growth may affect anyone irrespective of their age. It consequently might be alike for everyone. Tumors can destroy brain cells directly. The healthy cells can also be damaged indirectly by moving some brain parts, which results in “inflammation, brain swelling, and intracranial pressure” [1].

Brain tumors are classified into two types, the first is malignant type and the other is called benign. Malignant neoplasm is also called brain cancer, where malignant melorem hastily and often occupies or attracts the brain vigorous zones. By contrast, the benign brain tumors, which usually grow slowly, have no cancer cells at all [2].

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The result of using computer technology is widespread and comprehensive in many life applications or areas such as medical decision support covering a wide range of medical fields, such as cancer research, heart disease, gastrointestinal tract, and brain disease. In the last century, computer-aided diagnosis (CAD) has gradually become an essential area of intelligent systems [3]. CAD becomes very important in many applications such as detection or classification of diseases.

In general, unusual deviations happening in organs and tissues can be detected early by means of a number of diagnostic and imaging techniques such as CT scans, MRI, X-rays, and ultrasound [4]. "Magnetic resonance imaging (MRI)" is considered to be one of the basic remedial methods frequently used to portray the construction and function of the human body in which it provides rich information on excellent soft tissue variation and is particularly useful in neuroscience[5].
The fragmentation of medical images is a major step and an introductory stage in the use of computer assistance. The success of medical image analysis depends largely on micro-image fragmentation algorithms. It has become clear that the exact division of the medical image is essential in the planning of radiotherapy, clinical diagnosis and treatment planning [6].

Image Segmentation is very important stage in images and interpretation, processing, image segmentation consists of extracting one or several objects of interest from a given image [7].

Broadly, the techniques of image segmentation can be categorized into Region Based method, Threshold method, Clustering method Region and Edge based method [8],[9].

The threshold method is one of the important and vastly used methods which recognizes foreground objects from the background for medical image segmentation. This can be achieved by the similarity of gray levels. To select an appropriate threshold value T in between two peaks Thresholded image g(x,y) From the histogram of an image, can be defined as

$g(x, y) = 1$ if $f(x, y) > T$ (1)

0 else

where $f(x, y)$ is the input image

So, the values associated with gray level are to be categorized as "black (0)" if they are less than T, whereas the values exceeding T are to be the "white (1)". Threshold method is thereby is a means to obtain a binary image from gray level one being processed and altered [10].

The gold standard for performing segmentation is to manually delineate the object boundary [11]. Segmentation - as a means of medical capturing - plays a very paramount role in diagnosing apart from treating a lot of diseases. Still, [12] it is deemed a challenging job because of the slender dissimilarity along with "speckle noise" reflected in the photos.

To achieve this goal, several methods have been proposed for the use of image segmentation in the field of therapy, aiming to split the image into distinct areas so that determining the tumors will get possible [13] [14]. Some of authors depend on K-means clustering algorithm to extract the tumor such as [15]. The other uses the Threshold techniques [16], [17] or Histogram threshold technique that is achieved by the presumption that all pixels possess intensity rate below the predefined threshold's which belongs to a specific region.

This paper is structured as follows: section 1 to cover the introduction; section 2 to tackle the materials and methods; section 3 to discuss the results and finally, section 4 which is devoted to the conclusion.

2. MATERIALS & METHODS

This section is devoted to presenting the proposed method that is in Figure1; it is divided into a set of steps and each is responsible for a specific job:

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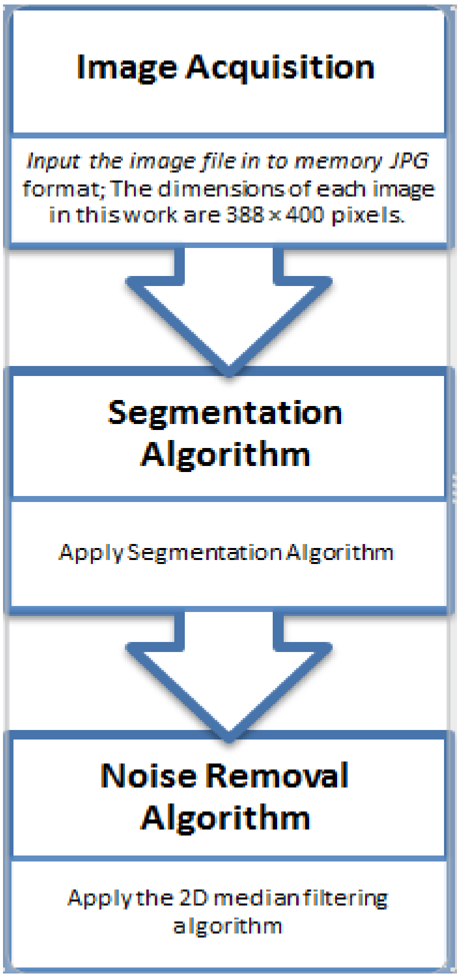

Figure 1: Main diagram of proposed method

2.1 Image Acquisition

At this procedure, the image file is uploaded to memory in *JPG* format. A digital image is composed of a finite number of elements, each of which has a particular location and values. The elements of a digital image are called pixels .The dimensions of each image in this work are 388 × 400 pixels.

2.2 Segmentation Algorithm

Subsequently, the said algorithm will apply to have the tumor extracted from the brain. This study, in effect, depends on the tumor colour whose crucial role is represented by identifying the objects. The segmentation approach hence is elucidated by the algorithm below:
Firstly: Read the coloured photo.
Secondly: Spilt the prime colour matrix to 3 matrices [R], [G] and [B].
Thirdly: Calculate the mean and the standard deviation for each row.
Fourthly: Match a normal distribution object to the data.
Fifthly: Calculate the threshold (95% confidence interval for the distribution parameters for each matrix Red, Green, and Blue).
Sixthly: Extract the (Red, Green, Blue) colours from the image by comparing each pixel for every matrix with threshold, if it is less than threshold, the pixel value is equal to zero, or else it is equal to one.
Seventhly: Apply noise removal algorithm.

2.3 Noise Removal Algorithm

Applying the “segmentation algorithm’ is associated with the appearance of some objects which, besides being noisy, do vary in sizes and shapes. Subsequently, “2D median filtering algorithm” [18] is necessary to strip away those intrusive objects which must be tested by means of windows prior to the elimination process, as long as they are located within the window boundary. The algorithm is as mentioned in the study [19].

2D Median Filtering (Huang & Yang, 1979) Algorithm:

```
allocate output Pixel Value[image width][image height];
allocate window[window width × window height];
edgex=(window width / 2) rounded down;
edgey=(windowheight / 2) rounded down;
for x from edge of image width - edge x
  for y from edgey to image height - edgey
    i = 0;
    for fx from 0 to window width
      for fy from 0 to window height
window[i]:=inputPixelValue[x+ fx - edgex][y+ fy - edgey];
i = i + 1;
    sort entries in window[];
output pixel value [x][y]=window[window width × window height/2];
```

Eighthly: Return the coloured pixels to the whole matrix (Red, Green, Blue).
Ninthly: Compare the three matrices with 8unit.

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3.Result and discussion

This part focuses on segmenting the tumor from the MRI brain image by using a novel algorithm that should run to extract the tumor .The proposed method implemented by using MATLAB application. In MATLAB application, in order to have the loaded MRI brain files of images read and stored in the memory, the user's border window should be opened immediately, as manifested in Figure 2:

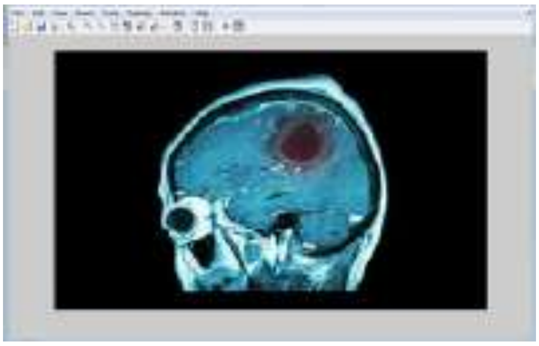

Figure 2: Represent The Original Image

This study depends on the colour of the tumor because it is a significant element to recognize the objects. Thus, after the images have been uploaded, the "segmentation algorithm" starts to extract the (Red, Green and Blue) colours from the image by comparing each pixel for every matrix with Threshold: if it is less than Threshold, the pixel value then is equal to zero or else it is equal to one. As shown in Figure 3, that clarifies the result of the segmentation procedure:

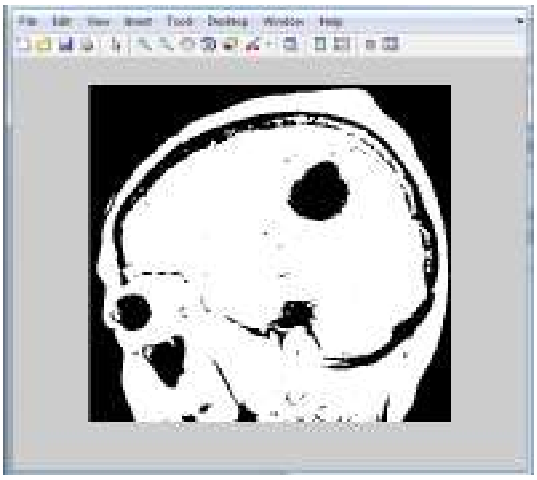

Figure 3: Apply segmentation algorithm.

In the subsequent process, the "2D median filtering algorithm" is used to eradicate the noisy objects due to applying the segmentation algorithm .This is achieved through using windows to test the objects causing noise in the image. Hence, figure 4 shows the result:

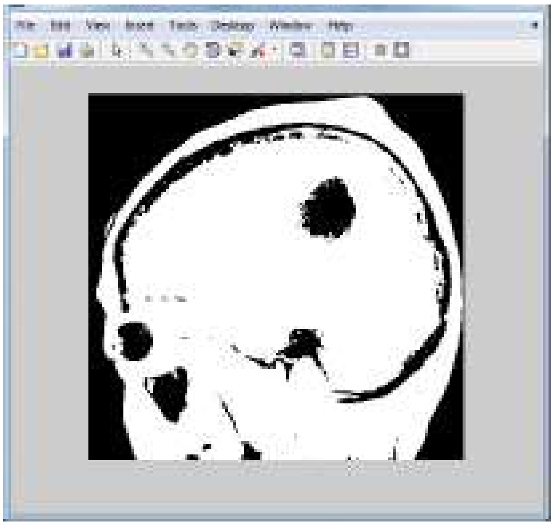

Figure 4: Noise removal.

In the end, a comparison should be made among the image pixels; all the pixels having (1) will maintain the original value of (Red, Green, and Blue) otherwise, they must have (0). Figure 5 displays that.

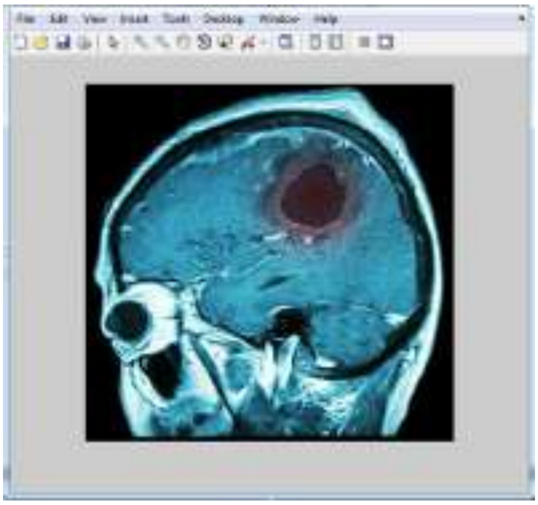

Figure 5: Tumor segmentation.

We were able to determine the tumor size, by calculating the diameter of the tumor .Therefore, a another results we can shown in Figure 6 :

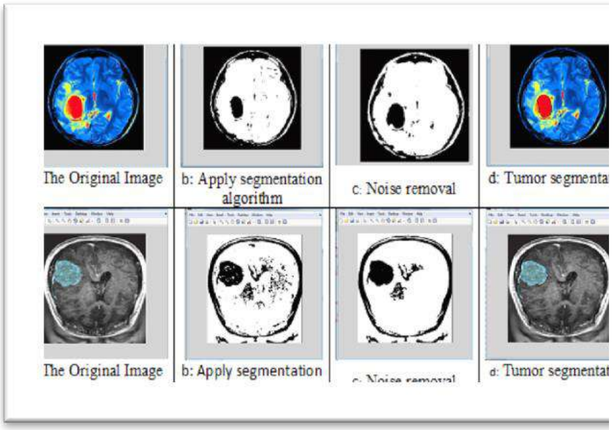

Figure 6: (a, b ,c ,d):Represent the result of the other images

As shown in Figure 6, the proposed method is applied in individual images to test it by using variant MRI brain images each one has got variant position and size of tumor in the brain, so in each case of MRI brain image, it is possible to determine and calculate the diameter of the tumor.

4. Conclusion

The proposed method is developed for segmenting the tumor which inflicts the brain from the brain MRI images. Therefore, the method that we have proposed is performed through multiple stages. The first stage is image acquisition, and with the second stage segmentation algorithm is to be applied; finally, we use the "2D median filtering algorithm" to get rid of the objects accompanied with noise. The proposed method applies on variant MRI brain images and each of these images contains a variant position and a specific size of the tumor. It is concluded that the proposed methods is effective and it can achieve promising results for determining the tumor and calculating its diameter.

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تجزئة ورم الدماغ بناء على اللون

اوفى حسن دخيل
كلية التربية الأساسية ، جامعة بابل ، العراق

Awfa_d@yahoo.com

المستخلص :
الهدف من هذه الدراسة هو تقديم طريقة تساعد في تشخيص الامراض في الدماغ بسبب الأورام من خلال تصوير الدماغ بالرنين المغناطيسي ، حيث يتم تقسيم الورم الدماغي باستخدام خوارزمية جديدة تعتمد على لون الصورة ثنائية الأبعاد. لذلك ، يتكون هذا العمل من ثلاث مراحل رئيسية ، أولها هو تحميل الصورة في الذاكرة ، ثم يتم تطبيق خوارزمية التجزئة. أخيرا ، من أجل محو كائن الضوضاء ، يتم إجراء خوارزمية متوسط 2D . بعد تطبيق الطريقة تظهر النتائج التي يتم الحصول عليها إخراج أفضل لتحديد للورم ، في الوقت نفسه يمكن حساب قطر الورم.

كلمات البحث: إزالة الضوضاء ، خوارزمية التقسيم ، خوارزمية متوسط 2D ، ورم.

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Monitoring software risks based on integrated AHP-ANN method

Jaber Ibrahim Naser
Msc(IS), Directorate of Education in Al-Qadisiya.
jaberalidami@gmail.com

Hussein Ali Ghadhban Alsalman
Msc(IS), Directorate of Educatio in Basrah.
Hussein.alsalman85@gmail.com

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Abstract

Software risk management refers to systematic process for analyzing and identifying the project risks. The present paper provides a hybrid method for IT software risks identification. Software projects possess different features which increase the project failure possibilities. Therefore, the present work integrate the Artificial Neural network with the Analytic Hierarchy Process (AHP-ANN) in order to solve the problem of software project estimation in early stage. The questionnaire developed to find out the risk functional model and provide the proposed method with proper data. The results observe a major common risk in software projects is the insufficient knowledge based on different software project life cycle stages. Also, there are some other important factors in software projects such as lack of good estimation in project scheduling, poor definition of project requirements which cause human errors.

Keywords: ANN, AHP, risk identification .

1. Introduction

Risk management can be defined as a method to identify the software project threats in order to enhance the software firms organization. The risks sources can be the erroneous strategic in project management or the external challenges. Therefore, there is a need for operation enhancement of software project in order to develop the software efficiency and flexibility [1]. Most of studies investigated the risk factors and provided some useful techniques to specify the effectiveness of them. The ranking based on risks importance is made in light of analysis, planning, maintenance, design and implementation [2]. Classifying risk factors can be considered risk attributes as the main issue in developing risk project. Development of risk management software can be classified into scheduling risks and quality risks. Also, it can be grouped into performance risks, cost risks support risks and schedule risks [1][3]. These classifications were very helpful in monitoring and controlling risks in software projects. More importantly, the top ten software risk factors in developing software were chosen and utilized for analysis [4].

Some authors apply Artificial Neural Networks to identify the risks and to develop an application for risks management during software development [5]. Many other techniques have been used in this field such as regression analysis, expert systems, stochastic models, Monte Carlo Simulation, Decision Tree and Analytic Hierarchy Process AHP [6]. finally, theresearchers develop some techniques to tag the same goal such as Singular Value Decomposition SVD technique [7]. In this work, Artificial Neural Networks have been integrated with Analytic Hierarchy Process (AHP) method for risk control as a tool for risk management.

2. Software Risk identification

Software risk identification is considered the activity of the potential risks which can effect on the project development and determination. The risk check list can be creadt based on the identified risk concepts [5]. It occurs when the organization faces uncertainties from limited capacity and costs in its pursuit for opportunities. In this regard, an effective risk management initiative coupled with suitable risk management strategies can help mitigate the cost and stress brought on by risk issues [8]. Risk identification is a critical process, the risk management mostly depends on identifying all possible risks that may face the project during development [1]. The result of software risk identification is the risk factor list. The identification of risk factors will be followed by risk analysis. The quantitative risk analysis simulates each critical risk effect. Elzamly in 2014 brought forward new methods using quantitative and mining methods to conduct comparisons among risk management methods in the lifecycle of software development [4].

Artificial Neural Networks (ANNs) were developed by Gandhi et al. in 2014 for the prediction of the level of risks in software projects, where risks were detected prior to the project implementation and the steps taken to mitigate them ensures higher rate for successful projects [5]. Hojjati and Noudehi in 2015 applied Monte Carlo simulation for risks assessment. The study evaluated project risks in the IT domain and utilized the Primavera Risk Analysis software to quantitatively analyze management [9]. Paraschivescu in 2016 brought forward integrated quality and risk management concepts resulting in an integrated management system risk that sheds light on new dimensions and perspectives. Also, Elzamly et al. in 2016 identify software risks and software development controls [10]. The study ranked the risk factors in software based on their importance and how often they occurred in a data source. The ANNs applicability was examined in Andreas's study in an attempt to analyze survey data concerning risk management practices effectiveness in the context of product development (PD) projects and forecasting of project outcomes [7]. They explained the relationships between risk management factors that influence successful PD project (e.g., cost). Salman in 2018 apply the maintenance risk factors in Singular Value Decomposition (SVD) correlated with the traditional risk factor calculations to estimate the software maintenance projects[7]. Based on the present review, the researchers specifies the main risk factors that can be used in this work as in the next section.

3. Selection of Risk Factors

Risk management is a process to develop strategies for identifying and estimating their impact. The steps taken for risk management process in the present work are as follows;

1. Risk identification, it represents the activity of detecting the effected potential risk in the project that affected the project development. In the present work, the researcher developed a questionnaire using the taxonomy based risk identification presented by Marvin J. Carr [11].
2. Risk analysis, it represents the process of understanding of where, when and why the risk appear. This process take place based on direct queries about the impact and probability of the risk elements. Traditional risk analysis focuses on the potential impacts to a human population due to the presence of an introduced substance or event, for example the presence of pesticides in a body of water used for human consumption, or an oil spill. A broad variety of techniques are used to evaluate risk in these situations. Risk analysis typically involves four steps: hazard identification, risk assessment, determining the significance of the risks, and risk communication. The traditional risk management in the present paper focuses on pure risk and refers to individual risks as if they don't interact (Simona-Iulia, 2014). Based on the present two bases in the software project risk management, the researchers listed the risk factors.

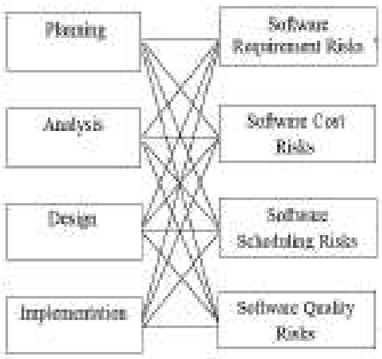
The ‘Top 10 software risk factors’ lists differ to some extent from author to author, but some essential software risk factors that appear almost on any list can be distinguished. These factors need to be addressed and thereafter need to be controlled. Consequently, the list consists of the 10 most serious risks of a software project ranked from one to ten, each risk's status, and the plan for addressing each risk [10] [4]. However, the software risk factors listed in Table 1 below are considered in this study. In addition, these factors are the most common factors used by researchers and experts when studying the software risk factors in software development lifecycle.

Hoodat and Rashid classify the software risks and specify the relations between these risks. They used the risk tree structure correlated with the probabilistic calculation. The analysis helps qualitative and quantitative assessment of risk of failure. Also, its help software risk management process [2]. Therefore, this classification used in this research as a base of study which is correlated with the Software Development Life Cycle. Figure 1 shows the project scheme.

Table 1. Illustrate Top Software Risk Factors in Software Project Lifecycle

Software Requirement Risks	A1	Poor definition of requirements
	A2	Inadequate of requirements
	A3	Invalid requirements
	A4	Lack of good estimation
	A5	Lack of accurate system domain definition
Software Cost Risks	B1	Unrealistic finance schedule
	B2	Lack of good cost estimation
	B3	Lack of monitoring
	B4	Complexity of architecture
	B5	Human errors
Software Scheduling Risks	C1	Inadequate knowledge about Techniques
	C2	Lack of accurate system domain definition
	C3	Lack of employment of manager experience
	C4	Lack of good estimation in project implementation
	C5	Lack of skill
Software Quality Risks	D1	Lack of skill
	D2	Lack of good estimation in projects
	D3	Human errors
	D4	Lack of employment of manager experience
	D5	Lack of project standard

Figure 1. The project scheme



This scheme will be used in ANN based on AHP technique. The risk parameters selected form Hoodat (2009) based on the top ten risks presented by Elzamly (2016) and Salman (2018) [2]'[7]'[10].

4. Risk Factors Evaluation

The study developed a questionnaire that comprised of questions relating to chosen 34 risks maintenance risk factors adopted from Lopez and Salmeron (2012). The questions were chosen with the hope of the works of Marvin (1993) and Webster (2006) and the risk factor values calculated two types of questions, positive and negative [11]'[13]'[6]. The former type represents the questions that had yes answers, while the latter type represented those that had no answers. The sum of the questionnaire list of questions for every type of risk can be represented by the following formula[5]'[7]

$$RF = \sum_{i=1}^{N} (Q_i W_i) \qquad (1)$$

Where *RF* = Risk Factor Value, *Q* =value of each question, W = weight

Thus, the boundary condition is represented as:

$$Q = \begin{cases} positive\ questions & \begin{matrix} if\ yes = 1 \\ if\ no\ = 0 \end{matrix} \\ negative\ questions & \begin{matrix} if\ yes = 0 \\ if\ no\ = 1 \end{matrix} \end{cases} \qquad (2)$$

Collection of data was conducted using questionnaire to determine the commonly occurring risks in majority of software projects in the software companies. The respondents were then presented with the 20 software risk factors. The study sample comprising of 150 persons worked in specific IT organizations in Iraq. These peoples represent the Software Life Cycle user areas. The collected data reflect the selected software risk factors which developed to be used in ANN.

5. Methodology of Risk Factor Specification

In order to specify the risk factors in software project life cycle, the researcher integrate the AHP technique with ANN as in the following:

a) Artificial Neural Networks (ANNs)

The neural network can be defined as a parallel distributed processor. The main processing unit is inspired by the way of biological nervous system, such as the process information of human brain. The potential system of ANN involves several layers developed by computing elements and called nodes. The system operation of neural network depends on the signal transmission.

When the nods receives the input signal from input representation of the system, it will transfer the signal to the next step node. This process will mining the transferred the data in order to find out the specific correlation in input data. The first layer represents the input layer and the last layer considered the output layer. The input layer is received the data of the case study which represent the statistical data. The last layer produce the solution of the problem which represent the predicted or identified data. In between, there are hidden layers which operate the complex data to identify proper pattern using system of specific formulas. The reason for using the neural network ire as in follows:

1. It must have the ability to learn the neural system how to do tasks. The tasks done based on the given tanning data.

2. It must have the ability to generalize the internal system operation. It must produces reasonable outputs without paying attention how deal with the internal processes.

b) AHP

Analytical Hierarchy Process (AHP) methodology has been applied to the evaluation of risk related to software project. Five risks are evaluated and defined in each project stage as presented in table 1. The criteria weights can be more precisely defined by the AHP methodology using “Saaty scale” than using the digital logic method. However, subjectivity is playing a great role in both of methods. Subjectivity is included to the comparison of alternatives by the original AHP methodology, also. Contrary, by using other method there is no subjectivity concerned of alternatives comparisons because of dealing with transformed values of criteria. The ranking of all alternatives can be performed, by obtaining the priorities. The weights present the relative importance of each criterion compared to the goal. Finally, alternatives present the group of feasible solutions of the decision problem.

6. Experimental results

The methodology of the present paper is to integrate the ANN with the AHP technique. The AHP will present a pattern to the ANN. Based on the results, the software project risks were important in the perspective of the project managers, whereas all controls are used most of the time, and often. The risks were ranked on importance in light of analysis, planning, design and implementation. In particular, top of software risk factors in software development Lifecycle were very important, aggregating the responses resulted in the following ranking of the importance of the listed risks. The AHP model in this study is formed to prioritize the various risks within the software project. The result observe the factor priority, for instance the software requirement results can be seen in table 2 and 3.

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Table 2. Analytical Heirarchy Process Matrix

	A1	A2	A3	A4	A5
A1	1	0.6	0.375	0.5	0.3
A2	1.666667	1	0.625	0.833333	0.5
A3	2.666667	1.6	1	1.333333	0.8
A4	2	1.2	0.75	1	0.6
A5	3.333333	2	1.25	1.666667	1
COL. TOTAL	10.66667	6.4	4	5.333333	3.2

Table 3. Normalized Score Table

A1	0.09	0.09	0.09	0.09	0.09	0.47	9.38
A2	0.16	0.16	0.16	0.16	0.16	0.78	15.63
A3	0.25	0.25	0.25	0.25	0.25	1.25	25.00
A4	0.19	0.19	0.19	0.19	0.19	0.94	18.75
A5	0.31	0.31	0.31	0.31	0.31	1.56	31.25
COL. TOTAL	1	1	1	1	1	5	

In the present method, the ANN trained based on the conjugate gradient backpropagation algorithm.It represents a proper choice for problem of classifications. It is used less memory requirements and provide faster response than gradient decent algorithms.

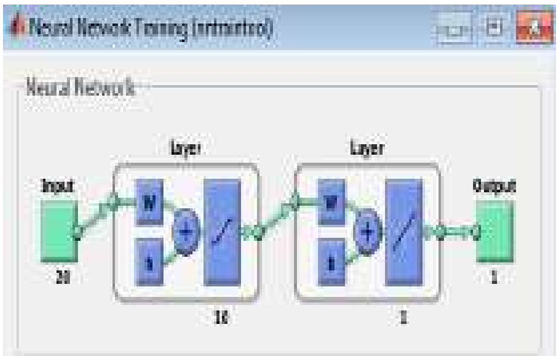

Figure 2. ANN with one hidden layer and ten hidden nodes

The plotted results shown in Figure 3 indicate that results observe the same responses of the risk factor effects.

The yellow bars represent the ANN results, while the blue bars represent the integrated method. The bar graph shows that the risk identification due to the present method. These valid results highlight the largest problem on IT software risk factors which represents inadequate knowledge/skills, insufficient expertise and Insufficient/inappropriate staffing. The results observe a big effect by the insufficient expertise in the applied software management as shown in figure 3.

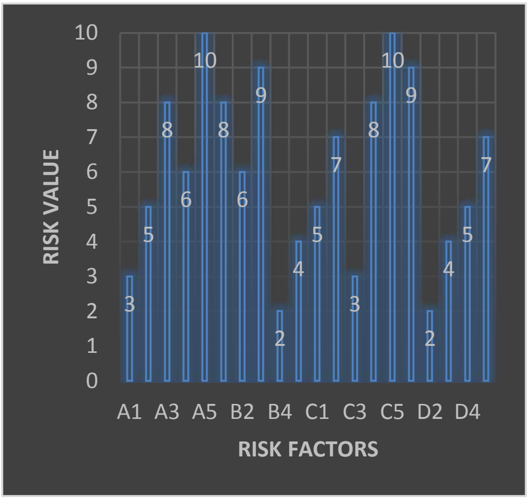

Figure 3: Identifying the Software Improvements Needs

This factor are critical in development the Risk Management. It can provide important information regarding that risk improvement and risk management practices. The higher risk in software project phases based on the four phases project life cycle came from identifying software improvements needs (phase 1) which observe 41%, while the other three risk groups observe 21% and 22% and 16% as shown in Figure 4.

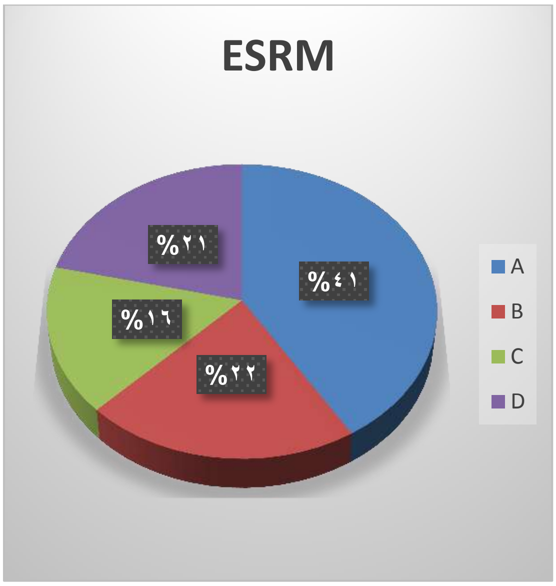

Figure 4: Identifying Risks in risk Phases

8. Conclusion

In this work we apply AHP technique with ANN to support Risk Management. The results show that that AHP technique is simple and efficient for total variance in common questionnaire of each of the software risk factors to model if they are effective in mitigating the occurrence of each risk factor. The result of AHP is presented as a pattern to ANN. As a conclusion, this method can be used effectively to identify the risk effect in all project phases. The used method specify the Risks in three reasons. It specifies the root of risk problem and the effective phase of project.

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مراقبة مخاطر البرمجيات بالاعتماد على طريقة دمج (الشبكة العصبية الاصطناعية – عملية التحليل الوراثي)

جابر ابراهيم ناصر
مديرية التربية في القادسية
jaberalidami@gmail.com

حسين علي غضبان
مديرية التربية في البصرة
Hussein.alsalman85@gmail.com

المستخلص :

ادارة مخاطر البرمجيات تشير الى المعالجة المنظمة لتحليل و تحديد مخاطر المشروع . هذا البحث يوفر طريقة هجينة لتحديد مخاطر برامجيات تكنولوجيا المعلومات . مشاريع البرامجيات تمتلك خصائص مختلفة تزيد من احتماليات فشل المشروع. لذلك فأن العمل الحالي يدمج (الشبكة العصبية الذكية) مع (عملية التحليل الوراثي) من اجل حل مشكلة تخمين مشروع البرامجيات في مرحلة متقدمة . طورت الاستبيانات لايجاد النموذج الوظيفي للخطر و كذلك توفير طريقة مقترحة مع بيانات مناسبة. النتائج رصدت الخطر الشائع و الرئيسي في مشاريع البرمجيات و هو المعرفة غير الكافية بالاعتماد على مراحل دورة حياة البرامجيات المختلفة. و كذلك، هنالك بعض العوامل المهمة الاخرى في مشاريع البرامجيات مثل الافتقار الى التخمين الجيد في جدولة المشروع ، ضعف تعريف متطلبات المشروع والتي تسبب اخطاء بشرية .

الكلمات المفتاحية : الشبكة العصبية الاصطناعية ، عملية التحليل الوراثي ، تحديد الخطر .

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Lightweight RC4 Algorithm

Mustafa M. Abd Zaid **Soukaena Hassan**

Department Of Computer Sciences , University Of Technology
mustafamajeed2014@gmail.com **Soukaena.Hassan@Yahoo.Com**

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Abstract.

As a significant number of applications in mobile transactions and wireless sensor networks are characterized by short duration sessions, security issues turn into a focal concern.

RC4 algorithm is a standout amongst the most broadly utilized stream ciphers which locates its application in numerous security conventions, for example, Wired Equivalence Privacy (WEP) and Wi-Fi Protocol Access (WPA).

In this paper, we suggest a lightweight variation of the well-known RC4 algorithm that is exceptionally appropriate for resources of computational compelled gadgets and energy in remote systems, when contrasted with RC4 and its variations like, HC128, Grain-128, and so on.,. We propose new PRGA which is replaced the PRGA keystream generation algorithm of RC4.

The proposed LRC4 execution is surveyed in terms of randomness test and time under an arrangement of analyses. The trial comes about demonstrate that the resulting stream are random, and the suggested algorithm quicker compared to standard RC4, the results indicate the average of speed improvement is about 54% in both encryption/decryption sides.

Keywords: Random Number Generator, Stream Cipher, Key Scheduling Algorithm, RC4, Lightweight cryptography .

1. Introduction

The computing devices utilized as a part of an extensive class of remote correspondence systems, for example, cell phones, Internet of Things (IoT), body area networks (BANs), remote sensor systems (WSNs), mobile ad hoc networks (MANETs), vehicular ad hoc networks (VANETs), and so on., are little and asset compelled. To guarantee the security of information correspondence sessions in such systems, Stream ciphers algorithms have been used. In particular, hardware stream ciphers and software stream ciphers are the two sorts of stream ciphers each of them contingent upon the stage most suited to their execution[1].

One of the stream ciphers algorithms, Ron Rivest outlined the RC4 algorithm in 1987 but the algorithm kept mystery until the point that it was as often as possible to the cypherpunks mailing list in 1994. RC4 is the most satisfactory stream cipher; it is utilized as a part of numerous web conventions, for example, Wireless Protected Access (WPA), Wired Equivalent Privacy (WEP), and Secure Socket Layer/ Transport Layer Security (SSL/ TLS) [1]. It is likewise utilized as a part of use, for example, Skype. RC4 proves its efficiency in both hardware and software and speed. It is extremely straightforward and quick equivalent to other encryption algorithms. RC4 algorithm predominantly comprises of two phases: the KSA (Key Scheduling Algorithm) to produce, from the key, an initial permutation of the S array and the PRGA (Pseudo Random Generation Algorithm) to create the key stream[2].

RC4 is as yet the most famous stream cipher algorithm because of its straightforwardness, speed, and simplicity of usage although more secure and efficient stream ciphers have been found after it [1].

2. Related Work

Numerous researchers have endeavored to upgrade the security of RC4 and make variation algorithms. However, this improvement impeded the execution speed. On the other hand, many researchers have attempted to improve algorithmic speed, but this caused a decrease in the randomness [3].

Jian et al, 2010 [3] introduced an enhanced RC4 in [3] . They improved the speed of RC4, and security. However, regardless of whether the enhanced RC4 in [3] has different escape clauses stays to be tried.

Weerasinghe, 2013 [4] proposed algorithm it is cost-effective than the first RC4 and different changes of RC4 utilized as a part of the examination. Since there are numerical qualities to portray the security level of the ciphers, anyone can get a major picture of the secrecy of the pertinent ciphers. Higher estimations of mystery areappeared by the new stream cipher, which implies the randomness of the cipher is higher than that of others, which is highlight of a decent cipher. The explanation for having a higher secrecy can be the expanded number of more activities and modifications in the PRGA.

Nishith et al, 2014 [5] The algorithm proposed in [5] enhanced the security of Improved RC4 algorithm by forcing substitution, along these lines changing over it into an item cipher. Time taken for encryption and decryption utilizing the proposed algorithm is hardly more than the Improved RC4 Algorithm.

Maytham et al, 2015[6] to solve the powerless keys issue of the RC4 utilizing a random introduction of inward state S. An arbitrary starting state (RRC4) was used to produce RC4 algorithm. Additionally, two state tables (RC4-2S) were used to propose RC4 algorithm. At long last, [6] they proposed RC4 algorithm with two state tables to create four keys (RC4-2S+) in each cycle which additionally upgrades randomness over RC4-2S and RRC4.

Sarab et al, 2016 [2] to overcome the weakness of the key scheduling algorithm of the original RC4, they presented a new modified key scheduling algorithm. The modified algorithm enhances the secrecy of the ciphertext especially when the key size is small and proves to be more random than the original RC4. Furthermore, the time of encryption of both algorithms is comparable.

Soumyadev et al, 2017 [1] They proposed a lightweight stream cipher algorithm. The suggested algorithm secure as Grain-128, orginal RC4 and other stream ciphers with regards of wireless applications that utilization short sessions.

3. Description of RC4

RC4 picks a cluster (S_{box}) and a secret key (K), the cluster known as Sbox which includes N (N=2^n) (N=256, where n=8). KSA and PRGA are two algorithms contained in RC4 algorithm [4].

A variable key length is used in RC4, which runs between (0-255) bytes for instating 256-byte array in the underlying state by components from S_{box} [0] to S_{box} [255] [3]. The KSA uses the symmetric key to permute an array S containing 256 entries. S array is initialized with identity permutation ranging from 0 to 255,(As suggested in [2][1] RC4 must utilize a key longer than 128 bytes). Then, a 256-iteration loop is utilized to produce a random permutation of the exhibit S, where the entries of the S array are continually swapped using the key value[2].

Algorithm of KSA:

set N ← 256
set ki to 0
while (true)
begin
S_{box} [ki] ← ki
ki←ki+1
end while
set kj←0
set ki←0
while (true)
begin
kj← (kj+S_{box} [ki]+k[ki]) Mod N;
swap(S_{box}[ki],S_{box}[kj])
ki←ki+1
end while

Figure 1: KSA of RC4

The objective of PRGA is to create a sequence of key stream. In the PRGA, two indices ki, kj are initialized to zero. In each iteration, ki is recomputed as (ki+1) and kj is recomputed as (kj+S_{box} [ki])mod 256 , and then a swap operation is conducted between S[ki],S[kj]. The key stream that is XORed with clear-text is generated as (S_{box} [(S_{box} [ki]+ S_{box} [kj]) mod 256) [1][2][6].PRGA steps show in figure2 :

Algorithm of PRGA:

set N←256
set ki←0
set kj← 0
while (generate key-stream)
begin
ki←ki+ mod N;
kj←kj+S_{box} [ki]mod N
swap(S_{box} [ki], S_{box} [kj])
Output←S_{box} [(S_{box} [ki]+ S_{box} [kj]) mod N]
end while

Figure 2: PRGA of RC4

4. Proposed Algorithm

In this paper, we produce an efficient stream cipher algorithm which is a lightweight compare to original RC4. The propose algorithm bring down cost of computational overhead when compared with the ordinary stream cipher like RC4. The suggest lightweight algorithm is sufficiently secure for use in many low term wireless communication application situations.

For creation of the random initial permutation S, utilize the KSA algorithm (first Algorithm) from RC4, but supplant the PRGA of RC4 new PRGA(lightweight PRGA). Lightweight PRGA is utilized for keystream creation from the (S_{box}) the input permutation (result from KSA). The new PRGA algorithm shows in figure3:

Proposed PRGA Algorithm:

set ki←0
set kj←255
set t←0
for ki←0 to N-1 **do**
begin
t← (S_{box}[ki]+ S_{box} [kj]+kj) mod 256
kj←ki
ki←S_{box} [ki]
S_{box} [kj] ←t
Output Z←S_{box} [ki] XOR S_{box} [kj]
end for

Figure 3: Proposed PRGA Algorithm

The yield of the PRGA algorithm is a key arrangement that will be XORed with cleartext/ciphertext) to get the ciphertext/cleartext.

5. Simple Example

Key= “password”

Plaintext= “mymessag”

Table1: Simple LRC4 example

S[i]	S[j]	Key-stream	Plaintext	Cipher-text
249	91	162	207	109
167	84	243	138	121
26	71	93	48	109
252	90	166	195	101
112	253	141	254	115
99	135	228	151	115
25	230	255	158	97
45	111	66	37	103

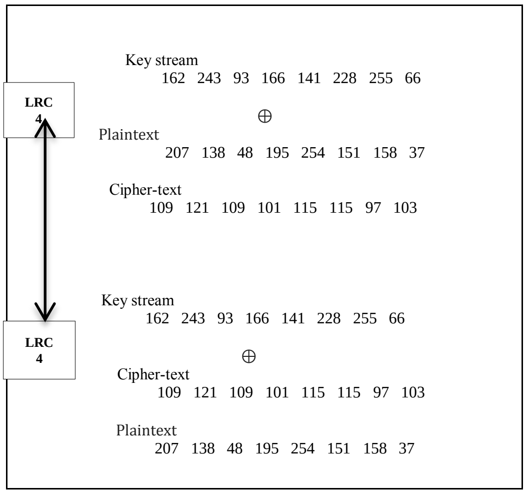

Figure 4:Simple example

6. Performance Evaluation

Different criteria can be used to measure the security level and performance of a given encryption algorithm. In this paper, two measurements: the randomness test of NIST (the National Institute of Standards and Technology) statistical test suite and time are utilized to assess the suggested algorithm.

6.1 Randomness Test

The statistical test suite (NIST) is the most broadly utilized one in the field of cryptography, which we have utilized for contrasting the standard RC4 and proposed LRC4 Algorithm. In this paper, three tests namely Approximate entropy, Run test and Linear complexity in the statistical test Suit are used to measure the randomness of the cipher-text created from RC4 and proposed lightweight RC4.In the wake of applying the NIST test suite, we use 10 random keys to test algorithms as showing in table2.

In this paper, the significance level, (p-value) is set to 0.01. The statistical tests suit (NIST) results indicates success of output (ciphertext) of all tested algorithms. In other words, all the test type of statistical test Suit are adequate and have good randomness for the two tested algorithms.

Table2: NIST Tests Applied to Standard and Modified RC4 Algorithms.

KEY	Standard RC4			Proposed LRC4		
	Approximate entropy	Run test	Linear complexity	Approximate entropy	Run test	Linear complexity
AEDW	0.021265	0.475773	0.955719	0.0553588	0.690328	0.6766
FmADDwerdf	0.075948	0.723729	0.808846	0.222083	0.166203	0.8088
0eey6tw453f15d2154f16a6883c	0.27209	0.7045	0.42319	0.09743	0.36818	0.8395
32881e0435a3137f6309807a88da234	0.099388	0.88353	0.398762	0.481929	0.678082	0.8088
2232d95de24a1b6b79fad3b37a427ea0	0.077827	0.523502	0.398762	0.062620	0.3722	0.1394
3040fa18f1908598656982223fa2dd8d	0.028795	0.523378	0.902774	0.418634	0.109129	0.3798
ndgekh77f1128598656982223ra2yt6d	0.018869	0.924253	0.320847	0.06648	0.497926	0.7306
34uiaff70eey67cpl6d2154f16a6441w	0.050375	0.533358	0.186466	0.609536	0.664365	0.1173
23uidd670eey655rt6d215wetr4a4kms2	0.2424	0.93857	0.46154	0.5826516	0.546237	0.7804
2b28ab097eaef7cf15d2154f16a6883c	0.408949	0.543813	0.962877	0.905865	0.925194	0.8929
Average	**0.129591**	**0.67744**	**0.581978**	**0.350259**	**0.50178**	**0.6174**

6.2 Encryption Time

We used different size of text files to test the speed of the proposed algorithms, and we compared the calculated time of both the standard RC4 with lightweight RC4.

Table3:Encryption time in second

File Size	Standard RC4	**Modified LRC4**
1.00 kb	0.00347	**0.00148**
2.01 kb	0.00433	**0.00262**
20.0 kb	0.04263	**0.024217**

In this evaluation step we have tested several files in order to prove that how fast the modified LRC4 algorithm than the standard RC4.

According to this test, we can indicate that the modified LRC4 (Lightweight RC4) algorithm is faster than standard RC4 algorithm and the results indicate the average of speed improvement is about 54% in encryption/decryption sides.

7. Conclusion

We have introduced a lightweight stream cipher algorithm and secure as original RC4. This paper presents a new modified PRGA algorithm to produce lightweight RC4 algorithm compared to the original RC4. Proposed algorithm is efficient; in other words, it is cost-effective than the standard RC4 and it is faster, The generated output sequences of proposed algorithm has passed the NIST suite of statistical tests. This makes the proposed LRC4 to a great degree appropriate for actualizing secure correspondence in a wide range of wireless applications like: Wi-Fi Protocol Access (WPA), where devices are compelled by either cost, energy or processing ability.

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خوارزمية ريفست 4 خفيفة الوزن

مصطفى مجيد عبدزيد سكينة حسن

قسم علوم الحاسوب ، الجامعة التكنولوجية

المستخلص:
نظرًا لتميز عدد كبير من التطبيقات في معاملات الجوّال وشبكات الاستشعار اللاسلكية بجلسات قصيرة المدة ، تتحول مشكلات الأمان إلى قلق محوري.

خوارزمية ريفست4 هي من ابرز الخوارزميات المستخدمة على نطاق واسع والذي يحدد تطبيقها في العديد من الاتفاقيات الأمنية ، على سبيل المثال Wired Equivalence Privacy (WEP) و Wi-Fi Protocol Access(WPA).

في هذه الورقة ، نقترح تباينًا خفيفًا لخوارزمية ريفست4 المعروفة والتي تعتبر مناسبة بشكل استثنائي لمصادر الأدوات الحاسوبية والطاقة في الأنظمة البعيدة ، عندما يقارن مع RC4 واقرانها مثل: HC128 ، Grain-128 ... الخ. اقتراحنا خوارزمية PRGA جديدة والتي تستبدل الخوارزمية الموجودة لتوليد المفتاح في ال RC4الاصلية.

تنفيذ الخوارزمية المقترحة تم فحصها من حيث اختبار العشوائية والوقت. اثبتت التجربة أن التدفق الناتج عشوائيًا للخوارزمية المقترحة ،وهي أسرع مقارنةً بالخوارزمية الاصلية ، حيث اثبتت النتائج ان معدل تحسن السرعة هو حوالي 54%.

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Images Analysis by Using Fuzzy Clustering

Shahla Hazim Ahmed Kharofa
Department of Dental Basic Sciences, College of Dentistry, University of Mosul, IRAQ
Shahlahazim@gmail.com

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Abstract:

The Fuzzy C-Mean algorithm is one of the most famous fuzzy clustering techniques. The process of fuzzy clustering is a useful method in analyzing many patterns and images. The Fuzzy C-Mean algorithm is widely used and based on the objective function reduction through adding membership values and the fuzzy coefficient. The Mean Absolute Error (MAE) was also measured in this research for each execution.

The research found that when the number of clusters increases, the mean absolute error value is reduced. When the number of clusters increased. The more details in the resulting image were not present in the original image. This helps in the analysis of the images.

In this research, medical images were treated and analyzed. The analysis helps physicians explain the patient's health status and also according to suggested algorithm helps them to diagnose the possibility of a particular disease or tumor. A Matlab program was created to perform the analysis.

Keywords:
Fuzzy clustering, Fuzzy C-Mean Algorithm, Matlab Language, Image analysis, Mean Absolute Error.

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Introduction:

Image analysis involves processing image data to determine the information needed to solve problems. The process of image analysis is the process of converting objects in the image data to quantitative information which derive and describe the object from its image and is usually flexible. The end result is high level information [1].

Medical images play a key role in helping to detect and diagnose many diseases. Medical imaging now provides advanced imaging techniques that enable the physician to see human bodies directly and monitor micro anatomical changes [2].

Methods:

In this research, Fuzzy C-Mean Algorithm (FCM) was used to analyze three medical images.

Fuzzy Clustering:

Fuzzy clustering is an extension of the analysis of the traditional techniques group, [3] and is usually used when there is no apparent grouping in the data set. The essence of the algorithm is the use of iterative processes because the number of steps to obtain the output is not predefined [4]. The objective of the fuzzy clustering method is to define each cluster by looking for its own membership function [5][6].

The performance of the clustering algorithm is affected by the initial values chosen at execution, so the algorithm is repeated a number of times to obtain the appropriate results[7]. The results obtained by the researchers showed that the results were of much higher quality than the use of traditional methods [8].

Fuzzy C-Mean Algorithm (FCM):

FCM is also called Fuzzy ISODATA. This method was developed by Dunn (1973) and improved by Bezdek (1981) [9]. FCM algorithm is one of the most effective algorithms of fuzzy clustering. It is based on the principle of fuzzy logic. It allows each data point to belong to the cluster at a membership degree, so that each data point can belong to several clusters at the same time and with different membership degrees between 0 and 1[10] [11].

The aim of FCM is to find cluster centers in the feature space that minimize an objective function. The objective function is associated with the optimization problem, which minimizes within class variation and maximizes variation between two classes [12].

This algorithm is widely used in image processing applications such as medical imaging and remote sensing. It is a local search optimization algorithm [13].

The FCM algorithm assign a membership for each data point. By calculating the distance between the cluster center and the data point. More the data is near to the cluster center more is its membership towards the particular cluster center. After each iteration membership and cluster centers are updated [14].

The steps of Fuzzy C-Means Algorithmic is:

1. Input original image

Let $X = \{x_1, x_2, x_3 ..., x_n\}$ be the set of pixels image and $V = \{v_1, v_2, v_3 ..., v_c\}$ be the set of centers.

2. Randomly select *'c'* cluster centers.

3. Calculate the fuzzy membership 'μ_{ij}' using equation (1):

$$\mu_{ij} = 1 / \sum_{k=1}^{c} (d_{ij} / d_{ik})^{(2/m-1)} \quad \text{--- (1)}$$

4. Compute the fuzzy centers 'v_j' using equation (2):

$$v_j = (\sum_{i=1}^{n} (\mu_{ij})^m x_i) / (\sum_{i=1}^{n} (\mu_{ij})^m) \quad \text{---(2)}$$

Where:

'*n*' is the number of data points.

'*vj*' represents the j^{th} cluster center

'*m*' is the fuzziness index m € [1, ∞].

'*c*' represents the number of cluster center.

5. Repeat step 3) and 4) until the minimum '*J*' value is achieved or $||U^{(k+1)} - U^{(k)}|| < \beta$.

Where:

'*k*' is the iteration step.

'β' is the termination criterion between [0, 1].

'$U = (\mu_{ij})_{n*c}$' is the fuzzy membership matrix.

'*J*' is the objective function.

6. end [15][16][17].

Evaluation Performance Factors:

- **Mean Absolute Error (MAE)**: It is used in statistic to measure the difference between two continuous variables. MAE is defined in equation (3) as followed:

---(3) $$\mathrm{MAE} = \frac{1}{n} \sum_{j=1}^{n} |y_j - \hat{y}_j|$$

Where: y_j=Original image, $\hat{y}_j$=Output image. Minimum value of MAE indicate that best result, because it include the minimum difference between the original image and output image [18].

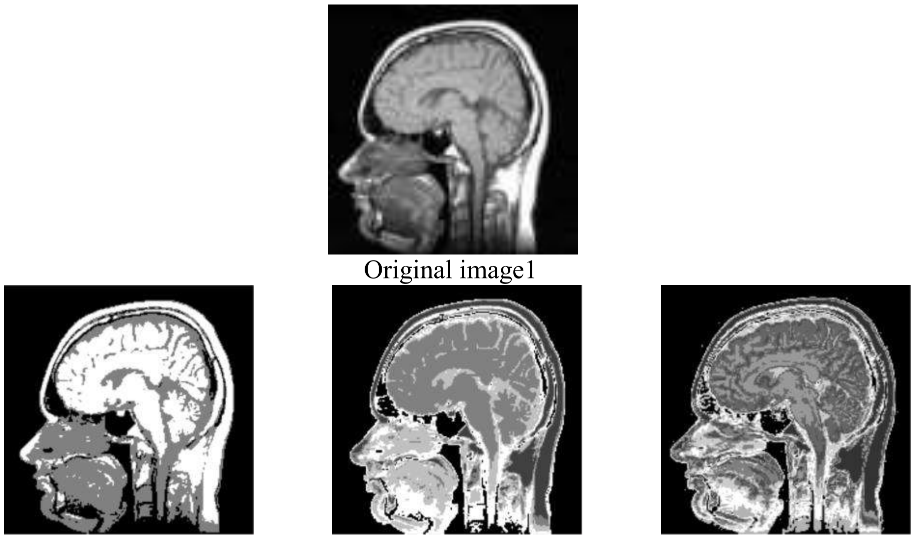
Suggested Algorithmic:

The following algorithm was proposed to execute Fuzzy C-Mean Algorithm and compute the Mean Absolute Error (MAE) values for medical images. This algorithm is:

1.Input medical image
2. Input number of clusters (in this research, number of clusters are equal 3 or 5 or 9)
3. Display original image
4. Randomly select *'c'* cluster centers.
5.Calculate the fuzzy membership 'μ_{ij}'
6. Compute the fuzzy centers 'v_j'
7. Repeat step 5) and 6) until the minimum 'J' value is achieved or $||U^{(k+1)} - U^{(k)}|| < \beta$.
8. Display output image
9. Compute Mean Absolute Error value (MAE) between original image and result image.
10. end

Results and Discussion:

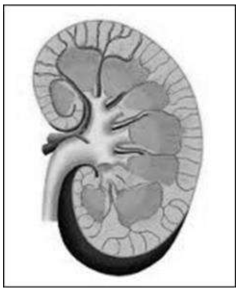
After applying the suggested algorithm to the three medical images, The details of result image were better than the original image. These details are increased by increasing the number of clusters entered. These details help the doctors to diagnose and analyze the disease. The results of executing the suggested algorithm were shown in (Figure 1-3) and (Table 1). The graph of MAE values for Image 1, Image 2 and Image 3 were show in (Figure 4-6). In each execution. The mean absolute error value was measured. It was observed that; the greater number of clusters, the lower of the mean absolute error value.



Original image1

Number of Cluster=3 Number of Cluster=5 Number of Cluster=9

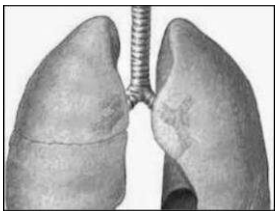
Figure 1: Many cases of image1 after execution FCM algorithm



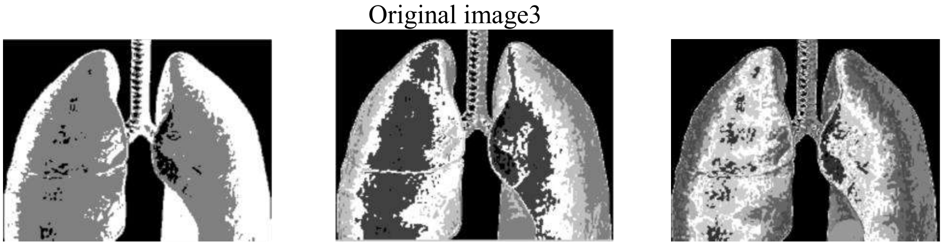
Original image2

Number of Cluster=3 Number of Cluster=5 Number of Cluster=9

Figure 2: Many cases of image2 after execution FCM algorithm



Original image3



Number of Cluster=3 Number of Cluster=5 Number of Cluster=9

Figure 3: Many cases of image3 after execution FCM algorithm

Table 1: Represent the results of execution the suggested algorithm and compare of MAE values in Image1, Image2 and Image3

Image Name	Size of Image	Number of Cluster	Center	Number of Iteration	Objective function values	Mean Absolute Error (MAE) Values
Image1	100, 100	3	105.2387 187.0227 8.9424	44	13180088.894	107
		5	69.6341 246.6715 167.5079 115.6076 6.3039	83	2863516.246	103
		9	249.0573 67.6441 93.2786 36.8996 181.8289 141.7170 163.9570 116.2323 5.1436	100	842195.516	93
Image2	100, 150	3	160.8961 41.9683 252.3240	32	11468903.771	137
		5	187.1260 108.0396 149.1992 28.6357 253.7071	100	2762756.947	96
		9	116.6543 24.0148 204.9843 73.3078 141.8892 242.4896 158.7911 184.0614 254.5403	100	1021515.375	86
Image3	150, 120	3	112.8440 177.8164 244.5405	40	11317475.414	121
		5	95.1411 131.3992 162.8970 192.6521 246.0111	100	3546875.560	107
		9	140.0731 192.1150 211.6600 117.5321 96.7130 158.6914 176.3608 246.6349 53.2349	100	1048110.728	98

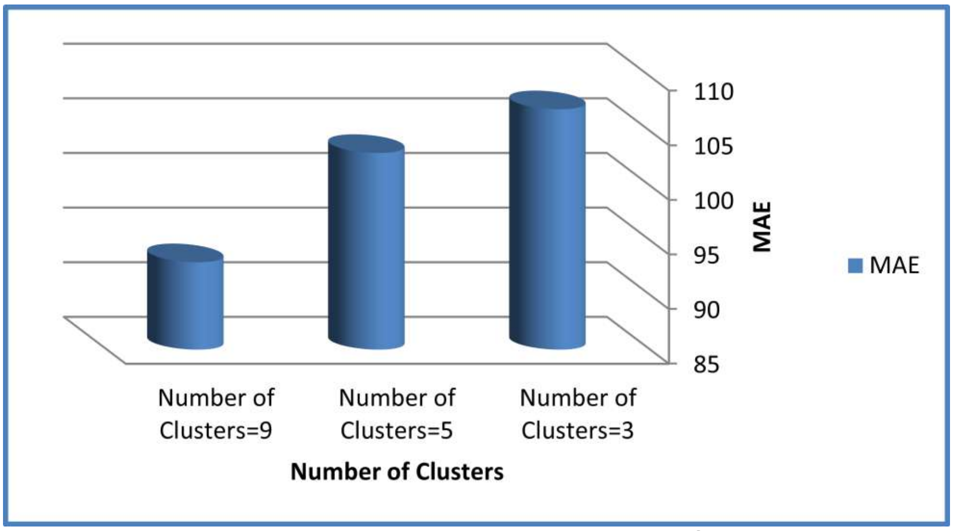


Figure 4: Mean Absolute Error (MAE) of Image 1

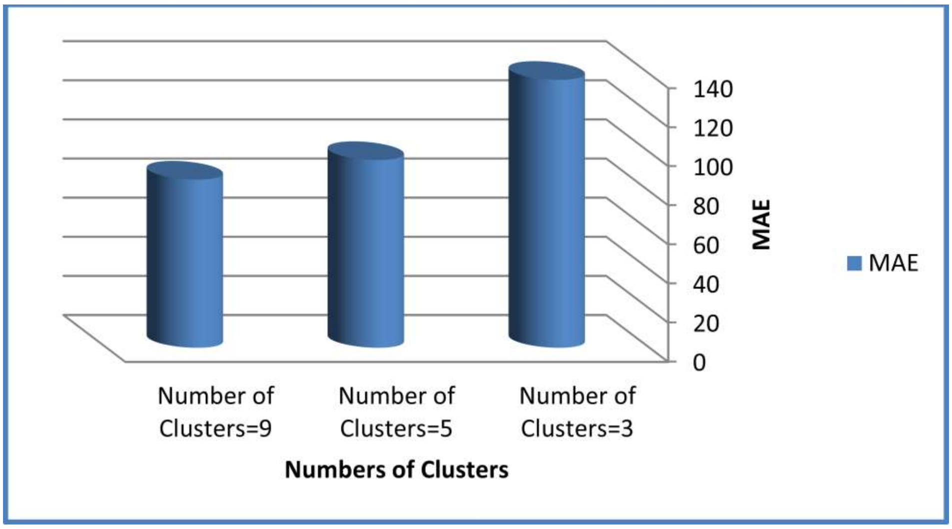


Figure 5: Mean Absolute Error (MAE) of Image 2

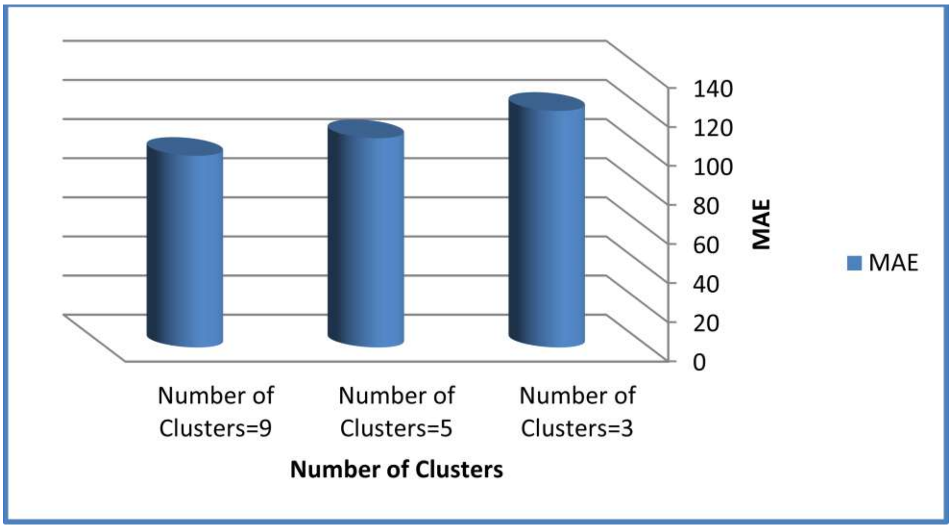


Figure 6: Mean Absolute Error (MAE) of Image 3

Conclusion:

Medical image analysis is currently an important subject in modern medicine. In view of the increasing number of patients as it helps the doctor or the person concerned to give a preliminary idea of the patient's condition without any operation or surgical intervention.

FCM algorithm is the most popular fuzzy clustering algorithm and extensively used in medical image. In this research, used FCM algorithm to analysis medical images. This algorithm provide few iterations steps already provide good approximation to the final solution.

After execution of the proposed algorithm, it was concluded that by increasing the number of clusters in the Fuzzy C-Mean algorithm, the Mean Absolute Error (MAE) values was reduced. The suggested algorithm gave good results in image analysis to help doctors to diagnose and identify the disease.

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تحليل الصور باستخدام العنقدة المضببة

شهلة حازم احمد خروفة

قسم علوم طب الأسنان الأساسية ، كلية طب الأسنان ، جامعة الموصل

المستخلص:

تم في هذا البحث استخدام خوارزمية (Fuzzy C-Mean) والتي تعد من أشهر تقنيات العنقدة المضببة. تعد عملية العنقدة بالمنطق المضبب طريقة مفيدة وجيدة في تحليل العديد من الأنماط والصور، وخوارزمية (Fuzzy C-Mean) تستخدم بشكل واسع وتكون مبنية على أساس تقليل الدالة القياسية وذلك بإضافة قيم العضوية ومعامل التضبب. كما تم قياس معدل الخطأ المطلق لكل حالة تنفيذ.

توصل البحث إلى أن عند زيادة عدد العناقيد المدخلة تقلل قيمة الخطأ المطلق المحسوبة، كما أنه بزيادة عدد العناقيد تظهر تفاصيل أكثر في الصورة الناتجة لم تكن موجودة في الصورة الأصلية وهذا يساعد في تحليل الصور.

تم التعامل مع الصور الطبية في البحث وتحليلها وفق الخوارزمية المقترحة. تساعد عملية التحليل الأطباء في تفسير الحالة الصحية للمريض وتساعده أيضاً على التشخيص كاحتمالية الإصابة بمرض معين أو ورم. كما تم عمل برنامج بلغة ماثلاب (Matlab) لتنفيذ عملية التحليل.

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Predicate the Ability of Extracorporeal Shock Wave Lithotripsy (ESWL) to treat the Kidney Stones by used Combined Classifier

Samera Shams Hussein Lubab Ahmed Tawfeeq Sukaina Sh Altyar

Department of Computer Science, College of Education for Pure Science, Baghdad University

sameramazn@yahoo.com lubabahmed@yahoo.com Sukaina_altyar@yahoo.com

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Abstract:

Extracorporeal Shock Wave Lithotripsy (ESWL) is the most commonplace remedy for kidney stone. Shock waves from outside the body frame are centered at a kidney stone inflicting the stone to fragment. The success of the (ESWL) treatment is based on some variables such as age, sex, stone quantity stone period and so on. Thus, the prediction the success of remedy by this method is so important for professionals to make a decision to continue using (ESWL) or to using another remedy technique. In this study, a prediction system for (ESWL) treatment by used three techniques of mixing classifiers, which is Product Rule (PR), Neural Network (NN) and the proposed classifier called Nested Combined Classifier (NCC). The samples had been taken from 2850 actual sufferers cases that had been treated at Urology and Nephrology center of Iraq. The results from three cases have been compared to actual treatment results of (ESWL) for trained and non-trained cases and compared the results of three models. The results show that (NCC) approach is the most accurate method in prediction the efficient of uses (ESWL) remedy in treatment the kidney stone.

Keywords: Extracorporeal Shock Wave Lithotripsy, Product Rule, Neural Network, ANN, PR.

Introduction:

Extracorporeal shock wave lithotripsy (ESWL) is a nonsurgical technique that makes use of high-energy surprise waves to interrupt a kidney stone into "stone dirt" or fragments that could more easily travel via the urinary tract and skip from the frame. ESWL was added into scientific practice within the 1980s, and due to the fact that then has become one of the principal treatment alternatives in patients with renal and/or ureteral calculi. but, the development of endourology and minimally invasive surgical procedures with their excessive achievement fees has reduced its applicability. From then on, it has emerged as necessary to search for the foremost technical parameters and careful selection of candidates for ESWL on the way to optimize its consequences and justify its indication. [1].

ESWL goes better with some stones than the others. Very big stones are unable to be treated by this technique. The shape and size of stone, wherein it is lodged inside urinary tract, patient health and his kidneys' health will probably be aspect of the decision to use it. Stones which are lower than 2cm in diameter are the ideal size for SWL. The treatment with SWL probably won't be successful in very big ones. SWL is more suitable for some people rather than others. Considering that shock waves and X-rays are required in SWL, pregnant women that have stones will not be treated by this way. People with severe skeletal abnormalities, infections, bleeding disorders, or who are morbidly obese also not commonly good candidates for SWL. In case patient kidneys have any other abnormalities, the doctor may possibly decide that the patient need to use another treatment. If patient come with a cardiac pacemaker, the heart specialist will decide if patient can be treated by ESWL [2].

A perfect estimation of the probability to eliminate the stone from individual’s kidney are required for appropriate treatment choice to figure out who will have optimum benefit from ESWL. Thus, to identity the prognostication factors that effect on clearing away stone from kidney by utilizing ESWL will be uses for predication result of treatment via utilized artificial intelligence techniques [3].

A range of computer models was developed in the field of machine learning and statistics which could be used for predicting medical results, such as decision trees, logistic regression (LR), Bayesian networks and artificial neural networks (ANNs). Perhaps the best commonly used methods are depending upon the statistical technique of regression. For researches with a binary endpoint (for example, yes/no, alive/ dead), the LR is used usually. For the testing of time to event data, the Cox proportional hazards regression is the standard. These methods are becoming standard because of their relative simplicity, the widely used availability of pc software to meet these models, the inference that permit by evaluate the fitted model coefficients, and the achieved statistical theory which supplements and supports their use [4]. During the last decade, a unique class of techniques called artificial neural networks (ANNs) have been proposed to be the alternative or supplement to standard statistical techniques. Artificial neural network is an important part of artificial intelligence which offer an "intelligent" method of predicting practical outcomes with higher efficiency and accuracy. ANN algorithm is dependent on the idealized design of a biological neuron (unit) and presents very good promise in conquering the complexities in actions of bio-systems/ materials that are otherwise hard to comprehend. Therefore, ANNs could be played as a model of human brain function, in which sets of data in the sort of input and output patterns are organized to train the ANN. The ANN classifiers can also be enhancing via combining their back class estimates with conventional language model likelihood ratios, by using a logistic regression combiner [5].

The aim of this study is to utilize a Nested Combined Classifiers (NCC), a method of combination in the classification area, hoping to increase the accuracy of classification in area of predication treatment the kidney stone by ESWL. NCC combined the results of combination using neural networks and the results of combination using product rule.

This study applied the following techniques in classification:

- Combination using Neural Networks.
- Combination using Product Rule.
- Nested Combination by Combination 1, 2.

Previous Works

ANN and LR have been utilized in a variety of domains in medical diagnosis. Currently, ANN have been applied for estimating risk in a wide range of application such as breast cancers. In other hand, LR has been utilized for estimation the disease risk in prostate cancer, breast cancer, coronary heart disease, postoperative complications), and stroke.

Hamid et al (2003) [6], had examine the ability of ANN to predict perfect renal stone fragmentation in people getting treatment by ESWL. The research used 82 patient's cases that have renal stones which they had been treated by ESWL. For training process, they used 60 patient's cases that got most effective fragmentation of stones by utilized ESWL. These data generally involved the settings of ESWL that been used, the 24h urinary variables, and the stone disease radiological features. The predication accuracy was tested on 22 non-trained patients, by providing the input parameters of the 22 patients towards the trained ANN and acquiring the predicted values. The tested results prove that the trained ANN forecast the optimum fragmentation in ≤13 000 shocks/stone in 17 patients and optimum fragmentation in>13 000 shocks/stone in the other 5 patients. The total correlation among the observed and predicted values was 75.5% in these 17 patients.

Goyal et al (2010) [7], compared the accuracy of multivariate regression analysis and ANN analysis for fragmentation of renal stone by ESWL. 276 total patients with renal calculus had been treated by ESWL at the time of (December 2001) to month of December in year of 2006. Of those, the 196 patient's cases have been used to build data that had been used for training the ANN. The predication accuracy of trained ANN was tested on 80 non-trained patients. The input data involve patient age, stone burden and size, urinary pH and number of sittings. For non-trained 80 patients, the input was examined and result had also determined by MVRA. The predicted value from both the methods had been compared and the results had been sketched. The observed and predicted number of shocks and values of shock power had been compared using 1:1 slope line. The results had been computed as coefficient of correlation. In summary, ANN gives better coefficient of correlation than MVRA, therefore is seen as a better tool to evaluate the perfect renal stone fragmentation by way of ESWL.

Seckiner et al (2010) [8], developed an artificial neural network model by making use of data from patients that have renal stone, to be able to predict stone-free situation and to make it possible for identifying treatment with ESWL for renal stones. The data had been collected from the 203 patients involves age, gender, stone size and density, stone size after ESWL, location of the stone, skin to stone distance, stone nature (single or multiple), and some other parameters. ANN method and regression analysis had been applied to estimate treatment success utilizing the same series of data. The consequently, patients had been divided into three groups by ANN software, to be able to implement the ANN which are: n=139 training group, n=32 validation group, and n=32 test group. The results show that the accuracy of the free stone rate was 99.25% in the training group, and it achieved 85.48% in the validation group, and it got 88.70% in the test group.

Theoretical Background

Logistic Regression (LR)

This method is inspects the relationship in between the binary outcome variable (dependent) like absence or presence of disease and predictor (independent or explanatory) variables like imaging findings or demographics of patient. For example, the absence or presence of breast cancer within a certain time period would possibly be predicted from information of the patient's breast density, age, genealogy and family history of breast cancer, and any previous breast procedures. The outcomes variables could be both categoric and continuous. If X1, X2, X3 to Xn denote n predictor variables (for example, calcification types, patient age, breast density, etc,), p denotes the possibility of disease existence, the equation defines the relationship between p and the predictor variables is given by [9]:

$$\text{Log}\left(\frac{p}{1-p}\right) = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \cdots + \beta_n X_n \quad (1)$$

Where: β_0 is a constant and $\beta_1, \beta_2, \ldots, \beta_n$ represent the regression coefficients for the predictor variables $X_1, \ldots, X_n$.

The regression coefficients β can be calculated from available data. Every regression coefficient represents the contribution size of the related predictor variable for the outcome.

The effects of the predictor variables on the output variable is generally determined by utilizing the odd ratio of the predictor variable that represents the factor wherein the odds of an outcome adjust for a one-unit modify in the predictor variable. The odds ratio can be estimated through taking the exponential of the coefficient (for example, exp (β_1). As an example, if β_1 is the coefficient of XFH variable (which represents the family history of breast cancer), and p presents the breast cancer probability, then the exp(β_1) is the odds ratio related to the family history of breast cancer.

In such a case, the odds ratio presents the factor where the odds of featuring breast cancer raise if the family of patient has a history of breast cancer and almost all remaining predictor variables keep unchanged. This means that, if the odds ratio related to the family history of patient with breast cancer is 2, hence the breast cancer may happen twice in women that have a family history of breast cancer than the women with no such family history. LR models usually involve only the variables which can be considered "important" in predicting the outcome. Through use of P values, the variables importance is described in relation to of the statistical value of the variables coefficients. The significance criterion $P \leq 0.05$ is generally used whenever testing for the statistical significance of variables; nevertheless, these types of criteria can vary based on the quantity of available data. As an example, if the observations number is very big, predictors with little effects on the outcome could also become significant. THE various techniques can yield a variety of regression models, that they generally work similarly. Often, medically important variables could be found to be statistically insignificant through the selection methods due to the fact their influence might be attenuated by the existence of other strong predictors. In these cases, these medically important variables can however be involved in the model regardless of their statistical significance level [9].

Neural Network

ANN is a computer models stimulated the system of biologic neural networks that is a part of the machine learning techniques in order to solve the complicated nonlinear systems in the realistic life. ANN have been widely used in numerous research areas ranging from marketing to medicine. Generally, in most cases the neural network is an adaptive system which modifies its structure throughout a learning phase. ANN can learn and identify correlated pattern between inputs and related outputs [10]. Fig. 1 illustrates a basic example of an ANN.

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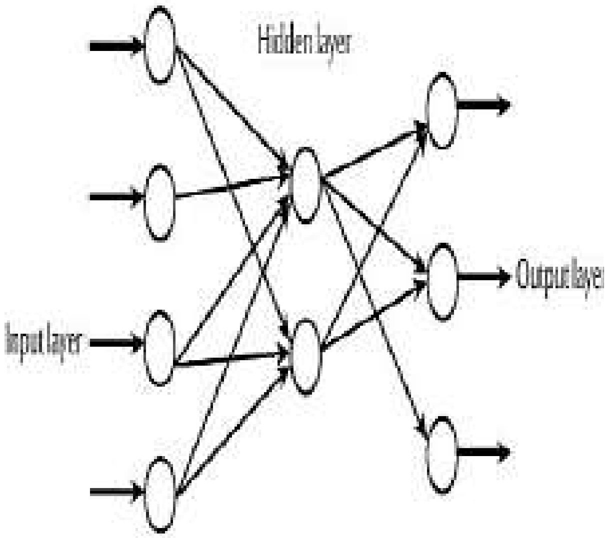

Figure 1: Typical structure of three-layer ANN which have four neurons in the input layer, two neurons in the hidden layer and three neurons in the output layer, without having direct connection from input layer to output layer [10]

As shown in figure 1, the interconnections don't loop back or skip any other neurons, this type of network is called feedforward [9]. In these networks, there are two functions concerning the behavior of a unit in a specific layer and influence the generalization of the model. One of those is input function and the second one is output function that is often known as the activation function. The equation of input function is given by [10]:

$$y = x_1 w_1 + \cdots . + x_n w_n + b \qquad (2)$$

Where n is the patterns number in the data set, x is the data points, which are called input variables or features with identified class memberships.Many non-linear functions have been used, and the most popular one is sigmoid function, because it is able to show both linear and non-linear property. The sigmoid function is given by Eq. (3) [10],

$$f(x) = \frac{1}{1 = e^{-x}} \qquad (3)$$

ANN are becoming highly popular with data mining practitioners, especially in marketing, finance and medical research. It is because they provide the major advantage of not being dependent on “a priori” assumptions and of enabling detection of links amongst factors that conventional statistical methods like LR might not be able to detect [10]. Comparing artificial neural network models with linear models of standard statistical generalized like LR is a significant step in the development method [11]. In case the results reveal that the gain of using a nonlinear model, like the ANN, is limited, then it should often choose the less complicated model.

Receiver Operating Characteristic (ROC) Curves

ROC curves is the method used to determine the predictive utility via displaying the trade-off between the false-positive rate and the true-positive rate that inherent in finding specific thresholds where predictions may be based. The area below this curve presents the likelihood that provided a negative and positive case, the output of classifier is going to be higher for the positive case and it isn't depending on the choosing of decision threshold. Using this method is less dependent on the malignancy frequency in the population and permits considering the and specificity of the model and the sensitivity at several probability levels. An effective one-class classifier should have both a mini fraction false negative as a mini fraction false positive. However, the ROC curve provides a very good summary of the efficiency of a one-class classifier, it is actually difficult to compare two ROC curves. The best way to summarize a ROC-curve with a single number often is the Area below the ROC Curve. This integrates the fraction false positive around ranging thresholds (or equivalently, ranging fraction false negative). Lesser values mean a better separation between out layer objects and the target. The graph in Figure 3 shows 3 ROC curves which represent excellent, and useless tests plotted on a single graph [11].

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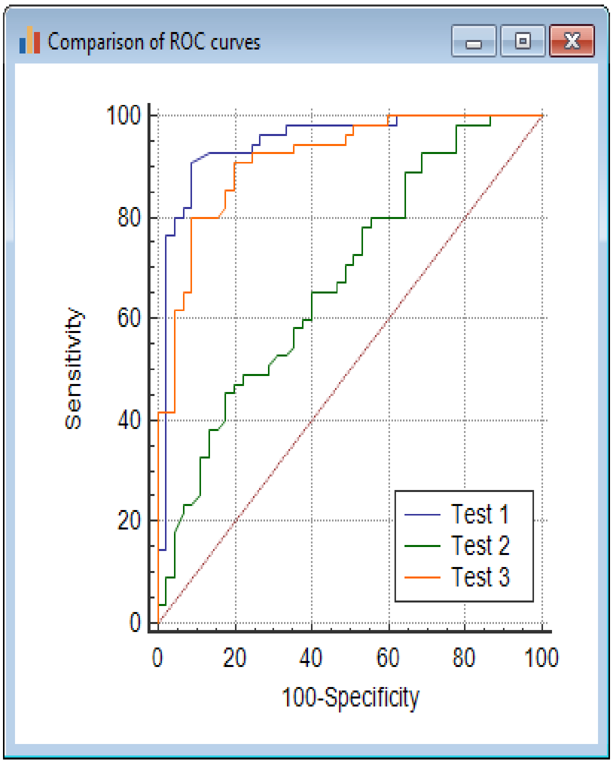

Figure 2: Comparing ROC Curves. Test 1 (Green) Worthless, Test 2(Orange) Good, and Test 3 (green) Excellent

The accuracy of the test is depending on how the approach separates the group that tested into those can and cannot treated by SWL. The Accuracy is measured via the area under the ROC curve.

The Proposed Approaches

Three approaches of hybrid classification rules have been proposed. The first approach is "Product Rule" termed (PR) which is based on combined neural network (NN) then logistic regression (LR) and the second approach is termed (NN1) which is adding outputs of the statistical techniques for training set. The third approach is Nested Combining Classifiers (NCC that based on adding the outputs combination using product rule (PR1) to the inputs combination using neural network (NN1), which presented an additional information that the improvement of network performance. In (NN+LR) approach the NN is the first combination classifier, while in (LR+NN) logistic regression is the first combination classifier and it doesn't need a specified condition in the medical data.

1- Acquiring the Data

The applied area of this study was Extracorporeal Shock Wave Lithotripsy (ESWL) for Renal Stones, where the renal stones represent the most important disorders which affect the Urinary tract. When we discover the present of a stone, it is treated by three ways:

- ESWL.
- Surgery.
- Ureteroscopy or Percutaneous Nephrolithotomy (PNL).

We use ESWL when the length of stone is less than 30 mm and the outcome of treatment is one of two:

- The patient becomes free from any fragments of stone.
- The patient becomes not free.

Where y is the dependent variable for the outputs of ESWL, Thus:

$$y = \begin{cases} 1 & if \quad the\ patient\ is\ free \\ 0 & if \quad the\ patient\ is\ not\ free \end{cases}$$

The model contains the following twelve independent variables, they are:

1- Age (x_1). 2- Stones Number (x_7).
3- Sex (x_2). 4- Stones Length (x_8).
5- Morphology (x_3). 6- Stones Site (x_9).
7- Anatomy (x_4). 8- Stones Nature (x_{10}).
9- Use of JJ Stent (x_5). 10- Side (x_{11}).
11- Solitary (x_6). 12- Stones Opacity (x_{12}).

The data has been taken from real cases at "Al Karama Teaching Hospital" and "Al Yarmuk General Teaching Hospital", where the patents data has been put in database. Figure 4 shows the samples of ESWL image and data includes patient information.

Figure 3: ESR Devise. Left, the kidney image and stone detected. On the right, the patient information

2.2 Combining Using Product Rule (PR1)

The product rule method is used to combine the Logistic Regression results which present the probability of belonging to class with Neural Networks results which represent the probability of belonging to class. The application of this method is used the testing sample and training sample.

2.3 Combining Using Neural Network (NN1)

In this method we are adding outputs of the statistical techniques for training set which present belonging the probabilities for classes and these probabilities are one of the inputs Neural Network. So, we will keep twelve Hidden Neurons to ensure that the improvement of improvement of Network performance due to the additional information which is presented in the outputs of statistical technique.

2.4 Nested Combining Classifiers (NCC):

In this study, we provide a new technique of combination in classification area, hoping to increase the accuracy of classification, which named Nested Combining Classifier in which we combine the results of combination using Neural Network and the results of combination using Product Rule, i.e. we combine between two results of combination. Methods of combining classifiers of the three types are shown in Figure 2.

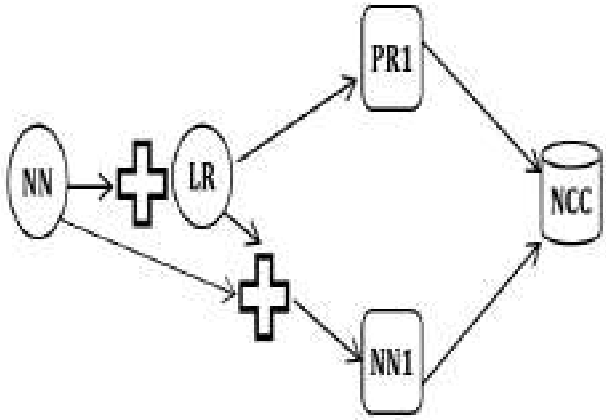

Figure 4: Methods of combining classifiers of the three types

Where: NN: is Neural Network, LR: is Logistic Regression, PR1: is combining using Product Rule, NN1: is combining using Neural Network, and NCC: is Nested Combined Classifier.

In the Nested Combined Classifier (NNC) method we combine between PR1 and NN1. To explain this method, we add the results of the product Rule to the inputs of the neural network (NN1), i.e. combined between two results of combination. In this section, we will explore the method that used to evaluate classification models.

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4- Results

In this section we test the three approaches for predicate the success of SWL in treatment the patient, where the samples had been taken from 3225 actual cases that had been treated at Urology and Nephrology center of Iraq ("Al Karama Teaching Hospital" and "Al Yarmuk General Teaching Hospital"). We selected 2800 cases for training and 425 select to be a non-trained cases. For trained data, there are 538 not free cases (failed to fee stone) and 2262 free cases (Successes in free stone). For no- trained, there are 47 not free cases (failed to fee stone) and 378 free cases (Successes in free stone). The results from three cases have been compared to actual treatment results of SWL for trained and non-trained cases and compered the results of three models.

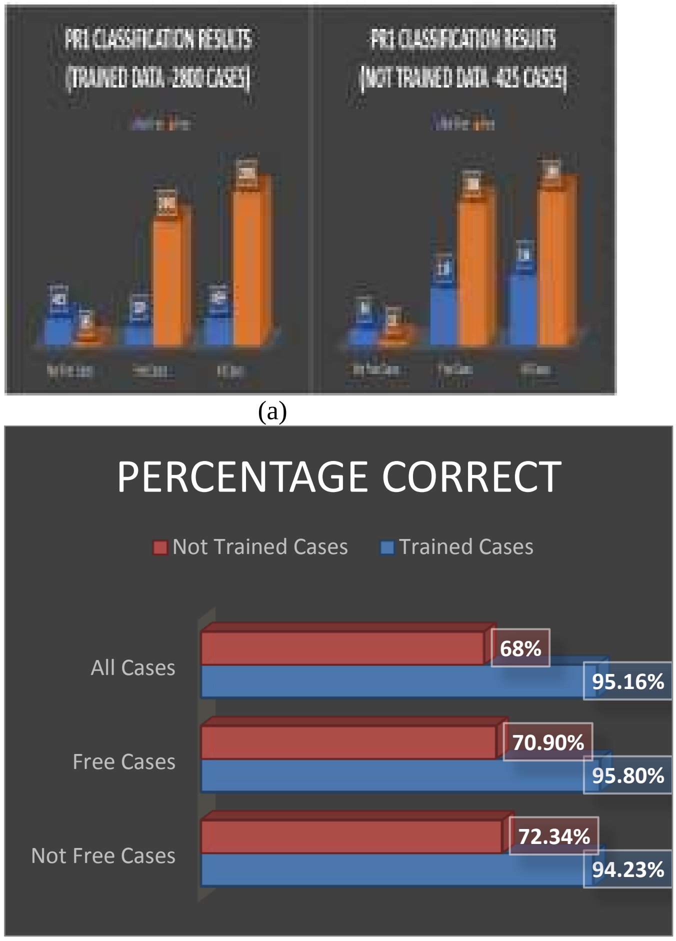
4-1 Results of Combination Using Product Rule (PR1):

The results of predicate SWL by PR1 are shown in Table 1, which shows the results for classification by using PR for each sample (testing & training).

Table 1: The results for classification by using PR1 for testing and training sample

Observation	Predicted: Trained Cases (2800) Not Free (538), Free (2262)			Non-Trained Cases (425) Not Free (47), Free (378)		
	y		Percentage Correct	y		Percentage Correct
	Not free	Free		Not free	Free	
y Not free	453	85	84.2%	34	13	72.34%
y Free	319	1943	85.9%	110	268	70.9%
Overall Percentage	459	2395	85.31%	136	289	68%

[ROC (Training Set) = 0.85, ROC (Testing Set) = = 0.72]



(a)

(b)

Figure 5: PR1 Classification results. (a) Classification Results for trained and not trained cases, (b) Percentage correct of classification

As shown from table 1 and figure 5 the system successes in detect of 453 non-free cases from 538 non-free cases that used in train which got (84.2%) accuracy, and for not trained data the system successes in detect of 34 non-free cases from 47 non-free cases that used in train which got (72.34%) accuracy. For free stone cases, the system successes in detect of 1843 free cases from 2262 free cases that used in train which got (81.47%) accuracy, and for not trained data the system successes in detect of 268 free cases from 378 free cases that used in train which got (70.9%) accuracy

4-2 Results Combination Using Neural Network (NN1):

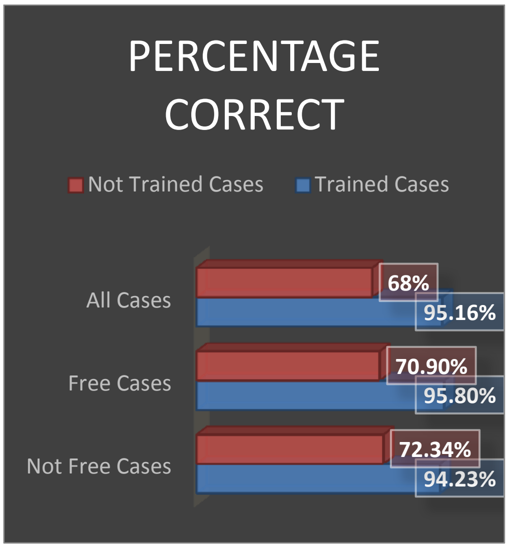
The result of predicate SWL by NN1 are shown in Table 2, which shows the results for classification by using NN for each sample (testing & training).

Table 2: The results for classification by using NN1 for testing and training sample

Observation	Predicted					
	Trained Cases (2800) Not Free (538), Free (2262)			**Non-Trained Cases (425)** Not Free (47), Free (378)		
	y		**Percentage Correct**	**y**		**Percentage Correct**
	Not free	**Free**		**Not free**	**Free**	
y Not free	507	31	94.23%	38	9	80.85%
y Free	95	2167	95.8%	56	322	85.18%
Overall Percentage	512	2288	95.16%	39	386	82.97%

[ROC (Training Set) = 0.96, ROC (Testing Set) = 0.81]

(a)



(b)

Figure 6: NN1 Classification results. (a) Classification Results for trained and not trained cases, (b) Percentage correct of classification

As shown from table 2 and figure 6 the system successes in detect of 507 non-free cases from 538 non-free cases that used in train which got (94.23%) accuracy, and for not trained data the system successes in detect of 38 non-free cases from 47 non-free cases that used in train which got (80.85%) accuracy. For free stone cases, the system successes in detect of 2167 free cases from 2262 free cases that used in train which got (95.16%) accuracy, and for not trained data the system successes in detect of 322 free cases from 387 free cases that used in train which got (85.18%) accuracy.

4-3 Results of the Nested Combined Classifier (NCC):

The Nested Combined Classifier method is carried out by adding the outputs combination using Product Rule (PR1) to the inputs combination using Neural Network (NN1), which present additional information that the improvement of network performance.

Table 3: The results for classification by using NCC for testing and training sample

Observation	Predicted					
	Trained Cases (2800) Not Free (538), Free (2262)			**Non-Trained Cases (425)** Not Free (47), Free (378)		
	y		**Percentage Correct**	**y**		**Percentage Correct**
	Not free	**Free**		**Not free**	**Free**	
y Not free	522	16	97.02%	41	9	87.23%
y Free	24	2167	98.93%	37	341	90.21%
Overall Percentage	527	2273	97.95%	43	382	91.5%

[ROC (Training Set) = 0.977, ROC (Testing Set) = 0.875]

(a)

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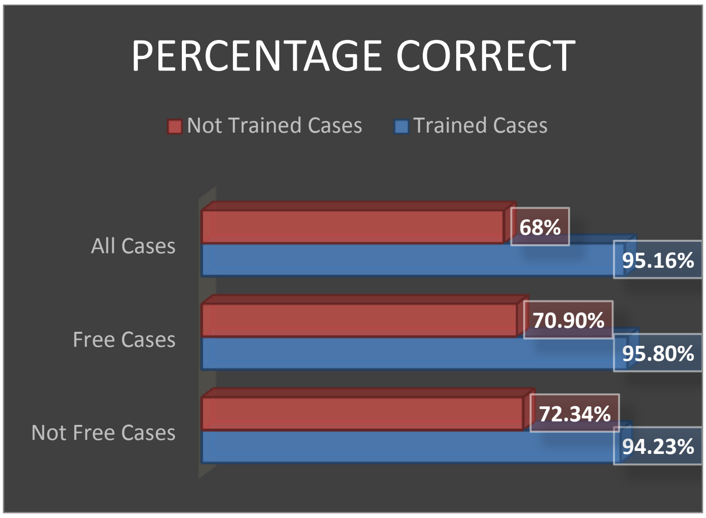

(b)

Figure 7: PR1 Classification results. (a) Classification Results for trained and not trained cases, (b) Percentage correct of classification

As shown from table 3 and figure 7 the system successes in detect of 522 non-free cases from 538 non-free cases that used in train which got (97.02%) accuracy, and for not trained data the system successes in detect of 41 non-free cases from 47 non-free cases that used in train which got (87.23%) accuracy. For free stone cases, the system successes in detect of 2167 free cases from 2262 free cases that used in train which got (98.93%) accuracy, and for not trained data the system successes in detect of 341 free cases from 378 free cases that used in train which got (90.21%) accuracy

4-4 Comparison between the Results:

The percentage results of the three classification techniques are displayed is table 4 as in follows:

Technique	Training Sample		Testing Sample	
	Overall Percentage	The Area under ROC Curve	Overall Percentage	The Area under ROC Curve
PR1	85.31%	**0.85**	68.00%	**0.72**
NN1	95.16%	**0.96**	82.97%	**0.81**
NCC	97.95%	**0.97**	91.50%	**0.87**

Table 4: Summary of the results for C techniques

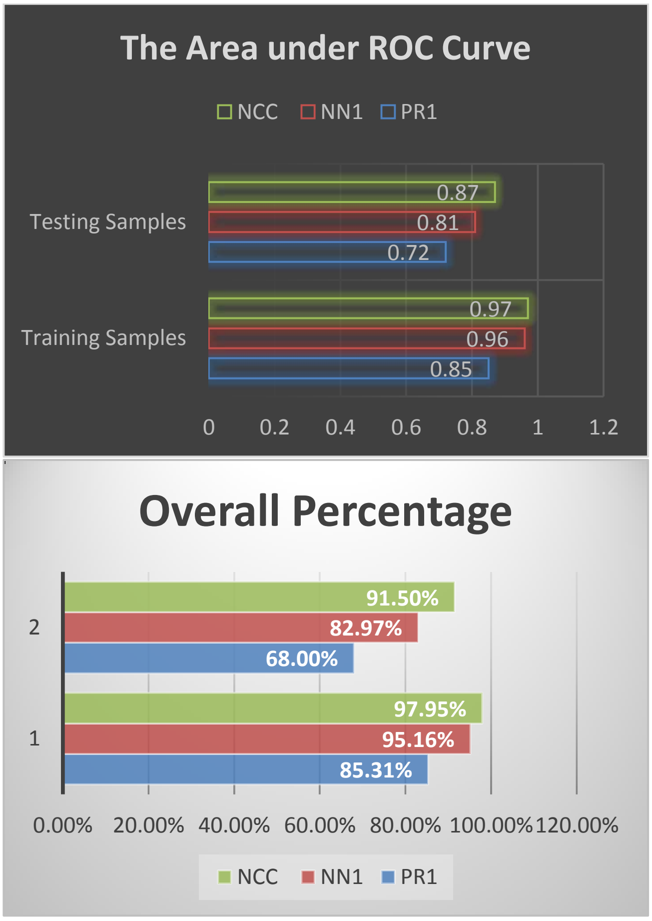

Figure 8: NN Classification results. (a) Classification Results for trained and not trained cases, (b) Percentage correct of classification

Table 4 and figure 8 shows summary of the results the overall percentage and the area under ROC curve for classification techniques of each testing sample and training sample. We compare between three techniques which are PR1, NN1, and NCC. As shown from table 4 the best technique for classification is the Nested Combined Classifier (NCC), where the overall percentage of the classification for training sample with NCC is (97.95%) which is higher than (85.31%) of PR1 and (95.16%) of NN1. Also, its shows that the overall percentage for testing sample of NCC is (91.5%) which is higher than the PR1 (68%) and NN1 (82.97%). The area under ROC curve of the NCC for training sample is (0.97) which is higher than (0.85) of PR1 and (0.96) of NN1, and the area under ROC curve of NCC for testing sample is (0.87) which is also higher than PR1 and NN1 which are (0.72 and 0.81) respectively.

By compared with previous works of [6] and [8], the results achieved higher accuracy than [6] in both NN and NCC where the accuracy of training result got about 22% accuracy improvement in training data detection and about 10% accuracy improvement in tested data. By compared with [8] the both got close accuracy result in detection of trained data where NCC got ~98% while their method achieved relatively higher accuracy of 99.25%, but for non-trained (tested) samples the proposed method (NCC) has achieved more accuracy reach to 91.5% which 3% higher accuracy than their method that achieved 88.7%.

Conclusions

Machine learning Techniques, such as the ANN, has been used widely in the medical field, as computer generated algorithms, that assist healthcare officials in clinical making decisions. One of medical application is to predicate the probability of success in desired treatment. For that purpose, three models to predicate the ability of ESWR to remove stone from kidney has been presented and tested, which based on used three techniques of mixing classifiers, PR, NN and NCC. We have been designed and develop a theoretical framework for combined classifiers and study the challenge of combining classifiers that uses different representations of the patterns for being classified. The testing shows that numerous existing schemes is often considered as unique cases of compound classification in which every pattern representations being used jointly to make a decision. The results show that the best technique for classification is the Nested Combined Classifier (NCC) when compared with other combination classifiers (NN and PR). Also, it should be emphasized that the analysis of result is dependent on a single experimentation for a single dataset. Thus, the conclusions can be summarized as in follows:

- Combining classifiers trained on a variety of feature sets is beneficial. Mostly in cases where these feature set probabilities are estimated via the classifier. On the other hand, combining the different classifiers trained with the same classifier might improve but is usually much less useful.
- The separated feature sets work well when used independently. Difficult datasets will not be thrown away; they include significant information! Using randomly selected feature sets seems to provide excellent results in our study.
- A resealing of all features sets to unit variance maybe enhance the accuracy of number of classifiers.

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توقع قدرة تقنية تَفْتيتُ الحَصاةِ بمَوجات صادِمَةٍ مِنْ خارِجِ الجسم (ESWL) على علاج حصى الكلى باستخدام المصنفات المدمجة

سميرة شمس حسين لباب احمد توفيق سكينه شكري محمود

قسم علوم الحاسبات، كلية التربية للعلوم الصرفة، جامعة بغداد

المستخلص :

تعد تقنية تَفْتيتُ الحَصاةِ بمَوجاتٍ صادِمَةٍ مِنْ خارِجِ الجسم (***ESWL***) من العلاجات الأكثر شيوعا لعلاج حصى الكلى. يتم تركيز الموجات الصادمة من خارج الجسم باتجاه حصى الكلى والتي تسبب تفتيت الحصى الى قطع صغيره. ان نجاح العلاج بتقنية (***ESWL***) يعتمد على عدة عوامل مثل: العمر، الجنس، حجم الحصى، عدد الحصى وغيرها. وبالتالي، فإن التنبؤ بنجاح العلاج بهذه الطريقة مهم للغاية للمهنيين لاتخاذ قرار بالاستمرار في استخدام (***ESWL***) أو استخدام تقنية علاج أخرى. في هذه الدراسة، تم تصميم نظام تنبؤ للمعالجة بطريقة (***ESWL***) من خلال استخدام ثلاثة أساليب لخلط المصنفات، وهو قاعدة المنتج (***PR***) والشبكة العصبية (***NN***) وطريقة تصنيف مقترحة تدعى المصنف المشترك المتداخل (***NCC***). تم أخذ عينات من 2850 حالة من الحالات التي تم علاجها في مركز أمراض الكلى والمسالك البولية في العراق. تمت مقارنة نتائج ثلاث حالات مع نتائج المعالجة الفعلية لـ (***ESWL***) للحالات المدربة وغير المدربة وقارنت نتائج ثلاثة نماذج. وأظهرت النتائج أن النهج (***NCC***)هو الأسلوب الأكثر دقة في التنبؤ بكفاءة في إمكانية استخدام طريقة (***ESWL***) في علاج حصى الكلى.

الكلمات المفتاحية: تَفْتيتُ الحَصاةِ بمَوجاتٍ صادِمَةٍ ، قاعدة المنتج، الشبكة العصبية ، ***ANN***، ***PR***

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A Comparative Study of KMCG Segmentation Based on YCbCr, RGB, and HSV Color Spaces

Fatin Sadiq Alkinani **Abdul Monem Salih Rahma**

Department of Computer Sciences , University of Technology

fatin.alkinani@sc.uobaghdad.edu.iq **monem.rahma@yahoo.com**

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Abstract:

Kekre Median Codebook Generation (KMCG) is a vector quantization algorithm. It is used for several purposes like image compression and segmentation. It has been applied by several application and shows its efficiency. This paper presents a comparison study of applying KMCG with three color models: RGB, YCbCr, and HSV for image segmentation. The experiments applied on five images, three of them are benchmarks. Two numerical metrics are utilized: E measure and Peak Signal to Noise Ratio (PSNR), in addition to the visual results. The final results show that KMCG conducts better segments when it is applied with the RGB color model. It returns more homogenies segments than using KMCG with YCbCr or HSV.

Introduction

Image segmentation is one of the most important steps in the analysis of processed image data. The main goal of segmentation is obtaining parts that have a strong correlation with objects areas of the real world in the image.

Segmentation accuracy is an important factor in determining the success or failure of computerized analysis procedures. For this reason, considerable care should be taken to improve the probability of accurate segmentation [1] [2]. Many methods of image segmentation have been proposed could be categorized generally as:

- **Edge**-based image segmentation like Active Contour
- **Region** base as Seed filling algorithm and clustering algorithms.
- **Thresholding** based image segmentation such as Otsu method.
- **Clustering** based image segmentation such as KMCG and K-mean
- **Combined** between the above category like watershed.

Literature Review

Generally, KMCG is a clustering algorithm found by Kekre in 2008 for data compression purpose. However; this algorithm enforces efficiency in several areas like segmentation and clustering. It summarized by the following steps [3]:

1. Distribute the data in equal size vectors.
2. Arrange vectors based on the first value.
3. Set the median of the array of vectors in the CB.
4. Repeat step 2 & 3 on the other vectors values until obtaining the desired codebook size as shown in Fig. (1).

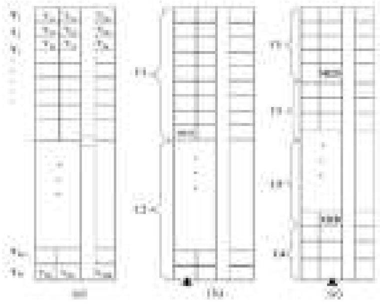

Figure 1: KMCG General Steps [5]

KMCG is applied to an image by dividing it's data to equal nonoverlapped windows. Each window color data is represented in a vector. As in gray level, the window size is four yields a vector length of 4 [4].

KMCG algorithm had been augmented, to decrease the required time in the clustering process of a gray image. In this method, vector size is increased to 6 columns, in which the last four columns are used to store original gray levels, which obtained from 2 x 2 blocks of the image. Further, averages of each of these blocks are done separately and stored in the second column in the respective vectors. The sequence number of the respective vectors has been stored in the first column [5].

KMCG used by different applications due to its homogeneity and efficiency of its segmentation. One of these applications is image cartooning. As in [6] shows that the consumed time of the cartoon production is less by using KMCG in comparison with other vector quantization methods. Also, it presents a better quality of segmentation of KMCG than the other methods.

Another application designed based on KMCG, which is fingerprint classification. It was observed that the method effectively improves the computation speed and provides high accuracy [7].

KMCG literature shows its importance and efficiency. Thus, this paper is to study and find the best color model to be applied with KMCG.

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Color Models

Color models in image processing provide valuable tools for objects recognition and extraction from the scene. It also enables the extension of the domain space compared with gray images. Color spaces are used for different applications such as; computer graphics, image processing, TV broadcasting, and computer vision. Several models had been introduced. YCbCr and HSV are two samples of them. All of the models are extracted from the original model of RGB. [8].

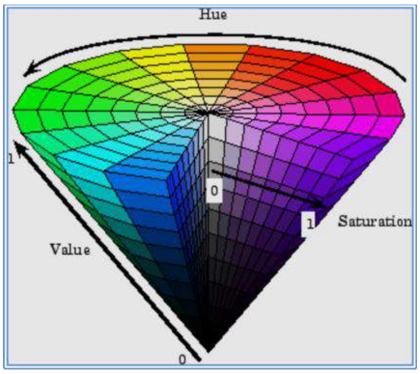

Figure 2: HSV color model single hex cone.

YCbCr Color model used for digital video. It defined in the ITU-R BT.601 standards of ITU (International Telecommunication Union) represents the encoding form of non-RGB signal. The transformation from RGB to YCbCr color model is given in the following equation [8]:

$$\begin{bmatrix} Y \\ Cb \\ Cr \end{bmatrix} = \begin{bmatrix} 0.299 & 0.587 & 0.114 \\ -0.169 & -0.331 & 0.500 \\ 0.500 & -0.419 & -0.081 \end{bmatrix} \begin{bmatrix} R \\ G \\ B \end{bmatrix} \quad (1)$$

The **HSV** color model based on the idea of the human visual system. It belongs to the HSI family of color models uses cylindrical coordinates for the representation of RGB points as shown in Fig.2. Hue (H) and saturation (S) which are the chrominance components. V is the maximum value of RGB.

HSV is used mainly for computer vision, and image analysis of the segmentation process. The conversion from RGB to HSV is summarized by the following steps [8].

1. Find the maximum and minimum values as in the following equation:

$$M = \max(R, G, B), m = \min(R, G, B) \quad (2)$$

2. Normalized the RGB values to be in the range [0, 1].

$$r = \frac{M-R}{M-m}, g = \frac{M-G}{M-m}, and\ b = \frac{M-B}{M-m} \quad 2(3)$$

3. Find V value.

$$V = Max(R, G, B) \quad (4)$$

4. Calculate S value.

$$S = \frac{M-m}{M} \quad (5)$$

However, if $M = 0$ then $S = 0\ and\ H = 180$.

5. Calculate H value as in equation (2.26)

$$H = H - 360\ when\ H \geq 360, H = H + 360\ when\ H < 0 \quad (6)$$

6. Normalize H to be in the range [0,360[as in the following:

$$H = 60(b - g)\ when\ R = M,\ H = 60(2 + r - b) when\ G = M,\ or\ H = 60(4 + g - r)\ when\ B = M \quad (7)$$

7. Return HSV

KMCG Algorithm

The detailed steps of KMCG are demonstrated by algorithm 1. The window represents the data that needed to be clustered. For instance, if the vector is of RGB image data, and the window size is 4, then the length of the vector is 12. As shown in Fig. 3.

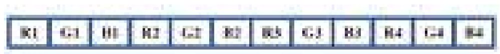

Figure 3 KMCG Vector Example

L is the length of the vector to be clustered. If the length is equal to 6, then the no. of clustering iterations is six, and the Codebook (CB) size is 64. Each iteration concludes sorting the vector of the whole image data based on one value of the vectors. For example: in the first iteration of the RGB image the vectors are sorted based on R1, which its position clarified by Fig. 3. Then after sorting the whole vectors, the data each cluster is separated into two clustered based on the median position of the vectors. Finally, the median vector is added into CB. This process is continued until the required length of the codebook is satisfied

Experiments and Results

The experiment is applied by testing KMCG with the three-color models: RGB, YCbCr, and HSV. Five samples of images that used. Three of them are standard images: Lena, Baboon, and the image of man is from a standard dataset called VidTIMIT [9].

The specification of the used computer and programming language is listed as the following:

- 12 G RAM, 512 SSD
- Intel ® Core™ i7_6500U CPU @ 2.50GHz
- System: Windows 10 64
- The programming language: C#

The size of the KMCG codebook which used by our experiment is 64.

Algorithm (1): Original KMCG

Input	A bitmap *Im* of size M*N, W (window size), L (required
Output	*CI* (Cartooned Image)
Begin	
Step1	**///Initialization** **Set** *lstVectors*← **FillVectors**(Im,W) **Set** *lstClusters* ←**InitializeClutsers**(*lstVectors*)
Step2	***///Clustering and Codebook Construction*** **for** *j* =1 **to** *L* **do** /* L is the required size of vectors values */ **foreach** *cluster* C in *lstClusters* **do** **SortAscending**(C,j) /* Sorting theCluster C based on value sequence j*/ **AddMediantoCB**(C) **SplitClusterAndAdd**(*lstTempClusters*,*C*) **end foreach** **Set** *lstClusters* ←*lstTempClusters* **end for**
Step3	***///Constructing Cartooned Image*** **foreach** *cluster* **in** *lstClusters* **do** **Set** *newRGB* ←**CalculateAverage(***cluster***)** **SetValueInImage(***C*,*newRGB*,*CI***)** **end foreach**
End	

Results

Table 1:PSNR of KMCG with RGB, YCbCr, and HSV

Image	KMCG RGB	KMCG YCbCr	KMCG HSV
Baboon	36.66	36.07	35.71
Lena	38.67	37.97	37.44
Man	42.85	39.21	40.18
Woman	42.02	38.33	38.08
Girl	42.76	38.39	38.8

Table 2: E measure of KMCG with RGB, YCbCr, and HSV

Image	**KMCG RGB**	**KMCG YCbCr**	**KMCG HSV**
Baboon	3.61	3.68	3.71
Lena	3.26	3.4	3.4
Man	2.8	2.9	2.94
Woman	2.95	3.15	3.14
Girl	2.92	3.2	3.2

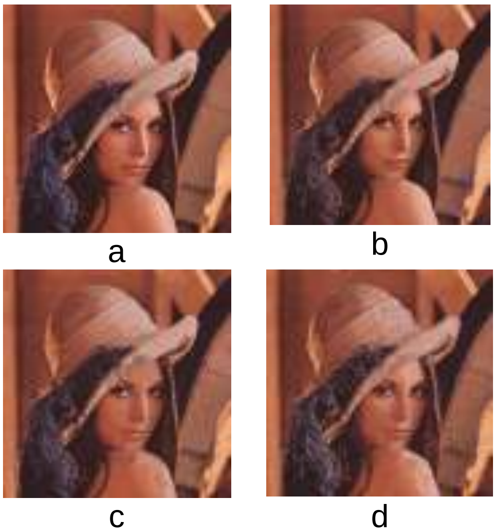

Figure 4: Lena a) Original b) RGB & KMCG c) YCbCr & KMCG d) HSV & KMCG

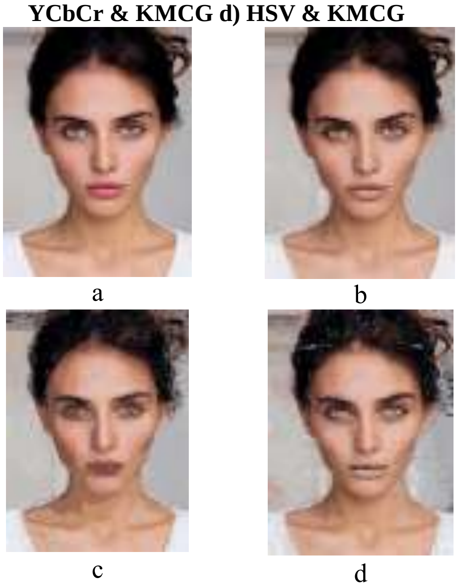

Figure 5: Woman a) Original b) RGB & KMCG c) YCbCr & KMCG d) HSV & KMCG

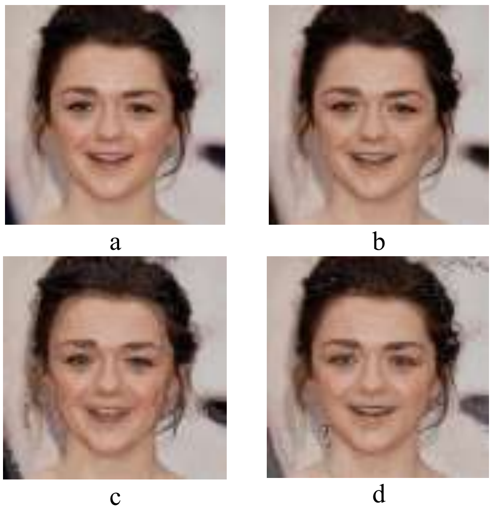

Figure 6: Girl a) Original b) RGB & KMCG c) YCbCr & KMCG d) HSV & KMCG

Discussion

The visual results are shown by Figs. 4,5,6. They reflect clearly that using KMCG with RGB color model conducts a better likeness and quality. While Applying KMCG with HSV shows the worst case. This remark is proved by utilizing the Peak Signal to Noise Ratio (PSNR) to evaluate and compare the precision of the KMCG with each color model as in Table 1.

PSNR is a good method to evaluate discrepancies between images. A high PSNR indicates an image of good quality. However, PSNR is not adequate for evaluating region homogeneity [10]. Thus, E measure is used to confirm PSNR results.

E is better at selecting images that agree with human evaluators. *E* balances region uniformity and the number of regions. A low *E* value indicates good internal uniformity of regions [11], [12]. Table 2 shows a functional impact of *E* measurement of the comparison between the three-color models. *E* indicates a good behavior balance of the KMCG with RGB. As shown by Table2 the E measure is the least in applying KMCG with RGB color model.

There is a slight difference between the results of using YCbCr and HSV. However, they don’t reach the quality of using RGB with KMCG.

Conclusion

The paper studied the best color model, which gives better segmentation results when it used with KMCG. The results show that KMCG gives a better visual and analytical results when it is used with RGB. We conclude that whatever the image is or the metrics used, using RGB with KMCG returns the best result. We recommend in the future work to compare KMCG with RGB segmentation with other clustering segmentation methods.

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مقارنة تقسيم الصورة بواسطة خوارزمية الـ KMCG بالاعتماد على HSV و YcbCr،RGB

فاتن صادق خزعل عبد المنعم صالح رحمة
قسم علوم الحاسوب ، الجامعة التكنولوجية

ألمستخلص :

KMCG هي احد انواع خوارزميات الـ Vector Quantization المستخدمة لاغراض الضغط و تقسيم الصورة. استخدمت في العديد من التطبيقات و اثبتت كفاءتها و سرعتها. هذه الورقة البحثية تعرض مقارنة في تطبيق خوارومية الـ KMCG مع ثلاثة انواع من الـ Color Models و المتضمنة RGB، YCbCr، و HSV. تم تطبيق تجربة المقارنة على خمسة صور، ثلاثة منها هي صور معيارية (Standard). و عدد المقاييس المستخدمة اثنين: PSNR و E measure ، بالاضافة الى الصور الناتجة المرئية. جميع المقاييس اثبتت ان استخدام الـ RGBمع الـ KMCG يعطي نتائج افضل في تقسييم الصور

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PCA Classification of vibration signals in WSN based oil pipeline monitoring system

Waleed F. Shareef **Nasheed F. Mossa**

Department of Control and System , University of Technology-Iraq, Baghdad, Iraq

60026@uotechnology.com **nasheed707@gmail.com**

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Abstract:

Using wireless sensor network technology in structure health monitoring applications results in generating large amount of data. To sift through this data and extract useful information an extensive data analysis should be applied. In this paper, a Wireless Sensor Network (WSNs) is proposed for the oil pipeline monitoring system with proposed method for event detection and classification. The method depends on the Principal Component Analysis (PCA). It applied to features extracted from vibration signals of the monitored pipeline. These vibration signals are collected while applying damage events (knocking and drilling) to the oil pipeline. PCA is applied to features extracted from both time domain and frequency domain. The results manifest that this method is able to detect the existence of damage and also to distinguish between the different levels of harmful events applied to the pipeline.

Keywords: WSN, Structural health monitoring, Oil pipeline.

1. Introduction

With the fast evolution of oil, manufacture oil transportation has become more important in economic expansion. In some cases, the most serious damage to transport networks is artificial damage such as ramming, drilling, steel pipe knocking, vehicle movement, etc. to steal oil. At present, the prevailing methods of judging damage are the process of observation of the difference between the pump station and the other, when the pressure is below the required level, then the guess is the existence of damage in oil pipe. Determine the damage area, depending on the human teams which in turn exploring along the pipe path [1]. Monitor the sensitivity of these methods are minimal and are obtained exclusively warning after the occurrence of the damage. The immediate problem to be solved in the detection and prevention interventions by pipeline damage or damage potential, but without causing any damage. This field currently is an active research area. Fang Wang et al., [2] suggested the method of detecting the leakage of the pipeline using statistical features in the time domain of acoustic sensors. These features are taken from the normal (not leakage) sample signals. The vector size of the extracted features is reduced with the PCA method. The model will be feeding Support Vector Data Description (SVDD) trainer features acoustic signal in real time. If the output function of a positive decision, it means that the input signal is abnormal otherwise, the signal is normal. In [3] authors offer the advantage of extracting the vibration signal detected by fiber optic along the crude oil pipeline and the Independent Component Analysis (ICA) warning system. It can be used for fiber-optic distributed along the pipeline to get the vibration signal as well as to determine the leak site and third-party intervention. ICA is applied to separate optical fiber vibration signals from each other, and the relevant signal is extracted. In [4], the authors offer the advantage of the extraction and integration of multiple sensor data in the system of monitoring and pre-warning to secure the pipeline based on multi-seismic sensors. The seismic signals are processed and extracted the features in respective modules. Empirical Mode Decomposition (EMD) was used to analyze signals. Test devices are composed of many seismic sensors and units of data acquisition and processing. The unit displays three typical target signals respectively, namely individual walking, car moving, and manual drilling. In [5], researchers provide an analysis of the effectiveness of statistical time domain features in determining the vibration signal error. The authors attempted to use the time domain feature to determine the characteristics of mechanical failure of the engine induction. Algorithms are used to determine the features to improve accuracy and reduce the burden of the arithmetic.

In this paper, we studied and developed a novel pipeline security monitoring and early warning system by using WSNs based on PCA. This system consists of data acquisition units that propagate along the pipeline. These sensors and units collect and analysis vibration signals resulting from different targets. The vibration signals are analyzed to obtain useful features. Usually, raw signals are not adequate to identify the existence of a damage; therefore, damage features are extracted from the time domain, frequency domain, or time-frequency domain analysis [7].

2. Architecture of oil pipeline monitoring system

The proposed oil pipeline monitoring system based on WSN illustrated in Figure 1. The system consists of two sensor nodes mounted at the end of 2m carbon steel oil pipeline, and one base station connected to the computer [8]. Base Station node (BS node) or "Monitoring Node": consists of MCU unit and ZigBee module connected via USB serial cable to the PC. End Points node (EP node) or Vibration Sensing Nodes: These nodes sense the vibration and send their readings to the monitoring node. Each one of these nodes consists of MCU unit, ZigBee module, vibration sensor (accelerometer) and Direct Current (DC) power supply. The distance between sensors is 1.8 m. The pipe mounted on one stand in each end. In this work, a real time vibration signals resulting from the damaging activity on pipe are captured by the accelerometer of each node, and transmitted wirelessly through base station to the computer. The position of (X-Y-Z) axis of the installed accelerometer for each node on the pipe shown in Figure 2.

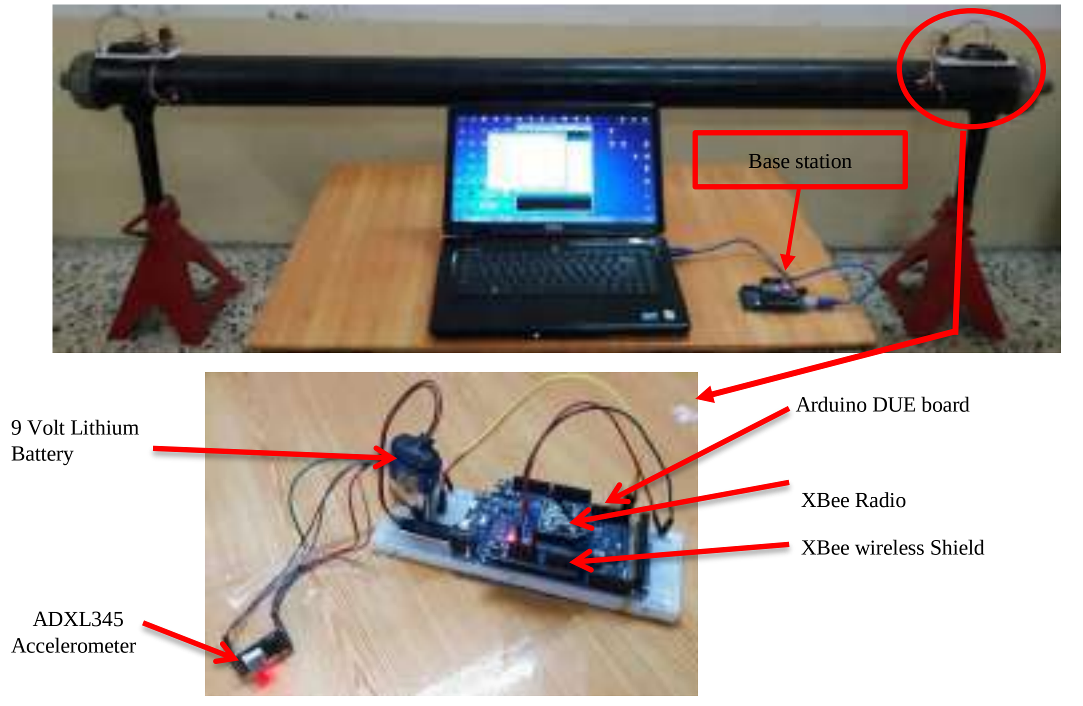

Fig. 1: Proposed system hardware and experimental setup

3. Test scenarios

For the purpose of this paper we subjected the pipe to both knocking and drilling events These events were applied using a hammer and a handheld electrical drill. To achieve consisting reading measurements of the two sensor nodes were averaged before sending the data to the base station. Figure 3 shows the real time vibration data captured by accelerometer without any damaging event (healthy state of the pipeline). It was observed that signals amplitude varied from one axis to another. This is due to the position on which the accelerometer is installed on the pipeline as we illustrated in Figure 2. The values of the amplitude of health status on the X- axis is 0.15 G and in Y-axis is -0.07 G, while in the Z-axes was shifted 1 G due to earth gravity. These values are the basic premise that distinguishes the health status of the pipeline, which we can compare the remaining damage events with them to know the nature and type of damage that can be exposed to the pipeline. Figure 4 illustrate the vibration data of knocking the pipe four times in average time equal to approximately (7 seconds).

For each case event four data set was captured, for the purpose of creating a basic database for each case which can be relied upon as input to the PCA system analysis for diagnosis the damage events. As shown in Figure 4, one note that the largest value of amplitude was on X-axis,

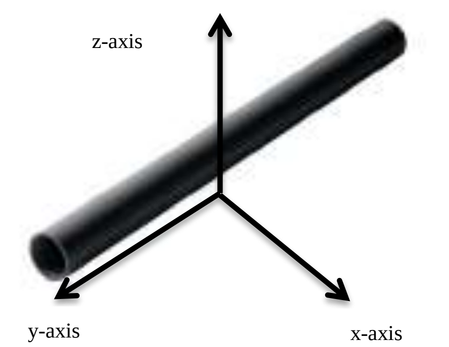

Fig. 2: Accelerometer layout relative to the pipe

where it reached ±2G and contains many components or information more than the rest of the other axes. In Y-axis, the amplitude values was approximately between -1.7 and 1.7 in addition that the information of signals less than X-axis. Lastly, the values of Z-axis amplitude reached 2G which are the minimum values from other axis. It is clear that the effect of knock case is very clear on the X-axis than the others.

Figure 5 shows that the process of drilling the pipe three times in total average time equal to (35 seconds). The amplitude values at the pipe-drilling attempt were between ±1G for X-axis and Y-axis, while in Z-axis were equal to 2 G. It is clear that the amplitude values in all three axes are convergent.

This is due to the fact that the vibration from the drilling process is very high, which results in the impact of the accelerometer on all axes and regardless of the axis direction relative to pipeline. The process of trying to drilling the pipe was done in different areas along the pipe and also vertically on the pipeline.

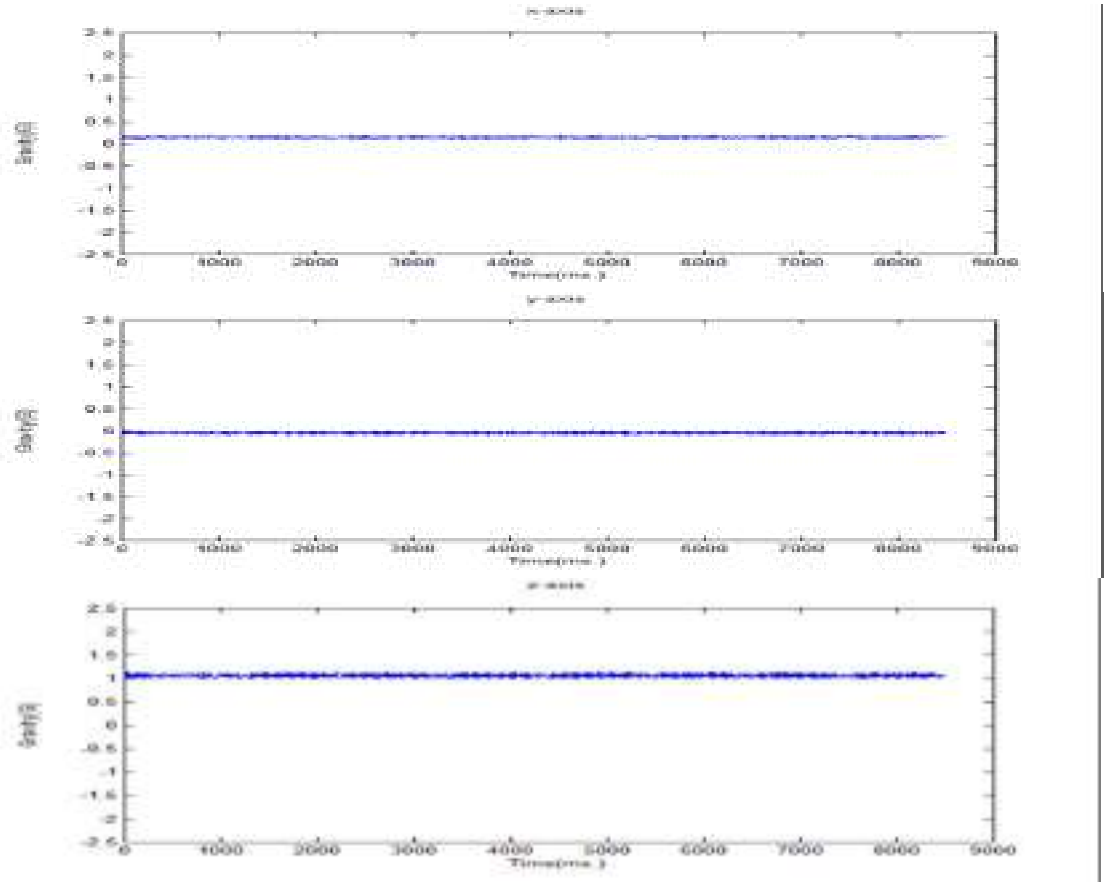

Fig. 3: The signals of the X, Y and Z in time-domain for normal case

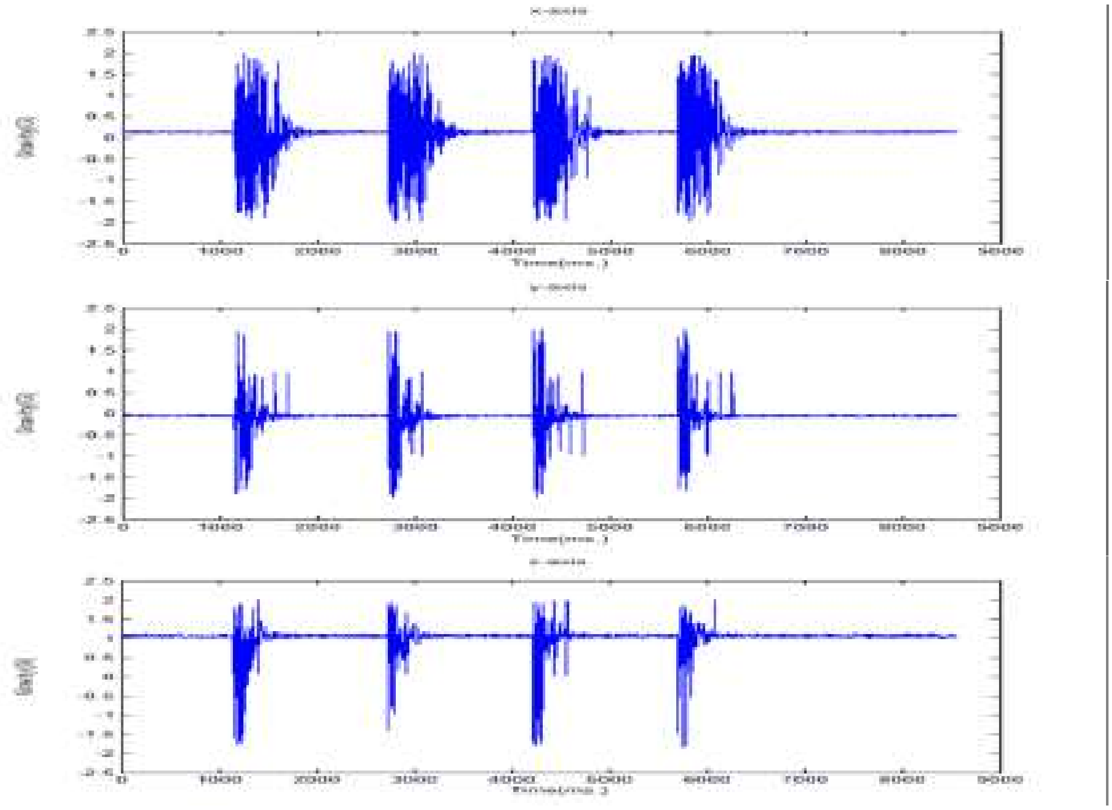

Fig. 4: The signals of the X, Y and Z in time-domain for knocking event case

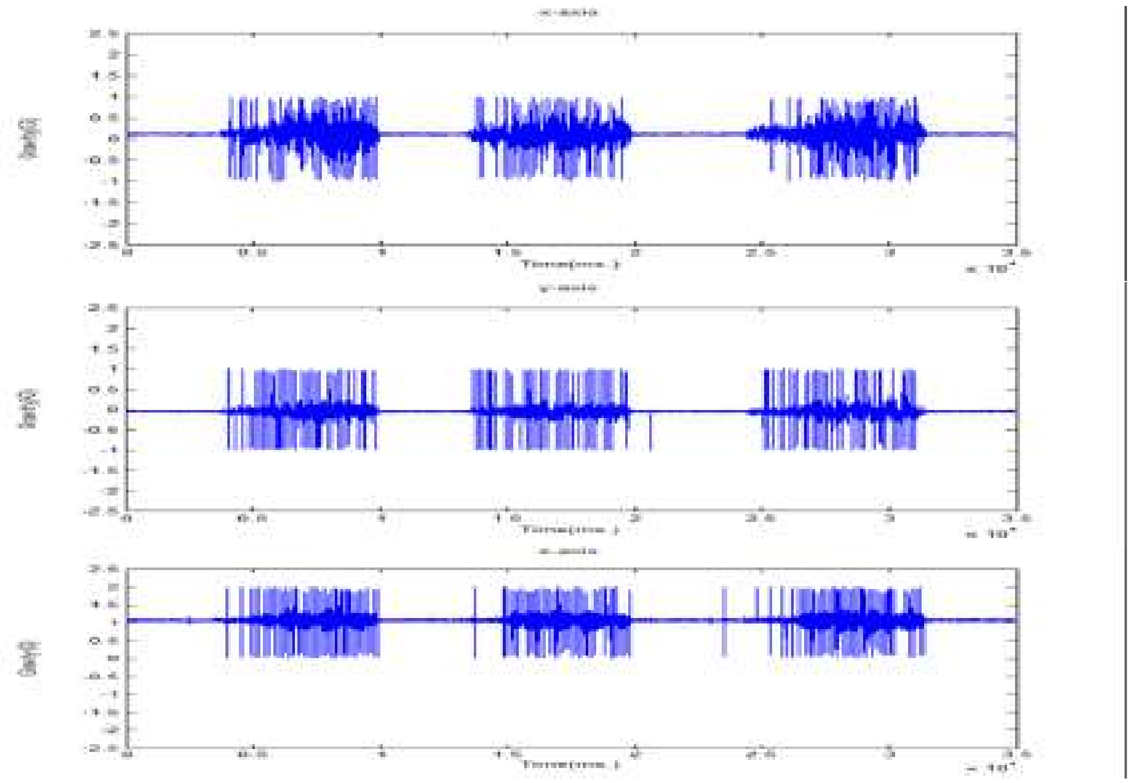

Fig. 5: The signals of the X, Y and Z in time-domain for drilling event case

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4. Peak to Peak(pk-pk)
Peak-to-peak refers to the difference between the highest positive and the lowest negative amplitude in a waveform.

4.2. Analysis in frequency domain

Frequency domain representation is the assessment of the strength of various frequency components (the power spectrum) of a time domain signal [6]. FFT is considered as more effective and efficient diagnosis technique to obtain the Fourier Transform of discretized time signals. This signal is considered for a finite time called the ‘‘frame’’ or ‘‘time window’’, which is then digitized and stored for feature extraction [6].
To detect the error, FFT can be used effectively, a known algorithm and a useful technique for signal analysis. Therefore, most researchers focused on FFT to detect damage in pipes [12]. Also for structural health monitoring(SHM) and in frequency domain analysis of acceleration measurements [14-16]. FFT is an improved algorithm for the performance of a Discrete Fourier Transform (DFT), which is an effective technique for static signal analysis [16]. DFT is defined by Equation (5). In this paper we propose to applying FFT algorithm on real time measurements data set, for the reasons mentions above and for optimizing the system. Once the FFT implementation complete, we calculate frequency domain features which we abbreviate them in other section.

$$\mathrm{X(K)} = \frac{1}{N}\sum_{n=0}^{N-1} X(n)\, W\frac{nk}{N} \quad for\ 0 \le k \le N\ and\ 0 \le n \le N \qquad (5)$$

Where:
X(n) is the discrete time signal
N is the sampling period

n and k are the discrete time and frequency indexes, respectively.
The transformation kernel Wnk N is given by Equation (6).

$$W\frac{nk}{N} = cos\left(\frac{2nk\pi}{N}\right) + j\,sin\left(\frac{2nk\pi}{N}\right) \qquad (6)$$

Figures 6, 7 and 8 shows the FFT results for one set of data, we notice that the power amplitude values different from one case to others, such as: the normal case power amplitude approximately equal to zero. Because of this case indicate the health state of the pipe (without applied damage events).

The effect of drilling case was on the z-axis greater than the others axis, where the maximum power amplitude equal to 150 at 50 Hz and in x-axis equal to 105 at 9 Hz then in y-axis equal to 55 at 159 Hz. The reason of the minimum power amplitude was in y-axis, as we mentioned the accelerometer installation in method that the y-axis in parallel to the pipe, while the z-axis perpendicular on the pipe. And the other reason that the electric drill used towards z-axis (in a vertical direction on pipe). The knocking case was record its maximum amplitude value in z-axis where equal to 107 in 2 Hz then in x-axis equal to 80 at 60 Hz and in y-axis 47 at 29 Hz.

Generally, the difference of power amplitude values at three axis for the knock cases and there effects on these axis, was not similar to effect of drilling case on the axis (in other word the effect of knock case not be maximum on z-axis and then x-axis and minimum effect on y-axis as in drilling case). Because of the knocking cases performed in different regions on the pipe. Once on vertical direction, the other once on horizontal direction, and once in arbitrary location and so on.

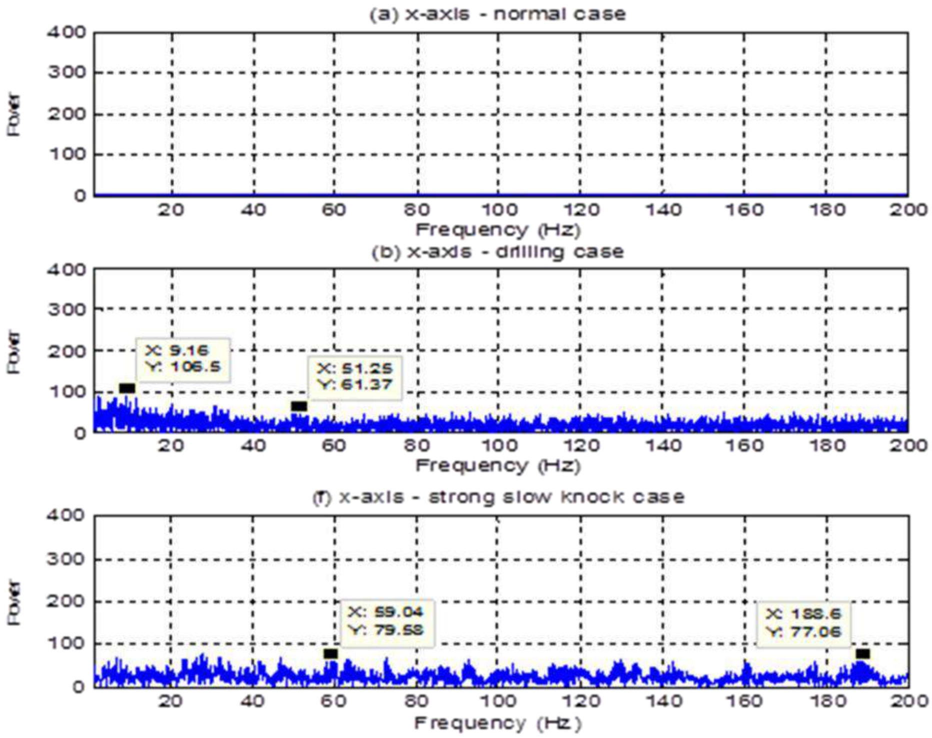

Fig. 6: FFT for x-axis of (a) normal case, (b) drilling case, (c) knocking case.

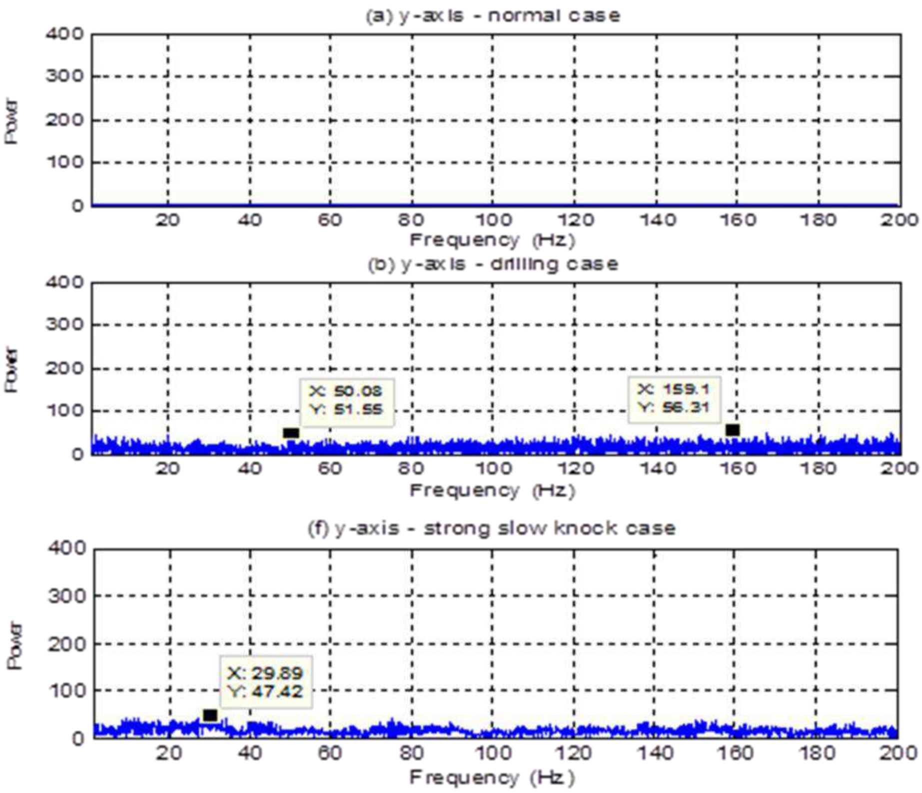

Fig. 7: FFT for y-axis of (a) normal case, (b) drilling case, (c) knocking case

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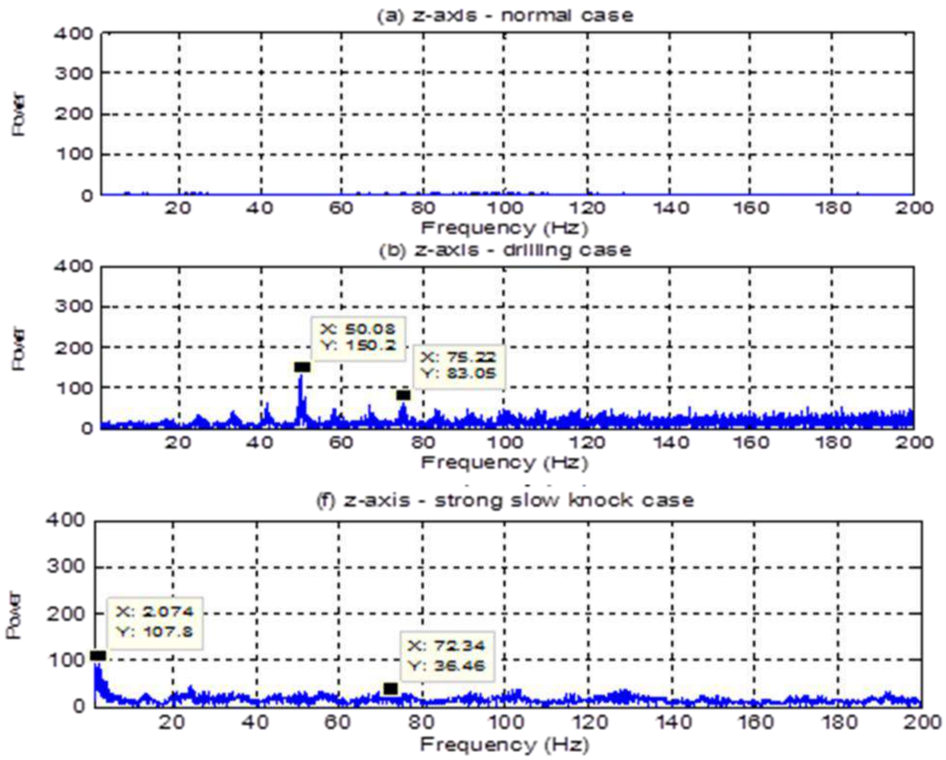

Fig. 8: FFT for z-axis of (a) normal case, (b) drilling case, (c) knocking case

5. Feature extraction in frequency domain

Frequency domain feature extraction is the process of extract useful information from the frequency domain representation of signals. This process is done using the methods discussed in this section. The most frequency-domain features obtained by FFT which considered by researcher are: energy [12], entropy in [11], frequency in [13]. In our proposed system, we propose to use energy and entropy as a features in frequency domain in addition to the time domain features. The energy and entropy are calculated as shown in equations below:

$$Energy = \frac{1}{N}\sum_{j=1}^{N} (|mj|)^2 \quad (7)$$

Where:

mj is FFT component

N is the length of FFT signal

$$Entropy = -\sum_{j=1}^{n} (pj * \log(pj)) \quad (8)$$

Where:

Pj is the probability of signal and must be between 0-1

6. Principal Component Analysis (PCA)

The first step in the analysis of high dimensional data is a dimensionality reduction [17]. Because of two reason; first, it is difficult to interpret multidimensional data sets, and their composition cannot be directly imagined. The second reason is that excessive variables create an empty space and computational problems. PCA is the most useful tool to solve these problems.

PCA is a multivariate statistical technique and powerful tool for analyzing data. It minimizes the feature space dimension by considering the difference of the input data without loss much of information in order to classify patterns in data [18]. The method determines the best projections for the representation of the input data structure. They are selected these projections in a way to enable them to get as much information as possible (i.e. the maximum variation) in a smaller number of dimensions of the region. In order to get the best variation in the data, the data is displayed on a partial space being built by eigenvector of data. In that sense, the eigenvalue corresponding to an eigenvector represents the amount of variance that eigenvector handles.

PCA has been used as a direct method of identifying, classifying and assigning damage as well as an essential step for other methods, and used for face recognition and image compression [17, 18]. PCA used for Structural Health Monitoring (SHM) and has received considerable attention over the past few years [18]. Also, In SHM, the natural frequencies depend not only on the damage but also on environmental conditions, such as temperature and humidity [19]. PCA is used to put this problem in mind because it allows the removal of external factors. As shown in Figure 9 we will explain the steps of implementing PCA algorithm [18].

Step-1 prepare the given data

In this step, the X matrix of size 12×18 (time domain and frequency domain features) was loaded to apply PCA on it.

Step-2 Data standardization

Since physical variables have different amounts and measurements, each data point is standardized. It means subtracting the sample mean from each observation, then dividing by the sample standard deviation. This procedure removes the differences between the scope of variables and giving it the same importance in the data analysis. In addition to the purpose of ensuring that the separation of values and achieving maximum variance between them. When performing this step, anew data matrix was generated (matrix B) in the same dimensions of the original data matrix.

Step-3 The calculation of the covariance matrix (coefficient) & eigenvalues of the covariance matrix (Latent) In this step, the function of PCA in MATLAB was implemented to the matrix B. This function return the coefficient and latent matrix. The result the 18×18 coefficient (loading) matrix and 18×18 eigenvalues (latent).

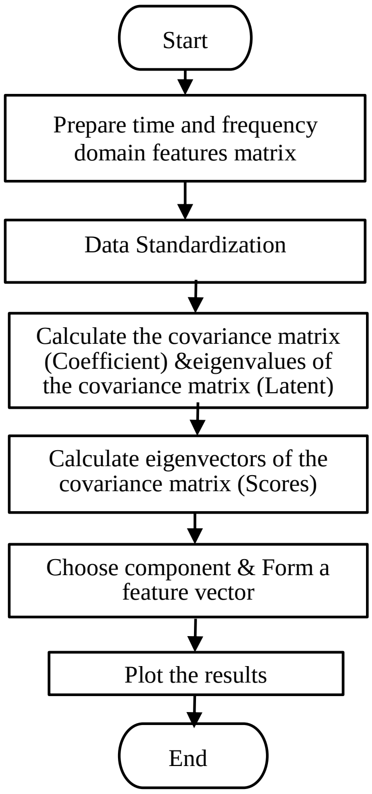

Fig 9: PCA algorithm steps

Step-4 The calculation of eigenvectors of the covariance matrix (Scores) This step, is the calculation of the eigenvectors of the covariance matrix (scores). by multiply the standardized data matrix B by the coefficient (loading) matrix. The result of this step is the eigenvectors of the covariance matrix (scores or in other word called a principal component matrix) with the new higher variance dimension than the original dimension of data 12×18 matrix.

Step-5 Selection of components and the formation of vector features

Here comes the concept of compressing data and reducing the dimensions. If we look at the eigenvectors and eigenvalues values of the previous steps, we will note that the values of eigenvalues are quite different. We notice that the eigenvectors (scores) matrix is arranged in ascending way, so its begin from the lowest significance vector and to the highest significance. In fact, each vector with the higher eigenvalue in latent vector is the principle component of the data set and this gives the components in order of significance.

7. Result and discussion

The result of PCA calculation is 18 principal components (from PCA1 to PCA18), which are time domain and frequency domain features for 3-axis measurements. In order to choose the principal components that separate the damage events clearly, Figure 10 show PCA1 and PCA2. The colored bubbles in the figure above indicate the scenarios of the damage events. Each event we recorded has four data set of measurements, so each four bubbles refer to a cluster of one damage event, where cluster one refers to case, cluster 2 refers to drilling case and cluster 3 refers to knocking case event.

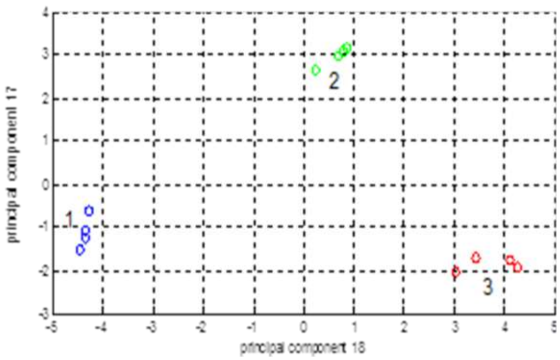

Fig. 12: PCA17 and PCA18

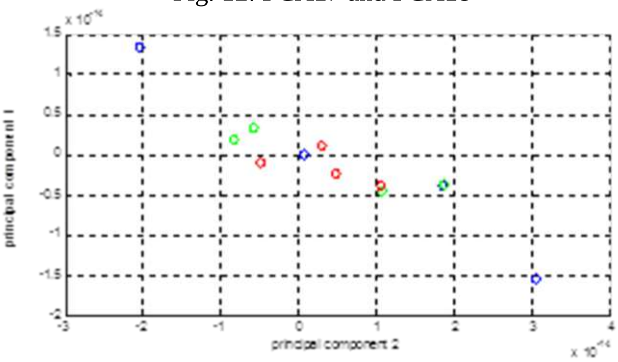

Fig. 10: PCA1 and PCA2

But it is noted that the data are distributed in a manner that cannot be separated or distinguish between them, and gives a state that the primary principal components have very little variance which does not give us the nature of the damage applied on the pipeline. We completed the testing process for others principals components gradually and we choose PCA9 and PCA13, PCA14 and PCA15, also give the same results.

The most significant PCA was PCA18,PCA17 and PCA16 with a percentage approximately equal to 67%, 28% and 3% respectively from the original data, this is make up 98 % from original data. And we can ignore the others PCA which have small variance and don't result to lose much information from original data. And we can ignore the others PCA which have small variance and don't result to lose much information from original data. Figure 11 show PCA16, PCA17 and PCA18.

If we ignore PCA16 which is make up 3% from the original data, the result will be less significant than in Figure 10, this is illustrated in Figure 12. It is clear that the cluster of damaging event are completely separated. This shows that the process of classification and discrimination between scenarios of damaging events can be achieved through our proposed methods .

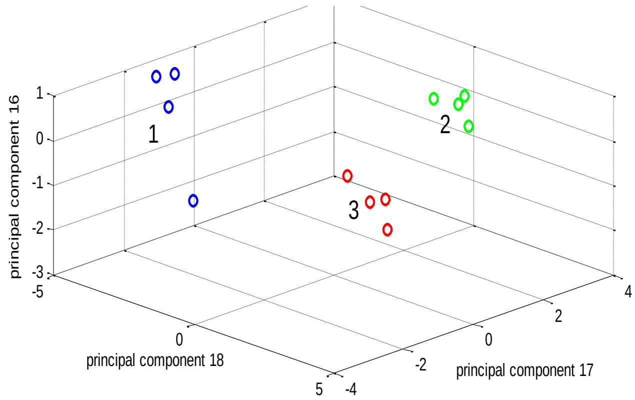

Fig. 11: PCA16,PCA17 and PCA18

8. Conclusions

In this paper, a wireless sensor network in monitoring and pre-warning system for security of oil pipeline based on statistical features and PCA is proposed. The detection model was created with the time domain and frequency domain feature extracted from the vibration signals, which are a result of damage activities on the carbon steel oil pipeline. The vibration signals was captured by accelerometers of wireless sensor nodes along the pipeline. These signals are relayed wirelessly through coordinator node to the PC. In time domain, these features are mean, root mean square, standard deviation and peak-to-peak amplitude. In frequency domain the energy and entropy of the signal spectrum in frequency are used as features. PCA has been applied to these features to analyze the data and reduce the dimension space, and as a result, diagnosis damage on the pipeline is done.

Using PCA with time domain features gives limited detection results comparing to using both time and frequency domain features. PCA was used as a way to identify patterns in data and to express data in a manner that highlighted the similarities and differences between them, in addition to reducing the number of dimensions in data without much loss of information. The results showed that the proposed system alongside with the proposed analysis is able to monitor the healthy state of the pipeline and detect several damaging events.

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تصنيف تحليل المكونات الرئيسية لإشارات الاهتزاز في نظام مراقبة خط أنابيب النفط القائم على شبكات الاستشعار اللاسلكية

وليد فواز شريف نشيد فيصل موسى
قسم هندسة السيطرة والنظم ، الجامعة التكنولوجية

المستخلص :
إن استخدام تقنية شبكة الاستشعار اللاسلكية في هيكلية تطبيقات المراقبة الصحية يؤدي إلى توليد كمية كبيرة من البيانات. لفحص هذه البيانات واستخراج المعلومات المفيدة ، يجب تطبيق تحليل شامل للبيانات. في هذا البحث ، تم اقتراح شبكة مستشعر لاسلكي (WSNs) لنظام مراقبة خط أنابيب النفط بالطريقة المقترحة لكشف الحدث وتصنيفه. تعتمد الطريقة على تحليل المكون الرئيسي (PCA). تم تطبيقه على الميزات المستخرجة من إشارات الاهتزاز الخاصة بمراقبة خط الأنابيب. يتم جمع إشارات الاهتزاز هذه أثناء تطبيق أحداث ضارة (الطرق والحفر) على خط أنابيب النفط. يتم تطبيق PCA على الميزات المستخرجة من النطاق الزمني ومجال التردد. تظهر النتائج أن هذه الطريقة قادرة على اكتشاف وجود الضرر وأيضاً للتمييز بين المستويات المختلفة للأحداث الضارة المطبقة على خط الأنابيب.

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Development cryptography protocol based on Magic Square and Linear Algebra System

Abdul Monem S. Rahma **Doaa Ayad Jabbar**

Department of Computer Science , University of Technology
monem.rahma@yahoo.com doaa.ayad87@gmail.com

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Abstract

Information security cryptographic protocols are very important in the modern era due to the development and advanced technology in internet applications and networks communications. In this paper, we proposed a protocol to save information from passive attacks when sending between two nodes over an insecure channel. This proposed protocol relies on magic square of size 3*3, linear equation system and finite field.

Keywords: magic square, linear algebra system, Gaussian elimination, and finite field .

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1.Introduction

information are sent from computer to another across an unsafe channel. This channel could be target to attack lead to steal the data or altered. For this reason, we require the sheltering of data transmitted through insecure channels. [1]. There are much methods or algorithms to encryption data by using magic square for example In 2014, A. Dharini, R.M. Saranya Devi, and I. Chandrasekar have introduced a new approach for secure data transmission through the cloud environment and sharing networks as well as during the Secure Socket Layer (SSL) by the RSA combined with magic square, to provide additional security layer to the cryptosystem[2].

Magic squares grew with "mathematics-based games like puzzles, Rubik and Sudoku games. a magic square is a n×n matrix (where n is the number of cells on each side) filled with distinct positive integers in the range 1,2,..,n^2 such that all cells are different from each other and the sum of the integers in each row, column and diagonal is equal. The sum is called the magic constant or magic sum of the magic square [3].

The Finite ,or Galois field, in mathematics, is a field that include a limited number of elements. It is a group on which the application of multiplication, addition, subtraction, and division are defined with satisfying the rules of arithmetic known as the field axioms [4]. The finite fields of prime order in which for each prime number p, denoted by GF(p). The integers modulo p is a finite field of order p and it is having the numbers$\{0,1,2,\ldots,p-1\}$ with addition and multiplication performed modulo p [5].

The Linear Algebra is a set of equations that give a unique solution. If those involved equations are linear then that collection is known as a system of linear equations. L.A.S are divided into two main classes: direct and indirect[6]. Each category include several elimination methods used for solving equations, one of these methods is the Gaussian elimination method which is a direct method for solving a system of linear equations[7].

2.The Proposed Protocol to Encryption Data

Until now no fixed or exclusive algorithm to build or construct all kind of magic squares. different approach for constructing magic squares have been developed through the ages. In our work, we used the protocol relies on the magic square. In this section explain the algorithm of encryption information. Algorithm 1 explain encryption data by magic square.

2.1: Encryption Algorithm

Input: Plaintext(in numerical data) and key.
Output: Ciphertext(summation of the magic square).

1. Divided plaintext(P) into blocks and length of each block equal six.
2. Define number of rounds(N), key and 3×3encryption mask that is part of the field GF(p)

M1	M2	M3
M4	M5	M6
M7	M8	M9

3. Build magic square of the size 3x3 and nine locations as follows:

In magic square select some locations of the key elements{k1,k2,k3} are (β_1, β_2, and β_5) and other locations of plaintext are($\beta 3$, β_4, β_6, $\beta 7$, $\beta 8$, and $\beta 9$), this sort gives a unique solution as follows:

Magic square		
β_1	β_2	β_3
β_4	β_5	β_6
β_7	β_8	β_9

key and plain text positions		
K1	K2	P1
P2	K3	P3
P4	P5	P6

4. Multiplication the magic square with encryption mask according to finite field rules.

K1	K2	P1
P2	K3	P3
P4	P5	P6

5. Calculate magic sum(MS) that result from previous step. By using the following equations:

$$\beta 1+\beta 2+\beta 3=\text{sum1} \quad (1)$$

$$\beta 7+\beta 8+\beta 9i=\text{sum2} \quad (2)$$

$$\beta 1+\beta 4+\beta 7=\text{sum3} \quad (3)$$

$$\beta 3+\beta 6+\beta 9=\text{sum4} \quad (4)$$

$$\beta 1+\beta 5+\beta 9=\text{sum5} \quad (5)$$

$$\beta 3+\beta 5+\beta 7=\text{sum6} \quad (6)$$

6. C_i = sum1,sum,2, …, sum6 and the last known values of k1,k2,k3
7. *end*

The decryption of the data used algorithm 2 as follows.

2.2: Decryption Algorithm

Input: Ciphertext(summation of the magic square) and N.

Output: Plaintext(in numerical data).

1. Build Augmented matrix(A) of linear equation system of magic square dependend on equations 1,2,….,6 as follows:

β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	β_9	
1	1	1	0	0	0	0	0	0	SUM1
0	0	0	0	0	0	1	1	1	SUM2
1	0	0	1	0	0	1	0	0	SUM3
0	0	1	0	0	1	0	0	1	SUM4
1	0	0	0	1	0	0	0	1	SUM5
0	0	1	0	1	0	1	0	0	SUM6

2. Update the summation of the matrix(A) as follows:

$$\text{sum}_i = \begin{cases} \text{sum}_i - k_1 & \text{if } \beta 1 = 1 \\ \text{sum}_i - k_2 & \text{if } \beta 2 = 1 \\ \text{sum}_i - k_3 & \text{if } \beta 5 = 1 \end{cases}$$

3. Reduce matrix(A), where remove columns(β_1, β_2, and β_5) and resort the matrix as follows:

β_3	β_4	β_6	β_7	β_8	β_9	
1	0	0	0	0	0	SUM1
0	1	0	1	0	0	SUM3
1	0	1	0	0	1	SUM4
1	0	0	1	0	0	SUM6
0	0	0	1	1	1	SUM2
0	0	0	0	0	1	SUM5

4. The matrix in step 3, solved by Gaussian elimination and relies on rules of the finite field , the result of this step as follows:

β_3	β_4	β_6	β_7	β_8	β_9	
1	0	0	0	0	0	P1
0	1	0	0	0	0	P2
0	0	1	0	0	0	P3
0	0	0	1	0	0	P4
0	0	0	0	1	0	P5
0	0	0	0	0	1	P6

5. Plaintext is (p1, p2, …, pN).

3.Example:

Plaintext is: This is just alittile test of my method Lets as try a couple new line characters

Ciphertext:

algorithm	Time encryption (M.S. ms)	Time decryption (M.S. ms)
Original-AES (Rijndael) 10 round	**1.166557**	**2.128282**
The proposal algorithm	**0.047686**	**0.059184**

4-.Analysis Study

This section explains the method of cryptanalysis.

4.1 Brute Force Attack

Brute force attack is a cryptanalytic attack used to attempt to decrypt for any ciphertext by trying all possible keys until the correct one is found. According to a brute force attack, the possibility of the key is 2^n . In our work n=3* no. of block.

4.2 Dictionary Attack

This type of attack depends on the block size where can apply to any type of block cipher for any design. If the block size is L then dictionary attack require 2^L different plain text to decrypt arbitrary message under uthe nknown key. In our work L=6 * no. of block.

5.Conclusion

In this work, we proposed an efficient cryptography algorithm to save data from attack. The algorithm is implemented for encryption and decryption by using magic square of size 3x3, linear algebra system and finite field . Also, this algorithm relies on divided data into blocks and sort with the key in a special location of magic square to give a ciphertext represented the summation of each row, column, and diagonals of the magic square, and using linear algebra system to retrieve the plain text.

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6. Suggestion Research

For future work, we can use a magic square with size 4x4 or exchange the binary field GF(2^n) instead of prime field , or used more rounds to encryption

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تطوير بروتوكول تشفير باستخدام المربع السحري ونظام المعادلات الخطية

عبد المنعم صالح رحمة   دعاء اياد جبار

قسم علوم الحاسبات ، الجامعة التكنولوجية

المستخلص:

بروتوكولات التشفير الأمنية المعلومات مهمة جدا في العصر الحديث بسبب التطور والتكنولوجيا المتقدمة في تطبيقات الإنترنت وشبكات الاتصالات. في هذا البحث ، اقترحنا بروتوكولًا لحفظ المعلومات من الهجمات السلبية عند الإرسال بين عقدتين على قناة غير آمنة. يعتمد هذا البروتوكول المقترح على المربع السحري لحجم 3 * 3 ، ونظام المعادلات الخطية والحقل المتناهي.

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Bayesian adaptive Lasso Tobit regression

Haider Kadhim Abbas **Rahim Jabbar Thaher**

Department of Statistics, College of Administration and Economics, University of Al-Qadisiyah, Iraq.
haiderhilali@gmail.com **rahim.alhamzawi@qu.edu.iq**

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Abstract :
In this paper, we introduce a new procedure for model selection in Tobit regression, we suggest the Bayesian adaptive Lasso Tobit regression (BALTR) for variable selection (VS) and coefficient estimation. We submitted a Bayesian hierarchical model and Gibbs sampler (GS) for our procedure. Our proposed procedure is clarified by means of simulations and a real data analysis. Results demonstrate our procedure performs well in comparison to further procedures.

Keywords: Tobit regression; Bayesian adaptive Lasso Tobit regression (BALTR); Variable selection (VS).

1.Introduction:

Tobit regression procedure (Tr) is proposed as a statistical model by Tobin (1958). This model is also known as left truncated regression. Tr has become important in many real-world applied sciences, such as econometric, agriculture, ecology, the environment and genetics. It is an excellent procedure to evaluate the relation along with outcome variable and a group of explanatory variables.

One of the most mainly important troubles in the regression when the number of explanatory variables is so large. It is then difficult to see which variables actually important. In addition to several problems appear when the statistical researchers are use some explanatory variables that are not important in regression. This leads to a regression model that will be unstable and so weak concerning of prediction. The selection process provides a perfect agent for estimating the parameters as well as the identification of important variables (Griffin and Brown, 2010). There occur several varieties of strategies for investigators to use in handling high dimensional data (very large of explanatory variables), including VS procedures, and data reduction techniques. Prior analysis has found that, in the existence of high dimensional data, these VS procedures can produce estimates with inflated errors for the coefficients (Hastie, Tibshirani, & Friedman, 2009). Some of the technique models that have proved beneficial in the condition of high dimensional data, these models known as regularization.

In 1996, Tibishrani suggested a procedure for VS and parameter estimation in linear models known be as Lasso model (Least Absolute Shrinkage and Selection Operator model). A lot of work has been devoted to the development of diverse of Bayesian organizational procedures for making VS in linear models. In 2006, Zou proposed the adaptive Lasso, who upgraded the Lasso way proposed by Tibshirani, permitting different penalty parameters to different regression coefficients. Zou proved that his proposed procedure had the characteristics of Oracle mentioned in Fan and Bing (2004) that Lasso does not have. Specifically, Zou indicates that his proposed procedure adopts the correct form of non-zero coefficients with the probability that he tends to one. Park and Casella suggested in 2008 the Lasso procedure based from a Bayesian point of sight. Likewise, Mallick and Yi (2014) suggested a new procedure known to be as new Bayesian Lasso regression for VS and coefficient estimation in linear regression.

In general, the last procedure observed results display that the Mallick procedure applied well compares with other Bayesian and non-Bayesian regression procedures.

The above results and good results reported in Mallick procedure motivate us to suggest a new Bayesian regression procedure. Subsequently, we submitted a Bayesian hierarchical for BALTR, and proposed a new Gibbs sampler (GS) for BALTR, that is set up on a theoretical derivation of the Laplace density (LD). Next, we implemented several simulated examples and analyzed real data by using BALTR with four Tobit regression procedures to compare the best results. These procedures include Tr, Bayesian Tobit regression (BTr), Tobit median regression, and BALTR. Both simulation and real analysis proved that BALTR results are excellent, and this procedure may be is a best of current procedures being compared.

2.Methods:

The Tobit regression is applied to estimate the relevance among an outcome variable (y_i) and explanatory variables ($\boldsymbol{X}$). Tobit regression assumes that there is a latent variable (y_i^*) depends linearly on the parameters ($\boldsymbol{\beta}$) which determines relevance between ($\boldsymbol{X}$) and (y_i^*), the formula of outcome variable is

$$y_i = \begin{cases} y_i^* & if \quad y_i^* > 0 \\ 0 & if \quad y_i^* \le 0 \end{cases}$$

$$\boldsymbol{y}^* = \boldsymbol{X\beta} + \boldsymbol{\varepsilon} \quad \cdots (1)$$

$$\boldsymbol{y}^* = (y_1, \cdots, y_n),$$

$$\boldsymbol{X} = \begin{bmatrix} 1 & x_{11} & \cdots & x_{1k} \\ \vdots & & \vdots & \vdots \\ 1 & x_{n1} & \cdots & x_{nk} \end{bmatrix},$$

$$\boldsymbol{\beta} = (\beta_0, \beta_1, \cdots, \beta_k) \,,$$

$$\boldsymbol{\varepsilon} = (\varepsilon_1, \cdots, \varepsilon_n),$$

$$\varepsilon_i \sim N(0, \sigma^2)$$

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2.1 Bayesian adaptive Lasso Tobit regression (BALTR):

It is well known, that the Lasso procedure gives biased estimates of considerable coefficients, so it might be below the required optimal level in terms of estimation risk. In 2006, Zou evidenced that the Lasso opts the incorrect model with non-fade the probability, despite the sample size and how λ is chosen. The event requires that coefficients not in the model aren't representable by coefficients in the real model. But this event is simply suffering because of the collinearity case between the coefficients. On the opposite hand, that the Lasso technique does not have Oracle properties. So, Zou suggested the adaptive Lasso technique who gives a consistent model for VS. Therefore, we consider BALTR procedure in this paper, the adaptive Lasso enjoys the oracle properties by utilizing the adaptably weighted Lasso penalty parameter, and leads to a near minimax optimum estimator. Additionally, the adaptive Lasso technique needs to initial estimates of the regression coefficients, when a sample size is less than of the covariates number, which is mostly not available in the high dimensional data. The estimator of adaptive Lasso is given by

$$\widehat{\boldsymbol{\beta}}_{alasso} = \underset{\boldsymbol{\beta}}{\operatorname{argmin}}(\boldsymbol{y}-\boldsymbol{X\beta})'(\boldsymbol{y}-\boldsymbol{X\beta}) + \sum_{j=1}^{k}\lambda_j\left|\beta_j\right| \quad where\ \lambda_j \geq 0$$

where varied penalty parameters are utilized for the regression coefficients. Surely, for the not important explanatory variables, we must place larger penalty λ_j on their matching coefficients.

We propose a BALTR procedure in this paper for coefficient estimation and VS. We submit a new practice of the adaptive Lasso form by using the scale mixture of a uniform represent of the LD. Following (Mallick& Yi, 2014), the Laplace representation can adaptive as

$$\frac{\lambda_j}{2}e^{-\lambda_j|\beta_j|} = \int_{s_j>|\beta_j|}\frac{1}{2s_j}\frac{\lambda_j{}^2}{\Gamma 2}\, s_j{}^{2-1}e^{-\lambda_j s_j}\, ds_j \cdots (2)$$

$$\frac{\lambda_j}{2}e^{-\lambda_j|\beta_j|} = \int_{s_j>|\beta_j|}\frac{\lambda_j{}^2}{2}\, e^{-\lambda_j s_j}\, ds_j,\ \lambda_j > 0$$

In this paper, we modify the above formula as follows:

Let $v_j = \lambda_j s_j \quad \Rightarrow \quad dv_j = \lambda_j ds_j$ then

$$\frac{\lambda_j}{2}e^{-\lambda_j|\beta_j|} = \frac{\lambda_j}{2}e^{-|\lambda_j\beta_j|}$$

$$= \int_{v_j>|\lambda_j\beta_j|}\frac{\lambda_j}{2v_j}\frac{\lambda_j^2}{\Gamma 2}\left(\frac{v_j}{\lambda_j}\right)^{2-1} e^{-v_j}\frac{1}{\lambda_j}dv_j$$

$$= \int_{v_j>|\lambda_j\beta_j|}\frac{\lambda_j}{2}e^{-v_j}dv_j \cdots (3)$$

In practice, this formula produces more tractable and efficient Gibbs sampler than the formula in 2.

2.2 Model Hierarchy and Prior Distributions of BALTR:

By using equation (1) and equation (3), the Bayesian hierarchical model can be formulated as follows:

$$\boldsymbol{y}^*|\boldsymbol{X},\boldsymbol{\beta},\sigma^2 \sim N_n(\boldsymbol{X\beta},\sigma^2 I_n) \ldots (4)$$

$$\boldsymbol{\beta}|\lambda \sim \prod_{j=1}^{k}\text{Uniform}(-\frac{1}{\lambda_j},\frac{1}{\lambda_j}) \ldots (5)$$

$$\boldsymbol{v} \sim \prod_{j=1}^{k}\text{Exp}(1) \ldots (6)$$

$$\sigma^2 \sim \text{Inverse Gamma}(a,b) \ldots (7)$$

$$\lambda_j \sim \text{Gamma}(f,g) \ldots (8)$$

where $\boldsymbol{v} = (v_1,\cdots,v_k)$

2.3 Full Conditional Posterior Distributions of BALTR:

Firstly, we can express the joint posterior distribution of all our procedure parameters as follows

$$\pi(\boldsymbol{\beta},\boldsymbol{v},\lambda,\sigma 2|\boldsymbol{y}^*,\boldsymbol{X}) \propto \pi(\boldsymbol{y}^*|\boldsymbol{X},\boldsymbol{\beta},\sigma 2)\pi(\boldsymbol{\beta}|\lambda)\pi(\boldsymbol{v})\pi(\lambda_j)\pi(\sigma 2)$$

Under the above posterior distribution, the posterior distribution of $\boldsymbol{\beta}$ is

$$\pi(\boldsymbol{\beta}|\boldsymbol{y}^*,\ \boldsymbol{X},\ \lambda) \propto \pi(\boldsymbol{y}^*|\boldsymbol{X},\boldsymbol{\beta},\sigma^2).\pi(\boldsymbol{\beta}|\lambda)$$

$$\propto exp\left\{-\tfrac{1}{2\sigma 2}(\boldsymbol{y}^*-\boldsymbol{X\beta})'(\boldsymbol{y}^*-\boldsymbol{X\beta})\right\}\prod_{j=1}^{k}I\left\{\left|\beta_j\right|<\tfrac{v_j}{\lambda_j}\right\}$$

$$\propto exp\left\{-\tfrac{1}{2\sigma 2}(-2\boldsymbol{y}^{*'}\boldsymbol{XB}+\boldsymbol{\beta}'\boldsymbol{X}'\boldsymbol{X\beta})\right\}\prod_{j=1}^{k}I\left\{\left|\beta_j\right|<\tfrac{v_j}{\lambda_j}\right\}$$

$$\propto exp\left\{-\tfrac{1}{2\sigma 2}(-2\boldsymbol{y}^{*'}\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-1}(\boldsymbol{X}'\boldsymbol{X})\boldsymbol{\beta}+\boldsymbol{\beta}'\boldsymbol{X}'\boldsymbol{X\beta})\right\}\prod_{j=1}^{k}I\left\{\left|\beta_j\right|<\tfrac{v_j}{\lambda_j}\right\}$$

$$\propto exp\left\{-\tfrac{1}{2\sigma 2}\left(-2\widehat{\boldsymbol{\beta}}'\boldsymbol{X}'\boldsymbol{X\beta}+\boldsymbol{\beta}'\boldsymbol{X}'\boldsymbol{X\beta}\right)\right\}\prod_{j=1}^{k}I\left\{-\tfrac{v_j}{\lambda_j}<\beta_j<\tfrac{v_j}{\lambda_j}\right\}$$

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$$\boldsymbol{\beta}|\boldsymbol{y}^*,\boldsymbol{X},\lambda \sim N_k(\widehat{\boldsymbol{\beta}}_{OLS},(\boldsymbol{X}'\boldsymbol{X})^{-1}\sigma 2)$$
$$\text{x}\prod_{j=1}^{k} I\left\{-\frac{v_j}{\lambda_j}<\beta_j<\frac{v_j}{\lambda_j}\right\}\cdots(9)$$

As well, the posterior distribution of v_j is

$$\pi(\boldsymbol{v}|\boldsymbol{y}^*,\boldsymbol{X},\boldsymbol{\beta},\lambda) \propto \pi(\boldsymbol{v})I\left\{v_j>\left|\lambda_j\beta_j\right|\right\}$$
$$\propto \prod_{j=1}^{k} e^{-v_j}I\left\{v_j>\left|\lambda_j\beta_j\right|\right\}$$
$$\boldsymbol{v}\sim\prod_{j=1}^{k} Exponential(1)\,I\left\{v_j>\left|\lambda_j\beta_j\right|\right\}\cdots(10)$$

Likewise, the posterior distribution of σ^2 is

$$\pi(\sigma^2|\boldsymbol{y}^*,\boldsymbol{X},\boldsymbol{\beta}) \propto \pi(\boldsymbol{y}^*|\ \boldsymbol{X},\boldsymbol{\beta},\sigma^2)\ \pi(\sigma^2)$$
$$\propto (\sigma^2)^{-\frac{n}{2}}exp\left\{-\frac{1}{2\sigma^2}(y^*-X\beta)'(y^*-X\beta)\right\}(\sigma^2)^{-a-1}exp\left\{-\frac{b}{\sigma^2}\right\}$$
$$\sigma^2|\boldsymbol{y}^*,\boldsymbol{X},\boldsymbol{\beta}\sim$$
$$\text{InvGamma}\left(\frac{n}{2}+a,\frac{1}{2}(y^*-X\beta)'(y^*-X\beta)+b\right)\cdots(11)$$

Lastly, the posterior distribution of λ is

$$\pi\left(\lambda_j\middle|\beta_j\right) \propto \pi\left(\beta_j\middle|\lambda_j\right).\pi(\lambda_j)$$
$$\pi\left(\lambda_j\middle|\beta_j,v_j\right) \propto \pi(\lambda_j)\lambda_j I\left\{\lambda_j<\frac{v_j}{\left|\beta_j\right|}\right\}$$
$$\propto \lambda_j^{(f+1)-1}\,exp\{-g\lambda_j\}\,I\left\{\lambda_j<\frac{v_j}{\left|\beta_j\right|}\right\}$$
$$\propto \text{Gamma}(f+1,g)I\left\{\lambda_j<\frac{v_j}{\left|\beta_j\right|}\right\}\dots(12)$$

Where the $I(.)$ is an indicator function in equation (9) and equation (12).

2.4 Computation:

In the computation section, we outline our Gibbs sampler as follows

- Updating $\boldsymbol{\beta}$:

We simulate the β_j from a truncated multivariate normal distribution in equation (9), the mean of this distribution is $(\widehat{\boldsymbol{\beta}}_{OLS})$ and the variance is $((\boldsymbol{X}'\boldsymbol{X})^{-1}\sigma 2)$.

- Updating $\boldsymbol{v}$:

We simulate the v_j from the left truncated exponential distribution in equation (10), by applying the inversion process, this simulate can be completed as follows:

1. Simulate $v_j{}^*$ from standard exponential distribution.
2. Set $v_j=v_j{}^*+\left|\lambda_j\beta_j\right|$

- Updating σ^2:

We simulate the σ^2 from Inverse Gamma distribution in equation (11), the shape parameter of this distribution is $(\frac{n}{2}+a)$ and the rate is

$$\left(\frac{1}{2}(y^*-X\beta)'(y^*-X\beta)+b\right)$$

- Updating λ :

We simulate λ_j from truncated Gamma distribution, the shape parameter of this distribution is $(f+1)$ and the rate parameter is (g).

3.Simulation Studies:

The performance of our procedure is evaluates in a simulation study in which the procedure for a BALTR is compared with, Tr procedure through using R language within package AER (Christian Kleiber, Achim Zeileis 2017), Bayesian Tobit regression procedure (BTr) through using R language within package MCMCpack (Jong Hee Park, 2018), and Bayesian Analysis of Quantile Regression Models (Bayesian Tobit quantile regression BTqr,and Bayesian adaptive Lasso Tobit quantile regression BALTqr); and Tau=0.5 by estimating the median through using R language within package Brq (Alhamzawi, R., & Alhamzawi, M. R., 2017) . For comparison, we draw 11,000 iterations of the GS, the first 1000 were ruled out as burn-in. The procedures are evaluated based on the median of mean absolute deviations (MMAD). The formula of MMAD is

$$\text{MMAD}=\text{median}(\text{mean}(\left|\boldsymbol{X}\widehat{\boldsymbol{\beta}}-\boldsymbol{X}\boldsymbol{\beta}^{true}\right|))$$

where $\widehat{\boldsymbol{\beta}}$ is the posterior mean of $\boldsymbol{\beta}$.

3.1 Independent and identically distributed random errors:

Here, simulation examples consider three cases (dense case, sparse case, and very sparse case), eight predictors $x_1.\cdots.x_8$ were simulated independently from a multivariate normal distribution with mean 0, and two values of the variance σ^2, the σ^2 is 1and 4.

3.1.1 Simulation example 1:

This example considers a dense case model, the true regression coefficients is

$$\boldsymbol{\beta}=(0,\underbrace{0.75,\dots,0.75}_{8})'$$

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The response variable was generated according to the model

$$y_i^* = \beta_0 + 0.75x_{1i} + 0.75x_{2i} + 0.75x_{3i} + 0.75x_{4i} + 0.75x_{5i} + 0.75x_{6i} + 0.75x_{7i} + 0.75x_{8i} + \varepsilon_i$$

We simulate 100 observations and $\beta_0 = 0$, the pair wise correlations between x_i and x_j is $0.5^{|i-j|}$.

Method	σ^2	MMAD	SD
BALTR	1	0.36193	0.10830
Tr		0.36779	0.12457
BTr		0.39495	0.15897
BTqr		0.41936	0.19493
BALTqr		0.38301	0.13498
BALTR	4	0.56246	0.11160
Tr		0.57281	0.12713
BTr		0.63695	0.19218
BTqr		0.63797	0.17940
BALTqr		0.60005	0.11972

Table 1: MMAD and SD for the dense case example

3.1.2 Simulation example 2:

This example considers a sparse case model, the setup is the same in simulation 1, except the number of observations is 150, and the true regression coefficients is

$$\boldsymbol{\beta} = (0, 2, 1, 0, 0, 2, 0, 0, 0)'$$

The response variable was generated according to the model

$$y_i^* = \beta_0 + 2x_{1i} + x_{2i} + 2x_{5i} + \varepsilon_i$$

Method	σ^2	MMAD	SD
BALTR	1	0.25350	0.08704
Tr		0.27300	0.08761
BTr		0.28327	0.10003
BTqr		0.29655	0.11001
BALTqr		0.25856	0.09530
BALTR	4	0.50916	0.12261
Tr		0.53753	0.14044
BTr		0.57080	0.17250
BTqr		0.59050	0.18781
BALTqr		0.51686	0.12951

Table 2: MMAD and SD list for the simulation 2

3.1.3 Simulation example 3:

This example considers a very sparse case model with high correlation. We simulate 200 observations and the pair wise correlations between x_i and x_j equals to 0.75**,** and the true regression coefficients is

$$\boldsymbol{\beta} = (0, 4, 0, 0, 0, 0, 0, 0, 0)'$$

The response variable was generated according to the model

$$y_i^* = \beta_0 + 4x_{1i} + \varepsilon_i$$

and intercept coefficient is 0 .

The response variable was generated according to the model

$$y_i^* = \beta_0 + 4x_{1i} + \varepsilon_i$$

and intercept coefficient is 0 .

Method	σ^2	MMAD	SD
BALTR	1	0.21634	0.06547
Tr		0.23154	0.06120
BTr		0.23800	0.07200
BTqr		0.26720	0.07970
BALTqr		0.22145	0.07329
BALTR	4	0.42391	0.12948
Tr		0.45243	0.11599
BTr		0.48257	0.12186
BTqr		0.52948	0.12676
BALTqr		0.43534	0.11626

Table 3: MMAD and SD list for the simulation 3

3.2 Simulation example 4:

This example considers a Difficult case model. We simulate 100 observations, four predictors $x_1.\cdots.x_4$ were simulated independently from a multivariate normal distribution with mean zero and variance σ^2. We consider three values of σ^2 (1, 4 and 9), and the pair wise correlations between x_i and x_j equal to (-0.4)**,** the true regression coefficients is

$$\boldsymbol{\beta} = (0, 5.5, 5.5, 5.5, 0)'$$

The response variable was simulated according to the model

$$y_i^* = 0 + 0.55x_{1i} + 0.55x_{2i} + 0.55x_{3i} + 0.55x_{4i} + \varepsilon_i$$

Method	σ^2	MMAD	SD
BALTR	1	0.22283	0.13737
Tr		0.23864	0.13851
BTr		0.23633	0.14456
BTqr		0.30271	0.17220
BALTqr		0.27722	0.15230
BALTR	4	0.49759	0.32372
Tr		0.51872	0.31529
BTr		0.53169	0.35028
BTqr		0.59351	0.40493
BALTqr		0.52709	0.33300
BALTR	9	0.62072	0.42722
Tr		0.65780	0.39645
BTr		0.67667	0.47057
BTqr		0.77867	0.48305
BALTqr		0.68023	0.34077

Table 4: MMAD and SD list for the simulation 4

From above tables 1, 2, 3 and 4, we noted that the BALTR procedure performs better than the other procedures in terms the median of mean absolute deviations.

3.3 Simulation example 5 (Heterogeneous random errors):

In this section, errors are considered to demonstrate the performance of our proposed procedure for VS. We simulated 100 observations from the model

$$y_i^* = x_i'\boldsymbol{\beta} + (1 + x_{3i})\varepsilon_i,$$
$$\varepsilon_i \sim \mathrm{N}(0,1) \text{ and } \boldsymbol{\beta} = (0,1,1,1,1,,0,0,0,0,0)'$$
$$\text{where } x_{1i} \sim \mathrm{N}(0,1),$$
$$x_{3i} \sim \mathrm{Uniform}[0,1],$$
$$x_{2i} = x_{1i} + x_{3i} + z_i, z_i \sim \mathrm{N}(0,1)$$

this process is often used to simulate data in the VS context (example of Wu and Liu, 2009 and Li et al., 2010). In this simulation, added 5 independent standard normal noise variables, $x_4 \cdots x_8$, were simulated. In this paper, we set $y_i = \max\{y_i^*, 0\}$

Method	MMAD	SD
BALTR	0.26923	0.06925
Tr	0.27969	0.06596
BTr	0.27911	0.07437
BTqr	0.32919	0.07278
BALTqr	0.29920	0.06916

Table 8: MMAD and SD list for the simulation 5

Table (8) reports MMADs and SDs of simulation example 5. The performance of BALTR procedure is excellent compared to the other procedures (Tr, BTr, BTqr, BALTqr).

4.Real Data Analysis:

In data analysis section, we implement our proposed procedure on wheat production data, we apply the four Tobit regression procedures in this data to compare in terms of the coefficient's estimation accuracy. The real data used for this study is taken from the national program for the development of wheat cultivation in Iraq - Qadisiyah governorate branch (2017). This real data contains 584 observations and are based on 10 explanatory variables. The outcome of interest in this dataset is (Percentage increase of wheat yield per dunam "2500 m^2").

The other ten variables (covariates) include fertilize the field with Urea (numeric variable coding the quantity of fertilizer in kilogram; "U"), the date of sowing wheat seeds (numeric variable coding date: 1 the ideal date, 2 early date, 3 late date; "Ds"), the quantity of sowing wheat seeds (numeric variable coding the quantity of sowing seeds in kilogram; "Qs"), laser field leveling technique (numeric variable coding date: 2 if there are used this technique; 1 otherwise; "LT"), fertilize the field with compound fertilizers "NPK" (numeric variable coding the quantity of fertilizer in kilogram; "NPK"), seed sowing machine technique (numeric variable coding date: 2 if there are used this technique; 1otherwise; "SMT"), planting successive mung bean crops (numeric variable coding type: 2 planting mung bean, 1 otherwise; "SC"), used herbicide for weed control (numeric variable coding the quantity of herbicide in milliliter; "H"), high Potassium fertilizer "Potash" (numeric variable coding the quantity of fertilizer in kilogram; "K") and Micro-Element fertilizer (numeric variable coding the quantity of fertilizer in gram; "ME").

Method	MSE
BALTR	0.4617
Tr	0.4784
BTr	0.4795
BTqr	0.4724
BALTqr	0.4685

Table 9: wheat production data analysis: Mean squared prediction errors (MSE) based on a test set with 584 observations.

Table (9) reports the mean squared errors for five Tobit regression procedures. We can observe that mean squared errors of BALTR procedure is lower than that of Tr, BTr, BTqr and BALTqr, that means BALTR procedure produces the lowest prediction errors.
that means BALTR procedure produces the lowest prediction errors.

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	β_0	U	Ds
	Estimate (25%, 95%)	Estimate (25%, 95%)	Estimate (25%, 95%)
BALTR	-0.039 (-0.402, 0.285)	0.021 (0.020, 0.023)	-0.672 (-0.749, -0.620)
Tr	-0.085 (-0.872, 0.702)	0.021 (0.014, 0.028)	-0.664 (-0.786, -0.541)
BTr	-0.082 (-0.899, 0.720)	0.021 (0.014, 0.028)	-0.666 (-0.791, -0.546)
BTqr	-1.228 (-1.815, -0.546)	0.024 (0.018, 0.031)	-0.654 (-0.806, -0.505)
BALTqr	-1.072 (-1.712, -0.263)	0.024 (0.017, 0.030)	-0.649 (-0.797, -0.498)
	Qs	LT	NPK
BALTR	-0.022 (-0.025, -0.020)	1.333 (1.012, 1.648)	0.005 (0.003, 0.007)
Tr	-0.022 (-0.035, 0.009)	1.357 (0.681, 2.034)	0.005 (-0.008, 0.017)
BTr	-0.022 (-0.035, -0.008)	1.358 (0.658, 2.035)	0.005 (-0.008, 0.018)
BTqr	-0.006 (-0.018, 0.004)	1.428 (0.459, 2.343)	-0.005 (-0.017, 0.007)
BALTqr	-0.008 (-0.022, 0.002)	1.441 (0.493, 2.181)	-0.004 (-0.016, 0.008)
	SMT	SC	H
BALTR	-0.090 (-0.409, 0.161)	0.925 (0.841, 1.003)	0.004 (0.004, 0.005)
Tr	-0.143 (-0.838, 0.553)	0.933 (0.611, 1.255)	0.004 (0.003, 0.006)
BTr	-0.148 (-0.840, 0.559)	0.931 (0.601, 1.259)	0.004 (0.003, 0.006)
BTqr	0.248 (-0.631, 1.204)	0.991 (0.651, 1.313)	0.005 (0.004, 0.007)
BALTqr	0.192 (-0.433, 1.132)	0.967 (0.622, 1.293)	0.005 (0.004, 0.007)
	K	ME	
BALTR	0.033 (0.032, 0.034)	0.006 (0.006, 0.006)	
Tr	0.033 (0.026, 0.040)	0.006 (0.005, 0.008)	
BTr	0.033 (0.026, 0.040)	0.006 (0.005, 0.008)	
BTqr	0.024 (0.014, 0.036)	0.008 (0.005, 0.010)	
BALTqr	0.025 (0.014, 0.036)	0.007 (0.005,0.0104)	

Table 10: Coefficients estimation and Credible intervals CIs (25%, 95%)

Although, our CIs in table (10) are narrower than the other methods, it is including all the estimations of other procedures.

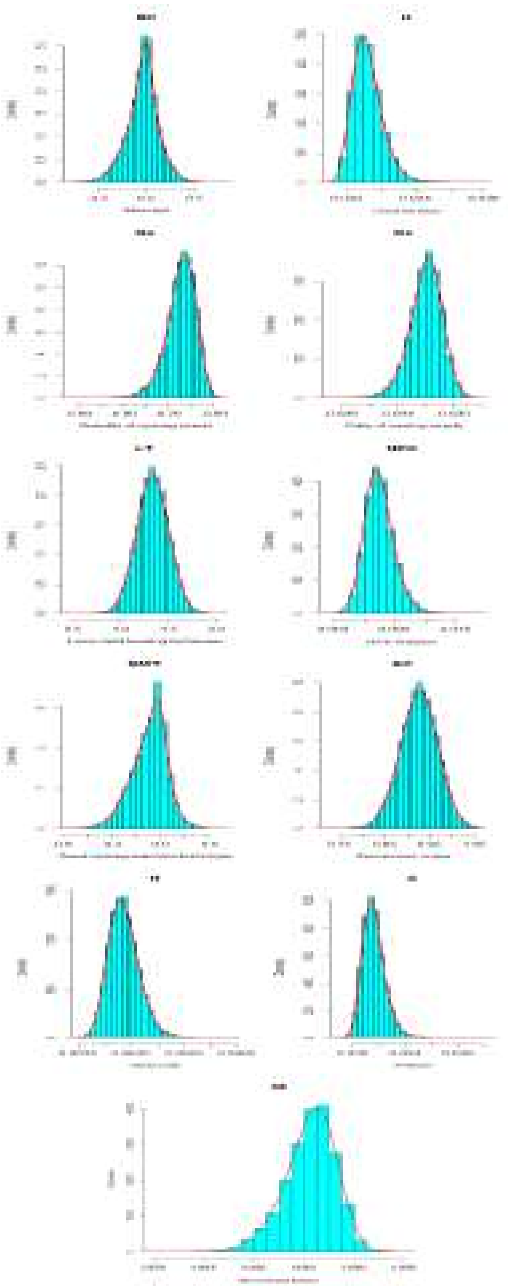


Figure 1: BALTR predictors histograms of wheat production data

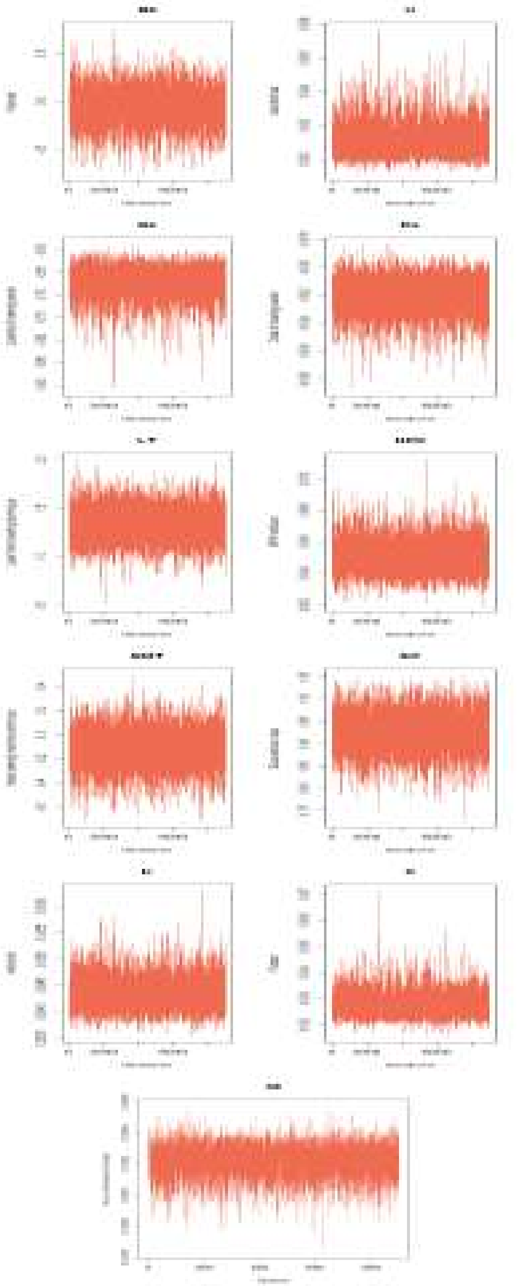
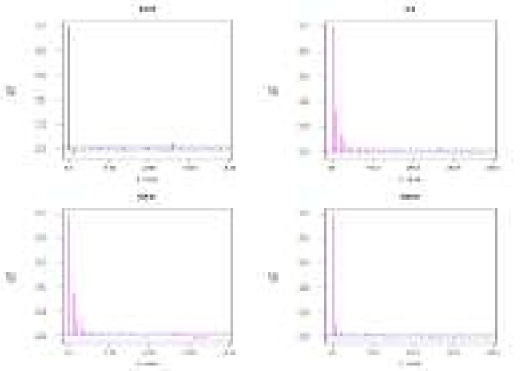


Figure 2: BALTR predictors trace plots of wheat production data



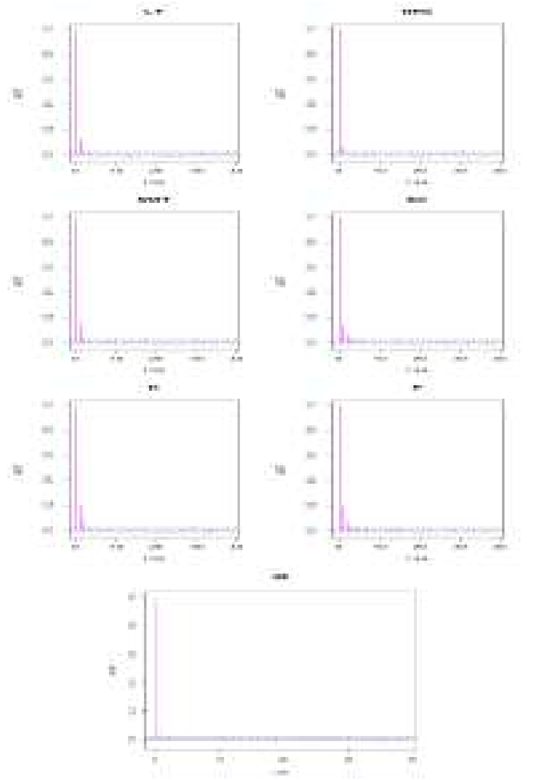


Figure 3: BALTR predictors autocorrelations of wheat production data

The predictors histograms of the wheat production based on posterior samples of 11,000 iterations are point up in figure 1, these histograms displayed that the conditional posteriors of wheat production data predictors are the preferred stationary truncated normal.

From figure 2, the trace plot indicates reasonably good convergence, and the noise does not appear to drift majorly. The chain has reached stable and the mean keeps relatively constant. it is mean that the chain is mixed well and converged.

From figure 3, the explanatory variables (covariates) in this real data are highly correlated and the mixing of the MCMC chain was reasonably good.

5.Conclusions:

This paper has introduced a new procedure for model selection of Tobit regression, we proposed BALTR for VS and coefficient estimation. Our proposed procedure depends on the scale mixture uniform as prior distribution. We advanced new Bayesian hierarchical models for BALTR. In addition, we introduced a Gibbs sampler for BALTR method. We clarified the features of the new procedure on both simulation studies and real data analysis. Results displayed that BALTR method performs very well in terms of VS and coefficient estimation.

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انحدار adaptive Lasso Tobit البيزي

حيدر كاظم عباس رحيم جبار ظاهر
كلية الادارة والاقتصاد ، جامعة القادسية

المستخلص :

في هذا البحث، نقدم طريقة جديدة لاختيار النموذج في انحدار Tobit ، حيث اقترحنا انحدار Lasso Tobit البيزي لاختيار المتغيرات (VS) وتقدير المعلمات لنموذج الانحدار. ونقدم في هذا البحث نموذجًا هرميًا جديدا و عينات Gibbs (GS) لطريقتنا المقترحة. وقد تم توضيح طريقتنا المقترحة عن طريق المحاكاة وتحليل حقيقي للبيانات. وقد ثبتت النتائج أن طريقتنا تحقق أداءً جيدًا بالمقارنة مع الطرق الأخرى.

قواعـد النشـر

- تعنى مجلة القادسية لعلوم الحاسوب والرياضيات بنشر البحوث العلمية الرصينة ذات العلاقة بعلوم الحاسبات ،الرياضيات ،الإحصاء والمعلوماتية والفيزياء الحاسوبية والتي لم تنشر أو تقدم للنشر سابقا .
- تخضع البحوث المقدمة للتقييم العلمي من لدن اختصاصين من داخل القطر وخارجه .
- يقدم البحث مطبوعا بنظام العمودين على ورق ابيض جيد قياس (A4) وبمسافة مضاعفة وبنظام الـ word حصرا وان يكون نظام office 2010 وان يكون حجم الخط المستخدم في طباعة البحث (10) ونوعه Time New Roman ماعدا العنوان واسم الباحث يكون حجم الخط (12)bold ونوعه Time New Roman ، أما الجداول والإشكال فيكون الخط bold ونوعه Time New Roman وعند وجود المعادلات في البحث يجب إضافتها باستخدام محرر المعادلات .
- على الباحث (أو الباحثيين) تقديم ملخص لبحثه باللغتين العربية والانكليزية يتضمن عنوان البحث واسم الباحث أو الباحثيين وعناوينهم بحدود (150-200) كلمة .
- على الباحث (أو الباحثيين) ادراج البريد الالكتروني ويفضل ان يكون بريد رسمي .
- استخدام الباحث (أو الباحثيين) ذات البيانات الخاصة به (اسم الباحث ، المرتبة العلمية ، جهة الانتساب ، البريد الالكتروني الرسمي) والمستخدمة في بحوثه السابقة .
- يرتب البحث كما يأتي الخلاصة ، المقدمة ، المواد وطرائق العمل ، النتائج والمناقشة ، الخلاصة باللغة الثانية تتضمن عنوان البحث، اسم الباحث ومكان عمله .
- يتم ذكر المصادر في البحث بإتباع أسلوب الترقيم حسب أسبقية ذكر المصدر وتذكر المصادر في النهاية على الوجه الأتي :
  اسم الباحث (أو الباحثيين) عنوان البحث اسم المجلة ، المجلد ، العدد ، رقم صفحتي بدء وانتهاء البحث ، سنة النشر بين قوسين .
- تنشر البحوث باللغة الانكليزية فقط وان يقدم الباحث أربع نسخ من البحث (ورقية + اقراص CD) .
- بعد الانتهاء من عملية التقييم والتصويبات وعند القبول النهائي يقدم البحث على قرص CD (office 2010 + pdf) مع نسخة ورقية نهائية .
- أن لا تزيد صفحات البحث المقدم للنشر عن عشر صفحات وبنظام العمودين وفي حالة تجاوز عدد صفحات البحث اكثر من ذلك يتم دفع خمسة الاف دينار عراقي لكل صفحة زيادة وان لايتجاوز العدد الاجمالي للبحث 20 صفحة .
- تعتمد المجلة تصنيف (Mathematics Subject Classificatio) في نشرها للبحوث العلمية .
- يقدم الباحث التصنيف المعتمد في المجلة لموضوع البحث .
- اجور التقييم والنشر للمجلة كالاتي :
  اولا :- اجور التقييم (30000) الف دينار عراقي .
  ثانيا :- اجور النشر حسب اللقب العلمي للباحث وكالاتي :
  1- المدرس المساعد والمدرس (50000) الف دينار عراقي .
  2- الاستاذ المساعد (75000) الف دينار عراقي .
  3- الاستاذ (100000) الف دينار عراقي .

ملاحظة : عند تقديم البحث يدفع الباحث مبلغ (35000) الف دينار عراقي غير قابل للرد وفي حالة قبول نشر بحثه في المجلة يدفع بقية الاجور حسب لقبه العلمي .
كما ويدفع مبلغ (10000) الاف دينار عراقي غير قابل للرد اجور استلال ، وفي حالة اعادة فحص الاستلال للبحث مرة اخرى يعاد دفع المبلغ (10000) كأجور اعادة استلال .

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المجلد (11) العدد (1) السنة (2019)

رقم الايداع في دار الكتب والوثائق في بغداد (1206) لسنة(2009)

كلية علوم الحاسوب وتكنولوجيا المعلومات - جامعة القادسية – ديوانية – جمهورية العراق

Journalcm@qu.edu.iq / البر يد الالكتروني   07829307902 / موبايل